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Conditions of general Z_2 symmetry and $\mathrm{TM}_{1,2}$ mixing for the minimal type-I seesaw mechanism in an arbitrary basis

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Abstract

In this paper, using a formula for the minimal type-I seesaw mechanism by LDL^T (or generalized Cholesky) decomposition, conditions of general Z_2-invariance for the neutrino mass matrix m are obtained in an arbitrary basis. The conditions are found to be $(M_{22}a_i^+ - M_{12}b_i^+)(M_{22}a_j^- - M_{12}b_j^-) = -\det M\, b_i^+ b_j^-$ for the Z_2-symmetric and -antisymmetric part of a Yukawa matrix $Y_{ij}^\pm \equiv (Y \pm TY)_{ij}/2 \equiv (a_j^\pm, b_j^\pm)$ and the right-handed neutrino mass matrix M_{ij}. In other words, the symmetric and antisymmetric part of b_i must be proportional to those of the quantity $\tilde{a}_i \equiv a_i - \frac{M_{12}}{M_{22}} b_i$. They are equivalent to the condition that m is block diagonalized by eigenvectors of the generator T.

These results are applied to three Z_2 symmetries, the $\mu - \tau$ symmetry, the TM_1 mixing, and the magic symmetry which predicts the TM_2 mixing. For the case of $\mathrm{TM}_{1,2}$, the symmetry conditions become $M_{22}^2\, \tilde{a}_1^{\mathrm{TBM}} \tilde{a}_2^{\mathrm{TBM}} = -\det M\, b_1^{\mathrm{TBM}} b_2^{\mathrm{TBM}}$ and $M_{22}^2\, \tilde{a}_{1,2}^{\mathrm{TBM}} \tilde{a}_3^{\mathrm{TBM}} = -\det M\, b_{1,2}^{\mathrm{TBM}} b_3^{\mathrm{TBM}}$ with components $\tilde{a}_i^{\mathrm{TBM}}$ and b_i^{TBM} in the TBM basis $\boldsymbol{v}_{1,2,3}$. In particular, for the TM_2 mixing, the magic (anti-)symmetric Yukawa matrix with $S_2 Y = \pm Y$ is phenomenologically excluded because it predicts $m_2 = 0$ or $m_1, m_3 = 0$. In the case where Yukawa is not (anti-)symmetric, the mass singular values are displayed without a root sign.

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1. Introduction

The structure of the lepton mixing matrix often involves a certain Z_2 symmetry of the neutrino mass matrix m. The bi-maximal mixing [1] accompanies the $\mu - \tau$ symmetry [2–32], and the trimaximal mixing [33–39] does the magic symmetry [40]. The tri-bi-maximal (TBM) mixing [41], the combination of these two mixings is realized by the Klein symmetry $K_4 \simeq Z_2 \times Z_2$ [42–44].

In this paper, using a recently discovered seesaw formula by LDL^T decomposition, we investigate conditions of a Z_2 symmetry in the mass of light neutrinos m for the minimal type-I seesaw mechanism [45–68] in an arbitrary basis. Such formulation can be applied to other generalized CP symmetries (GCP) [69–94] and seesaw mechanisms.

This paper is organized as follows. The next section gives a formula by LDL^T decomposition for the minimal type-I seesaw mechanism and conditions of general Z_2 symmetry in an arbitrary basis. In Sec. 3, we analyze eigensystems of the Z_2-symmetric m and its applications. The final section is devoted to a summary.

2. A formula for the minimal type-I seesaw model and conditions of general Z_2 symmetry in an arbitrary basis

Here we review a formula by LDL^T decomposition [95,96] in the minimal seesaw models [45–47]. The Yukawa matrix of neutrinos Y and the mass matrix of right-handed neutrinos M are defined as follows

$$Y = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \tag{1}$$

where a_i, b_i, and M_{ij} are general complex parameters. By setting the vacuum expectation value of the Higgs field to unity, the mass dimension of Y_{ij} becomes one. This M can be diagonalized by LDL^T (or generalized Cholesky) decomposition;

$$\tilde{M}^{-1} = L^{-1} M^{-1} (L^{-1})^T = \begin{pmatrix} \frac{M_{22}}{\det M} & 0 \\ 0 & \frac{1}{M_{22}} \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\tilde{M}_1} & 0 \\ 0 & \frac{1}{\tilde{M}_2} \end{pmatrix}, \tag{2}$$

where L is a lower unitriangular matrix that has all the diagonal entries equal to one

$$L = \begin{pmatrix} 1 & 0 \\ -\frac{M_{12}}{M_{22}} & 1 \end{pmatrix}, \quad L^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{M_{12}}{M_{22}} & 1 \end{pmatrix}. \tag{3}$$

By redefining the phase of the second right-handed neutrino ν_{R2}, we can choose a basis such that M_{22} is real-positive. A further phase transformation of the first right-handed neutrino $\nu'_{R1} = e^{i\phi}\nu_{R1}$ yields $\det M' = e^{2i\phi}(M_{11}M_{22} - M_{12}^2)$. Thus, the phase of $\det M$ can be absorbed and $\tilde{M}^{-1}$ can be chosen as real-positive. If M is strongly hierarchical, the absolute values of these diagonal elements coincide with the first approximation of the physical singular values of M. In other words, it corresponds to an approximate spectral decomposition based on diagonalization by L.

Thus, a deformed Yukawa matrix $\tilde{Y}$ defined as

$$\tilde{Y} \equiv \begin{pmatrix} \tilde{\boldsymbol{a}} & \boldsymbol{b} \end{pmatrix} \equiv \begin{pmatrix} \boldsymbol{a} - \boldsymbol{b}\frac{M_{12}}{M_{22}}, \boldsymbol{b} \end{pmatrix} = Y \begin{pmatrix} 1 & 0 \\ -\frac{M_{12}}{M_{22}} & 1 \end{pmatrix} \equiv YL, \tag{4}$$

yields a formula for the neutrino mass matrix m in an arbitrary basis,

$$m = \tilde{Y}\tilde{M}^{-1}\tilde{Y}^T = \frac{M_{22}}{\det M}\begin{pmatrix} \tilde{a}_1^2 & \tilde{a}_1\tilde{a}_2 & \tilde{a}_1\tilde{a}_3 \\ \tilde{a}_1\tilde{a}_2 & \tilde{a}_2^2 & \tilde{a}_2\tilde{a}_3 \\ \tilde{a}_1\tilde{a}_3 & \tilde{a}_2\tilde{a}_3 & \tilde{a}_3^2 \end{pmatrix} + \frac{1}{M_{22}}\begin{pmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_3 & b_2b_3 & b_3^2 \end{pmatrix} \tag{5}$$

$$\equiv \frac{1}{\tilde{M}_1}\tilde{\boldsymbol{a}} \otimes \tilde{\boldsymbol{a}}^T + \frac{1}{\tilde{M}_2}\boldsymbol{b} \otimes \boldsymbol{b}^T . \tag{6}$$

Although it does not correspond to a physical basis for not hierarchical M, this formula is valid in any basis.

If the two three-dimensional vectors $\tilde{\boldsymbol{a}} \equiv (\tilde{a}_i)$ and $\boldsymbol{b} \equiv (b_i)$ are linearly independent, the rank of m becomes two and a massless mode appears. The eigenvector belonging to the zero mode of $m\,m^\dagger$ is proportional to $\tilde{\boldsymbol{a}}^* \times \boldsymbol{b}^*$ (the complex conjugation of the cross product).

In this paper, we investigate general conditions under which m has a Z_2 symmetry, using the formula (6) and similar arguments as in the previous paper [96]. We obtain more general results than the previous analysis of CP-symmetry.

When a Lagrangian has a Z_2 symmetry, m satisfies $T\,m\,T^T = m$ by a matrix T. Such Z_2 symmetries include μ-τ symmetry [2–4] and magic symmetry [40], which has been studied extensively. T is unitary because the symmetry does not change the kinetic term, and T is also Hermitian because $T^2 = 1$ leads to $T = T^{-1} = T^\dagger$. Then the three eigenvectors of T belonging to the eigenvalues ± 1 form an orthonormal basis. The vectors $\tilde{\boldsymbol{a}}$ and $\boldsymbol{b}$ can be expanded by the eigenvectors as $\tilde{\boldsymbol{a}} = \tilde{\boldsymbol{a}}^+ + \tilde{\boldsymbol{a}}^-$ and $\boldsymbol{b} = \boldsymbol{b}^+ + \boldsymbol{b}^-$. This means that the vectors are divided into symmetric and antisymmetric parts under the transformation;

$$T\tilde{\boldsymbol{a}} = \tilde{\boldsymbol{a}}^+ - \tilde{\boldsymbol{a}}^- , \quad T\boldsymbol{b} = \boldsymbol{b}^+ - \boldsymbol{b}^- . \tag{7}$$

Note that either eigenspace is one-dimensional because the eigenvalues are $\{+1, +1, -1\}$ or $\{+1, -1, -1\}$ for a nontrivial T.

Since $\det T$ can be changed by redefinitions of fields, it does not lose generality by choosing $\det T = -1$. In this case, the normalized eigenvectors of T consist of $\boldsymbol{e}^-$ belonging to the eigenvalue -1 and $\boldsymbol{e}^{+1}, \boldsymbol{e}^{+2}$ belonging to $+1$. Using the vector $\boldsymbol{e}^-$, the generator can be written as $T = 1 - 2\boldsymbol{e}^- \otimes \boldsymbol{e}^{-\dagger}$. If $\boldsymbol{e}^{+1\dagger}\boldsymbol{e}^{+2} = 0$ holds, T is diagonalized by a unitary matrix $U = (\boldsymbol{e}^-, \boldsymbol{e}^{+1}, \boldsymbol{e}^{+2})$;

$$T' = U^\dagger T U = (\boldsymbol{e}^-, \boldsymbol{e}^{+1}, \boldsymbol{e}^{+2})^\dagger (1 - 2\boldsymbol{e}^- \otimes \boldsymbol{e}^{-\dagger})(\boldsymbol{e}^-, \boldsymbol{e}^{+1}, \boldsymbol{e}^{+2}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{8}$$

In this basis, a Z_2-symmetric m must have the following form,

$$m' = U^\dagger m\, U^* = \begin{pmatrix} m'_{11} & 0 & 0 \\ 0 & m'_{22} & m'_{23} \\ 0 & m'_{23} & m'_{33} \end{pmatrix} . \tag{9}$$

The conditions for m to be such a block diagonal matrix are

$$\frac{1}{\tilde{M}_1}(\boldsymbol{e}^-, \tilde{\boldsymbol{a}})(\boldsymbol{e}^{+1}, \tilde{\boldsymbol{a}}) = -\frac{1}{\tilde{M}_2}(\boldsymbol{e}^-, \boldsymbol{b})(\boldsymbol{e}^{+1}, \boldsymbol{b}) , \tag{10}$$

$$\frac{1}{\tilde{M}_1}(\boldsymbol{e}^-, \tilde{\boldsymbol{a}})(\boldsymbol{e}^{+2}, \tilde{\boldsymbol{a}}) = -\frac{1}{\tilde{M}_2}(\boldsymbol{e}^-, \boldsymbol{b})(\boldsymbol{e}^{+2}, \boldsymbol{b}) , \tag{11}$$

where $(\boldsymbol{u}, \boldsymbol{v}) \equiv \boldsymbol{u}^\dagger \boldsymbol{v}$ is the Hermitian inner product. By multiplying $\boldsymbol{e}^-$ and $(\boldsymbol{e}^{+1,2})^T$ from left and right of Eqs. (10) and (11) respectively, and adding two equations, conditions for m to have a Z_2 symmetry by T are summarized as

$$\frac{1}{\tilde{M}_1} \tilde{\boldsymbol{a}}^- \otimes \tilde{\boldsymbol{a}}^{+T} = -\frac{1}{\tilde{M}_2} \boldsymbol{b}^- \otimes \boldsymbol{b}^{+T} , \quad M_{22}^2 \tilde{a}_i^+ \tilde{a}_j^- = -\det M \, b_i^+ b_j^- . \tag{12}$$

Thus, $\tilde{\boldsymbol{a}}^+$ and $\tilde{\boldsymbol{a}}^-$ are proportional to $\boldsymbol{b}^+$ and $\boldsymbol{b}^-$ respectively and their coefficients are determined by Eq. (12). Otherwise, two of the four components will be zero vectors. This kind of alignment also seems to be necessary for naturalness in the seesaw mechanism [97].

If $\tilde{a}_i^\pm$ and $b_i^\pm \neq 0_i$ holds, we can find solutions to $\tilde{\boldsymbol{a}}^\pm$ for given $\boldsymbol{b}^\pm$;

$$(M_{22} \tilde{\boldsymbol{a}}^+ , \, M_{22} \tilde{\boldsymbol{a}}^-) = (r \sqrt{\det M} \, \boldsymbol{b}^+ , \, -\frac{1}{r} \sqrt{\det M} \, \boldsymbol{b}^-) . \tag{13}$$

Here, r is a complex constant defined by the non-zero $\tilde{a}_i^\pm$ and $b_j^\pm$,

$$r \equiv \frac{M_{22}}{\sqrt{\det M}} \frac{\tilde{a}_j^+}{b_j^+} = -\frac{\sqrt{\det M}}{M_{22}} \frac{b_i^-}{\tilde{a}_i^-} . \tag{14}$$

At first glance, the solution (13) seems to have a degree of freedom of sign $\pm$. Indeed, it is correct that some $\tilde{a}_i$ and $-\tilde{a}_i$ are solutions to each other. However, they are treated as independent solutions because all of $b_i^\pm$ and one of $\tilde{a}_i^\pm$ are required as input parameters to determine the solution uniquely. This fact is also manifest in representations (B.6) and (B.7) by orthogonal matrices, as seen in the Appendix.

There are four trivial cases with each of $\tilde{\boldsymbol{a}}^\pm$ and $\boldsymbol{b}^\pm$ are $\mathbf{0}$, in which the denominator of r or $1/r$ can not be defined and $\tilde{a}_i{}^\pm$ need not be proportional to $b_i^\pm$. In such cases, $\tilde{Y}$ itself has definite symmetry,

$$T \, \tilde{Y} = \{ \tilde{Y} \, , \tilde{Y} \sigma_3 \, , -\tilde{Y} \sigma_3 \, , -\tilde{Y} \} , \tag{15}$$

where $\sigma_3 \equiv \mathrm{diag}(1, -1)$. For example, if $\tilde{Y}$ is T-symmetric with $\tilde{a}_i^- = b_i^- = 0$, we obtain expressions for Y and a from Eq. (4);

$$a_i^+ = \tilde{a}_i^+ + \frac{M_{12}}{M_{22}} b_i^+ , \quad a_i^- = 0 . \tag{16}$$

That is, $T \, Y = Y$ holds and the original Y also has the T symmetry. Although the same is true for antisymmetric $\tilde{Y}$, the other two situations are somewhat different. For example, the case of $T \, \tilde{Y} = \tilde{Y} \sigma_3$, *i.e.* $\tilde{a}_i^- = b_i^+ = 0$, leads to

$$a_i^- = \frac{M_{12}}{M_{22}} b_i^- , \quad a_i^+ = \tilde{a}_i^+ . \tag{17}$$

Then, while the antisymmetric components of a_i and b_i must be proportional, a_i can have independent symmetric components. These different behavior influences representations of mass values in the analysis of eigensystems.

3. Analysis of eigensystem and its application

Due to the Z_2 symmetry, the mass values and eigenvectors can be formally determined for each solution (13) and (15). If Eq. (12), *i.e.* $M_{22}^2 \tilde{a}_i^+ \tilde{a}_j^- = -\det M \, b_i^+ b_j^-$ is satisfied, the mass matrix m becomes

$$m = \frac{1}{\tilde{M}_1}(\tilde{\boldsymbol{a}}^+ \otimes \tilde{\boldsymbol{a}}^{+T} + \tilde{\boldsymbol{a}}^- \otimes \tilde{\boldsymbol{a}}^{-T}) + \frac{1}{\tilde{M}_2}(\boldsymbol{b}^+ \otimes \boldsymbol{b}^{+T} + \boldsymbol{b}^- \otimes \boldsymbol{b}^{-T})\,. \tag{18}$$

Since the projective components $\tilde{\boldsymbol{a}}^\pm$ and $\boldsymbol{b}^\mp$ are orthogonal to each other, the sum of respective projections must be rank one unless $\tilde{\boldsymbol{a}}^\pm$ and $\boldsymbol{b}^\pm$ (or $\boldsymbol{b}^\mp$) are zero vectors. Thus, each of projections must be proportional to the others, as $\tilde{\boldsymbol{a}}^\pm \propto \boldsymbol{b}^\pm$. The behavior of the solutions can be classified into the following three cases.

a. $T\tilde{Y} = \pm Y\sigma_3$ First, if each of $\tilde{\boldsymbol{a}}^\mp$ and $\boldsymbol{b}^\pm$ is $\boldsymbol{0}$ in the trivial solutions (15), the mass matrix is

$$m = \frac{1}{\tilde{M}_1}\tilde{\boldsymbol{a}}^\pm \otimes \tilde{\boldsymbol{a}}^{\pm T} + \frac{1}{\tilde{M}_2}\boldsymbol{b}^\mp \otimes \boldsymbol{b}^{\mp T}\,. \tag{19}$$

The Hermitian matrix $m\,m^\dagger$ becomes

$$m\,m^\dagger = \left|\frac{\tilde{\boldsymbol{a}}^\pm}{\tilde{M}_1}\right|^2 \tilde{\boldsymbol{a}}^\pm \otimes \tilde{\boldsymbol{a}}^{\pm\dagger} + \left|\frac{\boldsymbol{b}^\mp}{\tilde{M}_2}\right|^2 \boldsymbol{b}^\mp \otimes \boldsymbol{b}^{\mp\dagger}\,, \tag{20}$$

where $|\boldsymbol{v}|^2 \equiv \sum_{i=1}^3 v_i^* v_i$.

The eigenvectors of $m\,m^\dagger$ are $\{(\tilde{\boldsymbol{a}}^\pm \times \boldsymbol{b}^\mp)^*, \tilde{\boldsymbol{a}}^\pm, \boldsymbol{b}^\mp\}$, and the corresponding mass singular values m_i are

$$m_i = \{0\,, \frac{|\tilde{\boldsymbol{a}}^\pm|^2}{|\tilde{M}_1|}\,, \frac{|\boldsymbol{b}^\mp|^2}{|\tilde{M}_2|}\}\,. \tag{21}$$

In this situation, a matrix $S = 1 - 2\boldsymbol{v}^+ \otimes \boldsymbol{v}^{+T}$ defined by the remaining $\boldsymbol{v}^+ = \tilde{\boldsymbol{a}}^+$ or $\boldsymbol{b}^+$ generates another Z_2 symmetry of m [42].

b. $T\tilde{Y} = \pm\tilde{Y}$ When $\tilde{Y}$ and Y are T-symmetric or antisymmetric in Eq. (15), the mass matrix is

$$m = \frac{1}{\tilde{M}_1}\tilde{\boldsymbol{a}}^\pm \otimes \tilde{\boldsymbol{a}}^{\pm T} + \frac{1}{\tilde{M}_2}\boldsymbol{b}^\pm \otimes \boldsymbol{b}^{\pm T}\,. \tag{22}$$

Since the eigenvalues of nontrivial T are $(+1, +1, -1)$ or $(-1, -1, +1)$, the solution of $TY = (\det T)Y$ is phenomenologically excluded because the rank of m is one. In the other solution, although the eigenvectors of $m\,m^\dagger$ are $\{(\tilde{\boldsymbol{a}}^\pm \times \boldsymbol{b}^\pm)^*, \tilde{\boldsymbol{a}}^\pm, \boldsymbol{b}^\pm\}$, there is no guarantee that the two vectors $\tilde{\boldsymbol{a}}^\pm$ and $\boldsymbol{b}^\pm$ are orthogonal. Thus the general representation of mass singular values becomes complicated expressions as displayed in Ref. [98]. This is because that $\tilde{\boldsymbol{a}}$ and $\boldsymbol{b}$ belong to the same eigenvalue of T. If we can specify another Z_2 symmetry by S, $\tilde{\boldsymbol{a}}$ and $\boldsymbol{b}$ can be decomposed into projections with different eigenvalues of S, so that mass singular values can be determined as in the previous case.

c. Nontrivial solutions In other general situations, by substituting the solution (13) into Eq. (18), the mass matrix m is

$$m = \frac{1}{M_{22}}[(1 + r^2)\boldsymbol{b}^+ \otimes \boldsymbol{b}^{+T} + (1 + \frac{1}{r^2})\boldsymbol{b}^- \otimes \boldsymbol{b}^{-T}]\,. \tag{23}$$

The eigenvectors of $m\,m^\dagger$ are $\{(\boldsymbol{b}^+ \times \boldsymbol{b}^-)^*, \boldsymbol{b}^+, \boldsymbol{b}^-\}$ and the mass singular values are

$$m_i = \{0\,, \left|\frac{1 + r^2}{M_{22}}\right| |\boldsymbol{b}^+|^2\,, \left|\frac{1 + \frac{1}{r^2}}{M_{22}}\right| |\boldsymbol{b}^-|^2\}\,. \tag{24}$$

There is another Z_2 symmetry generated by either $S^{\pm} = 1 - 2\boldsymbol{b}^{\pm} \otimes \boldsymbol{b}^{\pm T}$. In the following subsections, the results obtained above will be applied to three specific symmetries, the $\mu - \tau$ symmetry, the TM_1 mixing, and the TM_2 mixing predicted from the magic symmetry.

3.1. Bi-maximal mixing and $\mu - \tau$ symmetry

Although the exact $\mu - \tau$ symmetry [2] that predicts $\theta_{13} = 0$ has now been excluded, let us consider this as a simple example for practice. First, the basis of the TBM mixing is given by

$$\boldsymbol{v}_1 = \frac{1}{\sqrt{6}}(2, -1, -1)^T, \quad \boldsymbol{v}_2 = \frac{1}{\sqrt{3}}(1, 1, 1)^T, \quad \boldsymbol{v}_3 = \frac{1}{\sqrt{2}}(0, 1, -1)^T. \tag{25}$$

Expanding the vectors $\tilde{\boldsymbol{a}}$ and $\boldsymbol{b}$ in the TBM basis yields

$$\tilde{\boldsymbol{a}} = \boldsymbol{v}_1(\boldsymbol{v}_1, \tilde{\boldsymbol{a}}) + \boldsymbol{v}_2(\boldsymbol{v}_2, \tilde{\boldsymbol{a}}) + \boldsymbol{v}_3(\boldsymbol{v}_3, \tilde{\boldsymbol{a}}) \equiv \sum_{i=1}^{3} \tilde{a}_i^{\mathrm{TBM}} \boldsymbol{v}_i, \tag{26}$$

$$\boldsymbol{b} = \boldsymbol{v}_1(\boldsymbol{v}_1, \boldsymbol{b}) + \boldsymbol{v}_2(\boldsymbol{v}_2, \boldsymbol{b}) + \boldsymbol{v}_3(\boldsymbol{v}_3, \boldsymbol{b}) \equiv \sum_{i=1}^{3} b_i^{\mathrm{TBM}} \boldsymbol{v}_i, \tag{27}$$

where $\tilde{a}_i^{\mathrm{TBM}}$ and b_i^{TBM} are components in this basis.

The $\mu - \tau$ symmetry is defined as[1]

$$T\, m\, T = m, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = 1 - 2\boldsymbol{v}_3 \otimes \boldsymbol{v}_3^T. \tag{28}$$

The eigensystem of T is

$$T\boldsymbol{v}_3 = -\boldsymbol{v}_3, \quad T\boldsymbol{v}_{1,2} = +\boldsymbol{v}_{1,2}. \tag{29}$$

Since the third eigenvector of $m\, m^\dagger$ is fixed to v_3, m_3 can be formally determined as

$$m_3 = \frac{(\tilde{a}_3^{\mathrm{TBM}})^2}{\tilde{M}_1} + \frac{(b_3^{\mathrm{TBM}})^2}{\tilde{M}_2} = \frac{M_{22}}{\det M}\frac{(\tilde{a}_2 - \tilde{a}_3)^2}{2} + \frac{1}{M_{22}}\frac{(b_2 - b_3)^2}{2}. \tag{30}$$

This expression generally includes complex phases. However, the absolute value $|m_3|$ must coincide with the singular value because $\boldsymbol{v}_3$ belongs to one-dimensional eigenspace of T (28).

The symmetry condition (12) becomes

$$M_{22}^2\, \tilde{a}_3^{\mathrm{TBM}}(\tilde{a}_1^{\mathrm{TBM}}\boldsymbol{v}_1 + \tilde{a}_2^{\mathrm{TBM}}\boldsymbol{v}_2) = -\det M\, b_3^{\mathrm{TBM}}(b_1^{\mathrm{TBM}}\boldsymbol{v}_1 + b_2^{\mathrm{TBM}}\boldsymbol{v}_2). \tag{31}$$

Since $\boldsymbol{v}_{1,2}$ are orthogonal, the equal sign holds for each component;

$$M_{22}^2\, \tilde{a}_1^{\mathrm{TBM}}\tilde{a}_3^{\mathrm{TBM}} = -\det M\, b_1^{\mathrm{TBM}}b_3^{\mathrm{TBM}}, \quad M_{22}^2\, \tilde{a}_2^{\mathrm{TBM}}\tilde{a}_3^{\mathrm{TBM}} = -\det M\, b_2^{\mathrm{TBM}}b_3^{\mathrm{TBM}}. \tag{32}$$

This is equivalent to the block diagonalization conditions (10) and (11) in the TBM basis.

Let us examine the three types of solutions analyzed above. For the trivial solution (19) with $T\,\tilde{Y} = \pm\tilde{Y}\sigma_3$, the deformed Yukawa $\tilde{Y}$ is

[1] There is also a definition of T by $v_3' = \frac{1}{\sqrt{2}}(0, 1, 1)$, which is equivalent under the phase transformation $v_3' = v_3$ diag $(1, 1, -1)$.

$$\tilde{Y}=\begin{pmatrix}\tilde{a}_1 & 0\\ \tilde{a}_2 & b_2\\ \tilde{a}_2 & -b_2\end{pmatrix}\ \text{or}\ \begin{pmatrix}0 & b_1\\ \tilde{a}_2 & b_2\\ -\tilde{a}_2 & b_2\end{pmatrix}. \tag{33}$$

The mass matrix m is

$$m=\frac{M_{22}}{\det M}\begin{pmatrix}\tilde{a}_1^2 & \tilde{a}_1\tilde{a}_2 & \tilde{a}_1\tilde{a}_2\\ \tilde{a}_1\tilde{a}_2 & \tilde{a}_2^2 & \tilde{a}_2^2\\ \tilde{a}_1\tilde{a}_2 & \tilde{a}_2^2 & \tilde{a}_2^2\end{pmatrix}+\frac{1}{M_{22}}\begin{pmatrix}0 & 0 & 0\\ 0 & b_2^2 & -b_2^2\\ 0 & -b_2^2 & b_2^2\end{pmatrix}\ \text{or}\ (\tilde{a}_i\leftrightarrow b_i)\,. \tag{34}$$

The eigenvectors are $\{(\tilde{\boldsymbol{a}}^{\pm}\times\boldsymbol{b}^{\mp})^*, \tilde{\boldsymbol{a}}^{\pm}, \boldsymbol{b}^{\mp}\}$ and corresponding singular values m_i are

$$m_{1,2}=\{0\,, \frac{|\tilde{a}_1|^2+2\,|\tilde{a}_2|^2}{|\tilde{M}_1|}\},\quad m_3=\left|\frac{2b_2^2}{\tilde{M}_2}\right|, \tag{35}$$

or

$$m_{1,2}=\{0\,, \frac{|b_1|^2+2\,|b_2|^2}{|\tilde{M}_2|}\},\quad m_3=\left|\frac{2\tilde{a}_2^2}{\tilde{M}_1}\right|. \tag{36}$$

Since a finite m_3 is predicted, this is a normal hierarchy (NH) like solution with $\theta_{13}=0$. Such Yukawa matrices would be easily realized by flavons with vacuum expectation values $\langle\phi_1\rangle \propto (-2,\ 1,\ 1)\,, \langle\phi_2\rangle \propto (1,\ 1,\ 1)\,, \langle\phi_3\rangle \propto (0,\ 1,\ -1)$ [62].

Second, for the other trivial solution (22) in which $\tilde{\boldsymbol{a}}$ and $\boldsymbol{b}$ have the same eigenvalues, Y and $\tilde{Y}$ itself has $\mu-\tau$ (anti-)symmetry $T\,Y=\pm Y$.

$$\tilde{Y}=\begin{pmatrix}\tilde{a}_1 & b_1\\ \tilde{a}_2 & b_2\\ \tilde{a}_2 & b_2\end{pmatrix}\ \text{or}\ \begin{pmatrix}0 & 0\\ \tilde{a}_2 & b_2\\ -\tilde{a}_2 & -b_2\end{pmatrix}. \tag{37}$$

The latter is excluded because Y is also rank one. Since the eigenvector $\boldsymbol{v}_3=\frac{1}{\sqrt{2}}(0,1,-1)$ belongs to the zero eigenvalue, this solution is inverted hierarchy (IH) like. In this case, representations of two mass values would be cumbersome. However, it can be simpler by combining it with another Z_2 symmetry that accompanies $\mathrm{TM}_{1,2}$ mixing described in the next subsections.

Finally, the non-trivial solutions are explored. A solution with $\tilde{a}_1=b_1=0$ has no phenomenological interest because the first row and column of m are the zero vector. Thus, $\tilde{a}_1$ and b_1 are set to have nonzero values and the parameter r (14) is determined to be $r_3=\frac{M_{22}}{\sqrt{\det M}}\frac{\tilde{a}_1}{b_1}$. From Eq. (13) (or Eq. (32)) the following constraints are obtained; The symmetric part is

$$\tilde{\boldsymbol{a}}^{+}=\begin{pmatrix}\tilde{a}_1\\ (\tilde{a}_2+\tilde{a}_3)/2\\ (\tilde{a}_2+\tilde{a}_3)/2\end{pmatrix}=\frac{\tilde{a}_1}{b_1}\boldsymbol{b}^{+}=\frac{\tilde{a}_1}{b_1}\begin{pmatrix}b_1\\ (b_2+b_3)/2\\ (b_2+b_3)/2\end{pmatrix}, \tag{38}$$

and the antisymmetric part is

$$\tilde{\boldsymbol{a}}^{-}=\begin{pmatrix}0\\ (\tilde{a}_2-\tilde{a}_3)/2\\ -(\tilde{a}_2-\tilde{a}_3)/2\end{pmatrix}=-\frac{\tilde{M}_1}{\tilde{M}_2}\frac{b_1}{\tilde{a}_1}\boldsymbol{b}^{-}=-\frac{\tilde{M}_1}{\tilde{M}_2}\frac{b_1}{\tilde{a}_1}\begin{pmatrix}0\\ (b_2-b_3)/2\\ -(b_2-b_3)/2\end{pmatrix}. \tag{39}$$

Since there are two independent conditions, $\tilde{a}_{2,3}$ are determined for given $\tilde{a}_1$ and $b_{1,2,3}$;

$$\tilde{a}_2=\frac{\tilde{a}_1}{b_1}\frac{b_2+b_3}{2}-\frac{\tilde{M}_1}{\tilde{M}_2}\frac{b_1}{\tilde{a}_1}\frac{b_2-b_3}{2}, \tag{40}$$

$$\tilde{a}_3 = \frac{\tilde{a}_1}{b_1}\frac{b_2+b_3}{2} + \frac{\tilde{M}_1}{\tilde{M}_2}\frac{b_1}{\tilde{a}_1}\frac{b_2-b_3}{2}\,. \tag{41}$$

The original Yukawa matrix Y is obtained by the inverse "rotation" L^{-1}. Since Y and $\tilde{Y}$ have both symmetric and antisymmetric components, they do not have definite symmetry.

The mass matrix (23) is

$$m = \frac{(1+r_3^2)}{M_{22}}\begin{pmatrix} b_1^2 & b_1b_+ & b_1b_+ \\ b_1b_+ & b_+^2 & b_+^2 \\ b_1b_+ & b_+^2 & b_+^2 \end{pmatrix} + \frac{(1+\frac{1}{r_3^2})}{M_{22}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & b_-^2 & -b_-^2 \\ 0 & -b_-^2 & b_-^2 \end{pmatrix}, \tag{42}$$

where $b_\pm \equiv (b_2 \pm b_3)/2$. The eigenvectors are $\{(\boldsymbol{b}^+ \times \boldsymbol{b}^-)^*, \boldsymbol{b}^+, \boldsymbol{b}^-\}$ and the singular values are

$$m_{1,2} = \{0\,, \left|\frac{1+r_3^2}{M_{22}}\right|(|b_1|^2 + \frac{|b_2+b_3|^2}{2})\}\,, \quad m_3 = \left|\frac{1+\frac{1}{r_3^2}}{M_{22}}\right|\frac{|b_2-b_3|^2}{2}\,. \tag{43}$$

Since the v_3 direction has a nonzero singular value, this is also an NH-like solution.

3.2. TM_1 mixing

Recent observations of the non-zero θ_{13} stimulate studies of mixing matrices called $\mathrm{TM}_{1,2}$ [99–105];

$$U_{\mathrm{TM1}} = U_{\mathrm{TBM}}U_{23}\,, \quad U_{\mathrm{TM2}} = U_{\mathrm{TBM}}U_{13}\,, \tag{44}$$

where

$$U_{\mathrm{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \qquad U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta e^{-i\phi} \\ 0 & -s_\theta e^{i\phi} & c_\theta \end{pmatrix},$$
$$U_{13} = \begin{pmatrix} c_\theta & 0 & s_\theta e^{-i\phi} \\ 0 & 1 & 0 \\ -s_\theta e^{i\phi} & 0 & c_\theta \end{pmatrix}, \tag{45}$$

with $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$. The absolute values of $\sin\theta_{13}$ are

$$|\sin\theta_{13}^{\mathrm{TM}_1}| = \sin\theta/\sqrt{3}\,, \quad |\sin\theta_{13}^{\mathrm{TM}_2}| = \sqrt{2}\sin\theta/\sqrt{3}\,. \tag{46}$$

The following formalism is similar to Ref. [62] that gives a detailed analysis of TM_1 and TM_2 mixing in the minimal seesaw model. New points in this paper are that it is presented in an arbitrary basis and the existence of non-trivial solutions.

The matrix m that predicts TM_1 has a Z_2 symmetry by the following S_1;

$$S_1\, m\, S_1 = m\,, \quad S_1 = 1 - 2\boldsymbol{v}_1 \otimes \boldsymbol{v}_1^T = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}. \tag{47}$$

The eigensystem of S_1 are

$$S_1\boldsymbol{v}_1 = -\boldsymbol{v}_1\,, \quad S_1\boldsymbol{v}_{2,3} = +\boldsymbol{v}_{2,3}\,. \tag{48}$$

From this, the symmetry conditions (10) and (11) are

$$M_{22}^2\,\tilde{a}_1^{\rm TBM}\tilde{a}_{2,3}^{\rm TBM} = -\det M\, b_1^{\rm TBM} b_{2,3}^{\rm TBM}\,. \tag{49}$$

By transformation to the TBM basis with $U_{\rm TBM} = (\boldsymbol{v}_1\,,\boldsymbol{v}_2\,,\boldsymbol{v}_3)$, the mass matrix m with the symmetry conditions is

$$m^{\rm TBM} \equiv U_{\rm TBM}^T m\, U_{\rm TBM} = (\boldsymbol{v}_1\,,\boldsymbol{v}_2\,,\boldsymbol{v}_3)^T \left(\frac{1}{\tilde{M}_1}\tilde{\boldsymbol{a}}\otimes\tilde{\boldsymbol{a}}^T + \frac{1}{\tilde{M}_2}\boldsymbol{b}\otimes\boldsymbol{b}^T\right)(\boldsymbol{v}_1\,,\boldsymbol{v}_2\,,\boldsymbol{v}_3)\,, \tag{50}$$

$$= \frac{1}{\tilde{M}_1}\begin{pmatrix} (\tilde{a}_1^{\rm TBM})^2 & 0 & 0 \\ 0 & (\tilde{a}_2^{\rm TBM})^2 & \tilde{a}_2^{\rm TBM}\tilde{a}_3^{\rm TBM} \\ 0 & \tilde{a}_2^{\rm TBM}\tilde{a}_3^{\rm TBM} & (\tilde{a}_3^{\rm TBM})^2 \end{pmatrix} + \frac{1}{\tilde{M}_2}\begin{pmatrix} (b_1^{\rm TBM})^2 & 0 & 0 \\ 0 & (b_2^{\rm TBM})^2 & b_2^{\rm TBM}b_3^{\rm TBM} \\ 0 & b_2^{\rm TBM}b_3^{\rm TBM} & (b_3^{\rm TBM})^2 \end{pmatrix}. \tag{51}$$

This m is indeed symmetric under $S_1^{\rm TBM} = {\rm diag}(-1\,,1\,,1)$ in this basis. From this, θ_{13} and m_1 (with a complex phase) are formally obtained as

$$|\sin\theta_{13}^{\rm TM_1}| = \sin\theta_{\rm TM_1}/\sqrt{3}\,,$$

$$\tan 2\theta_{\rm TM_1} = \left|\frac{\frac{2\tilde{a}_2^{\rm TBM}\tilde{a}_3^{\rm TBM}}{\tilde{M}_1} + \frac{2b_2^{\rm TBM}b_3^{\rm TBM}}{\tilde{M}_2}}{\left|\frac{(\tilde{a}_3^{\rm TBM})^2}{\tilde{M}_1} + \frac{(b_3^{\rm TBM})^2}{\tilde{M}_2}\right| - \left|\frac{(\tilde{a}_2^{\rm TBM})^2}{\tilde{M}_1} + \frac{(b_2^{\rm TBM})^2}{\tilde{M}_2}\right|}\right|\,, \tag{52}$$

$$m_1 = \frac{M_{22}}{\det M}(\tilde{a}_1^{\rm TBM})^2 + \frac{1}{M_{22}}(b_1^{\rm TBM})^2$$
$$= \frac{M_{22}}{6\det M}(\tilde{2}a_1 - \tilde{a}_2 - \tilde{a}_3)^2 + \frac{1}{6M_{22}}(2b_1 - b_2 - b_3)^2\,. \tag{53}$$

Hereafter mass matrices will be omitted because of their complexity, and we will only focus on forms of Yukawa matrices and the mass singular values. There are four possibilities for the trivial solutions of $\rm TM_1$;

$$S_1\tilde{Y} = \pm\tilde{Y} \;\Rightarrow\; \tilde{Y} = (x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3,\, y_2\boldsymbol{v}_2 + y_3\boldsymbol{v}_3) \text{ or } (x_1\boldsymbol{v}_1,\, y_1\boldsymbol{v}_1) \tag{54}$$

$$= \begin{pmatrix} \frac{\tilde{a}_2+\tilde{a}_3}{2} & \frac{b_2+b_3}{2} \\ \tilde{a}_2 & b_2 \\ \tilde{a}_3 & b_3 \end{pmatrix} \text{ or } \begin{pmatrix} \tilde{a}_1 & b_1 \\ -\tilde{a}_1/2 & -b_1/2 \\ -\tilde{a}_1/2 & -b_1/2 \end{pmatrix}, \tag{55}$$

$$S_1\tilde{Y} = \pm\tilde{Y}\sigma_3 \;\Rightarrow\; \tilde{Y} = (x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3,\, y_1\boldsymbol{v}_1) \text{ or } (x_1\boldsymbol{v}_1,\, y_2\boldsymbol{v}_2 + y_3\boldsymbol{v}_3) \tag{56}$$

$$= \begin{pmatrix} \tilde{a}_1 & \frac{b_2+b_3}{2} \\ -\tilde{a}_1/2 & b_2 \\ -\tilde{a}_1/2 & b_3 \end{pmatrix} \text{ or } \begin{pmatrix} \frac{\tilde{a}_2+\tilde{a}_3}{2} & b_1 \\ \tilde{a}_2 & -b_1/2 \\ \tilde{a}_3 & -b_1/2 \end{pmatrix}, \tag{57}$$

where $x_{1,2,3}$ and $y_{1,2,3}$ are complex coefficients. Although $S_1\tilde{Y} = \tilde{Y}$ leads to $m_1 = 0$ and a NH solution, mass singular values cannot be displayed without solving a complicated quadratic equation. A solution with $S_1\tilde{Y} = -\tilde{Y}$ is excluded because it is rank one.

Solutions $S_1 \tilde{Y} = \pm \tilde{Y} \sigma_3$ lead to finite m_1 and IH. The mass singular values are

$$m_1 = \left| \frac{1}{M_{22}} \frac{3}{2} b_1^2 \right| , \quad m_{2,3} = \{0 , \left| \frac{M_{22}}{\det M} \right| (|\tilde{a}_2|^2 + \left| \frac{\tilde{a}_2 + \tilde{a}_3}{2} \right|^2 + |\tilde{a}_3|^2)\} , \tag{58}$$

$$\text{or} \quad m_1 = \left| \frac{M_{22}}{\det M} \frac{3}{2} \tilde{a}_1^2 \right| , \quad m_{2,3} = \{0 , \left| \frac{1}{M_{22}} \right| (|b_2|^2 + \left| \frac{b_2 + b_3}{2} \right|^2 + |b_3|^2)\} . \tag{59}$$

Next, nontrivial solutions are investigated. In order for the two singular values (24) to be non-zero, $\tilde{\boldsymbol{a}}$ and $\boldsymbol{b}$ must have $\boldsymbol{v}_1$ components. Thus we can set $(\boldsymbol{v}_1, \tilde{\boldsymbol{a}}) \neq 0$ and $(\boldsymbol{v}_1, \boldsymbol{b}) \neq 0$. The parameter r (14) can be determined as

$$r_1 = - \frac{\sqrt{\det M}}{M_{22}} \frac{b_1^{\text{TBM}}}{\tilde{a}_1^{\text{TBM}}} . \tag{60}$$

By Eq. (49), $\tilde{a}_{2,3}$ can be determined from the other components. Explicitly,

$$\tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 = - \frac{\det M}{M_{22}^2} \frac{2b_1 - b_2 - b_3}{2\tilde{a}_1 - \tilde{a}_2 - \tilde{a}_3} (b_1 + b_2 + b_3) , \tag{61}$$

$$\tilde{a}_2 - \tilde{a}_3 = - \frac{\det M}{M_{22}^2} \frac{2b_1 - b_2 - b_3}{2\tilde{a}_1 - \tilde{a}_2 - \tilde{a}_3} (b_2 - b_3) . \tag{62}$$

The mass singular values are obtained as

$$m_i = \{0 , \left| \frac{1 + r_1^2}{M_{22}} \right| (\left|b_2^{\text{TBM}}\right|^2 + \left|b_3^{\text{TBM}}\right|^2) , \left| \frac{1 + \frac{1}{r_1^2}}{M_{22}} \right| \left|b_1^{\text{TBM}}\right|^2\} \tag{63}$$

$$= \{0 , \left| \frac{1 + r_1^2}{M_{22}} \right| [\frac{|b_1 + b_2 + b_3|^2}{3} + \frac{|b_2 - b_3|^2}{2}] , \left| \frac{1 + \frac{1}{r_1^2}}{M_{22}} \right| \frac{|2b_1 - b_2 - b_3|^2}{6}\} . \tag{64}$$

3.3. TM_2 mixing and magic symmetry

Similarly, a mass matrix predicting TM_2 has a Z_2 symmetry generated by the following S_2;

$$S_2 \, m \, S_2 = m , \quad S_2 = 1 - 2\boldsymbol{v}_2 \otimes \boldsymbol{v}_2^T = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix} . \tag{65}$$

This symmetry is called the *magic* symmetry and the matrix m is called *magic* in which the row sums and the column sums are all equal to a number α [40]. The eigenvalues of S_2 are

$$S_2 \boldsymbol{v}_2 = -\boldsymbol{v}_2 , \quad S_2 \boldsymbol{v}_{1,3} = +\boldsymbol{v}_{1,3} . \tag{66}$$

From this, the symmetry conditions are

$$M_{22}^2 \, \tilde{a}_2^{\text{TBM}} \tilde{a}_{1,3}^{\text{TBM}} = - \det M \, b_2^{\text{TBM}} b_{1,3}^{\text{TBM}} . \tag{67}$$

In the TBM basis, the mass matrix m with these conditions is

$$m^{\text{TBM}} \equiv U_{\text{TBM}}^T m \, U_{\text{TBM}} \tag{68}$$

$$= \frac{1}{\tilde{M}_1}\begin{pmatrix} (\tilde{a}_1^{\rm TBM})^2 & 0 & \tilde{a}_1^{\rm TBM}\tilde{a}_3^{\rm TBM} \\ 0 & (\tilde{a}_2^{\rm TBM})^2 & 0 \\ \tilde{a}_1^{\rm TBM}\tilde{a}_3^{\rm TBM} & 0 & (\tilde{a}_3^{\rm TBM})^2 \end{pmatrix} + \frac{1}{\tilde{M}_2}\begin{pmatrix} (b_1^{\rm TBM})^2 & 0 & b_1^{\rm TBM}b_3^{\rm TBM} \\ 0 & (b_2^{\rm TBM})^2 & 0 \\ b_1^{\rm TBM}b_3^{\rm TBM} & 0 & (b_3^{\rm TBM})^2 \end{pmatrix}. \quad (69)$$

This is indeed symmetric under $S_2^{\rm TBM} = {\rm diag}(1\,, -1\,, 1)$ in this basis. From this, θ_{13} and m_2 are obtained as

$$|\sin\theta_{13}^{\rm TM_2}| = \sqrt{2}\sin\theta_{\rm TM_2}/\sqrt{3}\,,$$
$$\tan 2\theta_{\rm TM_2} = \left| \frac{\frac{2\tilde{a}_1^{\rm TBM}\tilde{a}_3^{\rm TBM}}{\tilde{M}_1} + \frac{2b_1^{\rm TBM}b_3^{\rm TBM}}{\tilde{M}_2}}{\left|\frac{(\tilde{a}_3^{\rm TBM})^2}{\tilde{M}_1} + \frac{(b_3^{\rm TBM})^2}{\tilde{M}_2}\right| - \left|\frac{(\tilde{a}_1^{\rm TBM})^2}{\tilde{M}_1} + \frac{(b_1^{\rm TBM})^2}{\tilde{M}_2}\right|} \right|, \quad (70)$$

$$m_2 = \frac{M_{22}}{\det M}(\tilde{a}_2^{\rm TBM})^2 + \frac{1}{M_{22}}(b_2^{\rm TBM})^2 = \frac{M_{22}}{3\det M}(\tilde{a}_1+\tilde{a}_2+\tilde{a}_3)^2 + \frac{1}{3M_{22}}(b_1+b_2+b_3)^2\,. \quad (71)$$

In the case of the magic symmetry, there are three types of solutions as well. The first four trivial solutions are

$$S_2\tilde{Y} = \pm\tilde{Y} \;\Rightarrow\; \tilde{Y} = (x_1\boldsymbol{v}_1 + x_3\boldsymbol{v}_3,\, y_1\boldsymbol{v}_1 + y_3\boldsymbol{v}_3) \text{ or } (x_2\boldsymbol{v}_2,\, y_2\boldsymbol{v}_2) \quad (72)$$
$$= \begin{pmatrix} -\tilde{a}_2-\tilde{a}_3 & -b_2-b_3 \\ \tilde{a}_2 & b_2 \\ \tilde{a}_3 & b_3 \end{pmatrix} \text{ or } \begin{pmatrix} \tilde{a}_1 & b_1 \\ \tilde{a}_1 & b_1 \\ \tilde{a}_1 & b_1 \end{pmatrix}, \quad (73)$$

$$S_2\tilde{Y} = \pm\tilde{Y}\sigma_3 \;\Rightarrow\; \tilde{Y} = (x_1\boldsymbol{v}_1 + x_3\boldsymbol{v}_3,\, y_2\boldsymbol{v}_2) \text{ or } (x_2\boldsymbol{v}_2,\, y_1\boldsymbol{v}_1 + y_3\boldsymbol{v}_3) \quad (74)$$
$$= \begin{pmatrix} -\tilde{a}_2-\tilde{a}_3 & b_1 \\ \tilde{a}_2 & b_1 \\ \tilde{a}_3 & b_1 \end{pmatrix} \text{ or } \begin{pmatrix} \tilde{a}_1 & -b_2-b_3 \\ \tilde{a}_1 & b_2 \\ \tilde{a}_1 & b_3 \end{pmatrix}. \quad (75)$$

The solutions with $S_2\tilde{Y} = \pm\tilde{Y}$ are phenomenologically excluded because they predict $m_2 = 0$ or $m_{1,3} = 0$. In other words, Y cannot have magic (anti-)symmetry in this meaning. In the case of $S_2\tilde{Y} = \pm\tilde{Y}\sigma_3$, $x_1, y_1 = 0$ ($x_3, y_3 = 0$) leads to a NH (IH) solution.

The mass singular values of the trivial solutions are

$$m_i = \{0\,, \frac{M_{22}}{\det M}(|\tilde{a}_2|^2 + |\tilde{a}_2+\tilde{a}_3|^2 + |\tilde{a}_3|^2)\,, \left|\frac{1}{M_{22}}3b_1^2\right|\} \quad (76)$$
$$\text{or } = \{0\,, \left|\frac{M_{22}}{\det M}3\tilde{a}_1^2\right|\,, \frac{1}{M_{22}}(|b_2|^2 + |b_2+b_3|^2 + |b_3|^2)\}\,. \quad (77)$$

Furthermore, if we impose the condition $\tilde{a}_2 = \pm\tilde{a}_3$ (or $b_2 = \pm b_3$), m has the $\mu - \tau$ symmetry and the respective signs correspond to IH and NH.

Finally, the non-trivial solution (13) is analysed. Since m_2 cannot be zero, we can set $\sqrt{3}\,\tilde{a}_2^{\rm TBM} = \sum_i \tilde{a}_i \neq 0$ and $\sqrt{3}\,b_2^{\rm TBM} = \sum_i b_i \neq 0$. From the expansions (26) and (27), the parameter r (14) is determined as

$$r_2 = -\frac{\sqrt{\det M}}{M_{22}} \frac{b_2^{\mathrm{TBM}}}{\tilde{a}_2^{\mathrm{TBM}}} \,. \tag{78}$$

From the symmetry conditions (67), $\tilde{a}_{1,3}$ are determined. Explicitly,

$$2\tilde{a}_1 - \tilde{a}_2 - \tilde{a}_3 = -\frac{\det M}{M_{22}^2} \frac{\sum_i b_i}{\sum_i \tilde{a}_i} (2b_1 - b_2 - b_3) \,, \tag{79}$$

$$\tilde{a}_2 - \tilde{a}_3 = -\frac{\det M}{M_{22}^2} \frac{\sum_i b_i}{\sum_i \tilde{a}_i} (b_2 - b_3) \,, \tag{80}$$

and the mass values are

$$m_i = \{0 \,, \left| \frac{1 + r_2^2}{M_{22}} \right| (|b_1^{\mathrm{TBM}}|^2 + |b_3^{\mathrm{TBM}}|^2) \,, \left| \frac{1 + \frac{1}{r_2^2}}{M_{22}} \right| |b_2^{\mathrm{TBM}}|^2 \} \tag{81}$$

$$= \{0 \,, \left| \frac{1 + r_2^2}{M_{22}} \right| (\frac{|2b_1 - b_2 - b_3|^2}{6} + \frac{|b_2 - b_3|^2}{2}) \,, \left| \frac{1 + \frac{1}{r_2^2}}{M_{22}} \right| \frac{|\sum_i b_i|^2}{3} \} \,. \tag{82}$$

If we further impose the condition $b_2 + b_3 = 2b_1$ ($b_2 = b_3$), m has the $\mu - \tau$ symmetry, corresponding to a NH (IH) solution.

4. Summary

In this paper, using a formula for the minimal type-I seesaw mechanism by LDL^T (or generalized Cholesky) decomposition, conditions of general Z_2-invariance of the neutrino mass matrix m are obtained in an arbitrary basis. The conditions are found to be $(M_{22}a_i^+ - M_{12}b_i^+)\,(M_{22}a_j^- - M_{12}b_j^-) = -\det M\, b_i^+ b_j^-$ for the Z_2-symmetric and -antisymmetric part of a Yukawa matrix $Y_{ij}^{\pm} \equiv (Y \pm TY)_{ij}/2 \equiv (a_j^{\pm}, b_j^{\pm})$ and the right-handed neutrino mass matrix M_{ij}. In other words, the symmetric and antisymmetric part of b_i must be proportional to those of the quantity $\tilde{a}_i \equiv a_i - \frac{M_{12}}{M_{22}} b_i$. They are equivalent to the condition that m is block diagonalized by eigenvectors $\boldsymbol{e}_i$ of the generator T.

Since $\boldsymbol{e}_i$ are orthogonal, we can analyze the eigensystem of $m m^\dagger$ for a Z_2-symmetric m. Two eigenvectors $\boldsymbol{u}_{1,2}$ of $m m^\dagger$ coincide with any of those of T, and the remaining one is a vector $(\boldsymbol{u}_1 \times \boldsymbol{u}_2)^*$ orthogonal to them. Furthermore, if the Yukawa matrix does not have the Z_2 symmetry, two nonzero neutrino masses are represented without a radical symbol. On the other hand, in the case of (anti-)symmetric Y with $TY = \pm Y$, the solution of $TY = (\det T)Y$ is phenomenologically rejected because the rank of m is one. In the other solution, the mass singular values cannot be expressed without solving a complicated quadratic equation. However, if the other Z_2 symmetry can be identified, the mass values can be concisely displayed.

These results are applied to three Z_2 symmetries, the $\mu - \tau$ symmetry, the TM_1 mixing, and the magic symmetry which predicts the TM_2 mixing. For the case of $\mathrm{TM}_{1,2}$, the symmetry conditions become $M_{22}^2\, \tilde{a}_1^{\mathrm{TBM}} \tilde{a}_2^{\mathrm{TBM}} = -\det M\, b_1^{\mathrm{TBM}} b_2^{\mathrm{TBM}}$ and $M_{22}^2\, \tilde{a}_{1,2}^{\mathrm{TBM}} \tilde{a}_3^{\mathrm{TBM}} = -\det M\, b_{1,2}^{\mathrm{TBM}} b_3^{\mathrm{TBM}}$ with components $\tilde{a}_i^{\mathrm{TBM}}$ and b_i^{TBM} in the TBM basis $\boldsymbol{v}_{1,2,3}$. In particular, for the TM_2 mixing, the magic (anti-)symmetric Yukawa matrix with $S_2 Y = \pm Y$ is phenomenologically excluded because it predicts $m_2 = 0$ or $m_1, m_3 = 0$.

CRediT authorship contribution statement

Masaki J.S. Yang: Methodology, Formal analysis, Writing

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Understanding from the original raw formula

Let us consider the condition (12) without the LDL^T decomposition. By rewriting $\tilde{\boldsymbol{a}}$ to $\boldsymbol{a}$,

$$(M_{22}a_i^+ - M_{12}b_i^+)\,(M_{22}a_j^- - M_{12}b_j^-) = -\det M\, b_i^+\, b_j^- \,. \tag{A.1}$$

By adding this equation with i and j interchanged,

$$\text{Asym}[(a_i M_{22} - b_i M_{12})a_j] = -\text{Asym}[(b_i M_{11} - a_i M_{12})b_j]\,, \tag{A.2}$$

where $\text{Asym}[x_i y_j] \equiv x_i^+ y_j^- + x_i^- y_j^+$ is the antisymmetric part for T.

They are identical to the conditions $TmT^T = m$ in the original raw formula. From Eq. (1), the mass matrix m is written by

$$m = YM^{-1}Y^T = \frac{1}{\det M}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}\begin{pmatrix} -(\boldsymbol{M}_2 \times \boldsymbol{Y}_1)_z & -(\boldsymbol{M}_2 \times \boldsymbol{Y}_2)_z & -(\boldsymbol{M}_2 \times \boldsymbol{Y}_3)_z \\ (\boldsymbol{M}_1 \times \boldsymbol{Y}_1)_z & (\boldsymbol{M}_1 \times \boldsymbol{Y}_2)_z & (\boldsymbol{M}_1 \times \boldsymbol{Y}_3)_z \end{pmatrix}, \tag{A.3}$$

$$m_{ij} = \frac{1}{\det M}[a_i(a_j M_{22} - b_j M_{21}) + b_i(b_j M_{11} - a_j M_{12})]\,, \tag{A.4}$$

where $(\boldsymbol{M}_i \times \boldsymbol{Y}_j)_z \equiv M_{i1}Y_{j2} - M_{i2}Y_{j1}$. Since the asymmetric parts of the two matrices with rank one must cancel for T-invariant m, Eq. (A.2) is obtained as the condition. Also, by considering these conditions from symmetries, the solution can be displayed by a complex orthogonal matrix O. This point is discussed in the next section.

Appendix B. Understanding from complex orthogonal matrices

The solutions (13) and (15) can also be understood from orthgonal matrices. By defining $X \equiv \tilde{Y}\sqrt{\tilde{M}^{-1}}$, the mass matrix m is written only in X;

$$X = \left(\sqrt{\frac{M_{22}}{\det M}}\tilde{\boldsymbol{a}} \quad \sqrt{\frac{1}{M_{22}}}\boldsymbol{b} \right), \quad m = XX^T\,. \tag{B.1}$$

In order for m to have T symmetry, $X = (u, v)$ must satisfy the following transformation with a complex orthogonal matrix O;

$$TX = (Tu\,,\,Tv) = \pm XO\,. \tag{B.2}$$

Since $T^2X = \pm TXO = XO^2 = X$ holds, the matrix O satisfies $O = O^{-1} = O^T$. In other words, O is a symmetric orthogonal matrix.

In the case of $\det O = +1$, only $O = \pm 1_3$ is allowed since $\sin z = 0$. This corresponds to T-(anti)symmetric $\tilde{Y}$ and Y in Eq. (15), respectively. On the other hand, in the case of $\det O = -1$,

$$O = \begin{pmatrix} \cos z & \sin z \\ \sin z & -\cos z \end{pmatrix}, \tag{B.3}$$

and there is no restriction on the complex parameter z. The T-invariant condition becomes

$$XO = (u, v)\begin{pmatrix} \cos z & \sin z \\ \sin z & -\cos z \end{pmatrix} = (c_z u + s_z v,\, s_z u - c_z v) = \pm(Tu\,,\,Tv) = \pm TX\,, \tag{B.4}$$

where $c_z \equiv \cos z$, $s_z \equiv \sin z$. From this,

$$u = \frac{1}{s_z}(\pm Tv + c_z v)\,, \quad Tu = \pm\frac{c_z}{s_z}(\pm Tv + c_z v) \pm s_z v = \frac{1}{s_z}(c_z Tv \pm v)\,. \tag{B.5}$$

From $u^\pm = \frac{1}{2}(u \pm Tu)$ and $v^\pm = \frac{1}{2}(v \pm Tv)$, we obtain

$$\sqrt{\frac{M_{22}}{\det M}}\tilde{\boldsymbol{a}}^+ = u^+ = \frac{c_z \pm 1}{s_z}v^+ = \{\cot\frac{z}{2}\,,\,-\tan\frac{z}{2}\}\sqrt{\frac{1}{M_{22}}}\boldsymbol{b}^+\,, \tag{B.6}$$

$$\sqrt{\frac{M_{22}}{\det M}}\tilde{\boldsymbol{a}}^- = u^- = \frac{c_z \mp 1}{s_z}v^- = \{-\tan\frac{z}{2}\,,\,\cot\frac{z}{2}\}\sqrt{\frac{1}{M_{22}}}\boldsymbol{b}^-\,. \tag{B.7}$$

Obviously, these coefficients satisfy

$$\frac{c_z \pm 1}{s_z} \times \frac{c_z \mp 1}{s_z} = -1, \tag{B.8}$$

and it corresponds to the solution (13). Note that the sign $\pm$ comes from the parity for T.

An understanding of these solutions by matrices is as follows. As in the previous paper [96], we consider a quantity $X(1 \pm O\sigma_3)$. This extra σ_3 has an effect of exchanging $\boldsymbol{b}^+$ and $\boldsymbol{b}^-$. For example, in the case of $XO = +TX$,

$$X(1 + O\sigma_3) = X + TX\sigma_3 = 2(\tilde{\boldsymbol{a}}^+\,, \boldsymbol{b}^-)\sqrt{\tilde{M}^{-1}}\,, \tag{B.9}$$

$$X(1 - O\sigma_3) = X - TX\sigma_3 = 2(\tilde{\boldsymbol{a}}^-\,, \boldsymbol{b}^+)\sqrt{\tilde{M}^{-1}}\,. \tag{B.10}$$

Since $O\sigma_3$ is written as

$$O\sigma_3 = \begin{pmatrix} c_z & -s_z \\ s_z & c_z \end{pmatrix} = c_z 1_2 - i\sigma_2 s_z\,, \tag{B.11}$$

the term is expressed as $1 \pm O\sigma_3 = 1_2(1 \pm c_z) \mp i\sigma_2 s_z$. From this,

$$X(1 \pm O\sigma_3)\frac{1 \mp c_z}{s_z} = X[1_2 s_z \mp i\sigma_2(1 \mp c_z)] = \mp X(1 \mp O\sigma_3)i\sigma_2\,. \tag{B.12}$$

For example, matrix elements of the upper sign become

$$(\tilde{\boldsymbol{a}}^+, \boldsymbol{b}^-)\sqrt{\tilde{M}^{-1}} = \frac{-s_z}{1-c_z}(\tilde{\boldsymbol{a}}^-, \boldsymbol{b}^+)\sqrt{\tilde{M}^{-1}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= \frac{s_z}{1-c_z}\left(\sqrt{\frac{1}{M_{22}}}\boldsymbol{b}^+, -\sqrt{\frac{M_{22}}{\det M}}\tilde{\boldsymbol{a}}^-\right), \tag{B.13}$$

and they correspond to Eqs. (B.6) and (B.7).

The symmetry condition (12) satisfied by these solutions, *i.e.* $M_{22}^2 \tilde{a}_i^+ \tilde{a}_j^- = -|M| b_i^+ b_j^-$ is obviously equivalent to $X^+X^{-T} = 0$ for $X^\pm \equiv (X \pm TX)/2$. This condition can be rewritten as

$$X^+X^{-T} = \frac{1}{4}X(1 \pm O)(1 \mp O)^T X^T = \pm\frac{1}{4i}X(O - O^T)X^T = 0, \tag{B.14}$$

and it is a solution because $O^T = O$ holds.

Furthermore, a matrix representation of the solution $TX = \pm XO$ is explored. In the case of $\det O = +1$, $TX_\pm = \pm X_\pm$ holds and X is symmetric or antisymmetric under T, as discussed above. In the case of $\det O = -1$, we consider the following solution by a complex orthogonal matrix Q and $\sigma_3 \equiv \mathrm{diag}(1, -1)$.

$$TX_1 = X_1\sigma_3, \quad X_1' = X_1 Q. \tag{B.15}$$

This $X_1 = (u^+, v^-)$ has T-symmetric and -antisymmetric vectors.[2] The T transformation for X_1' is

$$TX_1' = X_1\sigma_3 Q = X_1' Q^T \sigma_3 Q. \tag{B.16}$$

Thus, if we define $Q^T\sigma_3 Q \equiv O_1$, this is a symmetric orthogonal matrix with $\det O_1 = -1$ and X_1' satisfies the symmetry (B.2).

$$TX_1' = X_1' O_1. \tag{B.17}$$

Specifically,

$$Q \equiv \begin{pmatrix} \cos w & -\sin w \\ \sin w & \cos w \end{pmatrix} \Rightarrow O_1 = Q^T\sigma_3 Q = \begin{pmatrix} \cos 2w & -\sin 2w \\ -\sin 2w & -\cos 2w \end{pmatrix}. \tag{B.18}$$

Indeed O_1 is symmetric, and it agrees with Eq. (B.3) by a suitable redefinition.

Since we can write $X = X_\pm$ or $X_1 Q$, the general Yukawa matrix for a T-invariant m can be written by these matrices. In the case of $X = X_\pm$, Yukawa matrices Y and $\tilde{Y}$ are

$$\tilde{Y}_\pm = X_\pm\sqrt{\tilde{M}}, \quad Y_\pm = \tilde{Y}L^{-1} = X_\pm\sqrt{\tilde{M}}L^{-1}. \tag{B.19}$$

In other words, $Y_\pm$ is symmetric or antisymmetric under T, $TY_\pm = \pm Y$, and it can be expanded by eigenvectors of T with the same eigenvalues.

In the other case,

$$\tilde{Y} = X\sqrt{\tilde{M}} = X_1 Q\sqrt{\tilde{M}}, \quad Y = \tilde{Y}L^{-1} = X_1 Q\sqrt{\tilde{M}}L^{-1}. \tag{B.20}$$

Since X_1 has the form $X_1 = (u^-, v^+)$, it represents the solution (B.6) and (B.7).

From the diagonalization $L^{-1}M^{-1}(L^T)^{-1} = \tilde{M}^{-1}$ (2), we finally obtain

$$m = YM^{-1}Y^T = X_{\pm,1}QQ^T X_{\pm,1}^T, \tag{B.21}$$

[2] The other solution $X_2 = (u^+, v^-)$ can be reached by a permutation with $\det O = -1$ from X_1.

and m is T-invariant. Therefore, in order to predict T-invariant m, the Yukawa matrix Y must be T-symmetric or antisymmetric, or has degrees of freedom complex orthogonal matrix Q multiplied to T-covariant X_1 that satisfies $TX_1 = X_1\sigma_3$.

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