PTEP

Prog. Theor. Exp. Phys. **2023** 043B03(47 pages)
DOI: 10.1093/ptep/ptad037

Solitons in open $N = 2$ string theory

Masashi Hamanaka [1], Shan-Chi Huang [1,⋆], and Hiroaki Kanno [1,2]

[1] *Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan*
[2] *KMI, Nagoya University, Nagoya 464-8602, Japan*
⋆E-mail: x18003x@math.nagoya-u.ac.jp

Received January 25, 2023; Revised March 10, 2023; Accepted March 10, 2023; Published March 29, 2023

The open $N = 2$ string theory is defined on the 4D space-time with the split signature $(+, +, -, -)$. The string field theory action of the open $N = 2$ string theory is described by the 4D Wess–Zumino–Witten (WZW_4) model. The equation of motion of the WZW_4 model is the Yang equation, which is equivalent to the anti-self-dual Yang–Mills equation. In this paper, we study soliton-type classical solutions of the WZW_4 model in the split signature by calculating the action density of the WZW_4 model. We find that the action density of the one-soliton solutions is localized on a 3D hyperplane. This shows that there would be codimension-one-solitonic objects, or equivalently some kind of three-branes in the open $N = 2$ string theory. We also prove that, in the asymptotic region of the space-time, the action density of the n-soliton solutions is a "non-linear superposition" of n one-solitons. This suggests the existence of n intersecting three-branes in the $N = 2$ strings. Finally we make a reduction to a $(1 + 2)$D real space-time to calculate the energy densities of the soliton solutions. We can successfully evaluate the energy distribution for the two-soliton solutions and find that there is no singularity in the interacting region. This implies the existence of smooth intersecting codimension-one branes in the whole region. Soliton solutions in the Euclidean signature are also discussed.

Subject Index B28, B35, B81

1. Introduction

The anti-self-dual Yang–Mills equation in four dimensions is of great interest in elementary particle physics and mathematics. In the Euclidean signature, it has quite important soliton solutions, instantons that are crucial to reveal the non-perturbative aspects of quantum field theory. In the split signature $(+, +, -, -)$, it has a close relationship to integrable systems. It is well known that, by imposing appropriate reduction conditions for the gauge fields, the anti-self-dual Yang–Mills equations in the split signature can be reduced to various lower-dimensional integrable equations, such as the KdV equation, the non-linear Schrödinger equation, the Toda equations, and so on [1,2]. The integrability of these equations is well understood in the geometrical framework of twistor theory [1]. Soliton solutions are mostly of codimension-one in the sense that the energy density of the one-soliton solutions is localized on a codimension-one hyperplane in the space-time (see, e.g., Refs. [3,4]).

© The Author(s) 2023. Published by Oxford University Press on behalf of the Physical Society of Japan. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted reuse, distribution, and reproduction in any medium, provided the original work is properly cited.
Funded by SCOAP³. SCOAP3 and OUP support the goals of the International Year of Basic Sciences for Sustainable Development

The anti-self-dual Yang–Mills equation is realized in string theories that are classified according to the number N of the world-sheet supersymmetry. Under the condition that the critical dimension of the target space is positive and the string world-sheet theory has an appropriate conformal symmetry, the maximal number is found to be $N = 2$ (see Sect. 4.5 in Ref. [5] and references therein). In the case of the $N = 2$ string theories, the condition of conformal anomaly cancellation determines the critical dimension to be four, and the Virasoro constraints forbid any excited physical states except for massless scalars [5]. Hence, the string field theory can be reduced to the conventional field theory. The world-sheet $N = 2$ supersymmetry induces a complex structure on the 4D target space and hence the Minkowski target space is forbidden [6,7]. This is the reason why non-trivial string field theories are realized only when the metric has a split signature. (In the Euclidean case, the momentum of the massless scalar fields becomes identically zero.) Therefore the $N = 2$ string theory is closely related to the Ward conjecture and integrable systems.

The space-time action of the open $N = 2$ string theory is described by the 4D Wess–Zumino–Witten (WZW_4) model [6–10] (see also Refs. [11–16]). The equation of motion of this model is the Yang equation that is equivalent to the anti-self-dual Yang–Mills equation [17]. Hence solutions of the Yang equation are classical solutions of the open string field theory action of the $N = 2$ strings. It is surprising that the action of the string field theory is explicitly written down in terms of massless scalar fields only. Exact analysis of the classical solutions leads to exact analysis of classical aspects of the string field theory.

Recently a new type of soliton solution of the Yang equation has been constructed by using the Darboux transformation [18,19] in 4D flat spaces with all kinds of signatures, i.e., Euclidean, Minkowski, and split signatures [20]. These soliton solutions have localized Yang–Mills action densities on 3D hyperplanes, and hence can be interpreted as codimension-one solitons. Furthermore, an asymptotic analysis has also been undertaken in Refs. [17, 21, 22], which suggests the existence of intersecting three-branes. In the case of the split signature, these solutions are supposed to be relevant to the open $N = 2$ string theory. Therefore, analysis of the solitonic behavior (including the interacting region) for the WZW_4 action is much more appropriate than the Yang–Mills action.

In this paper, we study the classical soliton solutions in the WZW_4 model. The WZW_4 action (15) consists of the non-linear sigma model (NLσM) term and the Wess–Zumino (WZ) term [23]. We calculate the action densities of the NLσM model term and the Wess–Zumino term for the soliton solutions.[1] For the one-soliton solutions, we find that the WZW_4 action density is localized on a 3D hyperplane. This suggests that there would be a codimension-one solitonic object, or equivalently some kind of three-brane in the open $N = 2$ string theory. For the multi-soliton solutions, we clarify the asymptotic behavior and conclude that the n-soliton solution possesses n isolated and localized lumps of the action density, and can be interpreted as n intersecting soliton walls. More precisely, each lump of the action density is essentially the same as a one-soliton because it preserves its shape and “velocity” with a possible position shift (called the phase shift) of the peak in the scattering process. We evaluate the distribution of the NLσM term for the two-soliton solutions successfully and find that there is no singularity in the interacting region. This is consistent with the existence of smooth intersecting three-branes in the whole region. Finally, we make a reduction to a (1 + 2)D real space-time to calculate

[1] In Ref. [13], Parkes discussed similar problems by using the $SL(2, \mathbb{C})$ non-linear plane wave solutions [24] without calculating any action density.

the energy densities of the soliton solutions. In (2 + 2) dimensions, the concept of energy is ambiguous because of the existence of two time directions. This is the reason why in this paper we discuss action density instead of energy density. We compute the energy densities of the one-soliton and two-soliton solutions in (1 + 2) dimensions to confirm that they are localized on the same hyperplanes as the action densities. This suggests that the locus where the action density is localized could be considered as a physical object.

This paper is organized as follows. In Sect. 2, the WZW_4 model is introduced and our conventions are set up. In Sect. 3, soliton solutions of the Yang equation are reviewed and some properties of the solutions, such as the flip symmetry, singularities, and an asymptotic behavior, are discussed. In Sect. 4, the action density for the one- and two-soliton solutions is calculated. In Sect. 5, an asymptotic analysis of the n-soliton solution is given. In Sect. 6, we reduce the WZW_4 model from (2 + 2) dimensions to (1 + 2) dimensions and calculate the Hamiltonian density for the one- and two-soliton solutions. Section 7 is devoted to the conclusion and discussion. Appendix A is a brief review of the quasideterminant. In Appendix B, a statement in footnote 8 is proved (see Sect. 5). Appendix C is a proof of unitarity of the n-soliton solutions on the Euclidean space. Appendix D includes miscellaneous formulas and detailed calculations.

2. 4D Wess–Zumino–Witten model

In this section, we review the 4D Wess–Zumino–Witten (WZW_4) model. In order to treat it in a unified way, we introduce a 4D space with complex coordinates $(z, \widetilde{z}, w, \widetilde{w})$ and the flat metric:

$$ds^2 = g_{mn}dz^m dz^n = 2(dzd\widetilde{z} - dwd\widetilde{w}), \quad m, n = 1, 2, 3, 4,$$

$$\text{where } g_{mn} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \ (z^1, z^2, z^3, z^4) := (z, \widetilde{z}, w, \widetilde{w}). \tag{1}$$

The space $\mathbb{C}^4$ can be reduced to the three kinds of real spaces by imposing suitable reality conditions on $(z, \widetilde{z}, w, \widetilde{w})$. For example, the Euclidean real space $\mathbb{E}$ is given by $\widetilde{z} = \overline{z}$, $\widetilde{w} = -\overline{w}$, and the ultrahyperbolic real space $\mathbb{U}$ by (1) $z, \widetilde{z}, w, \widetilde{w} \in \mathbb{R}$ or (2) $\widetilde{z} = \overline{z}$, $\widetilde{w} = \overline{w}$, which are denoted respectively by $\mathbb{U}_1$ and $\mathbb{U}_2$. Our choices are shown in terms of real coordinates x^μ ($\mu = 1, 2, 3, 4$) as follows:

$$(\mathbb{E}) \begin{pmatrix} z & w \\ \widetilde{w} & \widetilde{z} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^1 + ix^2 & x^3 + ix^4 \\ -(x^3 - ix^4) & x^1 - ix^2 \end{pmatrix}, \tag{2}$$

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2. \tag{3}$$

$$(\mathbb{U}_1) \begin{pmatrix} z & w \\ \widetilde{w} & \widetilde{z} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^1 + x^3 & x^2 + x^4 \\ -(x^2 - x^4) & x^1 - x^3 \end{pmatrix}, \tag{4}$$

$$ds^2 = (dx^1)^2 + (dx^2)^2 - (dx^3)^2 - (dx^4)^2. \tag{5}$$

In this paper, we mainly consider the case of the ultrahyperbolic space $\mathbb{U}_1$.[2]

Let M_4 be a 4D flat space and σ be a map from M_4 to $G = GL(N, \mathbb{C})$ or its subgroup. The action of the WZW_4 model consists of two parts as follows:

[2]The case of $\mathbb{U}_2$ is not considered in this paper because the unitarity condition of σ leads to trivial action densities, as we will see in Appendix C. The case of the Euclidean space is discussed at the end of each (sub)section.

$$S_{\mathrm{WZW}_4} := S_\sigma + S_{\mathrm{WZ}}, \tag{6}$$

$$S_\sigma := \frac{i}{4\pi}\int_{M_4} \omega \wedge \mathrm{Tr}\left[(\partial\sigma)\,\sigma^{-1} \wedge (\widetilde{\partial}\sigma)\sigma^{-1}\right], \tag{7}$$

$$S_{\mathrm{WZ}} := -\frac{i}{12\pi}\int_{M_5} \omega \wedge \mathrm{Tr}\left[(d\widehat{\sigma})\,\widehat{\sigma}^{-1}\right]^3, \tag{8}$$

where $M_5 := M_4 \times [0, 1]$ and $\widehat{\sigma}(z, \widetilde{z}, w, \widetilde{w}, t)$, $t \in [0, 1]$ is a homotopy such that $\widehat{\sigma}(z, \widetilde{z}, w, \widetilde{w}, 0) =$ Id and $\widehat{\sigma}(z, \widetilde{z}, w, \widetilde{w}, 1) = \sigma(z, \widetilde{z}, w, \widetilde{w})$, and ω is the Kähler two-form on M_4 given by

$$\omega = \frac{i}{2}\,(dz \wedge d\widetilde{z} - dw \wedge d\widetilde{w})\,. \tag{9}$$

The exterior derivatives are defined as follows:

$$d := \partial + \widetilde{\partial} + dt\partial_t, \quad \partial := dw\partial_w + dz\partial_z, \quad \widetilde{\partial} := d\widetilde{w}\partial_{\widetilde{w}} + d\widetilde{z}\partial_{\widetilde{z}}. \tag{10}$$

The first part S_σ is called the non-linear sigma model (NLσM) term and the second part is called the Wess–Zumino (WZ) term. In the Wess–Zumino term, we use an abbreviated notation: $A^3 := A\wedge A\wedge A$ for a differential one-form A.

The equation of motion is

$$\widetilde{\partial}\left(\omega \wedge (\partial\sigma)\,\sigma^{-1}\right) = 0. \tag{11}$$

This is derived as follows. Let us consider an infinitesimal variation of the dynamical variable σ such that $\delta\sigma|_{\partial M_4} = 0$ and $d\delta = \delta d$. Then,

$$\delta((d\sigma)\sigma^{-1}) = d((\delta\sigma)\sigma^{-1}) - (d\sigma)\sigma^{-1}(\delta\sigma)\sigma^{-1} + (\delta\sigma)\sigma^{-1}(d\sigma)\sigma^{-1}.$$

Note that $(\delta\sigma)\sigma^{-1}$ is a $\mathfrak{g}$-valued zero-form while $(d\sigma)\sigma^{-1}$ is a $\mathfrak{g}$-valued one-form, where $\mathfrak{g}$ is the Lie algebra of G. The cyclic symmetry of trace implies

$$\delta\mathrm{Tr}\left[(d\widehat{\sigma})\widehat{\sigma}^{-1}\right]^3 = 3d\mathrm{Tr}\left[(\delta\widehat{\sigma})\widehat{\sigma}^{-1}\left((d\widehat{\sigma})\widehat{\sigma}^{-1}\right)^2\right].$$

Since $d\omega = 0$, we have

$$\delta S_{\mathrm{WZ}} = -\frac{i}{4\pi}\int_{M_5} \omega \wedge \delta\mathrm{Tr}\left[(d\widehat{\sigma})\,\widehat{\sigma}^{-1}\right]^3 = -\frac{i}{4\pi}\int_{M_4} \omega \wedge \mathrm{Tr}\left[(\delta\sigma)\sigma^{-1}\left((d\sigma)\sigma^{-1}\right)^2\right].$$

The variation of the sigma model term is

$$\begin{aligned}\delta S_\sigma &= \frac{i}{4\pi}\int_{M_4} \omega \wedge \delta\mathrm{Tr}\left[(\partial\sigma)\,\sigma^{-1} \wedge (\widetilde{\partial}\sigma)\sigma^{-1}\right] \\ &= \frac{i}{4\pi}\int_{M_4} \omega \wedge \mathrm{Tr}\Big[\partial\left((\delta\sigma)\sigma^{-1}(\widetilde{\partial}\sigma)\sigma^{-1}\right) - \widetilde{\partial}\left((\delta\sigma)\sigma^{-1}(\partial\sigma)\,\sigma^{-1}\right) \\ &\quad + (\delta\sigma)\sigma^{-1}\left(\widetilde{\partial}\left((\partial\sigma)\,\sigma^{-1}\right) - \partial\left((\widetilde{\partial}\sigma)\sigma^{-1}\right)\right)\Big],\end{aligned}$$

where we use $\partial\widetilde{\partial} + \widetilde{\partial}\partial = 0$ due to $\partial^2 = 0$, $\widetilde{\partial}^2 = 0$, and $d = \partial + \widetilde{\partial}$. The first and second terms become a surface integration due to $d\omega = 0$ and the fact that:

$$\begin{aligned}&\mathrm{Tr}\left[\partial\left((\delta\sigma)\sigma^{-1}(\widetilde{\partial}\sigma)\sigma^{-1}\right) - \widetilde{\partial}\left((\delta\sigma)\sigma^{-1}(\partial\sigma)\,\sigma^{-1}\right)\right] \\ &\quad = \mathrm{Tr}\left[d\left((\delta\sigma)\sigma^{-1}(\widetilde{\partial}\sigma)\sigma^{-1}\right) - d\left((\delta\sigma)\sigma^{-1}(\partial\sigma)\,\sigma^{-1}\right)\right].\end{aligned}$$

Therefore we get the final form of the total action variation and the equation of motion is obtained:

$$\delta S_{\mathrm{WZW}_4} = \frac{i}{2\pi}\int_{M_4} \omega \wedge \mathrm{Tr}\left[(\delta\sigma)\sigma^{-1}\widetilde{\partial}\left((\partial\sigma)\,\sigma^{-1}\right)\right].$$

Finally we rewrite the WZW_4 action without integration over M_5. By the cyclic property of the trace we have

$$d\mathrm{Tr}\left[(d\widehat{\sigma})\widehat{\sigma}^{-1}\right]^3 = -\mathrm{Tr}\left[(d\widehat{\sigma})\widehat{\sigma}^{-1}\right]^4 = 0. \tag{12}$$

The Kähler two-form ω is closed and $H^2(M_4, \mathbb{R}) = 0$ and hence there exists a one-form A on the flat space-time such that

$$\omega = dA. \tag{13}$$

Note that A is not uniquely determined and has ambiguity with respect to the following degree of freedom: $A \mapsto A + d\kappa$, where κ is an arbitrary zero-form.

The Wess–Zumino term is written as

$$\int_{M_5} d\left(A\wedge \mathrm{Tr}\left[(d\widehat{\sigma})\widehat{\sigma}^{-1}\right]^3\right) = \int_{M_4} A\wedge \mathrm{Tr}\left[(d\widehat{\sigma})\widehat{\sigma}^{-1}\right]^3\Big|_{t=1} - \int_{M_4} A\wedge \mathrm{Tr}\left[(d\widehat{\sigma})\widehat{\sigma}^{-1}\right]^3\Big|_{t=0}. \tag{14}$$

If there exists a homotopy such that $\widehat{\sigma}(t=0) = \mathrm{Id}$ and $\widehat{\sigma}(t=1) = \sigma$, the second term vanishes and we obtain

$$S_{\mathrm{WZW}_4} = \frac{i}{4\pi}\int_{M_4} \omega\wedge \mathrm{Tr}\left[(\partial\sigma)\sigma^{-1}\wedge(\widetilde{\partial}\sigma)\sigma^{-1}\right] - \frac{i}{12\pi}\int_{M_4} A\wedge \mathrm{Tr}\left[(d\sigma)\sigma^{-1}\right]^3. \tag{15}$$

From now on, we use this form of action. Since our soliton solutions allow such a homotopy as above, we can use Eq. (15) for computing the action density.

2.1. *Component representation of WZW_4 action density*

Let us write down explicit representations of the WZW_4 action density (15) in the flat 4D real spaces.

In terms of the local complex coordinates (1)–(4), the NLσM action is described as follows:

$$\begin{aligned} S_\sigma &= \frac{i}{4\pi}\int_{M_4} \omega\wedge \mathrm{Tr}\left[(\partial\sigma)\sigma^{-1}\wedge(\widetilde{\partial}\sigma)\sigma^{-1}\right] \\ &= -\frac{1}{16\pi}\int_{M_4} \mathrm{Tr}\left[(\partial_m\sigma)\sigma^{-1}(\partial^m\sigma)\sigma^{-1}\right] dz\wedge d\widetilde{z}\wedge dw\wedge d\widetilde{w}, \end{aligned} \tag{16}$$

where $\partial^m := g^{mn}\partial_n$ and the metric is given by Eq. (1). This can be represented explicitly in terms of real coordinates on $\mathbb{U}$, $\mathbb{E}$:

$$S_\sigma = -\frac{1}{16\pi}\int_{\mathbb{U} or \mathbb{E}} \mathrm{Tr}\left[\left(\partial_\mu\sigma\right)\sigma^{-1}(\partial^\mu\sigma)\sigma^{-1}\right] dx^1\wedge dx^2\wedge dx^3\wedge dx^4, \tag{17}$$

where the real space metrics are given in Eqs. (3) and (5). The NLσM action density is read from the integrand as $\mathcal{L}_\sigma := -(1/16\pi)\mathrm{Tr}\left[\left(\partial_\mu\sigma\right)\sigma^{-1}(\partial^\mu\sigma)\sigma^{-1}\right]$.

Similarly, the Wess–Zumino action is described as follows:

$$\begin{aligned} S_{\mathrm{WZ}} &= -\frac{i}{12\pi}\int_{M_4} A\wedge \mathrm{Tr}\left[(d\sigma)\sigma^{-1}\wedge(d\sigma)\sigma^{-1}\wedge(d\sigma)\sigma^{-1}\right] \\ &= \frac{1}{16\pi}\int_{M_4}\left\{\begin{matrix} \mathrm{Tr}\left(\theta_w\theta_z\theta_{\widetilde{z}} - \theta_w\theta_{\widetilde{z}}\theta_z\right) w \\ +\mathrm{Tr}\left(\theta_{\widetilde{w}}\theta_z\theta_{\widetilde{z}} - \theta_{\widetilde{w}}\theta_{\widetilde{z}}\theta_z\right)\widetilde{w} \\ -\mathrm{Tr}\left(\theta_z\theta_w\theta_{\widetilde{w}} - \theta_z\theta_{\widetilde{w}}\theta_w\right) z \\ -\mathrm{Tr}\left(\theta_{\widetilde{z}}\theta_w\theta_{\widetilde{w}} - \theta_{\widetilde{z}}\theta_{\widetilde{w}}\theta_w\right)\widetilde{z} \end{matrix}\right\} dz\wedge d\widetilde{z}\wedge dw\wedge d\widetilde{w}, \end{aligned} \tag{18}$$

where $\theta_m := (\partial_m\sigma)\sigma^{-1}$. Here we choose the potential one-form A as $A = (i/4)\left(z d\widetilde{z} - \widetilde{z}dz - w d\widetilde{w} + \widetilde{w}dw\right)$. This can be reduced to the three kinds of real spaces. For example, in the ultrahyperbolic space $\mathbb{U}_1$, it is

$$S_{\rm WZ} = -\frac{1}{16\pi}\int_{\mathbb{U}_1} \left\{ \begin{array}{c} {\rm Tr}\,(\theta_1\theta_2\theta_4 - \theta_1\theta_4\theta_2)\,x^1 \\ +{\rm Tr}\,(\theta_2\theta_1\theta_3 - \theta_2\theta_3\theta_1)\,x^2 \\ +{\rm Tr}\,(\theta_3\theta_2\theta_4 - \theta_3\theta_4\theta_2)\,x^3 \\ +{\rm Tr}\,(\theta_4\theta_1\theta_3 - \theta_4\theta_3\theta_1)\,x^4 \end{array} \right\} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4. \tag{19}$$

where $\theta_\mu := (\partial_\mu\sigma)\sigma^{-1}$. The Wess–Zumino action density $\mathscr{L}_{\rm WZ}$ can be read from the integrand.

2.2. *Useful formulas for $G = GL(2, \mathbb{C})$*

In this subsection, we focus on the case of $G = GL(2, \mathbb{C})$. Additionally we impose the condition $\partial_m|\sigma| = 0$ on σ because our soliton solutions σ obtained in Sect. 3.2 satisfy this condition. Then the WZW action density becomes a concise determinant form as follows. By Jacobi's formula, ${\rm Tr}[(\partial_m\sigma)\sigma^{-1}] = \partial_m|\sigma|/|\sigma| = \partial_m \log|\sigma|$, we find that the condition $\partial_m|\sigma| = 0$ is equivalent to the condition ${\rm Tr}[(\partial_m\sigma)\sigma^{-1}] = 0$, which can be expressed in terms of the matrix elements $(\partial_m\sigma_{11})\sigma_{22} - (\partial_m\sigma_{12})\sigma_{21} = -\{\sigma_{11}(\partial_m\sigma_{22}) - \sigma_{12}(\partial_m\sigma_{21})\}$. Therefore, we have

$${\rm Tr}\left[(\partial_m\sigma)\sigma^{-1}(\partial_n\sigma)\sigma^{-1}\right] = \frac{-1}{|\sigma|}\left(\begin{vmatrix} \partial_m\sigma_{11} & \partial_m\sigma_{12} \\ \partial_n\sigma_{21} & \partial_n\sigma_{22} \end{vmatrix} + \begin{vmatrix} \partial_n\sigma_{11} & \partial_n\sigma_{12} \\ \partial_m\sigma_{21} & \partial_m\sigma_{22} \end{vmatrix} \right). \tag{20}$$

In this paper, we always take the following parametrization for the soliton solution σ:

$$\sigma = \frac{-1}{\Delta}\begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}, \tag{21}$$

under the condition $\partial_m|\sigma| = 0$. Note that this reparametrization is not unique and there is a relation between the five variables:

$$\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21} = |\sigma|\,\Delta^2. \tag{22}$$

In this setting, the quadratic term (20) can be rewritten as

$$\begin{aligned} &{\rm Tr}\left[(\partial_m\sigma)\sigma^{-1}(\partial_n\sigma)\sigma^{-1}\right] \\ &= \frac{1}{|\sigma|\Delta^2}\left\{ \begin{vmatrix} \partial_m\Delta_{11} & \partial_m\Delta_{12} \\ \partial_n\Delta_{21} & \partial_n\Delta_{22} \end{vmatrix} + \begin{vmatrix} \partial_n\Delta_{11} & \partial_n\Delta_{12} \\ \partial_m\Delta_{21} & \partial_m\Delta_{22} \end{vmatrix} - 2|\sigma|(\partial_m\Delta)(\partial_n\Delta) \right\}. \end{aligned} \tag{23}$$

Similarly, the cubic term is:

$$\begin{aligned} &{\rm Tr}\left[(\partial_m\sigma)\sigma^{-1}(\partial_n\sigma)\sigma^{-1}(\partial_p\sigma)\sigma^{-1}\right] \\ &= \frac{1}{2|\sigma|^2\Delta^4}\begin{vmatrix} \Delta_{11} & \Delta_{12} & \Delta_{21} & \Delta_{22} \\ \partial_m\Delta_{11} & \partial_m\Delta_{12} & \partial_m\Delta_{21} & \partial_m\Delta_{22} \\ \partial_n\Delta_{11} & \partial_n\Delta_{12} & \partial_n\Delta_{21} & \partial_n\Delta_{22} \\ \partial_p\Delta_{11} & \partial_p\Delta_{12} & \partial_p\Delta_{21} & \partial_p\Delta_{22} \end{vmatrix} = \frac{1}{2|\sigma|^2\Delta^4}(A_{mnp} + A_{npm} + A_{pmn}), \end{aligned} \tag{24}$$

$$A_{mnp} := \begin{vmatrix} \Delta_{11} & \Delta_{22} \\ \partial_m\Delta_{11} & \partial_m\Delta_{22} \end{vmatrix}\begin{vmatrix} \partial_n\Delta_{12} & \partial_n\Delta_{21} \\ \partial_p\Delta_{12} & \partial_p\Delta_{21} \end{vmatrix} + \begin{vmatrix} \Delta_{12} & \Delta_{21} \\ \partial_m\Delta_{12} & \partial_m\Delta_{21} \end{vmatrix}\begin{vmatrix} \partial_n\Delta_{11} & \partial_n\Delta_{22} \\ \partial_p\Delta_{11} & \partial_p\Delta_{22} \end{vmatrix}. \tag{25}$$

By the permutation property of determinants (cf. Eq. (24)), we have

$${\rm Tr}\left[(\partial_m\sigma)\sigma^{-1}(\partial_n\sigma)\sigma^{-1}(\partial_p\sigma)\sigma^{-1}\right] = -{\rm Tr}\left[(\partial_m\sigma)\sigma^{-1}(\partial_p\sigma)\sigma^{-1}(\partial_n\sigma)\sigma^{-1}\right]. \tag{26}$$

Therefore under the condition $\partial_m|\sigma| = 0$, the Wess–Zumino term can be further simplified as

$$S_{\rm WZ} = -\frac{i}{12\pi}\int_{M_4} A \wedge {\rm Tr}\left[(d\sigma)\sigma^{-1}\wedge(d\sigma)\sigma^{-1}\wedge(d\sigma)\sigma^{-1}\right]$$
$$= \frac{1}{8\pi}\int_{M_4}\left\{\begin{array}{l}{\rm Tr}\,(\theta_w\theta_z\theta_{\tilde{z}})\,w + {\rm Tr}\,(\theta_{\tilde{w}}\theta_z\theta_{\tilde{z}})\,\tilde{w}\\ -{\rm Tr}\,(\theta_z\theta_w\theta_{\tilde{w}})\,z - {\rm Tr}\,(\theta_{\tilde{z}}\theta_w\theta_{\tilde{w}})\,\tilde{z}\end{array}\right\} dz\wedge d\tilde{z}\wedge dw\wedge d\tilde{w}$$
$$\overset{\mathbb{U}_1}{=} \int_{\mathbb{U}_1}\mathscr{L}_{\rm WZ}dx^1\wedge dx^2\wedge dx^3\wedge dx^4, \tag{27}$$

$$\mathscr{L}_{\rm WZ}\overset{\mathbb{U}_1}{=} -\frac{1}{8\pi}\left({\rm Tr}\,(\theta_1\theta_2\theta_4)\,x^1 + {\rm Tr}\,(\theta_2\theta_1\theta_3)\,x^2 + {\rm Tr}\,(\theta_3\theta_2\theta_4)\,x^3 + {\rm Tr}\,(\theta_4\theta_1\theta_3)\,x^4\right). \tag{28}$$

3. Darboux transformation and soliton solutions

In this section, we review the soliton solutions of the Yang equation, which are constructed by applying the Darboux transformation [18,19].

3.1. *Darboux transformation for the Yang equation*

Let us assume that $G = GL(N, \mathbb{C})$ in this subsection. The Yang equation (11) can be rewritten as the following differential equation:

$$\partial_{\tilde{z}}((\partial_z\sigma)\sigma^{-1}) - \partial_{\tilde{w}}((\partial_w\sigma)\sigma^{-1}) = 0. \tag{29}$$

There exists a Lax representation of Eq. (29) given by the following linear system [19]:

$$L(f) := \sigma\,\partial_w(\sigma^{-1}f) - (\partial_{\tilde{z}}f)\zeta = 0,$$
$$M(f) := \sigma\,\partial_z(\sigma^{-1}f) - (\partial_{\tilde{w}}f)\zeta = 0. \tag{30}$$

The spectral parameter ζ here must be generalized to an $N\times N$ constant matrix otherwise the Darboux transformation would be a trivial transformation. This is a key point to define a non-trivial Darboux transformation as we will see later.

It is not hard to verify that the compatibility condition $L(M(f)) - M(L(f)) = 0$ implies the Yang equation (29). The existence of N-independent solutions of the linear system (30) is an assumption here; however, we will show later that it actually exists for the soliton solution cases. Then, f can be rewritten as an $N\times N$ matrix that consists of the N-independent solutions as column vectors of length N.

The Darboux transformation is defined as an auto-Bäcklund transformation of the linear system (30). Firstly, we start with a solution σ of the Yang equation, and a solution $f = f(\zeta)$ of the linear system (30). Secondly, we prepare a special solution $\psi(\Lambda) := f(\Lambda)$ for a fixed spectral parameter matrix $\zeta = \Lambda$. Then the following Darboux transformation,

$$f' = f\zeta - \psi\Lambda\psi^{-1}f, \quad \sigma' = -\psi\Lambda\psi^{-1}\sigma, \tag{31}$$

keeps the linear system (30) invariant in form, i.e.,

$$L'(f') := \sigma'\partial_w(\sigma'^{-1}f') - (\partial_{\tilde{z}}f')\zeta = 0,$$
$$M'(f') := \sigma'\partial_z(\sigma'^{-1}f') - (\partial_{\tilde{w}}f')\zeta = 0. \tag{32}$$

As mentioned before, the transformation (31) becomes trivial if the spectral parameter is a scalar matrix where Λ commutes with ψ. The Darboux transformation maps the input data $(\sigma, f(\zeta), \psi(\Lambda))$ to the output data $(\sigma', f'(\zeta))$ and therefore we get a new solution σ' of the Yang equation successfully. In the same way, these output data can be reused as the next input data

$(\sigma', f'(\zeta), \psi'(\Lambda'))$ for the Darboux transformation. Here we define a special solution $\psi'(\Lambda') := f'(\Lambda')$ by choosing a suitable spectral parameter matrix $\zeta = \Lambda'$. Continuing this process, we get a series of input–output data: $(\sigma, f, \psi)\mapsto(\sigma', f', \psi')\mapsto\cdots$. Therefore by applying n iterations of the Darboux transformation, we can get n exact solutions $\sigma_{[j]}$ of the Yang equation and express them in terms of the quasideterminants in a compact form [18,19]. For our purposes in this paper, it is sufficient to choose a trivial seed solution $\sigma_{[1]} = 1$:

$$\sigma_{[n+1]} = \begin{vmatrix} \psi_1 & \cdots & \psi_n & 1 \\ \psi_1\Lambda_1 & \cdots & \psi_n\Lambda_n & 0 \\ \vdots & & \vdots & \vdots \\ \psi_1\Lambda_1^{n-1} & \cdots & \psi_n\Lambda_n^{n-1} & 0 \\ \psi_1\Lambda_1^{n} & \cdots & \psi_n\Lambda_n^{n} & \boxed{0} \end{vmatrix}, \quad n \in \mathbb{N} \tag{33}$$

where each $\psi_j = \psi_j(\Lambda_j)$ ($j = 1, 2,\ldots, n$) is a solution ($N \times N$ matrix) of the initial linear system ($\sigma = 1$):

$$\partial_w\psi_j = (\partial_{\widetilde{z}}\psi_j)\Lambda_j, \quad \partial_z\psi_j = (\partial_{\widetilde{w}}\psi_j)\Lambda_j. \tag{34}$$

Hence the problem of solving the Yang equation reduces to solving Eq. (34). (The label $_{[n+1]}$ in the n-soliton solution is omitted in most of this paper except for Appendix C.)

In the main part of this paper, we do not explain the details of the quasideterminant, but provide the definition and properties of the quasideterminant in Appendix A. The detailed computations can be found in Appendices B and C.

3.2. *Soliton solutions for $G = GL(2, \mathbb{C})$*

From now on, we focus only on the soliton solutions for $G = GL(2, \mathbb{C})$. An example of the multi-soliton solution is given by [21]:

$$\psi_j = \begin{pmatrix} e^{L_j} & e^{-\overline{L}_j} \\ -e^{-L_j} & e^{\overline{L}_j} \end{pmatrix}, \quad \Lambda_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & \mu_j \end{pmatrix}, \tag{35}$$

where the two kinds of spectral parameters λ_j, μ_j ($j = 1, 2,\ldots, n$) are complex constants with the following mutual relationship on each real space:

$$(\lambda_j, \mu_j) = \begin{cases} (\lambda_j, \overline{\lambda}_j) \text{ on } \mathbb{U}_1, \quad (\lambda_j, 1/\overline{\lambda}_j) \text{ on } \mathbb{U}_2, \\ (\lambda_j, -1/\overline{\lambda}_j) \text{ on } \mathbb{E}. \end{cases} \tag{36}$$

The powers L_j of the exponential function are linear in the complex coordinates: $L_j := \lambda_j\alpha_j z + \beta_j\widetilde{z} + \lambda_j\beta_j w + \alpha_j\widetilde{w}$, where $\alpha_j, \beta_j \in \mathbb{C}$. The representations of L_j in real coordinates are

$$L_j \overset{\mathbb{U}_1}{=} \frac{1}{\sqrt{2}}\left\{(\lambda_j\alpha_j + \beta_j)x^1 + (\lambda_j\beta_j - \alpha_j)x^2 + (\lambda_j\alpha_i - \beta_j)x^3 + (\lambda_j\beta_j + \alpha_j)x^4\right\}, \tag{37}$$

$$\overset{\mathbb{E}}{=} \frac{1}{\sqrt{2}}\left\{(\lambda_j\alpha_j + \beta_j)x^1 + i(\lambda_j\alpha_j - \beta_j)x^2 + (\lambda_j\beta_j - \alpha_j)x^3 + i(\lambda_j\beta_j + \alpha_j)x^4\right\}. \tag{38}$$

We use the notation $\ell_\mu^{(j)}$ to simplify the coefficients of L_j in the following sections, i.e., $L_j := \ell_\mu^{(j)}x^\mu$.

We remark that the determinant of the n-soliton solution σ is constant [17, 21]:

$$|\sigma| = \prod_{j=1}^{n}\lambda_j\mu_j, \tag{39}$$

which satisfies the requirement $\partial_\mu|\sigma| = 0$. Therefore, we can apply the formulas (23) and (24) to the n-soliton solutions. On the ultrahyperbolic space $\mathbb{U}_1$ in particular, the n-soliton solution σ satisfies $\sigma\sigma^\dagger = \sigma^\dagger\sigma = |\sigma|$ [17, 21] and hence after the scale transformation $\sigma \mapsto |\sigma|^{1/2}\sigma$, σ belongs to $SU(2)$. On the Euclidean space $\mathbb{E}$, σ can take values in $U(2)$, which is proved in Appendix C.

By the definition (A5) of the quasideterminant, the n-soliton solution σ (cf. Eqs. (33) and (35)) can be represented in the form of Eq. (21):

$$\Delta = \begin{vmatrix} \psi_1 & \cdots & \psi_n \\ \psi_1\Lambda_1 & \cdots & \psi_n\Lambda_n \\ \vdots & & \vdots \\ \psi_1\Lambda_1^{n-1} & \cdots & \psi_n\Lambda_n^{n-1} \end{vmatrix} = \begin{vmatrix} (\psi_1)_1 & \cdots & (\psi_n)_1 \\ (\psi_1)_2 & \cdots & (\psi_n)_2 \\ & \Psi_{(n-1)\times n} & \end{vmatrix},$$

$$\Delta_{11} = -\begin{vmatrix} \psi_1 & \cdots & \psi_n & \mathbf{e}_1 \\ \psi_1\Lambda_1 & \cdots & \psi_n\Lambda_n & \mathbf{0} \\ \vdots & & \vdots & \vdots \\ \psi_1\Lambda_1^{n-1} & \cdots & \psi_n\Lambda_n^{n-1} & \mathbf{0} \\ (\psi_1\Lambda_1^n)_1 & \cdots & (\psi_n\Lambda_n^n)_1 & 0 \end{vmatrix} = \begin{vmatrix} (\psi_1\Lambda_1^n)_1 & \cdots & (\psi_n\Lambda_n^n)_1 \\ (\psi_1)_2 & \cdots & (\psi_n)_2 \\ & \Psi_{(n-1)\times n} & \end{vmatrix},$$

$$\Delta_{12} = -\begin{vmatrix} \psi_1 & \cdots & \psi_n & \mathbf{e}_2 \\ \psi_1\Lambda_1 & \cdots & \psi_n\Lambda_n & \mathbf{0} \\ \vdots & & \vdots & \vdots \\ \psi_1\Lambda_1^{n-1} & \cdots & \psi_n\Lambda_n^{n-1} & \mathbf{0} \\ (\psi_1\Lambda_1^n)_1 & \cdots & (\psi_n\Lambda_n^n)_1 & 0 \end{vmatrix} = \begin{vmatrix} (\psi_1)_1 & \cdots & (\psi_n)_1 \\ (\psi_1\Lambda_1^n)_1 & \cdots & (\psi_n\Lambda_n^n)_1 \\ & \Psi_{(n-1)\times n} & \end{vmatrix},$$

$$\Delta_{21} = -\begin{vmatrix} \psi_1 & \cdots & \psi_n & \mathbf{e}_1 \\ \psi_1\Lambda_1 & \cdots & \psi_n\Lambda_n & \mathbf{0} \\ \vdots & & \vdots & \vdots \\ \psi_1\Lambda_1^{n-1} & \cdots & \psi_n\Lambda_n^{n-1} & \mathbf{0} \\ (\psi_1\Lambda_1^n)_2 & \cdots & (\psi_n\Lambda_n^n)_2 & 0 \end{vmatrix} = \begin{vmatrix} (\psi_1\Lambda_1^n)_2 & \cdots & (\psi_n\Lambda_n^n)_2 \\ (\psi_1)_2 & \cdots & (\psi_n)_2 \\ & \Psi_{(n-1)\times n} & \end{vmatrix},$$

$$\Delta_{22} = -\begin{vmatrix} \psi_1 & \cdots & \psi_n & \mathbf{e}_2 \\ \psi_1\Lambda_1 & \cdots & \psi_n\Lambda_n & \mathbf{0} \\ \vdots & & \vdots & \vdots \\ \psi_1\Lambda_1^{n-1} & \cdots & \psi_n\Lambda_n^{n-1} & \mathbf{0} \\ (\psi_1\Lambda_1^n)_2 & \cdots & (\psi_n\Lambda_n^n)_2 & 0 \end{vmatrix} = \begin{vmatrix} (\psi_1)_1 & \cdots & (\psi_n)_1 \\ (\psi_1\Lambda_1^n)_2 & \cdots & (\psi_n\Lambda_n^n)_2 \\ & \Psi_{(n-1)\times n} & \end{vmatrix}, \tag{40}$$

where

$$\Psi_{(n-1)\times n} := \begin{pmatrix} \psi_1\Lambda_1 & \cdots & \psi_n\Lambda_n \\ \vdots & & \vdots \\ \psi_1\Lambda_1^{n-1} & \cdots & \psi_n\Lambda_n^{n-1} \end{pmatrix},$$

and $\mathbf{e}_1 := (1, 0)^t$, $\mathbf{e}_2 := (0, 1)^t$, $\mathbf{0} := (0, 0)^t$, and $(A)_k$ is the kth row of a square matrix A. The data Δ and Δ_{jk} are determinants of $2n \times 2n$ matrices.

We remark that ψ_j can be decomposed into, for instance,

$$\psi_j = \begin{pmatrix} e^{L_j} & e^{-\overline{L}_j} \\ -e^{-L_j} & e^{\overline{L}_j} \end{pmatrix} = \begin{pmatrix} e^{X_j} & e^{i\Theta_j} \\ -e^{-i\Theta_j} & e^{X_j} \end{pmatrix} \begin{pmatrix} e^{-\overline{L}_j} & 0 \\ 0 & e^{-L_j} \end{pmatrix}, \tag{41}$$

where $X_j := L_j + \overline{L}_j$, $i\Theta_j := L_j - \overline{L}_j$. The second factor diag$(e^{-\overline{L}_j}, e^{-L_j})$ can be eliminated in the n-soliton solutions (33) due to the property of the quasideterminant (A7). Hence the n-soliton solutions (33) depend only on X_j and Θ_j. The expansion coefficients for the real coordinates are denoted by $X_j = r_\mu^{(j)} x^\mu$, $i\Theta_j = s_\mu^{(j)} x^\mu$, i.e., $r_\mu^{(j)} := \ell_\mu^{(j)} + \overline{\ell}_\mu^{(j)} \in \mathbb{R}$, $s_\mu^{(j)} := \ell_\mu^{(j)} - \overline{\ell}_\mu^{(j)} \in i\mathbb{R}$, where $L_j = \ell_\mu^{(j)} x^\mu$. It is obvious that

$$\partial_\mu e^{\pm X_j} = \pm r_\mu^{(j)} e^{\pm X_j}, \quad \partial_\mu e^{\pm i\Theta_j} = \pm s_\mu^{(j)} e^{\pm i\Theta_j}, \quad \partial_\mu e^{\pm i\Theta_{jk}} = \pm \left(s_\mu^{(j)} - s_\mu^{(k)}\right) e^{\pm i\Theta_{jk}}, \tag{42}$$

where $\Theta_{jk} := \Theta_j - \Theta_k$. Note that the flip of space-time coordinates $x \to -x \Leftrightarrow (x^1, x^2, x^3, x^4) \to (-x^1, -x^2, -x^3, -x^4)$ corresponds to the following flips of the new variables:

$$x \to -x \Longleftrightarrow L_j \to -L_j \Longleftrightarrow (X_j, \Theta_j) \to (-X_j, -\Theta_j). \tag{43}$$

Under this flip, we find the following symmetry (cf. Eq. (D1)):

$$(\Delta, \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})\Big|_{(X_j,\Theta_j)\to(-X_j,-\Theta_j)} = (\Delta, \Delta_{22}, \Delta_{21}, \Delta_{12}, \Delta_{11}), \tag{44}$$

$$\partial_\mu(\Delta, \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})\Big|_{(X_j,\Theta_j)\to(-X_j,-\Theta_j)} = -\partial_\mu(\Delta, \Delta_{22}, \Delta_{21}, \Delta_{12}, \Delta_{11}). \tag{45}$$

Here let us discuss the singularities of the solution σ. Under the decomposition (21), possible singularities correspond to zeros of Δ. For the one-soliton solution (cf. Eq. (D6)), $\Delta = 2\cosh X_1$ and hence there is no singularity.[3] For the two-soliton solution (cf. Eq. (D7)), we can evaluate the value of Δ on the ultrahyperbolic space $\mathbb{U}_1$ as follows:

$$\begin{aligned} \frac{1}{2}\Delta &= a\cosh(X_1 + X_2) + b\cosh(X_1 - X_2) + c\cos\Theta_{12} \\ &\geq |\lambda_1 - \lambda_2|^2 + \left|\lambda_1 - \overline{\lambda}_2\right|^2 - \left|\left(\lambda_1 - \overline{\lambda}_1\right)\left(\lambda_2 - \overline{\lambda}_2\right)\right| \\ &= \begin{cases} 2\left|\lambda_1 - \overline{\lambda}_2\right|^2 > 0 & \text{if } c > 0 \\ 2\left|\lambda_1 - \lambda_2\right|^2 > 0 & \text{if } c < 0 \end{cases}, \end{aligned} \tag{46}$$

where a, b, c are real constants defined in Table 1. Therefore, the denominator is positive anywhere and σ is proved to be non-singular on $\mathbb{U}_1$.

On the other hand, on the Euclidean space $\mathbb{E}$, σ has singularities because it has zero locus due to the fact that $\cosh(X_1 \pm X_2) \geq 1$, $|\cos\Theta_{12}| \leq 1$ and a, b have opposite signs (see Table 1). However, this problem can be solved successfully by choosing suitable initial data ψ, which will be discussed in Sect. 5.

Finally we comment on an asymptotic behavior in the region that $r^2 := (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$ is large enough in order to prove that the Wess–Zumino action density decays exponentially in the asymptotic region. We note that the n-soliton solution (33) is a meromorphic function of $(\xi_1,\ldots, \xi_n, \eta_1,\ldots, \eta_n)$ where $\xi_K := e^{X_K}$, $\eta_K := e^{i\Theta_K}$. Let us discuss the absolute value of the Wess–Zumino action density. In fact, we will see in Sect. 5 that the action density tends

[3] If the (2,1) component $-e^{-L_1}$ in the one-soliton solution ψ is replaced with $+e^{-L_1}$, then $\Delta = 2\sinh X_1$ and Δ has zero on $X_1 = 0$. This corresponds to the following singular one-soliton solution of the KP equation: $u = 2\partial_x^2 \log(e^X - e^{-X}) \propto \mathrm{csch}^2 X$, where X is a linear combination of the space-time coordinates t, x, y. On the other hand, a non-singular one-soliton solution is given by $u(t, x, y) = 2\partial_x^2 \log(e^X + e^{-X}) \propto \mathrm{sech}^2 X$.

Table 1. Summary of coefficients.

Space (Metric)	$\mathbb{U}_1$ $(+,+,-,-)$	$\mathbb{E}$ $(+,+,+,+)$
$a \in \mathbb{R}^+$	$\lvert\lambda_1-\lambda_2\rvert^2>0$	$\lvert\lambda_1-\lambda_2\rvert^2>0$
$b \in \mathbb{R}$	$\lvert\lambda_1-\overline{\lambda}_2\rvert^2>0$	$-\lvert\lambda_1\overline{\lambda}_2+1\rvert^2<0$
$c \in \mathbb{R}$	$(\lambda_1-\overline{\lambda}_1)(\lambda_2-\overline{\lambda}_2)$	$(\lvert\lambda_1\rvert^2+1)(\lvert\lambda_2\rvert^2+1)$
d_{jk} $(=\overline{d}_{kj})$	$\dfrac{(\alpha_j\overline{\beta}_k-\beta_j\overline{\alpha}_k)(\lambda_j-\overline{\lambda}_k)^3}{\lambda_j\overline{\lambda}_k}$	$\dfrac{(\alpha_j\overline{\alpha}_k+\beta_j\overline{\beta}_k)(\lambda_j\overline{\lambda}_k+1)^3}{\lambda_j\overline{\lambda}_k}$
e_{jk}	$\dfrac{(\alpha_j\beta_k-\beta_j\alpha_k)(\lambda_j-\lambda_k)^3}{\lambda_j\lambda_k}$	$\dfrac{(\alpha_j\beta_k-\beta_j\alpha_k)(\lambda_j-\lambda_k)^3}{\lambda_j\lambda_k}$

to zero in the asymptotic region. This implies that, for any variable ξ_K, the polynomial degree of the denominator is greater than that of the numerator. (η_K is not essential because of $|\eta_K| = 1$.) Let us consider a specific asymptotic direction where the most dominant terms are $\xi_1^{i_1}\cdots\xi_n^{i_n}$ in the numerator and $\xi_1^{j_1}\cdots\xi_n^{j_n}$ in the denominator where $i_k \leq j_k$. Then the action density behaves as $\mathcal{O}(\xi_1^{i_1-j_1}\cdots\xi_n^{i_n-j_n})$. At least for one K, $i_K < j_K$ and hence this implies that it decays exponentially.

Let us take the two-soliton case as an example. If we consider the asymptotic limit such that X_1 is finite and $|X_2| \gg 1$ (cf. Eq. (61)), the most dominant factor is $e^{\pm X_2}$ and the denominator and the numerator have the same order of $\xi_1 \equiv e^{\pm X_2}$. However, due to the identity (63), the most dominant term in the numerator vanishes and hence the Wess–Zumino action density is $\mathcal{O}(\xi_1^{-k})$, where k is some positive integer. Therefore we can conclude that the Wess–Zumino action density decays exponentially.

Therefore, on $\mathbb{U}_1$, the Wess–Zumino action converges for the one- and two-soliton solutions because it has no singularity and decays exponentially. For the n-soliton solution, this is an open problem, which is discussed in Sect. 7.

4. Evaluation of action density

In this section, we compute the action density of the $SU(2)$ WZW_4 model for the one- and two-soliton solutions and find that the corresponding action densities are real-valued on each space. We also find that, for the one-soliton solution, the NLσM action density is localized on a 3D hyperplane and the Wess–Zumino action density identically vanishes. For the two-soliton solution, we complete the calculation of the NLσM term, and reach a compact form. In particular, the two peaks of the action density are localized on two non-parallel 3D hyperplanes. As for the Wess–Zumino term, we show that the action density is asymptotic to zero. On the ultrahyperbolic space $\mathbb{U}_1$ in particular, no singularity appears in the action densities for the two-soliton case as indicated in Sect. 3.

4.1. *One-soliton solutions*

In this subsection, we compute the action densities of the $SU(2)$ WZW_4 model for the one-soliton solutions.

To calculate the NLσM action density explicitly, we substitute the data of one-soliton (D6) and Eq. (39) into Eq. (23) for $m = n = \mu$ and then obtain the following result:

$$\mathscr{L}_\sigma = -\frac{1}{16\pi}\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}\,(\partial^\mu\sigma)\,\sigma^{-1}\right] = \frac{1}{8\pi}d_{11}\,\mathrm{sech}^2 X_1, \tag{47}$$

where d_{11} is determined by Eq. (36) and Eqs. (37)–(38), for instance, $d_{11}^{\mathbb{U}_1} = (\alpha_1\overline{\beta}_1 - \overline{\alpha}_1\beta_1)(\lambda_1 - \overline{\lambda}_1)^3/|\lambda_1|^2$ and $d_{11}^{\mathbb{E}} = (|\alpha_1|^2 + |\beta_1|^2)(|\lambda_1|^2 + 1)^3/|\lambda_1|^2$ (cf. d_{jk} in Table 1). Hence d_{11} and $\mathscr{L}_\sigma$ are clearly real-valued. Note that the action density vanishes identically in the case of $\alpha_1, \beta_1, \lambda_1 \in \mathbb{R}$ on $\mathbb{U}_1$ and hence the result is trivial. For the non-trivial cases, the peak of the action density lies on the 3D hyperplane described by the linear equation $X_1 = 0$ on each space.

The Wess–Zumino action density can be calculated by substituting the data of one-soliton (D6) and Eq. (39) into Eq. (25) directly. Then we have

$$\begin{aligned}\begin{vmatrix}\Delta_{11} & \Delta_{22}\\ \partial_\mu\Delta_{11} & \partial_\mu\Delta_{22}\end{vmatrix} &= \begin{vmatrix}\lambda_1 e^{X_1} + \mu_1 e^{-X_1} & \mu_1 e^{X_1} + \lambda_1 e^{-X_1}\\ r_\mu^{(1)}(\lambda_1 e^{X_1} - \mu_1 e^{-X_1}) & r_\mu^{(1)}(\mu_1 e^{X_1} - \lambda_1 e^{-X_1})\end{vmatrix}\\ &= -2r_\mu^{(1)}(\lambda_1^2 - \mu_1^2),\end{aligned} \tag{48}$$

$$\begin{vmatrix}\partial_\nu\Delta_{11} & \partial_\nu\Delta_{22}\\ \partial_\rho\Delta_{11} & \partial_\rho\Delta_{22}\end{vmatrix} = \begin{vmatrix}r_\nu^{(1)}(\lambda_1 e^{X_1} - \mu_1 e^{-X_1}) & r_\nu^{(1)}(\mu_1 e^{X_1} - \lambda_1 e^{-X_1})\\ r_\rho^{(1)}(\lambda_1 e^{X_1} - \mu_1 e^{-X_1}) & r_\rho^{(1)}(\mu_1 e^{X_1} - \lambda_1 e^{-X_1})\end{vmatrix} = 0, \tag{49}$$

$$\begin{vmatrix}\Delta_{12} & \Delta_{21}\\ \partial_\mu\Delta_{12} & \partial_\mu\Delta_{21}\end{vmatrix} = (\lambda_1 - \mu_1)^2\begin{vmatrix}e^{i\Theta_1} & e^{-i\Theta_1}\\ s_\mu^{(1)}e^{i\Theta_1} & -s_\mu^{(1)}e^{-i\Theta_1}\end{vmatrix} = -2s_\mu^{(1)}(\lambda_1 - \mu_1)^2, \tag{50}$$

$$\begin{vmatrix}\partial_\nu\Delta_{12} & \partial_\nu\Delta_{21}\\ \partial_\rho\Delta_{12} & \partial_\rho\Delta_{21}\end{vmatrix} = (\lambda_1 - \mu_1)^2\begin{vmatrix}s_\nu^{(1)}e^{i\Theta_1} & -s_\nu^{(1)}e^{-i\Theta_1}\\ s_\rho^{(1)}e^{i\Theta_1} & -s_\rho^{(1)}e^{-i\Theta_1}\end{vmatrix} = 0. \tag{51}$$

These facts imply that

$$\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}(\partial_\nu\sigma)\sigma^{-1}(\partial_\rho\sigma)\sigma^{-1}\right] = 0, \tag{52}$$

and therefore the Wess–Zumino action density is identical to zero for the one-soliton case. In fact, the identity (52) holds even when σ is not a classical solution of the WZW_4 model because the condition (34) is not used in our discussion. More explicitly, as long as the power L_j of the exponential function in ψ_j is an arbitrary linear function of x^μ, the Wess–Zumino action density vanishes identically.

All the above results also hold in the case of $\mathbb{E}$.

4.2. *Two-soliton solutions*

In this subsection, we calculate the action densities of the $SU(2)$ WZW_4 model explicitly for the two-soliton solutions.

By the result of Appendix D.3 together with Eqs. (36)–(38), we get the following compact form of the NLσM action density for the two-soliton solution:

$$
\begin{aligned}
\mathscr{L}_\sigma &= -\frac{1}{16\pi}\mathrm{Tr}\left[\left(\partial_\mu\sigma\right)\sigma^{-1}\left(\partial^\mu\sigma\right)\sigma^{-1}\right] \\
&= \frac{\begin{Bmatrix} ab\left[d_{11}\cosh^2 X_2 + d_{22}\cosh^2 X_1\right] \\ +ac\left[d_{12}\cosh^2\left(\dfrac{X_1+X_2-i\Theta_{12}}{2}\right) + d_{21}\cosh^2\left(\dfrac{X_1+X_2+i\Theta_{12}}{2}\right)\right] \\ -bc\left[e_{12}\sinh^2\left(\dfrac{X_1-X_2-i\Theta_{12}}{2}\right) + \bar{e}_{12}\sinh^2\left(\dfrac{X_1-X_2+i\Theta_{12}}{2}\right)\right] \end{Bmatrix}}{2\pi\left[a\cosh(X_1+X_2)+b\cosh(X_1-X_2)+c\cos\Theta_{12}\right]^2} \\
&= \frac{\begin{Bmatrix} 2ab\left[d_{11}\cosh^2 X_2 + d_{22}\cosh^2 X_1\right] \\ +c\left[a(d_{12}+d_{21})-b(e_{12}+\bar{e}_{12})\right]\cosh X_1\cosh X_2\cos\Theta_{12} \\ +c\left[a(d_{12}+d_{21})+b(e_{12}+\bar{e}_{12})\right](\sinh X_1\sinh X_2+1)\cos\Theta_{12} \\ -ic\left[a(d_{12}-d_{21})+b(e_{12}-\bar{e}_{12})\right]\sinh X_1\cosh X_2\sin\Theta_{12} \\ -ic\left[a(d_{12}-d_{21})-b(e_{12}-\bar{e}_{12})\right]\cosh X_1\sinh X_2\sin\Theta_{12} \end{Bmatrix}}{4\pi\left[a\cosh(X_1+X_2)+b\cosh(X_1-X_2)+c\cos\Theta_{12}\right]^2}
\end{aligned} \tag{53}
$$

where a, b, c, d_{jk}, e_{jk} are defined in Table 1 for each space. (The difference between $\mathbb{E}$ and $\mathbb{U}_1$ appears only in the coefficients like the one-soliton case.) Note that the coefficients in Table 1 also guarantee the NLσM action density to be real-valued on $\mathbb{U}_1$ and $\mathbb{E}$.

Next, let us explain why Eq. (53) can be interpreted as two intersecting one-solitons in the asymptotic region. Due to the solitonic property, individual one-solitons will regain all their features (wave shape, velocity, amplitude, etc.) outside the scattering region except for respective differences of an additional position shift. Theoretically, each one-soliton can be separated completely from the other one-soliton in the asymptotic region in which it mainly dominates the asymptotic behavior. Therefore, we can consider the following type of asymptotic limit:

$$
\begin{cases} X_1 \text{ is finite} \\ |X_2| \gg |X_1| \end{cases} \tag{54}
$$

in which the first one-soliton, localized on the hyperplane $X_1 = 0$, mainly dominates the asymptotic behavior. Such asymptotics will be discussed more systematically in Sect. 5. In the asymptotic limit (54), the action density (53) is dominated by

$$
\begin{aligned}
&\mathrm{Tr}\left[\left(\partial_\mu\sigma\right)\sigma^{-1}\left(\partial^\mu\sigma\right)\sigma^{-1}\right]_{|X_2|\gg|X_1|} \\
&= \frac{8abd_{11}\cosh^2 X_2 + \mathcal{O}(\cosh X_2)}{\left[a\cosh(X_1+X_2)+b\cosh(X_1-X_2)+\mathcal{O}(1)\right]^2} \\
&= \frac{8abd_{11} + \mathcal{O}(\mathrm{sech} X_2)}{\left[(a+b)\cosh X_1 + (a-b)\sinh X_1\tanh X_2 + \mathcal{O}(\mathrm{sech} X_2)\right]^2}.
\end{aligned} \tag{55}
$$

Since $\sinh X_1$ and $\cosh X_1$ are finite and $\mathrm{sech} X_2 \to 0$ and $\tanh X_2 \to \pm 1$ as $X_2 \to \pm\infty$, we have

$$\mathrm{Tr}\left[\left(\partial_\mu \sigma\right)\sigma^{-1}\left(\partial^\mu \sigma\right)\sigma^{-1}\right] \overset{X_2\to\pm\infty}{\longrightarrow} \frac{8abd_{11}}{[(a+b)\cosh X_1 \pm (a-b)\sinh X_1]^2}$$
$$= \begin{cases} \dfrac{8abd_{11}}{(ae^{X_1}+be^{-X_1})^2} & \text{as } X_2 \longrightarrow +\infty \\ \dfrac{8abd_{11}}{(be^{X_1}+ae^{-X_1})^2} & \text{as } X_2 \longrightarrow -\infty \end{cases}. \tag{56}$$

Now we conclude that

$$-8\pi\mathscr{L}_\sigma \longrightarrow \begin{cases} (1)\ X_1 \text{ is finite},\ X_2 \to +\infty:\ d_{11}\,\mathrm{sech}^2\left(X_1+\delta_1\right) \\ (2)\ X_1 \text{ is finite},\ X_2 \to -\infty:\ d_{11}\,\mathrm{sech}^2\left(X_1-\delta_1\right) \\ (3)\ X_2 \text{ is finite},\ X_1 \to +\infty:\ d_{22}\,\mathrm{sech}^2\left(X_2+\delta_2\right) \\ (4)\ X_2 \text{ is finite},\ X_1 \to -\infty:\ d_{22}\,\mathrm{sech}^2\left(X_2-\delta_2\right) \end{cases}, \tag{57}$$

where the position shift factors (or the phase shift factors) are

$$\delta_1 \equiv \delta_2 := \frac{1}{2}\log\left(\frac{a}{b}\right) = \frac{1}{2}\log\left[\frac{a(1,1)}{a(1,-1)}\right] \quad \text{(cf. Eq. (D8))}. \tag{58}$$

Cases (3) and (4) are obtained by the same argument and we just skip the details here. By the above analysis, we find that the NLσM action density (53) has two peaks that are localized on two non-parallel 3D hyperplanes described by the linear equation $X_1 \pm \delta_1 = 0$ and $X_2 \pm \delta_2 = 0$. A more general discussion for n-soliton case is given in Appendix D.5.

As for other asymptotic regions that differ from cases (1)–(4), no solitonic effect contributes to the action density; i.e., the action density is asymptotic to zero in these regions. This will be proved in Sect. 5.

Let us proceed to calculate the Wess–Zumino action density for the two-soliton solution. By substituting Eqs. (39), (D7), and (42) into Eq. (25) for $(m, n, p) = (\mu, \nu, \rho)$, we have

$$\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}(\partial_\nu\sigma)\sigma^{-1}(\partial_\rho\sigma)\sigma^{-1}\right] = \frac{1}{2}(B_{\mu\nu\rho}+B_{\nu\rho\mu}+B_{\rho\mu\nu}),$$
$$B_{\mu\nu\rho} := \frac{1}{|\sigma|^2\Delta^4}\left(\begin{vmatrix}\Delta_{11} & \Delta_{22}\\ \partial_\mu\Delta_{11} & \partial_\mu\Delta_{22}\end{vmatrix}\begin{vmatrix}\partial_\nu\Delta_{12} & \partial_\nu\Delta_{21}\\ \partial_\rho\Delta_{12} & \partial_\rho\Delta_{21}\end{vmatrix} + \begin{vmatrix}\Delta_{12} & \Delta_{21}\\ \partial_\mu\Delta_{12} & \partial_\mu\Delta_{21}\end{vmatrix}\begin{vmatrix}\partial_\nu\Delta_{11} & \partial_\nu\Delta_{22}\\ \partial_\rho\Delta_{11} & \partial_\rho\Delta_{22}\end{vmatrix}\right). \tag{59}$$

Here each ingredient of $B_{\mu\nu\rho}$ can be calculated in the same way as the previous section. For example, the result of the first determinant factor in Eq. (59) is

$$\frac{1}{|\sigma|\Delta^2}\begin{vmatrix}\Delta_{11} & \Delta_{22}\\ \partial_\mu\Delta_{11} & \partial_\mu\Delta_{22}\end{vmatrix}$$
$$= \frac{-\left\{\begin{gathered} 2r_\mu^{(1)}ab\mathscr{D}_{11}\cosh(2X_2) + 2r_\mu^{(2)}ab\mathscr{D}_{22}\cosh(2X_1) \\ +\left[(r_\mu^{(1)}+r_\mu^{(2)})+(s_\mu^{(1)}-s_\mu^{(2)})\right]ac\mathscr{D}_{12}\cosh(X_1+X_2-i\Theta_{12}) \\ +\left[(r_\mu^{(1)}+r_\mu^{(2)})-(s_\mu^{(1)}-s_\mu^{(2)})\right]ac\mathscr{D}_{21}\cosh(X_1+X_2+i\Theta_{12}) \\ +\left[(r_\mu^{(1)}-r_\mu^{(2)})+(s_\mu^{(1)}-s_\mu^{(2)})\right]bc\mathscr{E}_{12}\cosh(X_1-X_2-i\Theta_{12}) \\ -\left[(r_\mu^{(1)}-r_\mu^{(2)})-(s_\mu^{(1)}-s_\mu^{(2)})\right]bc\widetilde{\mathscr{E}}_{12}\cosh(X_1-X_2+i\Theta_{12}) \end{gathered}\right\} - F}{2\left[a\cosh(X_1+X_2)+b\cosh(X_1-X_2)+c\cos\Theta_{12}\right]^2}. \tag{60}$$

The definition of the coefficients and the result of the remaining determinant factors can be found in Appendix D.4. Furthermore, we can also show that the Wess–Zumino action density is real-valued on $\mathbb{U}_1$ and $\mathbb{E}$ (cf. Appendix D.4).

By the same technique used in the previous section, we consider the asymptotic limit such that $|X_2| \gg |X_1|$ for finite X_1, and find that

$$B_{\mu\nu\rho} \overset{|X_2|\gg|X_1|}{\simeq} \frac{-4a^2b^2 C_{\mu\nu\rho}\mathscr{D}_{11}\mathscr{d}_{11}\tanh X_2 + \mathcal{O}(\mathrm{sech} X_2)}{[(a+b)\cosh X_1 + (a-b)\sinh X_1 \tanh X_2 + \mathcal{O}(\mathrm{sech} X_2)]^4}, \tag{61}$$

where $C_{\mu\nu\rho} := \left(r_\mu^{(1)}s_\nu^{(1)} + s_\mu^{(1)}r_\nu^{(1)}\right) r_\rho^{(2)} - \left(r_\mu^{(1)}s_\rho^{(1)} + s_\mu^{(1)}r_\rho^{(1)}\right) r_\nu^{(2)}$. This is asymptotic to

$$B_{\mu\nu\rho} \overset{X_2\to\pm\infty}{\longrightarrow} \mp 4C_{\mu\nu\rho}\mathscr{D}_{11}\mathscr{d}_{11}\mathrm{sech}^4(X_1 \pm \delta_1), \tag{62}$$

where the phase shift factor is $\delta_1 := (1/2)\log(a/b)$.

In fact, the coefficient $C_{\mu\nu\rho}$ in Eq. (62) satisfies the following relation:

$$C_{\mu\nu\rho} + C_{\nu\rho\mu} + C_{\rho\mu\nu} = 0. \tag{63}$$

Therefore the cubic term (59) identically vanishes in the asymptotic region, and the Wess–Zumino action density is asymptotic to zero for the two-soliton case:

$$\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}(\partial_\nu\sigma)\sigma^{-1}(\partial_\rho\sigma)\sigma^{-1}\right] \longrightarrow 0. \tag{64}$$

A more general discussion for the n-soliton case can be found in Appendix D.5.

The Wess–Zumino action density for the two-soliton is a smooth function and non-singular and is hence bounded. Moreover, it decays to zero exponentially as mentioned in Sect. 3.2. Therefore, we conjecture that the Wess–Zumino action S_{WZ} would be zero exactly.

5. Asymptotic analysis of *n*-soliton solutions

Due to the problem of the singularity of the two-soliton solution on the Euclidean space $\mathbb{E}$, in this section, we consider a modified n-soliton solution and discuss the corresponding asymptotic behaviors in a systematic way. The modified n-soliton solution is

$$\sigma = \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_n & 1 \\ \psi_1\Lambda_1 & \psi_2\Lambda_2 & \cdots & \psi_n\Lambda_n & 0 \\ \psi_1\Lambda_1^2 & \psi_2\Lambda_2^2 & \cdots & \psi_n\Lambda_n^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & \psi_2\Lambda_2^n & \cdots & \psi_n\Lambda_n^n & \boxed{0} \end{vmatrix}, \quad \begin{array}{l} \psi_j = \begin{pmatrix} e^{L_j} & e^{-\overline{L}_j} \\ -\epsilon e^{-L_j} & e^{\overline{L}_j} \end{pmatrix}, \; L_j = \ell_\mu^{(j)}x^\mu \\ \Lambda_j = \begin{pmatrix} \lambda_j^{(+)} & 0 \\ 0 & \lambda_j^{(-)} \end{pmatrix} \end{array}, \tag{65}$$

where the spectral parameters (λ_j, μ_j) are rewritten by $(\lambda_j^{(+)}, \lambda_j^{(-)})$ for later convenience. The slight difference between Eqs. (65) and (35) is an additional constant factor ϵ taking values in $\{\pm 1\}$. The case of $\epsilon = +1$ coincides with Eq. (35). We will show that the non-singular n-solitons can be constructed completely for all $n \in \mathbb{N}$ by suitable choices of the constant ϵ with respect to the ultrahyperbolic space $\mathbb{U}_1$ and the Euclidean space $\mathbb{E}$.

First of all, we define two types of the asymptotic region for the n-soliton solutions. Let us consider the asymptotic region of the 4D space where $r^2 = (x^1) + (x^2) + (x^3) + (x^4)$ is large enough for the spacial point $x = (x^1, x^2, x^3, x^4)$. The asymptotic region is divided into 2^n regions by the n hyperplanes $X_j = 0$ $(j = 1, 2,\ldots, n)$ depending on $X_j > 0$ or $X_j < 0$. In order to label these regions, it is convenient to introduce a new notation $\varepsilon_j \in \{\pm 1, 0\}$. Then the 2^n asymptotic regions can be denoted by $\mathscr{R}(\varepsilon_1, \ldots, \varepsilon_n)$ in which $\varepsilon_j = +1$, and $\varepsilon_j = -1$ correspond to the following cases (+) and (−) respectively:[4]

$$(+)X_j \gg +1 \Leftrightarrow \mathrm{Re}L_j \gg +1 \Leftrightarrow \left|e^{L_j}\right| \gg 1 \Leftrightarrow \left|e^{-L_j}\right| \ll 1,$$
$$(-)X_j \ll -1 \Leftrightarrow \mathrm{Re}L_j \ll -1 \Leftrightarrow \left|e^{L_j}\right| \ll 1 \Leftrightarrow \left|e^{-L_j}\right| \gg 1. \qquad (66)$$

Then we can unify the asymptotic regions as

$$\mathscr{R} := \bigcup_{\varepsilon_j=\pm 1} \mathscr{R}(\varepsilon_1, \ldots, \varepsilon_n). \qquad (67)$$

We will see that the Wess–Zumino action density vanishes in $\mathscr{R}$.

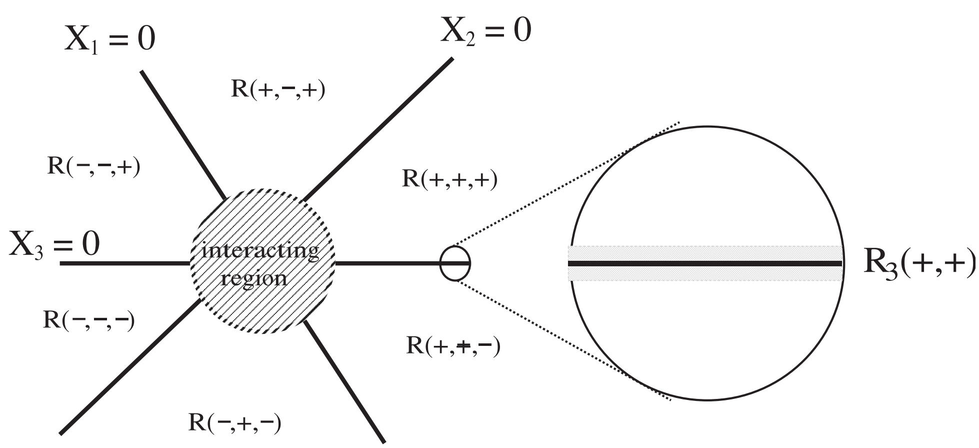
On the other hand, there is the other type of asymptotic region along the hyperplane X_j, which corresponds to the case of $\varepsilon_j = 0$. To make the asymptotic region 4D, let us define the asymptotic region along X_K as a tubular neighborhood of $\mathscr{R}(\varepsilon_1, \ldots, \varepsilon_K = 0, \ldots, \varepsilon_n)$, which is denoted by $\mathscr{R}_K(\varepsilon_1, \cdots, \varepsilon_n)$. In this region, the value of X_K is considered to be finite. We will see that the NLσM and the Wess–Zumino action densities coincide with a one-soliton configuration in $\mathscr{R}_K$.

The two type of asymptotic regions can be expressed in terms of the following sets:

$$\mathscr{R} := \left\{ x \in M_4 \,\middle|\, \begin{array}{l} x_1^2 + x_2^2 + x_3^2 + x_4^2 \text{ is large enough.} \\ X_j \text{ are all positive or negative } (j = 1, \ldots, n). \end{array} \right\}$$

$$\mathscr{R}_K := \left\{ x = k_1 e^1 + k_2 e^2 + k_3 e^3 + a \in M_4 \,\middle|\, \begin{array}{l} e^1, e^2, e^3 \text{ are linearly independent vectors} \\ \text{ tangent to the hyperplane: } X_K = 0. \\ a \text{ is a finite vector.} \\ k_1^2 + k_2^2 + k_3^2 \text{ is large enough.} \\ X_j (j \neq K) \text{ are all positive or negative.} \end{array} \right\}$$

A simple example of the asymptotic regions is shown in the figure below ($\mathscr{R}_3$ is shown in pink).



For the asymptotic regions of type $\mathscr{R}$, the behavior of Eq. (65) is dominated by all X_j for large enough $|X_j|$. By Eq. (66), we find that σ is asymptotic to a constant matrix for each asymptotic

[4]Here we suppose that $|X_j|$ are large enough in the asymptotic region and hence $X_j > 0$ implies $X_j \gg +1$.

region of the type $\mathscr{R}$:

$$\sigma \overset{\mathscr{R}}{\simeq} \begin{vmatrix} C_1^{(\pm)} & \cdots & C_n^{(\pm)} & 1 \\ C_1^{(\pm)}\Lambda_1 & \cdots & C_n^{(\pm)}\Lambda_n & 0 \\ \vdots & & \vdots & \vdots \\ C_1^{(\pm)}\Lambda_1^n & \cdots & C_n^{(\pm)}\Lambda_n^n & \boxed{0} \end{vmatrix}, \quad \text{where } C_i^{(+)} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ C_i^{(-)} := \begin{pmatrix} 0 & 1 \\ -\epsilon & 0 \end{pmatrix}.$$

The suffix $(\pm)$ in C_j corresponds to the signature of ε_j. Therefore, the action densities $\mathscr{L}_\sigma$ and $\mathscr{L}_{\mathrm{WZ}}$ identically vanish in the type $\mathscr{R}$ asymptotic regions.

On the other hand, since X_K is kept to be finite for the type $\mathscr{R}_K$ asymptotic regions, we have [21]

$$\sigma \overset{\mathscr{R}_K}{\simeq} \begin{vmatrix} C_1^{(\pm)} & \cdots & \psi_K & \cdots & C_n^{(\pm)} & 1 \\ C_1^{(\pm)}\Lambda_1 & \cdots & \psi_K\Lambda_K & \cdots & C_n^{(\pm)}\Lambda_n & 0 \\ \vdots & & \vdots & & \vdots & \vdots \\ C_1^{(\pm)}\Lambda_1^n & \cdots & \psi_K\Lambda_K^n & \cdots & C_n^{(\pm)}\Lambda_n^n & \boxed{0} \end{vmatrix}.$$

This actually leads to the following one-soliton-type solution [21]:[5]

$$\sigma \overset{\mathscr{R}_K}{\simeq} \begin{vmatrix} \check{\psi}_K & 1 \\ \check{\psi}_K\Lambda_K & \boxed{0} \end{vmatrix} D^{(k)}, \quad \text{where} \quad \begin{cases} \check{\psi}_K := \begin{pmatrix} a_K e^{L_K} & b_K e^{-\overline{L}_K} \\ -c_K e^{-L_K} & d_K e^{\overline{L}_K} \end{pmatrix} \\ D_K : \text{a constant matrix,} \end{cases} \tag{68}$$

and the coefficients a_K, b_K, c_K, d_K can be expressed in terms of the spectral parameters as:

$$\begin{aligned} a_K &= \prod_{j=1,\, j\neq K}^{n} \left(\lambda_K^{(+)} - \lambda_j^{(\pm)}\right), \quad b_K = \prod_{j=1,\, j\neq K}^{n} \left(\lambda_K^{(-)} - \lambda_j^{(\pm)}\right), \\ c_K &= \prod_{j=1,\, j\neq K}^{n} \left(\lambda_K^{(+)} - \lambda_j^{(\mp)}\right)\epsilon, \quad d_K = \prod_{j=1,\, j\neq K}^{n} \left(\lambda_K^{(-)} - \lambda_j^{(\mp)}\right). \end{aligned} \tag{69}$$

In fact, we will see later that the coefficients a_K, b_K, c_K, d_K determine the position shift (known as the phase shift) of the one-soliton solution (68) in each asymptotic region of type $\mathscr{R}_K$. Furthermore, these coefficients also determine whether the singularities of n-solitons exist in the action density of the WZW_4 model. First of all, we can calculate the asymptotic form of the following μth component of the quadratic term by using Eq. (68). (The summation is not taken

[5] We note that the two operations of taking the limit and taking the derivation do not commute in general; however, in our case, they do commute. This is proved in Appendix B.

over μ.) The result is

$$\begin{aligned}\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}\right]^2 &:= \mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}(\partial_\mu\sigma)\sigma^{-1}\right]\\ &\overset{\mathscr{R}_K}{\simeq} \frac{8\left|\ell_\mu^{(K)}\right|^2(\lambda_K^{(+)}-\lambda_K^{(-)})^2}{\lambda_K\mu_K}\cdot\frac{a_Kb_Kc_Kd_K}{(a_Kd_Ke^{X_K}+b_Kc_Ke^{-X_K})^2}\\ &= \frac{8\left|\ell_\mu^{(K)}\right|^2(\lambda_K^{(+)}-\lambda_K^{(-)})^2}{\lambda_K^{(+)}\lambda_K^{(-)}}\cdot\frac{1}{\left(\dfrac{a_Kd_K}{b_Kc_K}e^{2X_K}+\dfrac{b_Kc_K}{a_Kd_K}e^{-2X_K}+2\right)}\\ &= \frac{8\left|\ell_\mu^{(K)}\right|^2(\lambda_K^{(+)}-\lambda_K^{(-)})^2}{\lambda_K^{(+)}\lambda_K^{(-)}}\cdot\frac{1}{e^{i\varphi_K}\left(e^{X_K+\delta_K}+e^{-i\varphi_K}e^{-(X_K+\delta_K)}\right)^2},\end{aligned}$$

where $\delta_K := (1/2)\log r_K$, $r_K := |a_Kd_K/b_Kc_K|$ and the ratio $a_Kd_K/b_Kc_K := r_Ke^{i\varphi_K}$. In particular, $a_Kd_K/b_Kc_K \in \mathbb{R}^+$ if $\varphi_K = 0$ and $a_Kd_K/b_Kc_K \in \mathbb{R}^-$ if $\varphi_K = \pi$. This fact implies

$$\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}\right]^2 \overset{\mathscr{R}_K}{\simeq} \begin{cases} \dfrac{2\left|\ell_\mu^{(K)}\right|^2\left(\lambda_K^{(+)}-\lambda_K^{(-)}\right)^2}{\lambda_K^{(+)}\lambda_K^{(-)}}\,\mathrm{sech}^2(X_K+\delta_K) \ \text{ if } \ a_Kd_K/b_Kc_k \in \mathbb{R}^+ \\ \dfrac{-2\left|\ell_\mu^{(K)}\right|^2\left(\lambda_K^{(+)}-\lambda_K^{(-)}\right)^2}{\lambda_K^{(+)}\lambda_K^{(-)}}\,\mathrm{csch}^2(X_K+\delta_K) \ \text{ if } \ a_Kd_K/b_Kc_k \in \mathbb{R}^- \end{cases} \tag{70}$$

where $\mathrm{csch}\,x := 1/\sinh x$. Apparently, for $a_Kd_K/b_Kc_k < 0$, the singularities exist on the entire 3D hyperplane $X_K + \delta_K = 0$.

Now let us find out the condition such that the NLσM action density is non-singular. For the ultrahyperbolic space $\mathbb{U}_1$, the reality condition is $\lambda_j^{(-)} = \overline{\lambda}_j^{(+)}$. By Eq. (69), we have

$$\frac{a_Kd_K}{b_Kc_K} = \frac{1}{\epsilon}\prod_{j=1, j\neq K}^{n}\left|\frac{\lambda_K-\lambda_j}{\lambda_K-\overline{\lambda}_j}\right|^{2\varepsilon_j} \quad \text{on } \ \mathbb{U}_1. \tag{71}$$

Comparing this with Eq. (70), we can conclude that, in the case of $\epsilon = +1$, the NLσM action density of the n-soliton is definitely asymptotic to a non-singular one-soliton for the ultrahyperbolic signature. This fact implies that, for all $n \in \mathbb{N}$, the n-soliton solution (35) gives a class of non-singular NLσM action densities for the ultrahyperbolic signature.

Similarly, the reality condition of the Euclidean space $\mathbb{E}$: $\lambda_j^{(-)} = -1/\overline{\lambda}_j^{(+)}$ implies

$$\frac{a_Kd_K}{b_Kc_K} = \frac{(-1)^{n-1}}{\epsilon}\prod_{j=1, j\neq K}^{n}\left|\frac{\lambda_K-\lambda_j}{\lambda_K\overline{\lambda}_j+1}\right|^{2\varepsilon_j} \quad \text{on } \ \mathbb{E}. \tag{72}$$

The ratio (72) is positive in the following two cases: (1) n is odd and $\epsilon = +1$ or (2) n is even and $\epsilon = -1$. Then the NLσM action densities are non-singular. On the other hand, the ratio (72) is negative in the following two cases: (3) n is even and $\epsilon = +1$[6] or (4) n is odd and $\epsilon = -1$.[7] Then the NLσM action densities are singular. It is quite interesting that, in the Euclidean signature, singular and non-singular solutions are generated alternately by the Darboux transformations with respect to initial solutions ψ_j for $\epsilon = \pm 1$.

[6] Case (3) for $n = 2$ corresponds to the singular two-soliton solution on $\mathbb{E}$ (cf. Table 1).
[7] Case (4) for $n = 1$ corresponds to the singular solution in footnote 5.

In summary, for all $n \in \mathbb{N}$, non-singular NLσM action densities of the n-soliton can be constructed by taking $\epsilon = +1$ for all $n \in \mathbb{N}$ (cf. Eq. (71)) on the ultrahyperbolic space $\mathbb{U}_1$, and by taking $\epsilon = +1$ for all odd n and $\epsilon = -1$ for all even n (cf. Eq. (72) and cases (1) and (2)) on the Euclidean space $\mathbb{E}$. They would share the same asymptotic form in $\mathscr{R}_K$ on each real space:

$$\mathscr{L}_\sigma = -\frac{1}{16\pi}\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}(\partial^\mu\sigma)\sigma^{-1}\right] \overset{\mathscr{R}_K}{\simeq} -\frac{1}{8\pi}d_{KK}\,\mathrm{sech}^2\left(X_K + \delta_K\right), \tag{73}$$

where d_{KK} is defined in Table 1 (cf. $\mathbb{U}_1$ and $\mathbb{E}$) and the phase shift factor is

$$\delta_K = \frac{1}{2}\log\left(\frac{a_K d_K}{b_K c_K}\right) = \begin{cases} \displaystyle\sum_{j=1,\, j\neq K}^{n} \varepsilon_j \log\left|\frac{\lambda_K - \lambda_j}{\lambda_K - \bar{\lambda}_j}\right| & \text{on } \mathbb{U}_1 \\ \displaystyle\sum_{j=1,\, j\neq K}^{n} \varepsilon_j \log\left|\frac{\lambda_K - \lambda_j}{\lambda_K\bar{\lambda}_j + 1}\right| & \text{on } \mathbb{E} \end{cases}. \tag{74}$$

Since the result of Eq. (73) is valid for arbitrary K in $\{1, 2,\dots, n\}$, we can regard the behavior of non-singular n-soliton as a "non-linear superposition" of n non-singular and mutually non-parallel one-solitons on each real space in which each one-soliton in the asymptotic region $\mathscr{R}_K$ keeps its form invariant but is shifted by δ_K, called the phase shift factor, which results from a non-linear effect.

In conclusion, in the asymptotic region, the n-soliton solution possesses n isolated and localized lumps of the NLσM action density, and we can interpret this as n intersecting soliton walls. The phase shift factors are also obtained explicitly. The scattering process of the n-soliton solution is quite similar to that of the KP solitons [3,25,26]. On the other hand, the Wess–Zumino action density identically vanishes in the asymptotic region because in the asymptotic region $\mathscr{R}$, the action density identically vanishes, and in the asymptotic region $\mathscr{R}_K$, the n-soliton solution is reduced to the one-soliton (68) whose Wess–Zumino action density is identically zero as proved in Sect. 4.1.

6. Reduction to (1 + 2) dimensions

So far, we have discussed the action density of the WZW$_4$ model for n-soliton solutions and have found that it is localized on n non-parallel codimension-one hyperplanes in four dimensions. However, to understand better the physical meaning of our soliton solutions, it would be a good idea to calculate the energy density of the soliton solutions and compare it with the action density. For this purpose, we assume translation invariance in the x^2 direction.

The WZW$_4$ model Lagrangian is reduced to the following one (cf. Eqs. (17) and (27)):

$$\mathscr{L}_{\mathrm{tot}} = -\frac{1}{16\pi}\left(\mathrm{Tr}(\theta_t)^2 - \mathrm{Tr}(\theta_x)^2 - \mathrm{Tr}(\theta_y)^2 + 2\mathrm{Tr}\left(\theta_t\theta_x\theta_y\right)y\right), \tag{75}$$

where we reset $(t, x, y) := (x_1, x_3, x_4)$ and $\theta_\mu := (\partial_\mu\sigma)\sigma^{-1}$ ($\mu = t, x, y$). The equation of motion is the Ward chiral model [27] or the space-time monopole equation [28,29] in (1 + 2) dimensions in the Yang form. The n-soliton solution of Eq. (75) is obtained by imposing the condition $\alpha_j = \lambda_j\beta_j$ (cf. Eq. (37)) on the powers L_j of the n-soliton solution (35). Then, the powers of the n-soliton solution is actually reduced to $L_j = (\beta_j/\sqrt{2})\left[(\lambda_j^2 + 1)t + (\lambda_j^2 - 1)x + 2\lambda_j y\right]$ in the (1+2)D space-time.

Let us consider three angular coordinates $\phi_i(x)$ ($i = 1, 2, 3$) that parametrize $SU(2) \approx S^3$, where $\sigma(x)$ belongs. Then the Hamiltonian density can be obtained by the Legendre transformation of the Lagrangian:

$$\mathscr{H}_{\rm tot} = \sum_{i=1}^{3} \frac{\partial \mathscr{L}_{\rm tot}}{\partial(\partial_t \phi_i)} \partial_t \phi_i - \mathscr{L}_{\rm tot} = -\frac{1}{16\pi} \left({\rm Tr}(\theta_t)^2 + {\rm Tr}(\theta_x)^2 + {\rm Tr}(\theta_y)^2 \right).$$

This Hamiltonian physically makes sense because it is positive definite due to the fact that θ_μ is an anti-Hermitian matrix. This is a conserved energy density by definition. Note that the contribution of the Wess–Zumino term to $\mathscr{H}_{\rm tot}$ vanishes identically: $\mathscr{H}_{\rm WZ} = 0$, or equivalently $\mathscr{H}_{\rm tot} = \mathscr{H}_\sigma$.

Let us calculate the energy density of the reduced soliton solution from the Hamiltonian density $\mathscr{H}_{\rm tot}$. For the one-soliton solution, the Hamiltonian $\mathscr{H}_{\rm tot}$ is:

$$\mathscr{H}_{\rm tot} = -\frac{1}{8\pi} d_{11} {\rm sech}^2 X_1, \quad d_{11} := \frac{(|\lambda_1|^2+1)^2(\lambda_1 - \overline{\lambda}_1)^2}{|\lambda_1|^2} |\beta_1|^2. \tag{76}$$

This is in the same form as the reduced NLσM action density $\mathscr{L}_\sigma$ up to an overall coefficient (cf. Eq. (47)). Therefore, the peaks of $\mathscr{H}_{\rm tot}$ and $\mathscr{L}_\sigma$ are localized on the same 2D hyperplane $X = 0$ in the $(1+2)$D space-time. In this sense, $\mathscr{L}_\sigma$ can also be interpreted as an analogue of the energy density in physical reality.

For the two-soliton solution, the Hamiltonian density for the two-soliton solution is calculated by using the result of Appendix D.3; we have

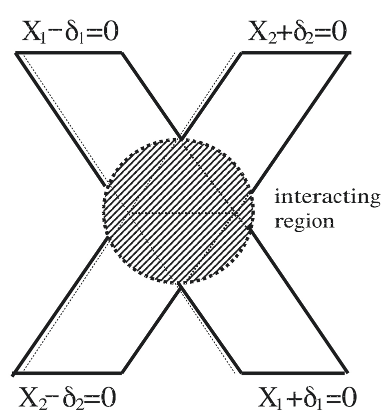
$$\mathscr{H}_{\rm tot} = -\frac{\left\{ \begin{aligned} & ab\left[d_{11}\cosh^2 X_2 + d_{22}\cosh^2 X_1\right] \\ & + ac\left[d_{12}\cosh^2\left(\frac{X_1+X_2-i\Theta_{12}}{2}\right) + d_{21}\cosh^2\left(\frac{X_1+X_2+i\Theta_{12}}{2}\right)\right] \\ & - bc\left[e_{12}\sinh^2\left(\frac{X_1-X_2-i\Theta_{12}}{2}\right) + \overline{e}_{12}\sinh^2\left(\frac{X_1-X_2+i\Theta_{12}}{2}\right)\right] \end{aligned} \right\}}{2\pi[a\cosh(X_1+X_2) + b\cosh(X_1-X_2) + c\cos\Theta_{12}]^2}, \tag{77}$$

where a, b, c are the same coefficients defined in Table 1 and

$$d_{jk} := \frac{(\lambda_j\overline{\lambda}_k+1)^2(\lambda_j-\overline{\lambda}_k)^2}{\lambda_j\overline{\lambda}_k}\beta_j\overline{\beta}_k, \quad e_{jk} := \frac{(\lambda_j\lambda_k+1)^2(\lambda_j-\lambda_k)^2}{\lambda_j\lambda_k}\beta_j\beta_k. \tag{78}$$

As for the NLσM term, the Hamiltonian density $\mathscr{H}_{\rm tot}$ for the two-soliton is also in the same form as the reduced NLσM action density $\mathscr{L}_\sigma$ up to the differences of the coefficients d_{ij} and e_{ij} (cf. Eq. (53)). Therefore, the two peaks of $\mathscr{H}_{\rm tot}$ are localized on the same hyperplanes $X_1 \pm \delta_1 = 0$ and $X_2 \pm \delta_2 = 0$ as those of $\mathscr{L}_\sigma$ (cf. Eq. (57)). The phase shift factors are also exactly the same. There is no singularity either. This result implies that there is no essential difference between $\mathscr{H}_{\rm tot}$ and $\mathscr{L}_\sigma$ for describing the solitonic properties.

The peaks of the energy density of the two-soliton solutions are shown in the figure below.



On the other hand, as for the Wess–Zumino term, there is a mismatch that the Hamiltonian density $\mathscr{H}_{\rm WZ}$ is identical to zero, while we cannot confirm whether the action density $\mathscr{L}_{\rm WZ}$ is. The physical meaning of this mismatch should be clarified in future work.

7. Conclusion and discussion

In this paper, we calculated the action density of the WZW_4 model for the classical soliton solutions. We found that, for the one-soliton solutions, the NLσM action density is localized on a 3D hyperplane and the Wess–Zumino action density identically vanishes. This suggests the existence of a three-brane in the open $N = 2$ string theory. For the two-soliton solutions, the NLσM action density has a beautiful compact form that represents two intersecting one-solitons. The Wess–Zumino action density does not vanish in the interaction region but does vanish in the asymptotic region. For the n-soliton solutions, we clarified the asymptotic behavior and found that the NLσM action density describes "non-linear superposition" of n intersecting one-solitons and the Wess–Zumino action density asymptotically vanishes. The non-linear interaction gives rise to phase shifts that were evaluated explicitly. We also calculated the Hamiltonian (energy) density of the one- and two-soliton solutions of the reduced WZW model in $(1 + 2)$ dimensions. We found that the energy density of the Wess–Zumino term identically vanishes, and the energy density of the NLσM term has the same profile as the action density for our soliton solutions. The peaks of the energy densities perfectly coincide with those of the action density, including the phase shift factor.

We also discussed whether singularities exist for the n-soliton solutions. For the one- and two-soliton solutions, we proved that there is no singularity. For the n-soliton solutions ($n \geq 3$), it is unsolved; however, we can argue as follows. The existence of singularities in the solution is equivalent to the existence of zeros in the data Δ, which is a polynomial of e^{X_j} and $e^{i\Theta_j}$. Because X_j and Θ_j are linear functions of the real coordinates, possible singularities would lie on the intersection of $X_j = C_j$ and/or $\Theta_j = D_j$ where C_j and D_j are constants. These possibilities are mostly forbidden because there is no singularity of the action density of the n-soliton solutions in the asymptotic region where the intersection still exists. The intersection of just four hyperplanes defined by $X_j = C_j$ or $\Theta_j = D_j$ ($j = 1, 2, 3, 4$) gives rise to an isolated singularity. This possibility might arise when the parameters in the solutions are appropriately tuned, which should be clarified in the future.

The next step is to clarify the roles and properties of the soliton solutions in the open $N = 2$ string theory. At least we can see that they are not D-branes because the number of solitons is not related to the rank of the gauge group. It is worth studying the topological charge and mass of the solitons, and explicit calculation of infinite conserved densities [8, 30–32] for the n-soliton solutions. It is also interesting to construct resonance solutions of the solitons that represent the three-brane reconnections or, in other words, annihilation and creation of the three-branes. Then a classification of the soliton solutions could be possible, like the positive Grassmannian description of the KP solitons by Kodama and Williams [33]. The moduli space of the n-soliton solutions could be described in a geometrical framework. Extension of the model to non-commutative spaces would allow the presence of background B-fields in the open $N = 2$ string theory [34,35,36,37]. The isolated singularities mentioned above might be resolved and new physical objects appear on the non-commutative spaces such as non-commutative $U(1)$ instantons [38]. Sen's conjecture on the tachyon condensation (for a review see Ref. [39]) could be confirmed by the solution-generating technique [40] in the context of the open $N = 2$ string theory.

Furthermore, the WZW_4 model can be realized in the context of the twistor string theory [41]. Recently, Bittleston and Skinner showed that a meromorphic Chern–Simons theory on the twistor space in six dimensions has a double fibration structure that gives rise to the WZW_4 model by solving along fibers in one direction and the 4D Chern–Simons theory by symmetry reduction in another direction [42]. These models are connected to each other and have a close relationship to integrable systems [43]. The KP equation has not yet obtained a symmetry reduction of the anti-self-dual Yang–Mills equation so far; however, this 6D Chern–Simons theory might give a "unified theory" of integrable systems including both the Sato theory [44] of the KP equation and the twistor descriptions of classical integrable systems. This might give a stringy viewpoint to various aspects of integrability and duality. The relation to mirror symmetry is also exciting [45].

Acknowledgements

M.H. thanks the string group members at Nagoya University for useful comments at the String Journal Club on July 21, 2022. M.H. is also grateful to the YITP at Kyoto University, where he had fruitful discussions at the conference on Strings and Fields 2022 (YITP-W-22-09) on August 19, 2022. The work of H.K. and S.C.H. is supported in part by a Grant-in-Aid for Scientific Research (#18K03274). The work of S.C.H. is supported by the Iwanami Fujukai Foundation.

Funding

Open Access funding: SCOAP^3.

A. Brief review of quasideterminants

In this subsection, we excerpt some necessary pre-knowledge of quasideterminants mentioned in Sect. 2 of the previous paper [21]. It is a brief review of the work of Gelfand and Retakh [46,47] (see also, e.g., Refs. [17, 48]).

The quasideterminant is defined for an $n \times n$ matrix X where matrix elements belong to a non-commutative ring. The quasideterminant is a non-commutative generalization of the matrix determinant in this sense; however, it rather has a direct relation to the inverse matrix of X.

Let $X = (x_{ij})$ be an $n \times n$ invertible matrix over a non-commutative ring and $Y = (y_{ij})$ be the inverse matrix of X: $XY = YX = 1$. The existence of Y is assumed. Then the (i, j)th quasideterminant of X is defined as the inverse of an element of $Y = X^{-1}$:

$$|X|_{ij} := y_{ji}^{-1}. \tag{A1}$$

This has a convenient expression as follows:

$$|X|_{ij} = \begin{vmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1n} \\ \vdots & & \vdots & & \vdots \\ x_{i1} & \cdots & \boxed{x_{ij}} & \cdots & x_{in} \\ \vdots & & \vdots & & \vdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nn} \end{vmatrix}. \tag{A2}$$

When the matrix elements belong to a commutative ring, e.g., $\mathbb{C}$, the quasideterminant can be represented as a ratio of ordinary determinants by virtue of the Laplace formula on inverse matrices:

$$|X|_{ij} = y_{ji}^{-1} = (-1)^{i+j} \frac{\det X}{\det X^{ij}}, \tag{A3}$$

where X^{ij} is a matrix obtained from X by deleting the ith row and jth column.

In order to find another representation of the quasideterminant, let us consider the inverse matrix formula for the 2×2 block matrix divided as follows:

$$X^{-1} = \begin{pmatrix} A & B \\ C & d \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}Bs^{-1}CA^{-1} & -A^{-1}Bs^{-1} \\ -s^{-1}CA^{-1} & s^{-1} \end{pmatrix},$$

where A is a square matrix, d is a single element, and $s := d - CA^{-1}B$ is called the Schur complement. The quantity s^{-1} is just the (n, n)-element of X^{-1} and hence the quasideterminant $|X|_{nn}$ is s. If we decompose X into a 2×2 block matrix where x_{ij} corresponds to the single element d, the (i, j)th quasideterminant can be expressed in the form of the Schur complement:

$$|X|_{ij} = x_{ij} - \sum_{k(\neq i), l(\neq j)} x_{ik} (X^{ij})^{-1}_{kl} x_{lj}, = x_{ij} - \sum_{k(\neq i), l(\neq j)} x_{ik} |X^{ij}|^{-1}_{lk} x_{lj}. \tag{A4}$$

By using this, explicit representations of the quasideterminants can be obtained iteratively.

We note that the quasideterminant is well defined in the case that each matrix element x_{ij} in Eq. (A2) take values in $GL(N, \mathbb{C})$. (Then, X is an $nN \times nN$ matrix.) The following example of the $N = 2$ case can be expressed finally by the ratios of determinants due to Eqs. (A4) and (A3):

$$\begin{vmatrix} M & C_1 & C_2 \\ R_1 & \boxed{a} & \boxed{b} \\ R_2 & \boxed{c} & \boxed{d} \end{vmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} M^{-1} \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{vmatrix} M & C_1 \\ R_1 & \boxed{a} \end{vmatrix} & \begin{vmatrix} M & C_2 \\ R_1 & \boxed{b} \end{vmatrix} \\ \begin{vmatrix} M & C_1 \\ R_2 & \boxed{c} \end{vmatrix} & \begin{vmatrix} M & C_2 \\ R_2 & \boxed{d} \end{vmatrix} \end{pmatrix} = \frac{1}{|M|} \begin{pmatrix} \begin{vmatrix} M & C_1 \\ R_1 & a \end{vmatrix} & \begin{vmatrix} M & C_2 \\ R_1 & b \end{vmatrix} \\ \begin{vmatrix} M & C_1 \\ R_2 & c \end{vmatrix} & \begin{vmatrix} M & C_2 \\ R_2 & d \end{vmatrix} \end{pmatrix}. \tag{A5}$$

The final form corresponds to the parametrization (21) of σ for the soliton solution (33):

$$\Delta = |M|, \; -\Delta_{11} = \begin{vmatrix} M & C_1 \\ R_1 & a \end{vmatrix}, \; -\Delta_{12} = \begin{vmatrix} M & C_2 \\ R_1 & b \end{vmatrix}, \; -\Delta_{21} = \begin{vmatrix} M & C_1 \\ R_2 & c \end{vmatrix}, \; -\Delta_{22} = \begin{vmatrix} M & C_2 \\ R_2 & d \end{vmatrix}, \tag{A6}$$

which leads to the soliton data (40).

Here we summarize some properties and identities of the quasideterminant, which are relevant to discussions in this paper.

Proposition A.1 *[17, 46–48]*

Let $A = (a_{ij})$ be a square matrix of order n in (1), while in (2) and (3) appropriate partitions are made so that all matrices in quasideterminants are square.

(1) The common multiplication of rows and columns
For any invertible elements Λ_j ($j = 1, \ldots, n$), we have

$$\begin{vmatrix} a_{1,1}\Lambda_1 & \cdots & a_{1,j}\Lambda_j & \cdots & a_{1,n}\Lambda_n \\ \vdots & & \vdots & & \vdots \\ a_{i,1}\Lambda_1 & \cdots & \boxed{a_{i,j}\Lambda_j} & \cdots & a_{i,n}\Lambda_n \\ \vdots & & \vdots & & \vdots \\ a_{n,1}\Lambda_1 & \cdots & a_{n,j}\Lambda_j & \cdots & a_{n,n}\Lambda_n \end{vmatrix} = \begin{vmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & & \vdots & & \vdots \\ a_{i,1} & \cdots & \boxed{a_{i,j}} & \cdots & a_{i,n} \\ \vdots & & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,j} & \cdots & a_{n,n} \end{vmatrix} \Lambda_j, \tag{A7}$$

$$\begin{vmatrix} \Lambda_1 a_{1,1} & \cdots & \Lambda_1 a_{1,j} & \cdots & \Lambda_1 a_{1,n} \\ \vdots & & \vdots & & \vdots \\ \Lambda_i a_{i,1} & \cdots & \boxed{\Lambda_i a_{i,j}} & \cdots & \Lambda_i a_{i,n} \\ \vdots & & \vdots & & \vdots \\ \Lambda_n a_{n,1} & \cdots & \Lambda_n a_{n,j} & \cdots & \Lambda_n a_{n,n} \end{vmatrix} = \Lambda_i \begin{vmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & & \vdots & & \vdots \\ a_{i,1} & \cdots & \boxed{a_{i,j}} & \cdots & a_{i,n} \\ \vdots & & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,j} & \cdots & a_{n,n} \end{vmatrix}. \tag{A8}$$

(2) Non-commutative Jacobi identity [49]

$$\begin{vmatrix} a & R & b \\ P & M & Q \\ c & S & \boxed{d} \end{vmatrix} = \begin{vmatrix} M & Q \\ S & \boxed{d} \end{vmatrix} - \begin{vmatrix} P & M \\ \boxed{c} & S \end{vmatrix} \begin{vmatrix} \boxed{a} & R \\ P & M \end{vmatrix}^{-1} \begin{vmatrix} R & \boxed{b} \\ M & Q \end{vmatrix}. \tag{A9}$$

(3) Homological relations [46, 47, 49]

$$\begin{aligned} \begin{vmatrix} a & R & b \\ P & M & Q \\ \boxed{c} & S & d \end{vmatrix} &= \begin{vmatrix} a & R & b \\ P & M & Q \\ c & S & \boxed{d} \end{vmatrix} \begin{vmatrix} a & R & b \\ P & M & Q \\ \boxed{0} & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} a & R & \boxed{b} \\ P & M & Q \\ c & S & d \end{vmatrix} \\ &= \begin{vmatrix} a & R & \boxed{0} \\ P & M & 0 \\ c & S & 1 \end{vmatrix} \begin{vmatrix} a & R & b \\ P & M & Q \\ c & S & \boxed{d} \end{vmatrix}. \end{aligned} \tag{A10}$$

B. Proof of statement in footnote 8

Proposition B.1 *Let σ be the n-soliton solution defined by*

$$\sigma = \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_n & 1 \\ \psi_1\Lambda_1 & \psi_2\Lambda_2 & \cdots & \psi_n\Lambda_n & 0 \\ \psi_1\Lambda_1^2 & \psi_2\Lambda_2^2 & \cdots & \psi_n\Lambda_n^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & \psi_2\Lambda_2^n & \cdots & \psi_n\Lambda_n^n & \boxed{0} \end{vmatrix}, \quad \begin{aligned} &\psi_j = \begin{pmatrix} e^{L_j} & e^{-\bar{L}_j} \\ -e^{-L_j} & e^{\bar{L}_j} \end{pmatrix}, \; L_j = \ell_\mu^{(j)} x^\mu \\ &\Lambda_j := \begin{pmatrix} \lambda_j & 0 \\ 0 & \mu_j \end{pmatrix} \end{aligned}, \tag{B1}$$

and $\mathscr{R}_K$ be the asymptotic region defined by the asymptotic limit

$$\begin{cases} \mathrm{Re}L_K \text{ is fixed} \\ \mathrm{Re}L_{j,j\neq K} \longrightarrow \pm\infty \end{cases}. \tag{B2}$$

Then the operation of the partial derivative ∂_μ commutes with the operation of the asymptotic limit (B2) for σ.

Proof. Without loss of generality, we consider the $K = 1$ case due to the fact that the quasideterminant $|\sigma|_{ij}$ does not depend on permutations of rows and columns in the matrix σ [46,47].

For $j \neq 1$, $\psi_j \Lambda_j^m$ can be decomposed into

$$\psi_j \Lambda_j^m = \begin{cases} \begin{pmatrix} 1 & e^{-2\overline{L}_j} \\ -e^{-2L_j} & 1 \end{pmatrix} \Lambda_j^m \begin{pmatrix} e^{L_j} & 0 \\ 0 & e^{\overline{L}_j} \end{pmatrix} =: f_j^{(+)} \Lambda_j^m E_j^{(+)} \\ \begin{pmatrix} -e^{2L_j} & 1 \\ 1 & e^{2\overline{L}_j} \end{pmatrix} \Lambda_j^m \begin{pmatrix} -e^{-L_j} & 0 \\ 0 & e^{-\overline{L}_j} \end{pmatrix} =: f_j^{(-)} \Lambda_j^m E_j^{(-)} \end{cases}, \tag{B3}$$

and

$$f_j^{(\pm)} \longrightarrow C_j^{(\pm)} \text{ as } \mathrm{Re} L_j \longrightarrow \pm\infty, \text{ where } \begin{cases} C_j^{(+)} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ C_j^{(-)} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{cases}. \tag{B4}$$

By Eqs. (B3) and (A7), the right common factors $E_j^{(\pm)}$ of each column of σ can be omitted completely, and hence

$$\sigma = \begin{vmatrix} \psi_1 & f_2^{(\pm)} & \cdots & f_n^{(\pm)} & 1 \\ \psi_1 \Lambda_1 & f_2^{(\pm)} \Lambda_2 & \cdots & f_n^{(\pm)} \Lambda_n & 0 \\ \psi_1 \Lambda_1^2 & f_2^{(\pm)} \Lambda_2^2 & \cdots & f_n^{(\pm)} \Lambda_n^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1 \Lambda_1^n & f_2^{(\pm)} \Lambda_2^n & \cdots & f_n^{(\pm)} \Lambda_n^n & \boxed{0} \end{vmatrix}, \tag{B5}$$

which is asymptotic to

$$\widetilde{\sigma} := \begin{vmatrix} \psi_1 & C_2^{(\pm)} & \cdots & C_n^{(\pm)} & 1 \\ \psi_1 \Lambda_1 & C_2^{(\pm)} \Lambda_2 & \cdots & C_n^{(\pm)} \Lambda_n & 0 \\ \psi_1 \Lambda_1^2 & C_2^{(\pm)} \Lambda_2^2 & \cdots & C_n^{(\pm)} \Lambda_n^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1 \Lambda_1^n & C_2^{(\pm)} \Lambda_2^n & \cdots & C_n^{(\pm)} \Lambda_n^n & \boxed{0} \end{vmatrix}. \tag{B6}$$

By the fact that

$$C_j^{(\pm)} \Lambda_j^m = \Lambda_j^{(\pm)m} C_j^{(\pm)}, \quad \begin{cases} \Lambda_j^{(+)} := \begin{pmatrix} \lambda_j & 0 \\ 0 & \mu_j \end{pmatrix} \\ \Lambda_j^{(-)} := \begin{pmatrix} \mu_j & 0 \\ 0 & \lambda_j \end{pmatrix} \end{cases}, \tag{B7}$$

and Eq. (A7), the right common factors $C_j^{(\pm)}$ of each column of $\widetilde{\sigma}$ can be omitted completely, and hence

$$\widetilde{\sigma} = \begin{vmatrix} \psi_1 & 1 & \cdots & 1 & 1 \\ \psi_1 \Lambda_1 & \Lambda_2^{(\pm)} & \cdots & \Lambda_n^{(\pm)} & 0 \\ \psi_1 \Lambda_1^2 & \Lambda_2^{(\pm)2} & \cdots & \Lambda_n^{(\pm)2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1 \Lambda_1^n & \Lambda_2^{(\pm)n} & \cdots & \Lambda_n^{(\pm)n} & \boxed{0} \end{vmatrix}, \tag{B8}$$

which is called the asymptotic form of the n-soliton solution σ. By the derivative formula [49] of the quasideterminant

$$\partial_\mu \begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} = \begin{vmatrix} A & \partial_\mu B \\ C & \boxed{\partial_\mu d} \end{vmatrix} + \sum_{j=1}^{n} \begin{vmatrix} A & \partial_\mu A_j \\ C & \boxed{\partial_\mu C_j} \end{vmatrix} \begin{vmatrix} A & B \\ E^j & \boxed{0} \end{vmatrix},$$

$$(A_j : j\text{th column of } A,\ E^j : j\text{th row of identity matrix } I) \qquad \text{(B9)}$$

we have

$$\partial_\mu \widetilde{\sigma} = \begin{vmatrix} \psi_1 & 1 & \cdots & 1 & \partial_\mu \psi_1 \\ \psi_1\Lambda_1 & \Lambda_2^{(\pm)} & \cdots & \Lambda_n^{(\pm)} & (\partial_\mu\psi_1)\Lambda_1 \\ \psi_1\Lambda_1^2 & \Lambda_2^{(\pm)2} & \cdots & \Lambda_n^{(\pm)2} & (\partial_\mu\psi_1)\Lambda_1^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & \Lambda_2^{(\pm)n} & \cdots & \Lambda_n^{(\pm)n} & \boxed{(\partial_\mu\psi_1)\Lambda_1^n} \end{vmatrix} \begin{vmatrix} \psi_1 & 1 & \cdots & 1 & 1 \\ \psi_1\Lambda_1 & \Lambda_2^{(\pm)} & \cdots & \Lambda_n^{(\pm)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^{n-1} & \Lambda_2^{(\pm)n-1} & \cdots & \Lambda_n^{(\pm)n-1} & 0 \\ 1 & 0 & \cdots & 0 & \boxed{0} \end{vmatrix}$$
$$+ \sum_{j=2}^{n} \underbrace{\begin{vmatrix} \psi_1 & 1 & \cdots & 1 & 0 \\ \psi_1\Lambda_1 & \Lambda_2^{(\pm)} & \cdots & \Lambda_n^{(\pm)} & 0 \\ \psi_1\Lambda_1^2 & \Lambda_2^{(\pm)2} & \cdots & \Lambda_n^{(\pm)2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & \Lambda_2^{(\pm)n} & \cdots & \Lambda_n^{(\pm)n} & \boxed{0} \end{vmatrix}} \begin{vmatrix} \psi_1 & \cdots & 1 & \cdots & 1 \\ \psi_1\Lambda_1 & \cdots & \Lambda_j^{(\pm)} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_1\Lambda_1^{n-1} & \cdots & \Lambda_j^{(\pm)n-1} & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & \boxed{0} \end{vmatrix}$$
$$= 0 \quad (\partial_\mu \Lambda_j^{(\pm)m} = 0 \text{ in the last column}). \qquad \text{(B10)}$$

Now we can conclude that

$$\partial_\mu \widetilde{\sigma} = \begin{vmatrix} \psi_1 & 1 & \cdots & 1 & \partial_\mu \psi_1 \\ \psi_1\Lambda_1 & \Lambda_2^{(\pm)} & \cdots & \Lambda_n^{(\pm)} & (\partial_\mu\psi_1)\Lambda_1 \\ \psi_1\Lambda_1^2 & \Lambda_2^{(\pm)2} & \cdots & \Lambda_n^{(\pm)2} & (\partial_\mu\psi_1)\Lambda_1^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & \Lambda_2^{(\pm)n} & \cdots & \Lambda_n^{(\pm)n} & \boxed{(\partial_\mu\psi_1)\Lambda_1^n} \end{vmatrix} \begin{vmatrix} \psi_1 & 1 & \cdots & 1 & 1 \\ \psi_1\Lambda_1 & \Lambda_2^{(\pm)} & \cdots & \Lambda_n^{(\pm)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^{n-1} & \Lambda_2^{(\pm)n-1} & \cdots & \Lambda_n^{(\pm)n-1} & 0 \\ 1 & 0 & \cdots & 0 & \boxed{0} \end{vmatrix}. \qquad \text{(B11)}$$

On the other hand, by the derivative formula of the quasideterminant on σ we have

$$\partial_\mu \sigma = \sum_{j=1}^{n} \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_n & \partial_\mu \psi_j \\ \psi_1\Lambda_1 & \psi_2\Lambda_2 & \cdots & \psi_n\Lambda_n & (\partial_\mu\psi_j)\Lambda_j \\ \psi_1\Lambda_1^2 & \psi_2\Lambda_2^2 & \cdots & \psi_n\Lambda_n^2 & (\partial_\mu\psi_j)\Lambda_j^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & \psi_2\Lambda_2^n & \cdots & \psi_n\Lambda_n^n & \boxed{(\partial_\mu\psi_j)\Lambda_j^n} \end{vmatrix} \begin{vmatrix} \psi_1 & \cdots & \psi_j & \cdots & 1 \\ \psi_1\Lambda_1 & \cdots & \psi_j\Lambda_j & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_1\Lambda_1^{n-1} & \cdots & \psi_j\Lambda_j^{n-1} & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & \boxed{0} \end{vmatrix}. \qquad \text{(B12)}$$

By a similar argument to Eqs. (B3), (B4), we have

$$(\partial_\mu\psi_j)\Lambda_j^m = \begin{cases} \begin{pmatrix} 1 & -e^{-2\overline{L}_j} \\ e^{-2L_j} & 1 \end{pmatrix} \Lambda_j^m \begin{pmatrix} \ell_\mu^{(j)} e^{L_j} & 0 \\ 0 & \overline{\ell}_\mu^{(j)} e^{\overline{L}_j} \end{pmatrix} =: \widetilde{f}_j^{(+)} \Lambda_j^m \widetilde{E}_j^{(+)} \\ \begin{pmatrix} e^{2L_j} & 1 \\ 1 & -e^{2\overline{L}_j} \end{pmatrix} \Lambda_j^m \begin{pmatrix} \ell_\mu^{(j)} e^{-L_j} & 0 \\ 0 & -\overline{\ell}_\mu^{(j)} e^{-\overline{L}_j} \end{pmatrix} =: \widetilde{f}_j^{(-)} \Lambda_j^m \widetilde{E}_j^{(-)} \end{cases}, \qquad \text{(B13)}$$

and

$$\widetilde{f}_j^{(\pm)} \longrightarrow C_j^{(\pm)} \text{ as } \mathrm{Re}L_j \longrightarrow \pm\infty, \text{ where } \begin{cases} C_j^{(+)} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ C_j^{(-)} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{cases}. \tag{B14}$$

Now by Eqs. (B3), (B13), and (A7), we can omit the right common factor $E_j^{(\pm)}$ from the jth column ($j = 2 - n$), and take the right common factor $\widetilde{E}_j^{(\pm)}$ of the last column out of the quasideterminant. Then we obtain

$$\begin{aligned}
&\partial_\mu \sigma \\
&\quad \begin{vmatrix} \psi_1 & f_2^{(\pm)} & \cdots & f_n^{(\pm)} & \partial_\mu \psi_1 \\ \psi_1 \Lambda_1 & f_2^{(\pm)}\Lambda_2 & \cdots & f_n^{(\pm)}\Lambda_n & (\partial_\mu\psi_1)\Lambda_1 \\ \psi_1\Lambda_1^2 & f_2^{(\pm)}\Lambda_2^2 & \cdots & f_n^{(\pm)}\Lambda_n^2 & (\partial_\mu\psi_1)\Lambda_1^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & f_2^{(\pm)}\Lambda_2^n & \cdots & f_n^{(\pm)}\Lambda_n^n & \boxed{(\partial_\mu\psi_1)\Lambda_1^n} \end{vmatrix} \begin{vmatrix} \psi_1 & \cdots & f_j^{(\pm)} & \cdots & 1 \\ \psi_1\Lambda_1 & \cdots & f_j^{(\pm)}\Lambda_j & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_1\Lambda_1^{n-1} & \cdots & f_j^{(\pm)}\Lambda_j^{n-1} & \cdots & 0 \\ 1 & \cdots & 0 & \cdots & \boxed{0} \end{vmatrix} \\
&= +\sum_{j=2}^{n} \begin{vmatrix} \psi_1 & f_2^{(\pm)} & \cdots & f_n^{(\pm)} & \widetilde{f}_j^{(\pm)} \\ \psi_1 \Lambda_1 & f_2^{(\pm)}\Lambda_2 & \cdots & f_n^{(\pm)}\Lambda_n & \widetilde{f}_j^{(\pm)}\Lambda_j \\ \psi_1\Lambda_1^2 & f_2^{(\pm)}\Lambda_2^2 & \cdots & f_n^{(\pm)}\Lambda_n^2 & \widetilde{f}_j^{(\pm)}\Lambda_j^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & f_2^{(\pm)}\Lambda_2^n & \cdots & f_n^{(\pm)}\Lambda_n^n & \boxed{(\widetilde{f}_j^{(\pm)}\Lambda_j^n} \end{vmatrix} \widetilde{E}_j^{(\pm)} \\
&\quad \times \begin{vmatrix} \psi_1 & f_2^{(\pm)} & \cdots & \psi_j & \cdots & f_n^{(\pm)} & 1 \\ \hline \psi_1\Lambda_1 & f_2^{(\pm)}\Lambda_2 & \cdots & \psi_j\Lambda_j & \cdots & f_n^{(\pm)}\Lambda_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^{n-1} & f_2^{(\pm)}\Lambda_2^{n-1} & \cdots & \psi_j\Lambda_j^{n-1} & \cdots & f_n^{(\pm)}\Lambda_n^{n-1} & 0 \\ \hline 0 & 0 & \cdots & 1 & \cdots & 0 & \boxed{0} \end{vmatrix}.
\end{aligned} \tag{B15}$$

By the Jacobi identity (A9), we have

$$\widetilde{E}_j^{(\pm)}\left|\begin{array}{ccccccc|c} \psi_1 & f_2^{(\pm)} & \cdots & \psi_j & \cdots & f_n^{(\pm)} & 1 \\ \hline \psi_1\Lambda_1 & f_2^{(\pm)}\Lambda_2 & \cdots & \psi_j\Lambda_j & \cdots & f_n^{(\pm)}\Lambda_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^{n-1} & f_2^{(\pm)}\Lambda_2^{n-1} & \cdots & \psi_j\Lambda_j^{n-1} & \cdots & f_n^{(\pm)}\Lambda_n^{n-1} & 0 \\ \hline 0 & 0 & \cdots & 1 & \cdots & 0 & \boxed{0} \end{array}\right| \tag{B16}$$

$$= -\widetilde{E}_j^{(\pm)}\begin{vmatrix} \psi_1 & f_2^{(\pm)} & \cdots & \boxed{\psi_j} & \cdots & f_n^{(\pm)} \\ \psi_1\Lambda_1 & f_2^{(\pm)}\Lambda_2 & \cdots & \psi_j\Lambda_j & \cdots & f_n^{(\pm)}\Lambda_n \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_1\Lambda_1^{n-1} & f_2^{(\pm)}\Lambda_2^{n-1} & \cdots & \psi_j\Lambda_j^{n-1} & \cdots & f_n^{(\pm)}\Lambda_n^{n-1} \end{vmatrix}^{-1} \tag{B17}$$

$$= -\widetilde{E}_j^{(\pm)}(E_j^{(\pm)})^{-1}\begin{vmatrix} \psi_1 & f_2^{(\pm)} & \cdots & \boxed{f_j^{(\pm)}} & \cdots & f_n^{(\pm)} \\ \psi_1\Lambda_1 & f_2^{(\pm)}\Lambda_2 & \cdots & f_j^{(\pm)}\Lambda_i & \cdots & f_n^{(\pm)}\Lambda_n \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_1\Lambda_1^{n-1} & f_2^{(\pm)}\Lambda_2^{n-1} & \cdots & f_j^{(\pm)}\Lambda_j^{n-1} & \cdots & f_n^{(\pm)}\Lambda_n^{n-1} \end{vmatrix}^{-1}.$$

(With Eqs. (B3) and (A7), we can take the right common factor $E_j^{(\pm)}$ out of the quasideterminant.) (B18)

By Eqs. (B3) and (B13),

$$\widetilde{E}_j^{(\pm)}(E_j^{(\pm)})^{-1} = \begin{cases} \begin{pmatrix} \ell_\mu^{(j)}e^{L_j} & 0 \\ 0 & \overline{\ell}_\mu^{(j)}e^{\overline{L}_j} \end{pmatrix}\begin{pmatrix} e^{-L_j} & 0 \\ 0 & e^{-\overline{L}_j} \end{pmatrix} \\ \begin{pmatrix} \ell_\mu^{(j)}e^{-L_j} & 0 \\ 0 & -\overline{\ell}_\mu^{(j)}e^{-\overline{L}_j} \end{pmatrix}\begin{pmatrix} -e^{L_j} & 0 \\ 0 & e^{\overline{L}_j} \end{pmatrix} \end{cases} = \pm\begin{pmatrix} \ell_\mu^{(j)} & 0 \\ 0 & \overline{\ell}_\mu^{(j)} \end{pmatrix} =: \widetilde{\Lambda}_j^{(\pm)}, \tag{B19}$$

which are constant matrices. Therefore, we can conclude that

$$\partial_\mu\sigma = \begin{vmatrix} \psi_1 & f_2^{(\pm)} & \cdots & f_n^{(\pm)} & \partial_\mu\psi_1 \\ \psi_1\Lambda_1 & f_2^{(\pm)}\Lambda_2 & \cdots & f_n^{(\pm)}\Lambda_n & (\partial_\mu\psi_1)\Lambda_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & f_2^{(\pm)}\Lambda_2^n & \cdots & f_n^{(\pm)}\Lambda_n^n & \boxed{(\partial_\mu\psi_1)\Lambda_1^n} \end{vmatrix}\begin{vmatrix} \psi_1 & \cdots & f_j^{(\pm)} & \cdots & 1 \\ \psi_1\Lambda_1 & \cdots & f_j^{(\pm)}\Lambda_j & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_1\Lambda_1^{n-1} & \cdots & f_j^{(\pm)}\Lambda_j^{n-1} & \cdots & 0 \\ 1 & \cdots & 0 & \cdots & \boxed{0} \end{vmatrix}$$

$$-\sum_{j=2}^{n}\begin{vmatrix} \psi_1 & f_2^{(\pm)} & \cdots & f_n^{(\pm)} & \widetilde{f}_j^{(\pm)} \\ \psi_1\Lambda_1 & f_2^{(\pm)}\Lambda_2 & \cdots & f_n^{(\pm)}\Lambda_n & \widetilde{f}_j^{(\pm)}\Lambda_j \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & f_2^{(\pm)}\Lambda_2^n & \cdots & f_n^{(\pm)}\Lambda_n^n & \boxed{\widetilde{f}_j^{(\pm)}\Lambda_j^n} \end{vmatrix}\widetilde{\Lambda}_j^{(\pm)}\begin{vmatrix} \psi_1 & f_2^{(\pm)} & \cdots & \boxed{f_j^{(\pm)}} & \cdots & f_n^{(\pm)} \\ \psi_1\Lambda_1 & f_2^{(\pm)}\Lambda_2 & \cdots & f_j^{(\pm)}\Lambda_i & \cdots & f_n^{(\pm)}\Lambda_n \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_1\Lambda_1^{n-1} & f_2^{(\pm)}\Lambda_2^{n-1} & \cdots & f_j^{(\pm)}\Lambda_j^{n-1} & \cdots & f_n^{(\pm)}\Lambda_n^{n-1} \end{vmatrix}^{-1},$$

which is asymptotic (cf. Eqs. (B4), (B14)) to

$$\widetilde{\partial_\mu\sigma} = \begin{vmatrix} \psi_1 & C_2^{(\pm)} & \cdots & C_n^{(\pm)} & \partial_\mu\psi_1 \\ \psi_1\Lambda_1 & C_2^{(\pm)}\Lambda_2 & \cdots & C_n^{(\pm)}\Lambda_n & (\partial_\mu\psi_1)\Lambda_1 \\ \psi_1\Lambda_1^2 & C_2^{(\pm)}\Lambda_2^2 & \cdots & C_n^{(\pm)}\Lambda_n^2 & (\partial_\mu\psi_1)\Lambda_1^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & C_2^{(\pm)}\Lambda_2^n & \cdots & C_n^{(\pm)}\Lambda_n^n & \boxed{(\partial_\mu\psi_1)\Lambda_1^n} \end{vmatrix} \begin{vmatrix} \psi_1 & \cdots & C_j^{(\pm)} & \cdots & 1 \\ \psi_1\Lambda_1 & \cdots & C_j^{(\pm)}\Lambda_j & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_1\Lambda_1^{n-1} & \cdots & C_j^{(\pm)}\Lambda_j^{n-1} & \cdots & 0 \\ 1 & \cdots & 0 & \cdots & \boxed{0} \end{vmatrix}$$
$$- \sum_{j=2}^{n} \underbrace{\begin{vmatrix} \psi_1 & C_2^{(\pm)} & \cdots & C_n^{(\pm)} & C_j^{(\pm)} \\ \psi_1\Lambda_1 & C_2^{(\pm)}\Lambda_2 & \cdots & C_n^{(\pm)}\Lambda_n & C_j^{(\pm)}\Lambda_j \\ \psi_1\Lambda_1^2 & C_2^{(\pm)}\Lambda_2^2 & \cdots & C_n^{(\pm)}\Lambda_n^2 & C_j^{(\pm)}\Lambda_j^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & C_2^{(\pm)}\Lambda_2^n & \cdots & C_n^{(\pm)}\Lambda_n^n & \boxed{(C_j^{(\pm)}\Lambda_j^n} \end{vmatrix}} \widetilde{\Lambda}_j^{(\pm)} \begin{vmatrix} \psi_1 & \cdots & \boxed{C_j^{(\pm)}} & \cdots & 1 \\ \psi_1\Lambda_1 & \cdots & C_j^{(\pm)}\Lambda_j & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_1\Lambda_1^{n-1} & \cdots & C_j^{(\pm)}\Lambda_j^{n-1} & \cdots & 0 \end{vmatrix}^{-1}$$
$$= 0 \ \text{(the } j\text{th column is identical to the last column).} \tag{B20}$$

By Eqs. (B7) and (A7), the right common factors $C_j^{(\pm)}$ of each column can be omitted completely, and hence

$$\widetilde{\partial_\mu\sigma} = \begin{vmatrix} \psi_1 & 1 & \cdots & 1 & \partial_\mu\psi_1 \\ \psi_1\Lambda_1 & \Lambda_2^{(\pm)} & \cdots & \Lambda_n^{(\pm)} & (\partial_\mu\psi_1)\Lambda_1 \\ \psi_1\Lambda_1^2 & \Lambda_2^{(\pm)2} & \cdots & \Lambda_n^{(\pm)2} & (\partial_\mu\psi_1)\Lambda_1^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & \Lambda_2^{(\pm)n} & \cdots & \Lambda_n^{(\pm)n} & \boxed{(\partial_\mu\psi_1)\Lambda_1^n} \end{vmatrix} \begin{vmatrix} \psi_1 & 1 & \cdots & 1 & 1 \\ \psi_1\Lambda_1 & \Lambda_2^{(\pm)} & \cdots & \Lambda_n^{(\pm)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^{n-1} & \Lambda_2^{(\pm)n-1} & \cdots & \Lambda_n^{(\pm)n-1} & 0 \\ 1 & 0 & \cdots & 0 & \boxed{0} \end{vmatrix} \tag{B21}$$
$$= \partial_\mu\widetilde{\sigma}. \tag{B22}$$

□

C. Proof of unitarity for *n*-soliton solutions on $\mathbb{E}$

Proposition C.1 *Let $\sigma_{[n+1]}$ be defined in Eq. (B1) with the reality condition (36) on $\mathbb{E}$. Then $\sigma_{[n+1]} \in U(2)$ on the Euclidean space if $|\lambda_j| = 1$.*

Proof. For $n = 1$, we have

$$\psi_1^\dagger\psi_1 = \psi_1\psi_1^\dagger = (e^{L_1+\overline{L}_1} + e^{-(L_1+\overline{L}_1)})I, \ \ \Lambda_1^\dagger\Lambda_1 = \Lambda_1\Lambda_1^\dagger = |\lambda_1|^2 I = I, \tag{C1}$$

which implies

$$\sigma_{[2]}^\dagger\sigma_{[2]} = (-\psi_1\Lambda_1\psi_1^{-1})^\dagger(-\psi_1\Lambda_1\psi_1^{-1}) = (\psi_1^\dagger\psi_1)(\psi_1\psi_1^\dagger)^{-1}(\Lambda_1^\dagger\Lambda_1) = I = \sigma_{[2]}\sigma_{[2]}^\dagger, \tag{C2}$$

i.e., the one-soliton solution $\sigma_{[2]} \in U(2)$. Assume that the *n*-soliton solution $\sigma_{[n+1]} \in U(2)$ for

$1 \le n \le k-1$. For $n = k$ and by the Darboux transformation [18], we have

$$\sigma_{[k+1]} = -\psi_{[k]}\Lambda_{[k]}\psi_{[k]}^{-1}\sigma_{[k]}, \tag{C3}$$

where

$$\psi_{[k]} := \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_{k-1} & \psi_k \\ \psi_1\Lambda_1 & \psi_2\Lambda_2 & \cdots & \psi_{k-1}\Lambda_{k-1} & \psi_k\Lambda_k \\ \vdots & \vdots & & \vdots & \vdots \\ \psi_1\Lambda_1^{k-2} & \psi_2\Lambda_2^{k-2} & \cdots & \psi_{k-1}\Lambda_{k-1}^{k-2} & \psi_k\Lambda_k^{k-2} \\ \psi_1\Lambda_1^{k-1} & \psi_2\Lambda_2^{k-1} & \cdots & \psi_{k-1}\Lambda_{k-1}^{k-1} & \boxed{\psi_k\Lambda_k^{k-1}} \end{vmatrix}. \tag{C4}$$

By using the Jacobi identity (A9) in the following two equalities marked by $\overset{\star}{=}$ notation, we have

$$\begin{aligned}
\psi_{[k]} &\overset{\star}{=} \begin{vmatrix} \psi_2\Lambda_2 & \cdots & \psi_k\Lambda_k \\ \vdots & & \vdots \\ \psi_2\Lambda_2^{k-1} & \cdots & \boxed{\psi_k\Lambda_k^{k-1}} \end{vmatrix} \\
&\quad - \begin{vmatrix} \psi_1\Lambda_1 & \cdots & \psi_{k-1}\Lambda_{k-1} \\ \vdots & & \vdots \\ \boxed{\psi_1\Lambda_1^{k-1}} & \cdots & \psi_{k-1}\Lambda_{k-1}^{k-1} \end{vmatrix} \begin{vmatrix} \boxed{\psi_1} & \cdots & \psi_{k-1} \\ \vdots & & \vdots \\ \psi_1\Lambda_1^{k-2} & \cdots & \psi_{k-1}\Lambda_{k-1}^{k-2} \end{vmatrix}^{-1} \begin{vmatrix} \psi_2 & \cdots & \boxed{\psi_k} \\ \vdots & & \vdots \\ \psi_2\Lambda_2^{k-2} & \cdots & \psi_k\Lambda_k^{k-2} \end{vmatrix} \\
&= \left[\begin{vmatrix} \psi_2\Lambda_2 & \cdots & \psi_k\Lambda_k \\ \vdots & & \vdots \\ \psi_2\Lambda_2^{k-1} & \cdots & \boxed{\psi_k\Lambda_k^{k-1}} \end{vmatrix} \begin{vmatrix} \psi_2 & \cdots & \boxed{\psi_k} \\ \vdots & & \vdots \\ \psi_2\Lambda_2^{k-2} & \cdots & \psi_k\Lambda_k^{k-2} \end{vmatrix}^{-1} \right. \\
&\qquad \left. - \begin{vmatrix} \psi_1\Lambda_1 & \cdots & \psi_{k-1}\Lambda_{k-1} \\ \vdots & & \vdots \\ \boxed{\psi_1\Lambda_1^{k-1}} & \cdots & \psi_{k-1}\Lambda_{k-1}^{k-1} \end{vmatrix} \begin{vmatrix} \boxed{\psi_1} & \cdots & \psi_{k-1} \\ \vdots & & \vdots \\ \psi_1\Lambda_1^{k-2} & \cdots & \psi_{k-1}\Lambda_{k-1}^{k-2} \end{vmatrix}^{-1} \right] \begin{vmatrix} \psi_2 & \cdots & \boxed{\psi_k} \\ \vdots & & \vdots \\ \psi_2\Lambda_2^{k-2} & \cdots & \psi_k\Lambda_k^{k-2} \end{vmatrix} \\
&\overset{\star}{=} \left[\begin{vmatrix} \psi_2 & \cdots & \psi_k & 1 \\ \psi_2\Lambda_2 & \cdots & \psi_k\Lambda_k & 0 \\ \vdots & & \vdots & \vdots \\ \psi_2\Lambda_2^{k-1} & \cdots & \psi_k\Lambda_k^{k-1} & \boxed{0} \end{vmatrix} - \begin{vmatrix} \psi_1 & \cdots & \psi_{k-1} & 1 \\ \psi_1\Lambda_1 & \cdots & \psi_{k-1}\Lambda_{k-1} & 0 \\ \vdots & & \vdots & \vdots \\ \psi_1\Lambda_1^{k-1} & \cdots & \psi_k\Lambda_{k-1}^{k-1} & \boxed{0} \end{vmatrix} \right] \begin{vmatrix} \psi_2 & \cdots & \boxed{\psi_k} \\ \vdots & & \vdots \\ \psi_2\Lambda_2^{k-2} & \cdots & \psi_k\Lambda_k^{k-2} \end{vmatrix} \\
&=: (\widetilde{\sigma}_{[k]} - \sigma_{[k]})\widetilde{\psi}_{[k-1]},
\end{aligned} \tag{C5}$$

where $\widetilde{\sigma}_{[k]} := \sigma_{[k]}\Big|_{(\psi_1,\Lambda_1)\to(\psi_k,\Lambda_k)} \in U(2)$. On the other hand,

$$
\begin{aligned}
&\widetilde{\psi}_{[k-1]}\Lambda_k\widetilde{\psi}^{-1}_{[k-1]}\\
&= -\underbrace{\begin{vmatrix} \psi_2 & \cdots & \boxed{\psi_k} \\ \vdots & & \vdots \\ \psi_2\Lambda_2^{k-2} & \cdots & \psi_k\Lambda_k^{k-2} \end{vmatrix}}_{\substack{\| \text{ By the homological relation (A10)} \\ \begin{vmatrix} \psi_2 & \cdots & \psi_{k-1} & \boxed{0} \\ \vdots & & \vdots & \vdots \\ \psi_2\Lambda_2^{k-2} & \cdots & \psi_{k-1}\Lambda_{k-1}^{k-2} & 1 \end{vmatrix} \begin{vmatrix} \psi_2 & \cdots & \psi_k \\ \vdots & & \vdots \\ \psi_2\Lambda_2^{k-2} & \cdots & \boxed{\psi_k\Lambda_k^{k-2}} \end{vmatrix}}} \Lambda_k \begin{vmatrix} \psi_2 & \cdots & \boxed{\psi_k} \\ \vdots & & \vdots \\ \psi_2\Lambda_2^{k-2} & \cdots & \psi_k\Lambda_k^{k-2} \end{vmatrix}^{-1}\\
&= \begin{vmatrix} \psi_2 & \cdots & \psi_{k-1} & 1 \\ \vdots & & \vdots & \vdots \\ \psi_2\Lambda_2^{k-2} & \cdots & \psi_{k-1}\Lambda_{k-1}^{k-2} & \boxed{0} \end{vmatrix}^{-1} \underbrace{\left\{ -\begin{vmatrix} \psi_2\Lambda_2 & \cdots & \psi_k\Lambda_k \\ \vdots & & \vdots \\ \psi_2\Lambda_2^{k-1} & \cdots & \boxed{\psi_k\Lambda_k^{k-1}} \end{vmatrix} \begin{vmatrix} \psi_2 & \cdots & \boxed{\psi_k} \\ \vdots & & \vdots \\ \psi_2\Lambda_2^{k-2} & \cdots & \psi_k\Lambda_k^{k-2} \end{vmatrix}^{-1} \right\}}_{\substack{\| \text{ By the Jacobi identity (A9)} \\ \begin{vmatrix} \psi_2 & \cdots & \psi_k & 1 \\ \vdots & & \vdots & \vdots \\ \psi_2\Lambda_2^{k-1} & \cdots & \psi_k\Lambda_k^{k-1} & \boxed{0} \end{vmatrix}}}\\
&= \widetilde{\sigma}^{-1}_{[k-1]}\widetilde{\sigma}_{[k]} \in U(2). \qquad \text{(C6)}
\end{aligned}
$$

Note that the second equality from the bottom is obtained by using the homological relation (A10) and (A7). By Eqs. (C3), (C5), and (C6), we can conclude that

$$
\sigma_{[k+1]} = (\widetilde{\sigma}_{[k]} - \sigma_{[k]})\widetilde{\sigma}^{-1}_{[k-1]}\widetilde{\sigma}_{[k]}(\widetilde{\sigma}_{[k]} - \sigma_{[k]})^{-1}\sigma_{[k]} \in U(2). \qquad \text{(C7)}
$$

□

D. Miscellaneous formulas

D.1. Flip symmetry of n-soliton solutions

Proposition D.1 *The data of the n-soliton solutions have the following symmetry:*

$$
(\Delta, \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})\Big|_{L_j\to -L_j} = (\Delta, \Delta_{22}, \Delta_{21}, \Delta_{12}, \Delta_{11}), \qquad \text{(D1)}
$$

$$
\partial_\mu\, (\Delta, \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})\Big|_{L_j\to -L_j} = \partial_\mu\, (\Delta, \Delta_{22}, \Delta_{21}, \Delta_{12}, \Delta_{11}). \qquad \text{(D2)}
$$

Proof. Let $\widetilde{\psi}_j$ be defined by $\widetilde{\psi}_j := \psi_j\Big|_{L_j\to -L_j} = \begin{pmatrix} e^{-L_j} & e^{\overline{L}_j} \\ -e^{L_j} & e^{-\overline{L}_j}\end{pmatrix}$, which satisfies

$$\widetilde{\psi}_j\Lambda_j^k = E(\psi_j\Lambda_j^k)F, \quad E := \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \quad F := \begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}.$$

Then

$$\begin{aligned}
\sigma\Big|_{L_j\to -L_j} &= \begin{vmatrix} \widetilde{\psi}_1 & \widetilde{\psi}_2 & \cdots & \widetilde{\psi}_n & 1 \\ \widetilde{\psi}_1\Lambda_1 & \widetilde{\psi}_2\Lambda_2 & \cdots & \widetilde{\psi}_n\Lambda_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \widetilde{\psi}_1\Lambda_1^n & \widetilde{\psi}_2\Lambda_2^n & \cdots & \widetilde{\psi}_n\Lambda_n^n & \boxed{0}\end{vmatrix} \\
&= \begin{vmatrix} E(\psi_1)F & E(\psi_2)F & \cdots & E(\psi_n)F & E1E \\ E(\psi_1\Lambda_1)F & E(\psi_2\Lambda_2)F & \cdots & E(\psi_n\Lambda_n)F & E0E \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ E(\psi_1\Lambda_1^n)F & E(\psi_2\Lambda_2^n)F & \cdots & E(\psi_n\Lambda_n^n)F & \boxed{E0E}\end{vmatrix} \\
&= E\begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_n & 1 \\ \psi_1\Lambda_1 & \psi_2\Lambda_2 & \cdots & \psi_n\Lambda_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & \psi_2\Lambda_2^n & \cdots & \psi_n\Lambda_n^n & \boxed{0}\end{vmatrix}E = E\sigma E
\end{aligned}$$

by the multiplicative rule (A7) and (A8) of the quasideterminant. (The common multiplier for the rows is E, the common multipliers for the columns are F and E.) Now we have

$$\sigma\Big|_{L_j\to -L_j} = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\frac{-1}{\Delta}\begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} = \frac{-1}{\Delta}\begin{pmatrix} \Delta_{22} & \Delta_{21} \\ \Delta_{12} & \Delta_{11}\end{pmatrix}.$$

Therefore, $(\Delta, \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})\Big|_{L_j\to -L_j} = (\Delta, \Delta_{22}, \Delta_{21}, \Delta_{12}, \Delta_{11})$.

By the fact that $\dfrac{df(-x)}{dx} = -\dfrac{df(x)}{dx}\Big|_{x\to -x}$, we have

$$\begin{aligned}
&\left(\partial_\mu\Delta, \partial_\mu\Delta_{11}, \partial_\mu\Delta_{12}, \partial_\mu\Delta_{21}, \partial_\mu\Delta_{22}\right)\Big|_{L_j\to -L_j} \\
&= -\partial_\mu\left(\Delta\big|_{L_j\to -L_j}, \Delta_{11}\big|_{L_j\to -L_j}, \Delta_{12}\big|_{L_j\to -L_j}, \Delta_{21}\big|_{L_j\to -L_j}, \Delta_{22}\big|_{L_j\to -L_j}\right) \\
&= -\partial_\mu\left(\Delta, \Delta_{22}, \Delta_{21}, \Delta_{12}, \Delta_{11}\right) \quad \text{by (D.1)} \\
&= -\left(\partial_\mu\Delta, \partial_\mu\Delta_{22}, \partial_\mu\Delta_{21}, \partial_\mu\Delta_{12}, \partial_\mu\Delta_{11}\right).
\end{aligned}$$

□

Corollary D.2 *Proposition D.1 implies that:*

(1) $Tr[(\partial_\mu\sigma)\sigma^{-1}(\partial_\nu\sigma)\sigma^{-1}]$ is an even function with respect to L_j.

(2) $Tr[(\partial_\mu\sigma)\sigma^{-1}(\partial_\nu\sigma)\sigma^{-1}(\partial_\rho\sigma)\sigma^{-1}]$ is an odd function with respect to L_j.

Proof. (1)

$$
\begin{aligned}
&\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}(\partial_\nu\sigma)\sigma^{-1}\right]\Big|_{L_j\to -L_j}\\
&=\frac{-1}{|\sigma|\Delta^2}\left\{\begin{vmatrix}\partial_\mu\Delta_{11} & \partial_\mu\Delta_{12}\\ \partial_\nu\Delta_{21} & \partial_\nu\Delta_{22}\end{vmatrix}+\begin{vmatrix}\partial_\nu\Delta_{11} & \partial_\nu\Delta_{12}\\ \partial_\mu\Delta_{21} & \partial_\mu\Delta_{22}\end{vmatrix}-2|\sigma|(\partial_\mu\Delta)(\partial_\nu\Delta)\right\}\Bigg|_{L_j\to -L_j}\\
&=\frac{-1}{|\sigma|\Delta^2}\left\{\begin{vmatrix}-\partial_\mu\Delta_{22} & -\partial_\mu\Delta_{21}\\ -\partial_\nu\Delta_{12} & -\partial_\nu\Delta_{11}\end{vmatrix}+\begin{vmatrix}-\partial_\nu\Delta_{22} & -\partial_\nu\Delta_{21}\\ -\partial_\mu\Delta_{12} & -\partial_\mu\Delta_{11}\end{vmatrix}-2|\sigma|(-\partial_\mu\Delta)(-\partial_\nu\Delta)\right\}\\
&=\frac{-1}{|\sigma|\Delta^2}\left\{\begin{vmatrix}\partial_\nu\Delta_{11} & \partial_\nu\Delta_{12}\\ \partial_\mu\Delta_{21} & \partial_\mu\Delta_{22}\end{vmatrix}+\begin{vmatrix}\partial_\mu\Delta_{11} & \partial_\mu\Delta_{12}\\ \partial_\nu\Delta_{21} & \partial_\nu\Delta_{22}\end{vmatrix}-2|\sigma|(\partial_\mu\Delta)(\partial_\nu\Delta)\right\}\\
&=\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}(\partial_\nu\sigma)\sigma^{-1}\right].
\end{aligned}
$$

(2)

$$
\begin{aligned}
&\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}(\partial_\nu\sigma)\sigma^{-1}(\partial_\rho\sigma)\sigma^{-1}\right]\Big|_{L_j\to -L_j}\\
&=\frac{1}{2|\sigma|^2\Delta^4}\begin{vmatrix}\Delta_{11} & \Delta_{12} & \Delta_{21} & \Delta_{22}\\ \partial_\mu\Delta_{11} & \partial_\mu\Delta_{12} & \partial_\mu\Delta_{21} & \partial_\mu\Delta_{22}\\ \partial_\nu\Delta_{11} & \partial_\nu\Delta_{12} & \partial_\nu\Delta_{21} & \partial_\nu\Delta_{22}\\ \partial_\rho\Delta_{11} & \partial_\rho\Delta_{12} & \partial_\rho\Delta_{21} & \partial_\rho\Delta_{22}\end{vmatrix}_{L_j\to -L_j}\\
&=\frac{1}{2|\sigma|^2\Delta^4}\begin{vmatrix}\Delta_{22} & \Delta_{21} & \Delta_{12} & \Delta_{11}\\ -\partial_\mu\Delta_{22} & -\partial_\mu\Delta_{21} & -\partial_\mu\Delta_{12} & -\partial_\mu\Delta_{11}\\ -\partial_\nu\Delta_{22} & -\partial_\nu\Delta_{21} & -\partial_\nu\Delta_{12} & -\partial_\nu\Delta_{11}\\ -\partial_\rho\Delta_{22} & -\partial_\rho\Delta_{21} & -\partial_\rho\Delta_{12} & -\partial_\rho\Delta_{11}\end{vmatrix}\\
&=\frac{(-1)^3}{2|\sigma|^2\Delta^4}\begin{vmatrix}\Delta_{22} & \Delta_{21} & \Delta_{12} & \Delta_{11}\\ \partial_\mu\Delta_{22} & \partial_\mu\Delta_{21} & \partial_\mu\Delta_{12} & \partial_\mu\Delta_{11}\\ \partial_\nu\Delta_{22} & \partial_\nu\Delta_{21} & \partial_\nu\Delta_{12} & \partial_\nu\Delta_{11}\\ \partial_\rho\Delta_{22} & \partial_\rho\Delta_{21} & \partial_\rho\Delta_{12} & \partial_\rho\Delta_{11}\end{vmatrix}=\frac{-1}{2|\sigma|^2\Delta^4}\begin{vmatrix}\Delta_{11} & \Delta_{12} & \Delta_{21} & \Delta_{22}\\ \partial_\mu\Delta_{11} & \partial_\mu\Delta_{12} & \partial_\mu\Delta_{21} & \partial_\mu\Delta_{22}\\ \partial_\nu\Delta_{11} & \partial_\nu\Delta_{12} & \partial_\nu\Delta_{21} & \partial_\nu\Delta_{22}\\ \partial_\rho\Delta_{11} & \partial_\rho\Delta_{12} & \partial_\rho\Delta_{21} & \partial_\rho\Delta_{22}\end{vmatrix}\\
&=-\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}(\partial_\nu\sigma)\sigma^{-1}(\partial_\rho\sigma)\sigma^{-1}\right].
\end{aligned}
$$

□

D.2. Data of n-soliton solutions

In this subsection we present the data of n-soliton solutions in terms of X_j, Θ_j. We introduce the convention ε_j, which takes values in $\{\pm 1, \pm i\}$, and define an informal symbol $\mathscr{P}\{(\varepsilon_1, \ldots, \varepsilon_n)\}$ to denote the set of all permutations of $(\varepsilon_1, \ldots, \varepsilon_n)$. The data of n-soliton solutions can be expressed formally as the following Δ, Δ_{ij} with some undetermined coefficients $a(\boldsymbol{\varepsilon}) := a(\varepsilon_1, \ldots, \varepsilon_n)$ and

$A(\boldsymbol{\varepsilon}) := A(\varepsilon_1, \ldots, \varepsilon_n)$:

$$
\begin{aligned}
\Delta = & \sum_{\substack{1 \le j \le n, \\ \varepsilon = \pm 1}} a(\boldsymbol{\varepsilon}) \exp\left(\sum_{j=1}^{n} \varepsilon_j X_j\right) \\
& + \sum_{\substack{1 \le k < \ell \le n, \\ \varepsilon_j = \pm 1,\, j \neq k, \ell \\ (\varepsilon_k, \varepsilon_\ell) \in \mathcal{P}\{(i, -i)\}}} a(\boldsymbol{\varepsilon}) \exp(\sum_{j \neq k, \ell}^{n} \varepsilon_j X_j + \varepsilon_k \Theta_k + \varepsilon_\ell \Theta_\ell) + \text{ terms involving more } i\Theta_j \\
= & \frac{1}{2} \sum_{\substack{1 \le j \le n, \\ \varepsilon_j = \pm 1}} \left[a(\boldsymbol{\varepsilon}) \exp(\sum_{j=1}^{n} \varepsilon_j X_j) + a(-\boldsymbol{\varepsilon}) \exp(-\sum_{j=1}^{n} \varepsilon_j X_j) \right] \\
& \underbrace{+ \frac{1}{2} \sum_{\substack{1 \le k < \ell \le n, \\ \varepsilon_j = \pm 1,\, j \neq k, \ell \\ (\varepsilon_k, \varepsilon_\ell) \in \mathcal{P}\{(i, -i)\}}} \left[\begin{array}{c} a(\boldsymbol{\varepsilon}) \exp(\sum_{j \neq k, \ell}^{n} \varepsilon_j X_j + \varepsilon_k \Theta_k + \varepsilon_\ell \Theta_\ell) \\ + a(-\boldsymbol{\varepsilon}) \exp(-\sum_{j \neq k, \ell}^{n} \varepsilon_j X_j - \varepsilon_k \Theta_k - \varepsilon_\ell \Theta_\ell) \end{array} \right] + \cdots}_{\displaystyle \sum_{\substack{1 \le k < \ell \le n, \\ \varepsilon_{j, j \neq k, \ell} = \pm 1}} \mathcal{O}\left(\cosh(\sum_{j=1, j \neq k, \ell}^{n} \varepsilon_j X_j)\right)} .
\end{aligned}
$$

By the symmetry (44), $\Delta\Big|_{(X_j, \Theta_j) \to (-X_j, \Theta_j)} = \Delta$ implies $a(\boldsymbol{\varepsilon}) = a(-\boldsymbol{\varepsilon})$, and hence

$$
\Delta = \sum_{\substack{1 \le j \le n, \\ \varepsilon_j = \pm 1}} a(\boldsymbol{\varepsilon}) \cosh\left(\sum_{j=1}^{n} \varepsilon_j X_j\right) + \sum_{\substack{1 \le k < \ell \le n, \\ \varepsilon_{j, j \neq k, \ell} = \pm 1}} \mathcal{O}\left(\cosh(\sum_{j=1, j \neq \ell, m}^{n} \varepsilon_j X_j)\right). \quad \text{(D3)}
$$

Similarly, we have

$$
\begin{aligned}
\Delta_{11} = & \sum_{\substack{1 \le j \le n, \\ \varepsilon_j = \pm 1}} A(\boldsymbol{\varepsilon}) \exp\left(\sum_{j=1}^{n} \varepsilon_j X_j\right) \\
& + \sum_{\substack{1 \le k < \ell \le n, \\ \varepsilon_j = \pm 1,\, j \ne k, \ell \\ (\varepsilon_k, \varepsilon_\ell) \in \mathcal{P}\{(i, -i)\}}} A(\boldsymbol{\varepsilon}) \exp(\sum_{j \ne k, \ell}^{n} \varepsilon_j X_j + \varepsilon_k \Theta_k + \varepsilon_\ell \Theta_\ell) + \text{terms involving more } i\Theta_j \\
= & \frac{1}{2} \sum_{\substack{1 \le j \le n, \\ \varepsilon_j = \pm 1}} \left[A(\boldsymbol{\varepsilon}) \exp(\sum_{j=1}^{n} \varepsilon_j X_j) + A(-\boldsymbol{\varepsilon}) \exp(-\sum_{j=1}^{n} \varepsilon_j X_j) \right] \\
& \underbrace{+ \frac{1}{2} \sum_{\substack{1 \le k < \ell \le n, \\ \varepsilon_j = \pm 1,\, j \ne k, \ell \\ (\varepsilon_k, \varepsilon_\ell) \in \mathcal{P}\{(i, -i)\}}} \left[\begin{array}{c} A(\boldsymbol{\varepsilon}) \exp(\sum_{j \ne k, \ell}^{n} \varepsilon_j X_j + \varepsilon_k \Theta_k + \varepsilon_\ell \Theta_\ell) \\ + A(-\boldsymbol{\varepsilon}) \exp(-\sum_{j \ne k, \ell}^{n} \varepsilon_j X_j - \varepsilon_k \Theta_k - \varepsilon_\ell \Theta_\ell) \end{array} \right] + \cdots}_{\displaystyle \sum_{\substack{1 \le k < \ell \le n, \\ \varepsilon_{j, j \ne k, \ell} = \pm 1}} \mathcal{O}\left(\cosh(\sum_{j=1, j \ne k, \ell}^{n} \varepsilon_j X_j) \right).}
\end{aligned}
\tag{D4}
$$

By the symmetry (44), $\Delta_{22} = \Delta_{11}\Big|_{(X_j,\Theta_j)\to(-X_j,\Theta_j)}$ implies that $\Delta_{22} = \Delta_{11}\Big|_{A(\boldsymbol{\varepsilon})\to A(-\boldsymbol{\varepsilon})}$:

$$\Delta_{12} = \sum_{\substack{1\le k\le n,\\ \varepsilon_j = \begin{cases} \pm 1,\ j\neq k \\ +i,\ j=k\end{cases}}} A(\boldsymbol{\varepsilon})\exp\Big(\sum_{j\neq k}^{n}\varepsilon_j X_j + \varepsilon_k\Theta_k\Big)$$

$$+\left\{\underbrace{\begin{array}{l} \displaystyle\sum_{\substack{1\le k<\ell<m\le n,\\ \varepsilon_j=\pm1,\ j\neq k,\ell,m \\ (\varepsilon_k,\varepsilon_\ell,\varepsilon_m)\in\mathcal{P}\{(i,i,-i)\}}} A(\boldsymbol{\varepsilon})\exp\Big(\sum_{j\neq k,\ell,m}^{n}\varepsilon_j X_j + \varepsilon_k\Theta_k + \varepsilon_\ell\Theta_\ell + \varepsilon_m\Theta_m\Big) \\ +\ \text{terms involving more } i\Theta_j \end{array}}_{\displaystyle\sum_{\substack{1\le k<\ell<m\le n,\\ \varepsilon_{j,j\neq k,\ell,m}=\pm1}} \mathcal{O}\left(\cosh\Big(\sum_{j=1,j\neq k,\ell,m}^{n}\varepsilon_j X_j\Big)\right)}\right\}. \tag{D5}$$

By the symmetry (44), $\Delta_{21} = \Delta_{12}\Big|_{(X_j,\Theta_j)\to(-X_j,\Theta_j)}$. Concrete examples are mentioned as follows:

- **The data of one-soliton solutions:**

$$\begin{aligned} &\Delta = a(1)\left(e^{X_1}+e^{-X_1}\right),\\ &\Delta_{11} = A(1)e^{X_1} + A(-1)e^{-X_1},\quad \Delta_{22} = A(-1)e^{X_1} + A(1)e^{-X_1},\ \Delta_{12} = A(i)e^{i\Theta_1},\\ &\Delta_{21} = A(i)e^{-i\Theta_1}, \end{aligned} \tag{D6}$$

where $a(1) = 1,\quad A(1) = \lambda_1,\quad A(-1) = \mu_1,\quad A(i) = -(\lambda_1 - \mu_1)$.

- **The data of two-soliton solutions ($\boldsymbol{\Theta_{jk} := \Theta_j - \Theta_k,\ j, k = 1, 2}$):**

$$\begin{aligned} \Delta &= \left\{\begin{array}{l} a(1,1)\left(e^{X_1+X_2}+e^{-(X_1+X_2)}\right) + a(1,-1)\left(e^{X_1-X_2}+e^{-(X_1-X_2)}\right) \\ +a(i,-i)\left(e^{i\Theta_{12}}+e^{-i\Theta_{12}}\right)\end{array}\right\},\\ \Delta_{11} &= \left\{\begin{array}{l} \left[A(1,1)e^{X_1+X_2} + A(-1,-1)e^{-(X_1+X_2)}\right] \\ +\left[A(1,-1)e^{X_1-X_2} + A(-1,1)e^{-(X_1-X_2)}\right] \\ +\left[A(i,-i)e^{i\Theta_{12}} + A(-i,i)e^{-i\Theta_{12}}\right]\end{array}\right\},\\ \Delta_{22} &= \Delta_{11}\Big|_{(X_j,\Theta_j)\to(-X_j,-\Theta_j)} = \Delta_{11}\Big|_{A(\varepsilon_1,\varepsilon_2)\to A(-\varepsilon_1,-\varepsilon_2)},\\ \Delta_{12} &= \left\{\begin{array}{l} A(1,i)e^{X_1+i\Theta_2} + A(-1,i)e^{-X_1+i\Theta_2} \\ +A(i,1)e^{X_2+i\Theta_1} + A(i,-1)e^{-X_2+i\Theta_1}\end{array}\right\},\\ \Delta_{21} &= \Delta_{12}\Big|_{(X_j,\Theta_j)\to(-X_j,-\Theta_j)}, \end{aligned} \tag{D7}$$

where

$$
\begin{aligned}
&a(1,1) = (\lambda_1 - \lambda_2)(\mu_1 - \mu_2), \quad a(1,-1) = (\lambda_1 - \mu_2)(\mu_1 - \lambda_2), a(i,-i) = (\lambda_1 - \mu_1)(\lambda_2 - \mu_2) \\
&A(1,1) = -(\lambda_1 - \lambda_2)(\mu_1 - \mu_2)\lambda_1\lambda_2, \quad A(-1,-1) = -(\lambda_1 - \lambda_2)(\mu_1 - \mu_2)\mu_1\mu_2, \\
&A(1,-1) = -(\lambda_1 - \mu_2)(\mu_1 - \lambda_2)\lambda_1\mu_2, \quad A(-1,1) = -(\lambda_1 - \mu_2)(\mu_1 - \lambda_2)\mu_1\lambda_2, \\
&A(i,-i) = -(\lambda_1 - \mu_1)(\lambda_2 - \mu_2)\lambda_1\mu_1, \quad A(-i,i) = -(\lambda_1 - \mu_1)(\lambda_2 - \mu_2)\lambda_2\mu_2, \\
&A(1,i) = (\lambda_1 - \lambda_2)(\lambda_1 - \mu_2)(\lambda_2 - \mu_2)\mu_1, A(-1,i) = (\mu_1 - \mu_2)(\mu_1 - \lambda_2)(\lambda_2 - \mu_2)\lambda_1, \\
&A(i,1) = (\lambda_1 - \lambda_2)(\lambda_1 - \mu_1)(\mu_1 - \lambda_2)\mu_2, \\
&A(i,-1) = (\mu_1 - \mu_2)(\lambda_1 - \mu_1)(\lambda_1 - \mu_2)\lambda_2.
\end{aligned}
\tag{D8}
$$

D.3. Exact calculation of the NLσM term (two-soliton)

For preparation, we introduce some symmetries between the coefficients of the soliton data before our calculation. Our observations are given in the following Remarks 1–3, which can be checked simply from Eqs. (D7) and (D8).

Remark 1

$$
\begin{aligned}
&A(1,1) = -\lambda_1\lambda_2\, a(1,1), \quad A(-1,-1) = -\mu_1\mu_2\, a(1,1), \\
&A(1,-1) = -\lambda_1\mu_2\, a(1,-1), \quad A(-1,1) = -\mu_1\lambda_2\, a(1,-1).
\end{aligned}
\tag{D9}
$$

Remark 2

$$
\begin{aligned}
&A(1,i)A(-1,i) = \lambda_1\mu_1(\lambda_2 - \mu_2)^2\, a(1,1)a(1,-1), \\
&A(i,1)A(i,-1) = \lambda_2\mu_2(\lambda_1 - \mu_1)^2\, a(1,1)a(1,-1), \\
&A(-1,i)A(i,1) = \lambda_1\mu_2(\mu_1 - \lambda_2)^2\, a(1,1)a(i,-i), \\
&A(1,i)A(i,-1) = \mu_1\lambda_2(\lambda_1 - \mu_2)^2\, a(1,1)a(i,-i), \\
&A(-1,i)A(i,-1) = \lambda_1\lambda_2(\mu_1 - \mu_2)^2\, a(1,-1)a(i,-i), \\
&A(1,i)A(i,1) = \mu_1\mu_2(\lambda_1 - \lambda_2)^2\, a(1,-1)a(i,-i).
\end{aligned}
\tag{D10}
$$

Remark 3

$$
\begin{aligned}
&A(1,i) - A(-1,i) = A(i,-1) - A(i,1) = (\lambda_1\mu_1 - \lambda_2\mu_2)\, a(i,-i), \\
&A(i,1) - A(-1,i) = A(i,-1) - A(1,i) = (\lambda_1\mu_2 - \mu_1\lambda_2)\, a(1,-1), \\
&A(1,i) + A(i,1) = A(-1,i) + A(i,-1) = (\lambda_1\lambda_2 - \mu_1\mu_2)\, a(1,1),
\end{aligned}
\tag{D11}
$$

which implies the relation

$$
A(1,i) - A(-1,i) + A(i,1) - A(i,-1) = 0 \tag{D12}
$$

directly by taking some simple addition and subtraction over Eq. (D11). On the other hand, by taking the sum of squares over the left- and right-hand sides of Eq. (D11) and using the relation (D12), we get the following non-trivial identity.

Remark 4

$$
\begin{aligned}
&(\lambda_1\lambda_2 - \mu_1\mu_2)^2 a^2(1,1) + (\lambda_1\mu_2 - \mu_1\lambda_2)^2 a^2(1,-1) + (\lambda_1\mu_1 - \lambda_2\mu_2)^2 a^2(i,-i) \\
&\quad = A^2(1,i) + A^2(-1,i) + A^2(i,1) + A^2(i,-1).
\end{aligned}
\tag{D13}
$$

Now let us start our main calculation of the two-soliton NL sigma model term. By using the soliton data (D7) and after a slightly tedious calculation, we can conclude that

$$
\begin{aligned}
&\partial_\mu \Delta_{11} \partial_\mu \Delta_{22} - \partial_\mu \Delta_{12} \partial_\mu \Delta_{21} - |\sigma|(\partial_\mu \Delta)^2 \\
&\qquad (r_\mu^{(1)} + r_\mu^{(2)})^2 \left[A(1,1)A(-1,-1) - |\sigma| a^2(1,1)\right] \left[e^{2(X_1+X_2)} + e^{-2(X_1+X_2)}\right] \\
&= +(r_\mu^{(1)} - r_\mu^{(2)})^2 \left[A(1,-1)A(-1,1) - |\sigma| a^2(1,-1)\right] \left[e^{2(X_1-X_2)} + e^{-2(X_1-X_2)}\right] \\
&\qquad +(s_\mu^{(1)} - s_\mu^{(2)})^2 \left[A(i,-i)A(-i,i) - |\sigma| a^2(i,-i)\right] \left(e^{2\Theta_{12}} + e^{-2\Theta_{12}}\right) \\
&\qquad + \left\{ \begin{array}{l} (r_\mu^{(1)} + r_\mu^{(2)})(r_\mu^{(1)} - r_\mu^{(2)}) \left[\begin{array}{l} A(1,1)A(-1,1) + A(1,-1)A(-1,-1) \\ -2|\sigma| a(1,1) a(1,-1) \end{array} \right] \\ -(r_\mu^{(1)} + s_\mu^{(2)})(r_\mu^{(1)} - s_\mu^{(2)}) A(1,i)A(-1,i) \end{array} \right\} \left[e^{2X_1} + e^{-2X_1}\right] \\
&\qquad - \left\{ \begin{array}{l} (r_\mu^{(1)} + r_\mu^{(2)})(r_\mu^{(1)} - r_\mu^{(2)}) \left[\begin{array}{l} A(1,1)A(1,-1) + A(-1,1)A(-1,-1) \\ -2|\sigma| a(1,1) a(1,-1) \end{array} \right] \\ -(s_\mu^{(1)} + r_\mu^{(2)})(s_\mu^{(1)} - r_\mu^{(2)}) A(i,1)A(i,-1) \end{array} \right\} \left[e^{2X_2} + e^{-2X_2}\right] \\
&\qquad + \left\{ \begin{array}{l} (r_\mu^{(1)} + r_\mu^{(2)})(s_\mu^{(1)} - s_\mu^{(2)}) \left[\begin{array}{l} A(1,1)A(-i,i) + A(-1,-1)A(i,-i) \\ -2|\sigma| a(1,1) a(i,-i) \end{array} \right] \\ -(r_\mu^{(1)} - s_\mu^{(2)})(s_\mu^{(1)} + r_\mu^{(2)}) A(-1,i)A(i,1) \end{array} \right\} \left[\begin{array}{l} e^{X_1 - X_2 + i\Theta_{12}} \\ +e^{-(X_1 - X_2 + i\Theta_{12})} \end{array} \right] \\
&\qquad - \left\{ \begin{array}{l} (r_\mu^{(1)} + r_\mu^{(2)})(s_\mu^{(1)} - s_\mu^{(2)}) \left[\begin{array}{l} A(1,1)A(i,-i) + A(-1,-1)A(-i,i) \\ -2|\sigma| a(1,1) a(i,-i) \end{array} \right] \\ -(r_\mu^{(1)} + s_\mu^{(2)})(s_\mu^{(1)} - r_\mu^{(2)}) A(1,i)A(i,-1) \end{array} \right\} \left[\begin{array}{l} e^{X_1 + X_2 - i\Theta_{12}} \\ +e^{-(X_1 + X_2 - i\Theta_{12})} \end{array} \right] \\
&\qquad + \left\{ \begin{array}{l} (r_\mu^{(1)} - r_\mu^{(2)})(s_\mu^{(1)} - s_\mu^{(2)}) \left[\begin{array}{l} A(1,-1)A(-i,i) + A(-1,1)A(i,-i) \\ -2|\sigma| a(1,-1) a(i,-i) \end{array} \right] \\ -(r_\mu^{(1)} - s_\mu^{(2)})(s_\mu^{(1)} - r_\mu^{(2)}) A(-1,i)A(i,-1) \end{array} \right\} \left[\begin{array}{l} e^{X_1 - X_2 + i\Theta_{12}} \\ +e^{-(X_1 - X_2 + i\Theta_{12})} \end{array} \right] \\
&\qquad - \left\{ \begin{array}{l} (r_\mu^{(1)} - r_\mu^{(2)})(s_\mu^{(1)} - s_\mu^{(2)}) \left[\begin{array}{l} A(1,-1)A(i,-i) + A(-1,1)A(-i,i) \\ -2|\sigma| a(1,-1) a(i,-i) \end{array} \right] \\ -(r_\mu^{(1)} + s_\mu^{(2)})(s_\mu^{(1)} + r_\mu^{(2)}) A(1,i)A(i,1) \end{array} \right\} \left[\begin{array}{l} e^{X_1 - X_2 - i\Theta_{12}} \\ +e^{-(X_1 - X_2 - i\Theta_{12})} \end{array} \right] \\
&\qquad - \left\{ \begin{array}{l} (r_\mu^{(1)} + r_\mu^{(2)})^2 \left[A^2(1,1) + A^2(-1,-1) - 2|\sigma| a^2(1,1)\right] \\ +(r_\mu^{(1)} - r_\mu^{(2)})^2 \left[A^2(1,-1) + A^2(-1,1) - 2|\sigma| a^2(1,-1)\right] \\ +(s_\mu^{(1)} - s_\mu^{(2)})^2 \left[A^2(i,-i) + A^2(-i,i) - 2|\sigma| a^2(i,-i)\right] \\ -(r_\mu^{(1)} + s_\mu^{(2)})^2 A^2(1,i) - (r_\mu^{(1)} - s_\mu^{(2)})^2 A^2(-1,i) \\ -(s_\mu^{(1)} + r_\mu^{(2)})^2 A^2(i,1) - (s_\mu^{(1)} - r_\mu^{(2)})^2 A^2(i,-1) \end{array} \right\}.
\end{aligned}
$$

By Eqs. (D9) and (D10), we find that the coefficients of the leading terms $\exp(\pm 2(X_1 \pm X_2))$ are identical to zero and the remaining terms can be rewritten as

$$
\begin{aligned}
&\partial_\mu \Delta_{11}\partial_\mu \Delta_{22} - \partial_\mu \Delta_{12}\partial_\mu \Delta_{21} - |\sigma|(\partial_\mu \Delta)^2 \\
&= \left\{
\begin{array}{l}
-(r_\mu^{(2)} + s_\mu^{(2)})(r_\mu^{(2)} - s_\mu^{(2)})A(1,i)A(-1,i)\left(e^{2X_1} + e^{-2X_1}\right) \\
-(r_\mu^{(1)} + s_\mu^{(1)})(r_\mu^{(1)} - s_\mu^{(1)})A(i,1)A(i,-1)\left(e^{2X_2} + e^{-2X_2}\right) \\
-(r_\mu^{(1)} - s_\mu^{(1)})(r_\mu^{(2)} + s_\mu^{(2)})A(-1,i)A(i,1)\left(e^{X_1+X_2+i\Theta_{12}} + e^{-(X_1+X_2+i\Theta_{12})}\right) \\
-(r_\mu^{(1)} + s_\mu^{(1)})(r_\mu^{(2)} - s_\mu^{(2)})A(1,i)A(i,-1)\left(e^{X_1+X_2-i\Theta_{12}} + e^{-(X_1+X_2-i\Theta_{12})}\right) \\
+(r_\mu^{(1)} - s_\mu^{(1)})(r_\mu^{(2)} - s_\mu^{(2)})A(-1,i)A(i,-1)\left(e^{X_1-X_2+i\Theta_{12}} + e^{-(X_1-X_2+i\Theta_{12})}\right) \\
+(r_\mu^{(1)} + s_\mu^{(1)})(r_\mu^{(2)} + s_\mu^{(2)})A(1,i)A(i,1)\left(e^{X_1-X_2-i\Theta_{12}} + e^{-(X_1-X_2-i\Theta_{12})}\right)
\end{array}
\right\} \\
&\quad - \underbrace{\left\{
\begin{array}{l}
(r_\mu^{(1)} + r_\mu^{(2)})^2(\lambda_1\lambda_2 - \mu_1\mu_2)^2\, a^2(1,1) \\
+(r_\mu^{(1)} - r_\mu^{(2)})^2(\lambda_1\mu_2 - \mu_1\lambda_2)^2\, a^2(1,-1) \\
+(s_\mu^{(1)} - s_\mu^{(2)})^2(\lambda_1\mu_1 - \lambda_2\mu_2)^2\, a^2(i,-i) \\
-(r_\mu^{(1)} + s_\mu^{(2)})^2A^2(1,i) - (r_\mu^{(1)} - s_\mu^{(2)})^2A^2(-1,i) \\
-(s_\mu^{(1)} + r_\mu^{(2)})^2A^2(i,1) - (s_\mu^{(1)} - r_\mu^{(2)})^2A^2(i,-1)
\end{array}
\right\}} \\
&=: \Xi. \qquad \text{(D14)}
\end{aligned}
$$

Next, we want to show that the constant term Ξ above can be absorbed completely into the non-constant terms. By the definition of $r_\mu^{(j)}, s_\mu^{(j)}$ above Eq. (42), we can replace $r_\mu^{(j)}, s_\mu^{(j)}$ with $\ell_\mu^{(j)}$ and obtain

$$
\Xi = \left\{
\begin{array}{l}
\left(\ell_\mu^{(1)} + \overline{\ell}_\mu^{(1)} + \ell_\mu^{(2)} + \overline{\ell}_\mu^{(2)}\right)^2 (\lambda_1\lambda_2 - \mu_1\mu_2)^2\, a^2(1,1) \\
+\left(\ell_\mu^{(1)} + \overline{\ell}_\mu^{(1)} - \ell_\mu^{(2)} - \overline{\ell}_\mu^{(2)}\right)^2 (\lambda_1\mu_2 - \mu_1\lambda_2)^2\, a^2(1,-1) \\
+\left(\ell_\mu^{(1)} - \overline{\ell}_\mu^{(1)} - \ell_\mu^{(2)} + \overline{\ell}_\mu^{(2)}\right)^2 (\lambda_1\mu_1 - \lambda_2\mu_2)^2\, a^2(i,-i) \\
-\left(\ell_\mu^{(1)} + \overline{\ell}_\mu^{(1)} + \ell_\mu^{(2)} - \overline{\ell}_\mu^{(2)}\right)^2 A^2(1,i) - \left(\ell_\mu^{(1)} + \overline{\ell}_\mu^{(1)} - \ell_\mu^{(2)} + \overline{\ell}_\mu^{(2)}\right)^2 A^2(-1,i) \\
-\left(\ell_\mu^{(1)} - \overline{\ell}_\mu^{(1)} + \ell_\mu^{(2)} + \overline{\ell}_\mu^{(2)}\right)^2 A^2(i,1) - \left(\ell_\mu^{(1)} - \overline{\ell}_\mu^{(1)} - \ell_\mu^{(2)} - \overline{\ell}_\mu^{(2)}\right)^2 A^2(i,-1)
\end{array}
\right\}.
$$

By Eqs. (D13) and (D11),

$$\Xi = -4\left\{\begin{array}{l} \ell_\mu^{(1)}\overline{\ell}_\mu^{(1)}\left[-(\lambda_1\mu_1-\lambda_2\mu_2)^2a^2(i,-i)+A^2(i,1)+A^2(i,-1)\right] \\ +\ell_\mu^{(2)}\overline{\ell}_\mu^{(2)}\left[-(\lambda_1\mu_1-\lambda_2\mu_2)^2a^2(i,-i)+A^2(1,i)+A^2(-1,i)\right] \\ +\ell_\mu^{(1)}\overline{\ell}_\mu^{(2)}\left[-(\lambda_1\mu_2-\mu_1\lambda_2)^2a^2(1,-1)+A^2(1,i)+A^2(i,-1)\right] \\ +\overline{\ell}_\mu^{(1)}\ell_\mu^{(2)}\left[-(\lambda_1\mu_2-\mu_1\lambda_2)^2a^2(1,-1)+A^2(-1,i)+A^2(i,1)\right] \\ -\ell_\mu^{(1)}\ell_\mu^{(2)}\left[-(\lambda_1\lambda_2-\mu_1\mu_2)^2a^2(1,1)+A^2(1,i)+A^2(i,1)\right] \\ -\overline{\ell}_\mu^{(1)}\overline{\ell}_\mu^{(2)}\left[-(\lambda_1\lambda_2-\mu_1\mu_2)^2a^2(1,1)+A^2(-1,i)+A^2(i,-1)\right] \end{array}\right\}$$

$$= -8\left\{\begin{array}{l} \ell_\mu^{(1)}\overline{\ell}_\mu^{(1)}A(i,1)A(i,-1)+\ell_\mu^{(2)}\overline{\ell}_\mu^{(2)}A(1,i)A(-1,i) \\ +\ell_\mu^{(1)}\overline{\ell}_\mu^{(2)}A(1,i)A(i,-1)+\overline{\ell}_\mu^{(1)}\ell_\mu^{(2)}A(-1,i)A(i,1) \\ +\ell_\mu^{(1)}\ell_\mu^{(2)}A(1,i)A(i,1)+\overline{\ell}_\mu^{(1)}\overline{\ell}_\mu^{(2)}A(-1,i)A(i,-1) \end{array}\right\}.$$

Comparing this with Eq. (D14), we have

$$\begin{aligned} &\partial_\mu\Delta_{11}\partial_\mu\Delta_{22}-\partial_\mu\Delta_{12}\partial_\mu\Delta_{21}-|\sigma|(\partial_\mu\Delta)^2 \\ &= -4\left\{\begin{array}{l} \ell_\mu^{(1)}\overline{\ell}_\mu^{(1)}A(i,1)A(i,-1)\left(e^{X_2}+e^{-X_2}\right)^2 \\ +\ell_\mu^{(2)}\overline{\ell}_\mu^{(2)}A(1,i)A(-1,i)\left(e^{X_1}+e^{-X_1}\right)^2 \\ +\ell_\mu^{(1)}\overline{\ell}_\mu^{(2)}A(1,i)A(i,-1)\left(e^{\frac{X_1+X_2-i\Theta_{12}}{2}}+e^{-\frac{(X_1+X_2-i\Theta_{12})}{2}}\right)^2 \\ +\overline{\ell}_\mu^{(1)}\ell_\mu^{(2)}A(-1,i)A(i,1)\left(e^{\frac{X_1+X_2+i\Theta_{12}}{2}}+e^{-\frac{(X_1+X_2+i\Theta_{12})}{2}}\right)^2 \\ -\ell_\mu^{(1)}\ell_\mu^{(2)}A(1,i)A(i,1)\left(e^{\frac{X_1-X_2-i\Theta_{12}}{2}}-e^{-\frac{(X_1-X_2-i\Theta_{12})}{2}}\right)^2 \\ -\overline{\ell}_\mu^{(1)}\overline{\ell}_\mu^{(2)}A(-1,i)A(i,-1)\left(e^{\frac{X_1-X_2+i\Theta_{12}}{2}}-e^{-\frac{(X_1-X_2+i\Theta_{12})}{2}}\right)^2 \end{array}\right\}. \end{aligned}$$

By Eqs. (23) and (D10), we can conclude that

$$\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}\right]^2 = \frac{-8\left\{\begin{array}{l} a(1,1)a(1,-1)\left[\ell_\mu^{(1)}\overline{\ell}_\mu^{(1)}\Lambda_{11}\cosh^2X_2+\ell_\mu^{(2)}\overline{\ell}_\mu^{(2)}\Lambda_{22}\cosh^2X_1\right] \\ +a(1,1)a(i,-i)\left[\begin{array}{l} \ell_\mu^{(1)}\overline{\ell}_\mu^{(2)}\Lambda_{12}\cosh^2\left(\dfrac{X_1+X_2-i\Theta_{12}}{2}\right) \\ +\overline{\ell}_\mu^{(1)}\ell_\mu^{(2)}\Lambda_{21}\cosh^2\left(\dfrac{X_1+X_2+i\Theta_{12}}{2}\right) \end{array}\right] \\ -a(1,-1)a(i,-i)\left[\begin{array}{l} \ell_\mu^{(1)}\ell_\mu^{(2)}\Omega_{12}\sinh^2\left(\dfrac{X_1-X_2-i\Theta_{12}}{2}\right) \\ +\overline{\ell}_\mu^{(1)}\overline{\ell}_\mu^{(2)}\widetilde{\Omega}_{12}\sinh^2\left(\dfrac{X_1-X_2+i\Theta_{12}}{2}\right) \end{array}\right] \end{array}\right\}}{\left[a(1,1)\cosh(X_1+X_2)+a(1,-1)\cosh(X_1-X_2)+a(i,-i)\cos\Theta_{12}\right]^2},$$

where

$$\Lambda_{jk} := \frac{(\lambda_j-\mu_k)^2}{\lambda_j\mu_k},\quad \Omega_{jk} := \frac{(\lambda_j-\lambda_k)^2}{\lambda_j\lambda_k},\quad \widetilde{\Omega}_{jk} := \frac{(\mu_j-\mu_k)^2}{\mu_j\mu_k}.$$

D.4. Exact calculation of the Wess–Zumino action density (two-soliton)

By substituting Eqs. (39), (D7), and (42) into Eq. (25) for $(m, n, p) = (\mu, \nu, \rho)$, we have

$$\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}(\partial_\nu\sigma)\sigma^{-1}(\partial_\rho\sigma)\sigma^{-1}\right] = \frac{1}{2}(B_{\mu\nu\rho} + B_{\nu\rho\mu} + B_{\rho\mu\nu}),$$

$$B_{\mu\nu\rho} := \frac{1}{|\sigma|^2\Delta^4}\left(\begin{vmatrix}\Delta_{11} & \Delta_{22}\\ \partial_\mu\Delta_{11} & \partial_\mu\Delta_{22}\end{vmatrix}\begin{vmatrix}\partial_\nu\Delta_{12} & \partial_\nu\Delta_{21}\\ \partial_\rho\Delta_{12} & \partial_\rho\Delta_{21}\end{vmatrix} + \begin{vmatrix}\Delta_{12} & \Delta_{21}\\ \partial_\mu\Delta_{12} & \partial_\mu\Delta_{21}\end{vmatrix}\begin{vmatrix}\partial_\nu\Delta_{11} & \partial_\nu\Delta_{22}\\ \partial_\rho\Delta_{11} & \partial_\rho\Delta_{22}\end{vmatrix}\right), \tag{D15}$$

where

$$\frac{1}{|\sigma|\Delta^2}\begin{vmatrix}\Delta_{11} & \Delta_{22}\\ \partial_\mu\Delta_{11} & \partial_\mu\Delta_{22}\end{vmatrix} = \frac{-\left\{\begin{array}{l} 2r_\mu^{(1)}ab\mathscr{D}_{11}\cosh(2X_2) + 2r_\mu^{(2)}ab\mathscr{D}_{22}\cosh(2X_1)\\ +\left[(r_\mu^{(1)}+r_\mu^{(2)}) + (s_\mu^{(1)}-s_\mu^{(2)})\right]ac\mathscr{D}_{12}\cosh(X_1+X_2-i\Theta_{12})\\ +\left[(r_\mu^{(1)}+r_\mu^{(2)}) - (s_\mu^{(1)}-s_\mu^{(2)})\right]ac\mathscr{D}_{21}\cosh(X_1+X_2+i\Theta_{12})\\ +\left[(r_\mu^{(1)}-r_\mu^{(2)}) + (s_\mu^{(1)}-s_\mu^{(2)})\right]bc\mathscr{E}_{12}\cosh(X_1-X_2-i\Theta_{12})\\ -\left[(r_\mu^{(1)}-r_\mu^{(2)}) - (s_\mu^{(1)}-s_\mu^{(2)})\right]bc\widetilde{\mathscr{E}}_{12}\cosh(X_1-X_2+i\Theta_{12}) \end{array}\right\} - F}{2\left[a\cosh(X_1+X_2) + b\cosh(X_1-X_2) + c\cos\Theta_{12}\right]^2}, \tag{D16}$$

$$\frac{1}{|\sigma|\Delta^2}\begin{vmatrix}\Delta_{12} & \Delta_{21}\\ \partial_\mu\Delta_{12} & \partial_\mu\Delta_{21}\end{vmatrix} = \frac{-\left\{\begin{array}{l} 2s_\mu^{(1)}ab\mathscr{d}_{11}\cosh(2X_2) + 2s_\mu^{(2)}ab\mathscr{d}_{22}\cosh(2X_1)\\ +\left[(r_\mu^{(1)}-r_\mu^{(2)}) + (s_\mu^{(1)}+s_\mu^{(2)})\right]ac\mathscr{d}_{12}\cosh(X_1+X_2-i\Theta_{12})\\ -\left[(r_\mu^{(1)}-r_\mu^{(2)}) - (s_\mu^{(1)}+s_\mu^{(2)})\right]ac\mathscr{d}_{21}\cosh(X_1+X_2+i\Theta_{12})\\ +\left[(r_\mu^{(1)}+r_\mu^{(2)}) + (s_\mu^{(1)}+s_\mu^{(2)})\right]bc\mathscr{e}_{12}\cosh(X_1-X_2-i\Theta_{12})\\ -\left[(r_\mu^{(1)}+r_\mu^{(2)}) - (s_\mu^{(1)}+s_\mu^{(2)})\right]bc\widetilde{\mathscr{e}}_{12}\cosh(X_1-X_2+i\Theta_{12}) \end{array}\right\} + f}{2\left[a\cosh(X_1+X_2) + b\cosh(X_1-X_2) + c\cos\Theta_{12}\right]^2}, \tag{D17}$$

$$\frac{1}{|\sigma|\Delta^2}\begin{vmatrix}\partial_\nu\Delta_{11} & \partial_\nu\Delta_{22}\\ \partial_\rho\Delta_{11} & \partial_\rho\Delta_{22}\end{vmatrix} = \frac{\left\{\begin{array}{l} 2\left[r_\nu^{(1)}r_\rho^{(2)} - r_\rho^{(1)}r_\nu^{(2)}\right]ab\left[\mathscr{D}_{11}\sinh(2X_2) - \mathscr{D}_{22}\sinh(2X_1)\right]\\ -\left[(r_\nu^{(1)}+r_\nu^{(2)})(s_\rho^{(1)}-s_\rho^{(2)}) - (r_\rho^{(1)}+r_\rho^{(2)})(s_\nu^{(1)}-s_\nu^{(2)})\right]\\ \cdot\ ac\left[\mathscr{D}_{12}\sinh(X_1+X_2-i\Theta_{12}) - \mathscr{D}_{21}\sinh(X_1+X_2+i\Theta_{12})\right]\\ -\left[(r_\nu^{(1)}-r_\nu^{(2)})(s_\rho^{(1)}-s_\rho^{(2)}) - (r_\rho^{(1)}-r_\rho^{(2)})(s_\nu^{(1)}-s_\nu^{(2)})\right]\cdot\\ bc\left[\mathscr{E}_{12}\sinh(X_1-X_2-i\Theta_{12}) + \widetilde{\mathscr{E}}_{12}\sinh(X_1-X_2+i\Theta_{12})\right] \end{array}\right\}}{2\left[a\cosh(X_1+X_2) + b\cosh(X_1-X_2) + c\cos\Theta_{12}\right]^2}, \tag{D18}$$

Table D1. Summary of coefficients.

Space (Metric)	$\mathbb{U}_1$ $(+,+,-,-)$	$\mathbb{E}$ $(+,+,+,+)$
$\mathscr{D}_{jk}$	$\dfrac{(\lambda_j-\overline{\lambda}_k)(\lambda_j+\overline{\lambda}_k)}{\lambda_j\overline{\lambda}_k}$	$\dfrac{-(\lambda_j\overline{\lambda}_k-1)(\lambda_j\overline{\lambda}_k+1)}{\lambda_j\overline{\lambda}_k}$
$\mathscr{E}_{jk}$	$\dfrac{(\lambda_j-\lambda_k)(\lambda_j+\lambda_k)}{\lambda_j\lambda_k}$	$\dfrac{(\lambda_j-\lambda_k)(\lambda_j+\lambda_k)}{\lambda_j\lambda_k}$
$\widetilde{\mathscr{E}}_{jk}$	$\dfrac{(\overline{\lambda}_j-\overline{\lambda}_k)(\overline{\lambda}_j+\overline{\lambda}_k)}{\overline{\lambda}_j\overline{\lambda}_k}$	$\dfrac{-(\overline{\lambda}_j-\overline{\lambda}_k)(\overline{\lambda}_j+\overline{\lambda}_k)}{\overline{\lambda}_j\overline{\lambda}_k}$
$\mathscr{d}_{jk}$	$\dfrac{(\lambda_j-\overline{\lambda}_k)^2}{\lambda_j\overline{\lambda}_k}$	$\dfrac{-(\lambda_j\overline{\lambda}_k+1)^2}{\lambda_j\overline{\lambda}_k}$
$\mathscr{e}_{jk}$	$\dfrac{(\lambda_j-\lambda_k)^2}{\lambda_j\lambda_k}$	$\dfrac{(\lambda_j-\lambda_k)^2}{\lambda_j\lambda_k}$
$\widetilde{\mathscr{e}}_{jk}$	$\dfrac{(\overline{\lambda}_j-\overline{\lambda}_k)^2}{\overline{\lambda}_j\overline{\lambda}_k}$	$\dfrac{(\overline{\lambda}_j-\overline{\lambda}_k)^2}{\overline{\lambda}_j\overline{\lambda}_k}$

$$
\frac{1}{|\sigma|\Delta^2}\begin{vmatrix}\partial_\nu\Delta_{12} & \partial_\nu\Delta_{21}\\ \partial_\rho\Delta_{12} & \partial_\rho\Delta_{21}\end{vmatrix}
$$
$$
=\frac{\left\{\begin{array}{l}
2\Big[\ (s_\nu^{(1)}r_\rho^{(2)}-s_\rho^{(1)}r_\nu^{(2)})ab\mathscr{d}_{11}\sinh(2X_2)-(r_\nu^{(1)}s_\rho^{(2)}-r_\rho^{(1)}s_\nu^{(2)})ab\mathscr{d}_{22}\sinh(2X_1)\Big]\\
-\Big[\ (r_\nu^{(1)}+s_\nu^{(2)})(s_\rho^{(1)}-r_\rho^{(2)})-(r_\rho^{(1)}+s_\rho^{(2)})(s_\nu^{(1)}-r_\nu^{(2)})\Big]ac\mathscr{d}_{12}\sinh(X_1+X_2-i\Theta_{12})\\
-\Big[\ (r_\nu^{(1)}-s_\nu^{(2)})(s_\rho^{(1)}+r_\rho^{(2)})-(r_\rho^{(1)}-s_\rho^{(2)})(s_\nu^{(1)}+r_\nu^{(2)})\Big]ac\mathscr{d}_{21}\sinh(X_1+X_2+i\Theta_{12})\\
-\Big[\ (r_\nu^{(1)}+s_\nu^{(2)})(s_\rho^{(1)}+r_\rho^{(2)})-(r_\rho^{(1)}+s_\rho^{(2)})(s_\nu^{(1)}+r_\nu^{(2)})\Big]bc\mathscr{e}_{12}\sinh(X_1-X_2-i\Theta_{12})\\
-\Big[\ (r_\nu^{(1)}-s_\nu^{(2)})(s_\rho^{(1)}-r_\rho^{(2)})-(r_\rho^{(1)}-s_\rho^{(2)})(s_\nu^{(1)}-r_\nu^{(2)})\Big]bc\widetilde{\mathscr{e}}_{12}\sinh(X_1-X_2+i\Theta_{12})
\end{array}\right\}}{2\Big[a\cosh(X_1+X_2)+b\cosh(X_1+X_2)+c\cos\Theta_{12}\Big]^2}\,. \tag{D19}
$$

a, b, c are defined in Table 1 and $\mathscr{D}_{jk}$, $\mathscr{d}_{jk}$, $\mathscr{E}_{jk}$, $\widetilde{\mathscr{E}}_{jk}$, $\mathscr{e}_{jk}$ are defined in Table D1 for each real space respectively. F and f are some constants. Note that

$$
(\overline{\mathscr{D}}_{jk},\ \overline{\mathscr{E}}_{jk})=\Big\{(-\mathscr{D}_{kj},\ \widetilde{\mathscr{E}}_{jk})\quad\text{on }\mathbb{U}_1(\mathscr{D}_{kj},\ -\widetilde{\mathscr{E}}_{jk})\quad\text{on }\mathbb{E}, \tag{D20}
$$

$$
(\overline{\mathscr{d}}_{jk},\ \overline{\mathscr{e}}_{jk})=(\mathscr{d}_{kj},\ \widetilde{\mathscr{e}}_{jk})\qquad\text{on }\mathbb{U}_1,\mathbb{E}, \tag{D21}
$$

which implies that Eqs. (D16)–(D19) are all pure imaginary functions on $\mathbb{U}_1$ and hence Eq. (D15) is real-valued on $\mathbb{U}_1$. By Eq. (28), the Wess–Zumino term is real-valued on $\mathbb{U}_1$. On the other hand, we find that Eqs. (D16), (D18) are real functions and Eqs. (D17), (D19) are pure imaginary functions on $\mathbb{E}$. Therefore, Eq. (D15) is a pure imaginary function on $\mathbb{E}$. This implies that the Wess–Zumino term is real-valued on $\mathbb{E}$ because of (cf. Eq. (28))

$$
\mathscr{L}_{\mathrm{WZ}}\overset{\mathbb{E}}{=}-\frac{i}{8\pi}\left(\mathrm{Tr}\,(\theta_1\theta_3\theta_4)\,x^1+\mathrm{Tr}\,(\theta_2\theta_3\theta_4)\,x^2-\mathrm{Tr}\,(\theta_3\theta_1\theta_2)\,x^3-\mathrm{Tr}\,(\theta_4\theta_1\theta_2)\,x^4\right). \tag{D22}
$$

For convenience, to discuss the asymptotic behavior of Eq. (D15), we consider the asymptotic limit

$$\left\{ X_1 \text{ is a finite real number} |X_2| \gg |X_1| \right. . \tag{D23}$$

By the same calculation as mentioned in Sect. 4.2, we have

$$\begin{aligned}
\frac{1}{|\sigma|\Delta^2}\begin{vmatrix}\Delta_{11} & \Delta_{22}\\ \partial_\mu\Delta_{11} & \partial_\mu\Delta_{22}\end{vmatrix}_{|X_2|\gg|X_1|} &= \frac{-2r_\mu^{(1)}ab\mathscr{D}_{11}\cosh^2 X_2 + \mathscr{O}(\cosh X_2)}{\Big[a\cosh(X_1+X_2)+b\cosh(X_1+X_2)+\mathscr{O}(1)\Big]^2}\\
&= \frac{-2r_\mu^{(1)}ab\mathscr{D}_{11} + \mathscr{O}(\mathrm{sech}X_2)}{[(a+b)\cosh X_1 + (a-b)\sinh X_1 \tanh X_2 + \mathscr{O}(\mathrm{sech}X_2)]^2},\\
\frac{1}{|\sigma|\Delta^2}\begin{vmatrix}\Delta_{12} & \Delta_{21}\\ \partial_\mu\Delta_{12} & \partial_\mu\Delta_{21}\end{vmatrix}_{|X_2|\gg|X_1|} &= \frac{-2s_\mu^{(1)}ab\mathscr{d}_{11}\cosh^2 X_2 + \mathscr{O}(\cosh X_2)}{\Big[a\cosh(X_1+X_2)+b\cosh(X_1+X_2)+\mathscr{O}(1)\Big]^2}\\
&= \frac{-2s_\mu^{(1)}ab\mathscr{d}_{11} + \mathscr{O}(\mathrm{sech}X_2)}{[(a+b)\cosh X_1 + (a-b)\sinh X_1 \tanh X_2 + \mathscr{O}(\mathrm{sech}X_2)]^2},\\
\frac{1}{|\sigma|\Delta^2}\begin{vmatrix}\partial_\nu\Delta_{11} & \partial_\nu\Delta_{22}\\ \partial_\rho\Delta_{11} & \partial_\rho\Delta_{22}\end{vmatrix}_{|X_2|\gg|X_1|} &= \frac{2\left(r_\nu^{(1)}r_\rho^{(2)} - r_\rho^{(1)}r_\nu^{(2)}\right)ab\mathscr{D}_{11}\cosh X_2\sinh X_2 + \mathscr{O}(\cosh X_2)}{\Big[a\cosh(X_1+X_2)+b\cosh(X_1+X_2)+\mathscr{O}(1)\Big]^2}\\
&= \frac{2\left(r_\nu^{(1)}r_\rho^{(2)} - r_\rho^{(1)}r_\nu^{(2)}\right)ab\mathscr{D}_{11}\tanh X_2 + \mathscr{O}(\mathrm{sech}X_2)}{[(a+b)\cosh X_1 + (a-b)\sinh X_1 \tanh X_2 + \mathscr{O}(\mathrm{sech}X_2)]^2},\\
\frac{1}{|\sigma|\Delta^2}\begin{vmatrix}\partial_\nu\Delta_{12} & \partial_\nu\Delta_{21}\\ \partial_\rho\Delta_{12} & \partial_\rho\Delta_{21}\end{vmatrix}_{|X_2|\gg|X_1|} &= \frac{2\left(s_\nu^{(1)}r_\rho^{(2)} - s_\rho^{(1)}r_\nu^{(2)}\right)ab\mathscr{d}_{11}\cosh X_2\sinh X_2 + \mathscr{O}(\cosh X_2)}{\Big[a\cosh(X_1+X_2)+b\cosh(X_1+X_2)+\mathscr{O}(1)\Big]^2}\\
&= \frac{2\left(s_\nu^{(1)}r_\rho^{(2)} - s_\rho^{(1)}r_\nu^{(2)}\right)ab\mathscr{d}_{11}\tanh X_2 + \mathscr{O}(\mathrm{sech}X_2)}{[(a+b)\cosh X_1 + (a-b)\sinh X_1 \tanh X_2 + \mathscr{O}(\mathrm{sech}X_2)]^2}.
\end{aligned}$$

Therefore, for fixed X_1 and $|X_2| \gg |X_1|$, we can conclude that

$$\begin{aligned}
B_{\mu\nu\rho} &\overset{|X_2|\gg|X_1|}{\simeq} \frac{-4a^2b^2C_{\mu\nu\rho}\mathscr{D}_{11}\mathscr{d}_{11}\tanh X_2 + \mathscr{O}(\mathrm{sech}X_2)}{[(a+b)\cosh X_1 + (a-b)\sinh X_1\tanh X_2 + \mathscr{O}(\mathrm{sech}X_2)]^4}\\
&\overset{X_2\to\pm\infty}{\longrightarrow} \frac{\mp 32a^2b^2C_{\mu\nu\rho}\mathscr{D}_{11}\mathscr{d}_{11}}{(ae^{X_1}+be^{X_1})^4} = \mp 2C_{\mu\nu\rho}D_{11}d_{11}\mathrm{sech}^4(X_1+\delta_1) \quad \text{for fixed } X_1,
\end{aligned}$$

where the phase shift factor is $\delta_1 := (1/2)\log(a/b)$. and $C_{\mu\nu\rho} := \left(r_\mu^{(1)}s_\nu^{(1)} + s_\mu^{(1)}r_\nu^{(1)}\right)r_\rho^{(2)} - \left(r_\mu^{(1)}s_\rho^{(1)} + s_\mu^{(1)}r_\rho^{(1)}\right)r_\nu^{(2)}$.

D.5. Asymptotic form of WZW_4 action density (n-soliton)

Without loss of generality, we consider one of the asymptotic regions of type $\mathscr{R}_K$, which is labeled by (the other cases are equivalent to this one)

$$\varepsilon_j = \begin{cases} +1, & j = 1, \ldots, K-1 \\ -1, & j = K+1, \ldots, n \end{cases}, \tag{D24}$$

and define

$$X_{\widehat{K}} := \sum_{j=1, j\neq K}^{n} \varepsilon_j X_j \overset{(D25)}{=} X_1 + \cdots + X_{K-1} - X_{K+1} + \cdots + X_n.$$

By Eqs. (D3), (D4), and (D5), we can conclude that

$$\begin{aligned}
\Delta &\overset{\mathscr{R}_K}{\simeq} 2\left\{\begin{array}{l} a(\mathbf{1},\ \varepsilon_K=+1,\ \mathbf{-1})\cosh(\boldsymbol{X}_{\widehat{K}}+X_K) \\ +a(\mathbf{1},\ \varepsilon_K=-1,\ \mathbf{-1})\cosh(\boldsymbol{X}_{\widehat{K}}-X_K)\end{array}\right\} + \sum_{j=1, j\neq K}^{n} \mathcal{O}\left(\cosh(\boldsymbol{X}_{\widehat{K}}-\varepsilon_j X_j)\right),\\
\Delta_{11} &\overset{\mathscr{R}_K}{\simeq} \left\{\begin{array}{l} A(\mathbf{1},\ \varepsilon_K=+1,\ \mathbf{-1})\exp(\boldsymbol{X}_{\widehat{K}}+X_K) \\ +A(\mathbf{1},\ \varepsilon_K=-1,\ \mathbf{-1})\exp(\boldsymbol{X}_{\widehat{K}}-X_K) \\ +A(\mathbf{-1},\ \varepsilon_K=+1,\ \mathbf{1})\exp(-\boldsymbol{X}_{\widehat{K}}+X_K) \\ +A(\mathbf{-1},\ \varepsilon_K=-1,\ \mathbf{1})\exp(-\boldsymbol{X}_{\widehat{K}}-X_K)\end{array}\right\} + \sum_{j=1, j\neq J}^{n} \mathcal{O}\left(\cosh(\boldsymbol{X}_{\widehat{K}}-\varepsilon_j X_j)\right),\\
\Delta_{22}(\mathscr{R}_K) &= \Delta_{11}(\mathscr{R}_K)\Big|_{A(\varepsilon_1,\ldots,\varepsilon_n)\,\to\, A(-\varepsilon_1,\ldots,-\varepsilon_n)}, \qquad\qquad (D25)\\
\Delta_{12} &\overset{\mathscr{R}_K}{\simeq} \left\{\begin{array}{l} A(\mathbf{1},\ \varepsilon_K=+i,\ \mathbf{-1})\exp(\boldsymbol{X}_{\widehat{K}}+i\Theta_K) \\ +A(\mathbf{-1},\ \varepsilon_K=+i,\ \mathbf{1})\exp(-\boldsymbol{X}_{\widehat{K}}+i\Theta_K)\end{array}\right\} + \sum_{j=1, j\neq K}^{n} \mathcal{O}\left(\cosh(\boldsymbol{X}_{\widehat{K}}-\varepsilon_j X_j)\right),\\
\Delta_{21}(\mathscr{R}_K) &= \Delta_{12}(\mathscr{R}_K)\Big|_{(X_j,\Theta_K)\,\to\,(-X_j,-\Theta_K)}.
\end{aligned}$$

(D26)

By Eqs. (D25), (42), and direct calculation, we have the asymptotic form of the NLσM action density:

$$\begin{aligned}
&\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}(\partial^\mu\sigma)\sigma^{-1}\right]\\
&\overset{\mathscr{R}_K}{\simeq} \frac{\left[\begin{array}{c} 4d_{KK}\, a(\mathbf{1}, \varepsilon_K=+1, \mathbf{-1})\cosh(2\boldsymbol{X}_{\widehat{K}}) \\ +\displaystyle\sum_{j=1, j\neq K}^{n} \mathcal{O}\left(\cosh(2\boldsymbol{X}_{\widehat{K}}-\varepsilon_j X_j)\right)\end{array}\right]}{\left\{\left[\begin{array}{l} a(\mathbf{1},\ \varepsilon_K=+1,\ \mathbf{-1})\cosh(\boldsymbol{X}_{\widehat{K}}+X_K) \\ +a(\mathbf{1},\ \varepsilon_K=-1,\ \mathbf{-1})\cosh(\boldsymbol{X}_{\widehat{K}}-X_K)\end{array}\right] + \displaystyle\sum_{j=1, j\neq K}^{n} \mathcal{O}\left(\cosh(\boldsymbol{X}_{\widehat{K}}-\varepsilon_j X_j)\right)\right\}^2}\\
&= \frac{8d_{KK}\, a(\mathbf{1}, \varepsilon_K=+1, \mathbf{-1})a(\mathbf{1}, \varepsilon_K=-1, \mathbf{-1}) + \displaystyle\sum_{j=1, j\neq K}^{n} \mathcal{O}\left(\mathrm{sech} X_j\right)}{\left\{\begin{array}{l} [a(\mathbf{1},\ \varepsilon_K=+1,\ \mathbf{-1})+a(\mathbf{1},\ \varepsilon_K=-1,\ \mathbf{-1})]\cosh X_K \\ +[a(\mathbf{1},\ \varepsilon_K=+1,\ \mathbf{-1})-a(\mathbf{1},\ \varepsilon_K=-1,\ \mathbf{-1})]\tanh \boldsymbol{X}_{\widehat{K}}\ \sinh X_K \\ +\displaystyle\sum_{j=1, j\neq K}^{n} \mathcal{O}\left(\mathrm{sech} X_j\right)\end{array}\right\}^2}\\
&\overset{\boldsymbol{X}_{\widehat{K}}\to\pm\infty}{\longrightarrow} 2d_{KK}\mathrm{sech}^2(X_K \pm \delta_K) \ \text{ for fixed real number } X_K,
\end{aligned}$$

where the phase shift factor is $\delta_K := \frac{1}{2}\log\left[\frac{a(\mathbf{1},\ \varepsilon_K=+1,\ \mathbf{-1})}{a(\mathbf{1},\ \varepsilon_K=-1,\ \mathbf{-1})}\right]$ and d_{KK} is defined in Table 1.

By Eqs. (D25), (42), and direct calculation, we have the asymptotic form of the Wess–Zumino action density:

$$B_{\mu\nu\rho} \overset{\mathscr{R}_K}{\simeq} \frac{-4\mathscr{A}\widetilde{\mathscr{A}}C^{(K)}_{\mu\nu\rho}\tanh \boldsymbol{X}_{\widehat{K}} + \sum_{j=1,\, j\neq K}^{n} \mathbb{O}\left(\mathrm{sech} X_j\right)}{|\sigma|^2 \left\{\begin{array}{l} [a(\mathbf{1},\, \varepsilon_K=+1,\, -\mathbf{1}) + a(\mathbf{1},\, \varepsilon_K=-1,\, -\mathbf{1})]\cosh X_K \\ +[a(\mathbf{1},\, \varepsilon_K=+1,\, -\mathbf{1}) - a(\mathbf{1},\, \varepsilon_K=-1,\, -\mathbf{1})]\tanh \boldsymbol{X}_{\widehat{K}} \sinh X_K \\ +\displaystyle\sum_{j=1,\, j\neq K}^{n} \mathbb{O}\left(\mathrm{sech} X_j\right) \end{array}\right\}^4},$$

where

$$C^{(K)}_{\mu\nu\rho} := \left(r^{(K)}_\mu s^{(K)}_\nu + s^{(K)}_\mu r^{(K)}_\nu\right)\boldsymbol{r}^{(\widehat{K})}_\rho - \left(r^{(K)}_\mu s^{(K)}_\rho + s^{(K)}_\mu r^{(K)}_\rho\right)\boldsymbol{r}^{(\widehat{K})}_\nu),\ \ r^{(K)}_\rho := \sum_{j=1,\, j\neq K}^{n} \varepsilon_j r^{(j)}_\rho,$$

and

$$\mathscr{A} := \begin{bmatrix} A(\mathbf{1},\ \varepsilon_K=+1,\ -\mathbf{1})A(-\mathbf{1},\ \varepsilon_K=+1,\ \mathbf{1}) \\ -A(\mathbf{1},\ \varepsilon_K=-1,\ -\mathbf{1})A(-\mathbf{1},\ \varepsilon_K=-1,\ \mathbf{1}) \end{bmatrix},$$

$$\widetilde{\mathscr{A}} := A(\mathbf{1},\ \varepsilon_K=+i,\ -\mathbf{1})A(-\mathbf{1},\ \varepsilon_K=+i,\ \mathbf{1}).$$

By Eq. (59) and the fact that

$$C^{(K)}_{\mu\nu\rho} + C^{(K)}_{\nu\rho\mu} + C^{(K)}_{\rho\mu\nu} = 0,$$

we have

$$\mathrm{Tr}\left[(\partial_\mu\sigma)\sigma^{-1}(\partial_\nu\sigma)\sigma^{-1}(\partial_\rho\sigma)\sigma^{-1}\right] \longrightarrow 0$$

in the asymptotic region. Therefore, the Wess–Zumino term is asymptotic to zero for the n-soliton case.

References

1 L. J. Mason and N. M. Woodhouse, Integrability, Self-Duality, and Twistor Theory (Oxford University Press, Oxford, UK, 1996).
2 R. S. Ward, Phil. Trans. R. Soc. Lond. A **315**, 451 (1985).
3 Y. Kodama, KP Solitons and the Grassmannians (Springer, Berlin, 2017).
4 V. B. Matveev and M. A. Salle, Darboux Transformations and Solitons (Springer, Berlin, 1991).
5 M. Green, J. Schwarz, and E. Witten, Superstring Theory (Cambridge University Press, Cambridge, UK, 1987), Vol. 1.
6 H. Ooguri and C. Vafa, Nucl. Phys. B **361**, 469 (1991).
7 H. Ooguri and C. Vafa, Nucl. Phys. B **367**, 83 (1991).
8 T. Inami, H. Kanno, and T. Ueno, Mod. Phys. Lett. A **12**, 2757 (1997) [arXiv:hep-th/9704010] [Search inSPIRE].
9 T. Inami, H. Kanno, T. Ueno, and C. S. Xiong, Phys. Lett. B **399**, 97 (1997) [arXiv:hep-th/9610187] [Search inSPIRE].
10 A. Losev, G. W. Moore, N. Nekrasov, and S. Shatashvili, Nucl. Phys. B Proc. Suppl. **46**, 130 (1996) [arXiv:hep-th/9509151] [Search inSPIRE].
11 V. P. Nair, Kahler-Chen-Simons theory, STRINGS, 479, (World Scientific, 1991) [arXiv:hep-th/9110042] [Search inSPIRE].
12 V. P. Nair and J. Schiff, Phys. Lett. B **246**, 423 (1990).
13 A. Parkes, Nucl. Phys. B **376**, 279 (1992) [arXiv:hep-th/9110075] [Search inSPIRE].
14 V. P. Nair and J. Schiff, Nucl. Phys. B **371**, 329 (1992).
15 N. Marcus, Nucl. Phys. B **387**, 263 (1992) [arXiv:hep-th/9207024] [Search inSPIRE].

16 N. Marcus, A tour through N=2 strings,String Theory, Quantum Gravity and the Unification of the Fundamental Interactions,391–413 (World Scientific,1993) [arXiv:hep-th/9211059] [Search inSPIR E].
17 S. C. Huang, On soliton solutions of the anti-self-dual Yang-Mills equations from the perspective of integrable systems, Ph.D. Thesis, Nagoya University (2021) [arXiv:2112.10702 [hep-th]] [Search inSPIRE].
18 C. R. Gilson, M. Hamanaka, S. C. Huang, and J. J. C. Nimmo, J. Phys. A **53**, 404002 (2020) [arXiv:2004.01718 [nlin.SI]] [Search inSPIRE].
19 J. J. C. Nimmo, C. R. Gilson, and Y. Ohta, Theor. Math. Phys. **122**, 239 (2000) [Teor. Mat. Fiz. **122**, 284 (2000)].
20 M. Hamanaka and S. C. Huang, J. High Energy Phys. **2010**, 101 (2020) [arXiv:2004.09248 [hep-th]] [Search inSPIRE].
21 M. Hamanaka and S. C. Huang, J. High Energy Phys. **2201**, 039 (2022) [arXiv:2106.01353 [hep-th]] [Search inSPIRE].
22 S. C. Huang, Proc. East Asia Joint Symp. on Fields and Strings 2021, p. 33 (2022) [arXiv:2201.13318 [hep-th]] [Search inSPIRE].
23 S. K. Donaldson, Proc. Lond. Math. Soc. **3**, 1 (1985).
24 H. J. de Vega, Commun. Math. Phys. **116**, 659 (1988).
25 Y. Kodama, Solitons in Two-Dimensional Shallow Water (SIAM, Philadelphia, PA, 2018).
26 K. Ohkuma and M. Wadati, J. Phys. Soc. Jpn. **52**, 749 (1983).
27 R. Ward , Soliton solutions in an integrable chiral model in 2+1 dimensions, J. Math. Phys. **29**, 386 (1988)https://aip.scitation.org/doi/10.1063/1.528078
28 B. Dai, C.-L. Terng, and K. Uhlenbeck, On the space-time monopole equation, in Surveys in Differential Geometry (International Press, Somerville, MA, 2005), Vol. 10 [arXiv:math/0602607][Search inSPIRE].
29 D. Gluck, Y. Oz, and T. Sakai, J. High Energy Phys. **0308**, 055 (2003) [arXiv:hep-th/0306112] [Sear ch inSPIRE].
30 L. L. Chau, M. L. Ge, A. Sinha, and Y. S. Wu, Phys. Lett. B **121**, 391 (1983).
31 L. Dolan, Phys. Lett. B **113**, 387 (1982).
32 T. A. Ivanova and O. Lechtenfeld, Int. J. Mod. Phys. A **16**, 303 (2001) [arXiv:hep-th/0007049] [Sear ch inSPIRE].
33 Y. Kodama and L. Williams, Invent. Math. **198**, 637 (2014) [arXiv:1106.0023 [math.CO]] [Search in SPIRE].
34 M. Hamanaka, Nucl. Phys. B **741**, 368 (2006) [arXiv:hep-th/0601209] [Search inSPIRE].
35 M. Hamanaka and K. Toda, Phys. Lett. A **316**, 77 (2003) [arXiv:hep-th/0211148] [Search inSPIRE].
36 O. Lechtenfeld, A. D. Popov, and B. Spendig, J. High Energy Phys. **0106**, 011 (2001) [arXiv:hep-th/010396] [Search inSPIRE].
37 N. Seiberg and E. Witten, J. High Energy Phys. **9909**, 032 (1999) [arXiv:hep-th/9908142] [Search in SPIRE].
38 N. Nekrasov and A. Schwarz, Commun. Math. Phys. **198**, 689 (1998) [arXiv:hep-th/9802068] [Sear ch inSPIRE].
39 K. Ohmori, A review on tachyon condensation in open string field theories, Master thesis, University of Tokyo (2001) [arXiv:hep-th/0102085] [Search inSPIRE].
40 J. A. Harvey, P. Kraus, and F. Larsen, J. High Energy Phys. **0012**, 024 (2000) [arXiv:hep-th/0010060] [Search inSPIRE].
41 E. Witten, Commun. Math. Phys. **252**, 189 (2004) [arXiv:hep-th/0312171] [Search inSPIRE].
42 R. Bittleston and D. Skinner, JHEP **02**,227 (2023) [arXiv:2011.04638 [hep-th]] [Search inSPIRE].
43 K. Costello and M. Yamazaki, [arXiv:1908.02289 [hep-th]] [Search inSPIRE].
44 M. Sato, RIMSKokyuroku, (1981); M. Sato and Y. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifold, in Nonlinear Partial Differential Equations in Applied Sciences (North-Holland, Amsterdam, 1983), p. **439**, 259.
45 A. Neitzke and C. Vafa, [arXiv:hep-th/0402128][Search inSPIRE].
46 I. Gelfand and V. Retakh, Funct. Anal. Appl. **25**, 91 (1991).
47 I. Gelfand and V. Retakh, Funct. Anal. Appl. **26**, 231 (1992).

48 I. Gelfand, S. Gelfand, V. Retakh, and R. Wilson, Adv. Math. **193**, 56 (2005) [arXiv:math.QA/0208146] [Search inSPIRE].
49 C. R. Gilson and J. J. C. Nimmo, J. Phys. A **40**, 3839 (2007) [arXiv:nlin.si/0701027] [Search inSPIRE].