An approach to BPS black hole microstate counting in an $N = 2$ STU model

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ABSTRACT: We consider four-dimensional dyonic single-center BPS black holes in the $N = 2$ STU model of Sen and Vafa. By working in a region of moduli space where the real part of two of the three complex scalars S, T, U are taken to be large, we evaluate the quantum entropy function for these BPS black holes. In this regime, the subleading corrections point to a microstate counting formula partly based on a Siegel modular form of weight two. This is supplemented by another modular object that takes into account the dependence on Y^0, a complex scalar field belonging to one of the four off-shell vector multiplets of the underlying supergravity theory. We also observe interesting connections to the rational Calogero model and to formal deformation of a Poisson algebra, and suggest a string web picture of our counting proposal.

Contents

1 Introduction

A highly active area of focused research in string theory aims at a formulation of the microscopic degeneracy of states of black holes to arrive at a statistical description of black hole thermodynamics. This is a critical first step in gleaning insights into the organization of the degrees of freedom in any purported theory of quantum gravity. A series of steady definitive advancements has been accomplished in the last decades in this field, specifically w.r.t. the exact counting of microstates of $\frac{1}{4}$ BPS and $\frac{1}{8}$ BPS black holes in four-dimensional $N = 4$ and $N = 8$ string theories, respectively [1–19]. In this paper, we address the problem of black hole microstate counting for $\frac{1}{2}$ BPS asymptotically flat black hole backgrounds. We do this in a specific four-dimensional $N = 2$ string theory model, namely the STU model of Sen and Vafa (example D in [20]), for which the holomorphic function F, which encodes (part of) the Wilsonian effective action, has been determined recently [21]. The function F encodes an infinite set of gravitational coupling functions $\omega^{(n)}$ ($n \geq 1$). A proposal for the counting of microstates in this model has been made in [22], by only taking into account the gravitational coupling function $\omega^{(1)}$. In this work, by using the detailed knowledge about F given in [21], we arrive at a different proposal. While doing so, we will work in a certain region of moduli space, as we will explain below.

In four-dimensional $N = 2$ string models, extremal black hole backgrounds are characterized by electric and magnetic charges (q_I, p^I) w.r.t. the various U(1) gauge fields in these models. These backgrounds, which at weak string coupling correspond to BPS states, possess a near-horizon AdS_2 geometry [23]. This geometry is an exact solution of the equations of motions that is decoupled from the asymptotics of the background. The decoupling ensures that the values of scalar fields $\{\phi\}$ in the near-horizon black hole background as well as all relevant length scales, such as the AdS_2 radius, are solely fixed in terms of the charges and are independent of the asymptotic data. The near-horizon geometry, in effect, acts as an attractor [24–26] for the scalar fields $\{\phi\}$ in the black hole background: the attractor values of these scalars determine the near-horizon length scale, and hence the black hole entropy $\mathcal{S}_{\text{BH}}$. This, in turn, implies that the quantum gravity partition function $Z_{AdS_2}(\phi, p)$ in this attractor background, evaluated in a canonical ensemble defined at fixed magnetic charges, is the generator of the dimension of the black hole microstate Hilbert space, i.e. of the black hole microstate degeneracy $d_{\text{BH}}(q, p)$, graded by electric and magnetic charges. Hence, formally, one has

$$Z_{AdS_2}(\phi, p) = \sum_q d_{\text{BH}}(q, p) e^{\pi q \cdot \phi} \,. \tag{1.1}$$

The SL$(2, \mathbb{R})$-isometries of AdS_2 indicate that the generator of degeneracies has modular properties, an expectation that has been extensively borne out in all cases where the exact microscopic degeneracy is known, such as in four-dimensional $N = 4$ and $N = 8$ string theories. Here the microstate degeneracy $d_{\text{BH}}(q, p)$ is related to the macroscopic entropy of the black hole by the Boltzmann relation

$$\mathcal{S}_{\text{BH}} = \ln d_{\text{BH}}(q, p) \,. \tag{1.2}$$

The macroscopic entropy $\mathcal{S}_{\mathrm{BH}}$ has an expansion in terms of the dimensionless black hole area A_{BH} as[1]

$$\mathcal{S}_{\mathrm{BH}} = \frac{A_{\mathrm{BH}}}{4} + c_1 \ln A_{\mathrm{BH}} + \frac{c_2}{A_{\mathrm{BH}}} + \dots . \tag{1.3}$$

For extremal black holes with a near-horizon AdS_2 geometry, a proposal for their exact macroscopic entropy was put forward in [28, 29]. This proposal is called the quantum entropy function, and is defined in terms of a regulated path integral over AdS_2 spacetimes with an insertion of a Wilson line corresponding to electric charges. This in turn provides an operational definition for (1.1).

For BPS black holes in four-dimensional $N = 2$ string theories, the quantum entropy function $W(q, p)$ can be expressed as [30] (when strictly restricted to smooth configurations)

$$W(q, p) = \int_{\mathcal{C}} d\phi \, \mu(\phi) \, Z_{1-\mathrm{loop}}(\phi) \, e^{\pi[4 \operatorname{Im} F(\phi + ip) - q \cdot \phi]} . \tag{1.4}$$

The various ingredients that go into this formula will be reviewed in section 2. Here we note that the holomorphic function F appearing in the exponent is the holomorphic function mentioned above that encodes (part of) the Wilsonian effective action. The function F can be decomposed as $F = F^{(0)} + 2i\Omega$, where $F^{(0)}$ denotes the prepotential of the model, whereas Ω encodes all the gravitational coupling functions $\omega^{(n)}$, as a series expansion in powers of $\Upsilon/(Y^0)^2$. Here, (Υ, Y^0) denote rescaled supergravity fields (A, X^0), where A denotes the lowest component of the square of a chiral superfield that describes the Weyl multiplet, and where X^0 denotes a complex scalar field belonging to one of the off-shell vector multiplets in the associated supergravity theory.

The STU model of Sen and Vafa (example D in [20]) is based on four off-shell vector multiplets, and Ω will therefore depend on four complex scalar fields Y^I that reside in these multiplets. Introducing projective coordinates $S = -iY^1/Y^0, T = -iY^2/Y^0, U = -iY^3/Y^0$, Ω has the series expansion

$$\Omega(Y, \Upsilon) = \Upsilon \sum_{n=0}^{\infty} \left(\frac{\Upsilon}{(Y^0)^2} \right)^n \omega^{(n+1)}(S, T, U) . \tag{1.5}$$

This STU model possesses duality symmetries, namely $[\Gamma_0(2)]^3$ symmetry and triality symmetry. The former refers to the duality symmetry $\Gamma_0(2)_S \times \Gamma_0(2)_T \times \Gamma_0(2)_U$, associated with each of the moduli S, T, U. Here, $\Gamma_0(2)$ denotes a certain congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$. In addition, the model possesses triality symmetry, i.e. invariance under the exchange of S, T and U. Using all these symmetries, it was demonstrated in [21] that the gravitational coupling functions $\omega^{(n+1)}(S, T, U)$ are expressed in terms of one single function ω, and derivatives thereof with respect to S, T, U. The duality symmetries were implemented by adding to Ω in (1.5) a term proportional to $\Upsilon \ln Y^0$. However, such a term is, strictly speaking, not part of the Wilsonian effective action. Since we will take Ω as an input in (1.4), we will refrain from adding this term to Ω. This term will be effectively generated from $Z_{1-\mathrm{loop}}$ in (1.4), as already noted in [31].

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[1]Additionally, a constant contribution may be present. We will neglect such a contribution when computing the quantum entropy function. For a recent development in this regard see [27].

The explicit expressions for the $\omega^{(n+1)}(S,T,U)$ in the STU model are complicated. However, in [21] it was observed that they dramatically simplify in the limit where the real part of two of the three moduli S,T,U are taken to be large. This is the limit in which we will work in this paper. We will thus fix two of the moduli, say T and U, to large values, which we will denote by T_0, U_0. In this limit, Ω becomes effectively replaced by

$$\widehat{\Omega}(Y^0, S, \Upsilon) = \Upsilon \left(\omega(S) + \frac{\Upsilon}{(Y^0)^2} \frac{\alpha}{\gamma} \frac{\partial \omega(S)}{\partial S} + \Xi(Y^0, S, \Upsilon) \right), \tag{1.6}$$

where α, γ denotes constants, and where Ξ encodes the higher gravitational couplings $\omega^{(n+1)}$ with $n \geq 2$. The latter are expressed in terms of modular forms $I_n(S)$ and products thereof. The I_n are themselves products of Eisenstein series for $\Gamma_0(2)$, see (B.38). We are thus led to an expansion of Ξ in terms of I_n as

$$\Xi(Y^0, S, \Upsilon) = \sum_{n=2}^{\infty} \left(\frac{\Upsilon}{(Y^0)^2} \right)^n \omega^{(n+1)}(S) \tag{1.7}$$

$$= \sum_{n=2}^{\infty} \alpha_n \left(\frac{\Upsilon}{(Y^0)^2} \right)^n I_n(S) + \sum_{m,n=2}^{\infty} \alpha_{m,n} \left(\frac{\Upsilon}{(Y^0)^2} \right)^{m+n} I_m(S) \, I_n(S) + \dots \, .$$

Here, the right hand side is organised in powers of I_n. We call the first sector (which is linear in I_n) the monomial sector, the second sector (which is quadratic in I_n) the binomial sector, and so on. In [21], explicit expressions were only given for the coefficients of the first two sectors: the α_n were determined fully, and partial expressions for the $\alpha_{m,n}$ were given. Notwithstanding this limited knowledge about the expansion coefficients, there are several lessons that one can draw from the power series expansion (1.6), (1.7). Firstly, under $\Gamma_0(2)$-transformations, $\widehat{\Omega}$ transforms in the same manner as the logarithm of $\vartheta_2^8(\tau, z)$, where $\vartheta_2^8(\tau, z)$ denotes one of the Jacobi theta functions, and where $\tau = iS$, $z = 1/Y^0$. Moreover, the series expansion (1.6), (1.7) is reminiscent of the Taylor series expansion around $z = 0$ of $\ln \vartheta_2^8(\tau, z)$, see (D.21). In this expansion, the coefficients α_n in the monomial sector behave as $1/(2n)!$, whereas in (1.7) they behave as $1/n!$. Secondly, each of the I_n-sectors in the expansion of $\ln \vartheta_2^8(\tau, z)$ defines a function that is given as a series expansion in z, and the sum over all these functions gives rise to $\ln \vartheta_2^8(\tau, z)$, which is naturally expressed in terms of variables $q = \exp(2\pi i \tau)$ and $y = \exp(2\pi i z)$. It is tempting to conclude that a similar story may apply to $\widehat{\Omega}(Y^0, S, \Upsilon)$. Thirdly, $\ln \vartheta_2^8(\tau, z)$ is a solution to a non-linear PDE derived from the heat equation. So we may similarly ask whether Ξ given in (1.7) arises as solution to a non-linear PDE. The answer is affirmative, and we give a PDE whose solution yields the monomial and binomial sectors displayed in (1.7) for the expressions of the coefficients given in [21] ([21] obtained exact expressions for α_n and partial expressions for $\alpha_{m,n}$). The higher sectors in (1.7), whose expansion coefficients have not been determined in [21], will lead to modifications of this PDE, and we give a candidate for the modified PDE. Note that Ξ has weight 0 under $\Gamma_0(2)$-transformations. Finally, we note that the monomial sector exhibits a connection with the two-particle rational Calogero model [32–34], as well as a relation with Serre-Rankin-Cohen brackets which is suggestive of formal deformation [35].

In this paper we will compute the quantum entropy function (1.4) for large single-center BPS black holes in the STU model. We follow [36, 37] and add a boundary term to

the quantum entropy function, so as to bring it into a manifestly duality invariant form. This modified form is what we thenceforth call the quantum entropy function. It requires specifying an integration contour $\mathcal{C}$, which we take to be the one constructed in [38] in the context of microstate counting for $N = 4$ BPS black holes. The quantum entropy function will depend on the three ingredients displayed in (1.6), namely $\omega(S)$, $I_1(S) \equiv \partial\omega/\partial S$ and $\Xi(Y^0, S)$. Since $\widehat{\Omega}(Y^0, S, \Upsilon)$ depends on Y^0, so will the quantum entropy function. The dependence on Y^0 brings in an explicit dependence on the magnetic charges (p^0, p^1), as we will see. A microstate counting formula that reproduces the quantum entropy function will have to take this dependence on Y^0 into account. Microstate counting formulae for BPS black holes in four-dimensional $N = 4, 8$ string theories [1, 3, 5, 9] are based on modular objects. In the $N = 4$ context, these are Siegel modular forms, and [22, 39] indicates that they should also play a role in $N = 2$ theories. However, in the latter case, microstate counting formulae cannot be solely based on Siegel modular forms due to the dependence of the quantum entropy function on the additional modulus Y^0.

Let us briefly describe our microstate proposal. We find that $\omega(S)$, which is proportional to $\ln\vartheta_2^8(S)$, together with subleading corrections in the quantum entropy function, point to a dependence of the microstate counting formula on a Siegel modular form Φ_2 of weight 2. This Siegel modular form can be constructed by applying a Hecke lift [5] to a specific Jacobi form constructed from the seed ϑ_2^8, and it differs from the Siegel modular form proposed in [22], which did not take into account the subleading corrections just mentioned. Due to the dependence on Y^0, this needs to be supplemented by a modular object that depends on Y^0. By focusing attention onto the leading divisor of the Siegel modular form Φ_2, we give an expression for this modular object: one of the ingredients that goes into it is Ξ, which we view as a solution to the aforementioned non-linear PDE. We then verify that on the leading divisor, the proposed microstate counting formula is invariant under the subgroup $H \subset Sp(4, \mathbb{Z})$ that acts on the Siegel upper half plane and implements the $\Gamma_0(2)_S$-symmetry of the STU model. We stress that the various approximations that we have implemented (such as working in a certain region of moduli space, or focussing attention on the leading divisor of Φ_2), render the microstate proposal to be only an approximate one.

The paper is structured as follows. In section 2, we give a brief review of the quantum entropy function for BPS black holes in $N = 2$ supergravity theories in four dimensions. In section 3 we evaluate the quantum entropy function for large single-center BPS black holes in the STU model. While doing so, we work in a certain region of moduli space in which the function $F = F^{(0)} + 2i\Omega$ that encodes the Wilsonian effective action simplifies: Ω becomes effectively replaced by (1.6). The modular invariant function Ξ in (1.6) arises as a solution to a non-linear PDE, and exhibits a relation with the two-particle rational Calogero model, as well as with Serre-Rankin-Cohen brackets and formal deformation. We discuss our integration contour $\mathcal{C}$, which we take to be the one constructed in [38]. The result for the quantum entropy function is based on several assumptions and approximations which we explain. In section 4, we propose a microstate counting formula that reproduces this result for the quantum entropy function. It is based on the Siegel modular form Φ_2 as well as on another modular object that captures the dependence on the complex scalar Y^0. Given

the various approximations that went into computing the quantum entropy function, the proposed counting function will be an approximation to the (as yet unkown) exact counting function. In section 5, we conclude with a brief summary and a few observations, and we suggest a string web picture of our counting proposal. In appendices A–G, we collect results about modular forms for $SL(2, \mathbb{Z})$ and $\Gamma_0(2)$, Jacobi forms, Siegel modular forms, Rankin-Cohen brackets and Hecke lifts.

2   Quantum entropy function for BPS black holes in $N = 2$ supergravity theories

2.1   Generic structure

The equations of motion of $N = 2$ supergravity coupled to n_V Abelian vector multiplets in four dimensions admit dyonic single-center BPS black hole solutions [24]. These are static, extremal black hole solutions that carry electric/magnetic charges (q_I, p^I), where the index $I = 0, \ldots, n_V$ labels the Maxwell fields in the theory. The near-horizon geometry of these solutions contains an AdS_2 factor. These BPS black holes are supported by complex scalar fields Y^I that reside in the vector multiplets. At the horizon, these scalar fields are fixed in terms of specific values that are entirely specified by the charges carried by the black hole.

In the presence of higher-derivative terms proportional to the square of the Weyl tensor, the associated $N = 2$ Wilsonian effective action is encoded in a holomorphic function $F(Y, \Upsilon)$ [40], where the complex field Υ is related to the lowest component of the square of the Weyl superfield. At the horizon of a BPS black hole, the field Υ takes the real value $\Upsilon = -64$, and Wald's entropy of the BPS black hole [41] can be written as [42]

$$\mathcal{S}_{\mathrm{BH}}(q, p) = \pi \left[4 \operatorname{Im} F(Y, \Upsilon) - q_I(Y^I + \bar{Y}^I) \right], \tag{2.1}$$

where

$$Y^I = \frac{1}{2}\left(\phi^I + ip^I\right), \quad \Upsilon = -64 \tag{2.2}$$

are evaluated at the horizon. The horizon value of ϕ^I is determined by extremizing the right hand side of (2.1) with respect to ϕ^I,

$$4 \frac{\partial}{\partial \phi^I} \operatorname{Im} F(\phi + ip, -64) = q_I. \tag{2.3}$$

Wald's entropy (2.1) constitutes the semi-classical approximation to the exact macroscopic entropy of a BPS black hole. For extremal black holes whose near-horizon geometry contains an AdS_2 factor, a proposal for the exact macroscopic entropy of extremal black holes has been put forward in [28, 29]. This proposal is called the quantum entropy function, and it is a regulated partition function for quantum gravity in a near-horizon AdS_2 space-time.

The quantum entropy function is a functional integral, which for the subclass of BPS black holes can be defined by means of equivariant localization techniques [30, 36, 37, 43–46] that reduce the infinite dimensional functional integral to a finite dimensional integral

over a bosonic localization manifold. When restricting to smooth field configurations in near-horizon AdS_2 backgrounds, the resulting expression for the quantum entropy function $W(q, p)$ for BPS black holes takes the form

$$W(q,p) = \int_{\mathcal{C}} d\phi \, \mu(\phi) \, Z_{1-\text{loop}}(\phi) \, e^{\pi[4\,\text{Im}F(\phi+ip)-q\cdot\phi]} \,, \tag{2.4}$$

where $q \cdot \phi = q_I \phi^I$, with $I = 0, \ldots, n_V$. The localization manifold is labelled by $n_V + 1$ parameters $\{\phi^I\}$. Note that the integral in (2.4) requires specifying a contour $\mathcal{C}$. The measure $\mu(\phi)$ arises as a result of the localization procedure implemented on the field configuration space. The term $Z_{1-\text{loop}}(\phi)$ describes the semi-classical correction, giving rise to a super determinant, that arises when performing the Gaussian integration over terms quadratic in quantum fluctuations around the localization manifold. The function F entering in (2.4) is the holomorphic function that defines the $N = 2$ Wilsonian effective action (with $\Upsilon = -64$).

Let us describe the quantities that enter in (2.4) in more detail. The Wilsonian function F can be decomposed into

$$F(Y, \Upsilon) = F^{(0)}(Y) + 2i\,\Omega(Y, \Upsilon), \tag{2.5}$$

where the dependence on the field Υ is solely contained in Ω. The first term $F^{(0)}(Y)$ is the so-called prepotential of the $N = 2$ model. Local supersymmetry requires both $F^{(0)}$ and Ω to be homogeneous of degree 2 under complex rescalings [47],

$$F^{(0)}(\lambda Y) = \lambda^2\, F^{(0)}(Y)\,,$$
$$\Omega(\lambda Y, \lambda^2 \Upsilon) = \lambda^2\, \Omega(Y, \Upsilon), \quad \lambda \in \mathbb{C}\backslash\{0\}\,, \tag{2.6}$$

and hence they satisfy the homogeneity relations

$$2F^{(0)} = Y^I\, F_I^{(0)}\,,$$
$$2\Omega = Y^I\, \Omega_I + 2\Upsilon\, \Omega_\Upsilon\,. \tag{2.7}$$

Here, we have introduced the notation $F_I = \partial F/\partial Y^I$, $\Omega_\Upsilon = \partial\Omega/\partial\Upsilon$.

The Wilsonian effective action is based on a function F where $\Omega(Y, \Upsilon)$ is given in terms of a power series expansion in Υ. Choosing $\lambda = 1/Y^0$ in the homogeneity relation (2.7) implies that $\Omega(Y, \Upsilon)$ takes the form

$$\Omega(Y^0, z^A, \Upsilon) = \Upsilon \sum_{n=0}^{\infty} \left(\frac{\Upsilon}{(Y^0)^2}\right)^n \omega^{(n+1)}(z^A)\,. \tag{2.8}$$

Here, the gravitational coupling functions $\omega^{(n+1)}$ only depend on the projective coordinates $z^A = Y^A/Y^0$ ($A = 1, \ldots, n_V$). In a generic $N = 2$ model, this perturbative series in $\Upsilon/(Y^0)^2$ may not be convergent.

Next, let us discuss the measure factor μ. While this measure factor has not yet been worked out from first principles,[2] an approximate expression for it can be obtained by demanding consistency under electric-magnetic duality transformations as well as consistency with semi-classical results for BPS entropy, as follows.

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[2]A formalism for carrying out localization calculations in the presence of a supersymmetric background has recently been given in [45, 46].

Electric-magnetic duality is a characteristic feature of systems with $N = 2$ Abelian vector multiplets in four dimensions, also in the presence of a chiral background field [40]. Duality transformations act as symplectic transformations on the Abelian field strengths, and hence, on the associated charge vector (p^I, q_I), as well as on the vector (Y^I, F_I) [47]. The semi-classical entropy (2.1) of a BPS black hole transforms as a function under symplectic transformations, that is, under changes of the duality frame, and this should also be the case for the exact macroscopic entropy $W(q, p)$ of the BPS black hole. To ensure that (2.4) transforms as a function under symplectic transformations, the measure factor has to be proportional to [43, 48]

$$\sqrt{\det \mathrm{Im} F_{KL}}\,, \tag{2.9}$$

as we will review in the next subsection. Here, F_{KL} denotes the second derivative of the Wilsonian function F (2.5) with respect to the Y^I.

In addition, the measure factor μ should also include non-holomorphic terms, which we will denote by Σ, that have their origin in the holomorphic anomaly equations for the free energies of perturbative topological string theory, and are needed for consistency with semi-classical results [49]. Keeping only non-holomorphic terms associated with the topological free energy $F^{(1)}$, Σ takes the form

$$\Sigma = -\frac{1}{2} \ln |\det \mathrm{Im} F_{IJ}^{(0)}| + \left(\frac{\chi}{24} - 1\right) \ln \left(\frac{e^{-\mathcal{K}^{(0)}}}{|Y^0|^2}\right)\,, \tag{2.10}$$

where $\chi = 2(n_V - n_H + 1)$, which is determined in terms of the number of vector and hyper multiplets of the $N = 2$ model (n_V and n_H, respectively), denotes the Euler number of the Calabi-Yau threefold underlying the model, and

$$e^{-\mathcal{K}^{(0)}} = i\left(\bar{Y}^I F_I^{(0)} - Y^I \bar{F}_I^{(0)}\right)\,. \tag{2.11}$$

We are thus led to consider a measure factor of the form

$$\mu = \sqrt{\det \mathrm{Im} F_{KL}}\, e^{\Sigma}\,, \tag{2.12}$$

which, when approximating F by $F^{(0)}$ takes the form (cf. eq. (4.21) in [49])

$$\mu \approx \left(\frac{e^{-\mathcal{K}^{(0)}}}{|Y^0|^2}\right)^{\chi/24-1}\,. \tag{2.13}$$

The expressions (2.12) and (2.13) are evaluated at (2.2). We stress that the measure factor given in (2.13) is an approximate measure factor that will receive further corrections stemming from Ω in (2.5). We will, in due course, make use of this observation. Note that in this approximation, μ only depends on the projective coordinates $z^A, \bar{z}^A$.

Now let us turn to the 1-loop determinant $Z_{1-\text{loop}}$, which reads [43, 44, 46]

$$Z_{1-\text{loop}} = e^{-(2-\chi/24)\mathcal{K}}\,. \tag{2.14}$$

Here, the quantity

$$e^{-\mathcal{K}} = i\left(\bar{Y}^I F_I - Y^I \bar{F}_I\right) \tag{2.15}$$

is computed from the Wilsonian function $F(Y, \Upsilon)$, and not just from the prepotential $F^{(0)}(Y)$. The factor 2 in the exponent of (2.14) denotes the contribution from fluctuations of the Weyl multiplet. Note that the expression for $Z_{1-\text{loop}}$ depends on Y^0 and z^A (and their complex conjugates), and that it is a symplectic function. $Z_{1-\text{loop}}$ is again evaluated at (2.2).

The quantum entropy function (2.4) requires a choice of integration contour $\mathcal{C}$, which we will specify in subsection 3.5 for the specific $N = 2$ model under consideration.

Let us close this subsection by mentioning three checks that one can perform on the proposed approximate measure factor μ and on $Z_{1-\text{loop}}$. Firstly, when replacing $\mathcal{K}$ by $\mathcal{K}^{(0)}$ in $Z_{1-\text{loop}}$, one infers [43, 44] that the macroscopic entropy $\mathcal{S}_{\text{BH}}(q, p) = \ln W(q, p)$ receives a logarithmic correction given by

$$\Delta \mathcal{S}_{\text{BH}} = \ln Z_{1-\text{loop}}|_* = (2 - \chi/24) \ln e^{-\mathcal{K}^{(0)}}|_* . \tag{2.16}$$

Here, $*$ indicates that the expression is evaluated at the attractor values (2.3). For a large supersymmetric black hole, $\pi e^{-\mathcal{K}^{(0)}}|_*$ equals the area A_{BH} of the event horizon at the two-derivative level [50], and hence

$$\Delta \mathcal{S}_{\text{BH}} = (2 - \chi/24) \ln A_{\text{BH}} , \tag{2.17}$$

which reproduces the logarithmic area correction to the BPS entropy of a large black hole computed in [31].

Secondly, approximating $\mathcal{K}$ by $\mathcal{K}^{(0)}$ in (2.14) results in [31]

$$\mu \, Z_{1-\text{loop}} = |Y^0|^{(2-\chi/12)} \, e^{-\mathcal{K}^{(0)}} , \tag{2.18}$$

which is the combination that plays the role of a measure factor in the study [51] of the OSV conjecture [42].

Thirdly, as mentioned above, the measure factor μ is required to ensure that the quantum entropy function is a symplectic function. To verify this, we will now compute $W(q, p)$ in saddle-point approximation.

2.2 Saddle-point approximation

Let us denote the exponent in (2.4) by

$$H(\phi, p, q) = 4 \, \text{Im} F(\phi + ip) - q \cdot \phi . \tag{2.19}$$

Then, equations (2.3) follow by extremizing $H(\phi, p, q)$,

$$\frac{\partial H(\phi, p, q)}{\partial \phi^I} = 0 . \tag{2.20}$$

Let us assume that for a given set of black hole charges (q_I, p^I) there exists only one non-inflective critical point ϕ_*, corresponding to BPS attractor values such that $H(\phi_*, p, q) > 0$. Expanding $H(\phi, p, q)$ around the critical point ϕ_*, we obtain

$$\begin{aligned} H(\phi, p, q) &= H(\phi_*, p, q) + \frac{1}{2} \frac{\partial^2 H}{\partial \phi^I \partial \phi^J}|_{\phi_*} \, \delta\phi^I \, \delta\phi^J + \mathcal{O}((\delta\phi)^3) , \\ &= H(\phi_*, p, q) + \frac{1}{2} \frac{\partial^2 (4 \, \text{Im} F)}{\partial \phi^I \partial \phi^J}|_{\phi_*} \, \delta\phi^I \, \delta\phi^J + \mathcal{O}((\delta\phi)^3) . \end{aligned} \tag{2.21}$$

Using (2.2), this equals

$$H(\phi, p, q) = H(\phi_*, p, q) + \frac{1}{2} \operatorname{Im} F_{IJ}|_{\phi_*} \, \delta\phi^I \, \delta\phi^J + \mathcal{O}((\delta\phi)^3) \,. \tag{2.22}$$

Next, we evaluate (2.4) in saddle point approximation by taking $\mu \, Z_{1-\text{loop}}$ at the attractor point,

$$\mu \, Z_{1-\text{loop}}|_{\phi_*} \,, \tag{2.23}$$

and by integrating over the fluctuations $\delta\phi^I \in \mathbb{C}$ in Gaussian approximation,

$$\begin{aligned} W(q, p) &\approx \left(\frac{\mu \, Z_{1-\text{loop}}}{\sqrt{\det \operatorname{Im} F_{KL}}} \right) |_{\phi_*} \, e^{\pi H(\phi_*, p, q)} \\ &= \left(e^\Sigma \, Z_{1-\text{loop}} \right) |_{\phi_*} \, e^{\pi H(\phi_*, p, q)} \,. \end{aligned} \tag{2.24}$$

We note that the factor $\sqrt{\det \operatorname{Im} F_{KL}}$ has cancelled out against the corresponding factor in (2.12).

In the absence of non-holomorphic corrections ($\Sigma = 0$), H is a symplectic function [48], and hence, the result (2.24) is a symplectic function. This justifies the presence of the factor $\sqrt{\det \operatorname{Im} F_{KL}}$ in (2.12). In the presence of non-holomorphic corrections ($\Sigma \neq 0$), the discussion of symplectic covariance is more involved [48, 49, 52], but it can be shown that the combination $\Sigma + \pi H$ in (2.24) is a symplectic function.

In this paper, we will focus on a specific $N = 2$ model, namely the STU model of Sen and Vafa (example D in [20]). This is a model for which the duality symmetries are known exactly. We will analyze the expression (2.24) for this model in subsection 3.4.1, and verify that it is consistent with the duality invariance of the model.

Finally, we recall that the attractor equations (2.20) take the form

$$F_I - \bar{F}_I = i q_I \,, \quad I = 0, \ldots, n_V \,, \tag{2.25}$$

with $Y^I = \frac{1}{2}(\phi^I + i p^I)$. The attractor value $H(\phi_*, p, q)$ can be expressed as [48]

$$H(\phi_*, p, q) = \left[i \left(\bar{Y}^I F_I - Y^I \bar{F}_I \right) - 2i \left(\Upsilon F_\Upsilon - \bar{\Upsilon} \, \bar{F}_{\bar\Upsilon} \right) \right] \big|_{Y = \frac{1}{2}(\phi_* + i p)} \tag{2.26}$$

by means of homogeneity relations (2.7).

3 Quantum entropy function for the $N = 2$ STU model of Sen and Vafa

Next, we specialize to the $N = 2$ STU model of Sen and Vafa (example D in [20]). This is a model with duality symmetries which are so restrictive that they have recently led to the determination [21] of Ω in (2.5). We will consider large BPS black holes in this model, and we will evaluate the quantum entropy function (2.4) for these black holes. In doing so, we will work in a certain region of moduli space of the STU model in which the function F simplifies. We begin by reviewing the form of F for this model.

3.1 The $N = 2$ STU model with $\chi = 0$

The $N = 2$ STU model of [20] (example D), obtained by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold compactification of type II string theory, is a model with $n_V = 3$ vector multiplets and $n_H = 4$ hyper multiplets, and hence vanishing Euler number $\chi = 2(n_V - n_H + 1)$. The complex scalar fields residing in the three vector multiplets are denoted by S, T and U,

$$S = -iz^1 = -i\frac{Y^1}{Y^0}, \quad T = -iz^2 = -i\frac{Y^2}{Y^0}, \quad U = -iz^3 = -i\frac{Y^3}{Y^0}. \tag{3.1}$$

The prepotential $F^{(0)}(Y)$ is exact and given by

$$F^{(0)}(Y) = -\frac{Y^1 Y^2 Y^3}{Y^0}. \tag{3.2}$$

The model possesses symmetries, in particular a duality symmetry $\Gamma_0(2)_S \times \Gamma_0(2)_T \times \Gamma_0(2)_U$, where $\Gamma_0(2)$ denotes a congruence subgroup of the group $\mathrm{SL}(2, \mathbb{Z})$,

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{2} \right\}, \tag{3.3}$$

where $*$ can take any value in $\mathbb{Z}$.

Duality transformations act as symplectic transformations on the vector (Y^I, F_I). Under $\Gamma_0(2)_S$ transformations, the complex scalars S and Y^0 transform as

$$S \to \frac{aS - ib}{icS + d}, \quad Y^0 \to (icS + d) Y^0, \tag{3.4}$$

while the scalars T, U transform as [49]

$$T \to T + \frac{2ic}{(icS + d)(Y^0)^2} \frac{\partial \Omega}{\partial U},$$
$$U \to U + \frac{2ic}{(icS + d)(Y^0)^2} \frac{\partial \Omega}{\partial T}, \tag{3.5}$$

and hence are not inert in the presence of Ω. Similar transformation rules apply under $\Gamma_0(2)_{T,U}$ transformations.

The symmetries of the model, namely $\Gamma_0(2)_S \times \Gamma_0(2)_T \times \Gamma_0(2)_U$ and triality symmetry under exchange of S, T and U, are very restrictive, and have recently been used [21] to determine the coupling functions $\omega^{(n+1)}(z^A)$ in Ω, cf. (2.8). This was achieved by adding to Ω the term $2\Upsilon \gamma \ln Y^0$, with the constant γ given by (3.7). This term, which is not of Wilsonian type, since it is not of the power law type, was crucial to implement the duality symmetries of the STU model while maintaining holomorphy. It was found that the higher gravitational coupling functions $\omega^{(n+1)}(z^A)$ (with $n \geq 1$) are determined in terms of the first gravitational coupling function $\omega^{(1)}(z^A)$. The latter takes the form [53]

$$\omega^{(1)}(S, T, U) = \omega(S) + \omega(T) + \omega(U), \tag{3.6}$$

where

$$\omega(S) = -\frac{1}{2}\gamma \ln \vartheta_2^8(S), \quad \gamma = -\frac{1}{256\pi}, \tag{3.7}$$

with $\vartheta_2(S) = 2\eta^2(2S)/\eta(S)$, and likewise for $\omega(T)$ and $\omega(U)$. Here, $\vartheta_2^8(S)$ denotes a modular form of weight 4 under $\Gamma_0(2)_S$ (with trivial multiplier system [54]),

$$\vartheta_2^8\left(\frac{aS - ib}{icS + d}\right) = \Delta^4(S)\,\vartheta_2^8(S)\,, \tag{3.8}$$

where

$$\Delta(S) = icS + d\,. \tag{3.9}$$

We refer to appendices A and B for a brief review of modular forms. We pick a single-valued analytic branch of $\ln \vartheta_2^8(S)$ such that under $\Gamma_0(2)_S$, $\omega(S)$ transforms into

$$\omega(S) \longrightarrow \omega(S) - 2\gamma\,\ln\Delta(S)\,. \tag{3.10}$$

Now let us describe the region in moduli space in which we will work. It was observed in [21] that the coupling functions $\omega^{(n+1)}(z^A)$ in Ω simplify dramatically when working in a regime where two of the three moduli S, T, U (say T and U) are taken to be large, i.e. $\operatorname{Re} T, \operatorname{Re} U \gg 1$. In this limit,

$$\omega(T) \propto T\,, \quad \omega(U) \propto U\,, \tag{3.11}$$

that is,

$$\frac{\partial\omega(T)}{\partial T} = -\frac{1}{256}\,, \quad \frac{\partial\omega(U)}{\partial U} = -\frac{1}{256}\,. \tag{3.12}$$

We introduce the combination

$$\alpha \equiv \frac{\partial\omega(T)}{\partial T}\,\frac{\partial\omega(U)}{\partial U} = \frac{1}{(256)^2}\,, \tag{3.13}$$

which is constant in this limit. In this region of moduli space, the coupling functions $\omega^{(n+1)}$ with $n \geq 1$ in (2.8) are functions of S only, and are moreover expressed in terms of derivatives of $\omega(S)$, as we now describe.

In this limit, the coupling function $\omega^{(2)}$ is given by

$$\omega^{(2)}(S) = \frac{\alpha}{\gamma}\,\frac{\partial\omega(S)}{\partial S}\,, \tag{3.14}$$

and the Wilsonian function $\Omega(Y, \Upsilon)$ in (2.8) takes the form[3]

$$\Omega(Y^0, S, T, U, \Upsilon) = \Upsilon\,(\omega(T) + \omega(U)) + \widehat\Omega(Y^0, S, \Upsilon)\,,$$
$$\widehat\Omega(Y^0, S, \Upsilon) = \Upsilon\left[\omega(S) + \frac{\Upsilon}{(Y^0)^2}\,\frac{\alpha}{\gamma}\,\frac{\partial\omega(S)}{\partial S} + \Xi(Y^0, S, \Upsilon)\right]\,, \tag{3.15}$$

where we used (3.6). Here, Ξ contains the coupling functions $\omega^{(n+1)}(S)$ with $n \geq 2$, and it denotes a function that is invariant under $\Gamma_0(2)_S$ transformations. This can be established as follows.

---

[3]Note that the Wilsonian function Ω used here differs from the function Ω used in [21] by a term $\Upsilon\,\gamma\,\ln[(Y^0)^2/\Upsilon]$.

Instead of working with variables S, Y^0, we find it convenient to work with variables S, z, where[4]

$$z \equiv 1/Y^0 \,. \tag{3.16}$$

Under $\Gamma_0(2)_S$ transformations, these variables transform according to (3.4), and hence, $z \mapsto z/\Delta$. The derivatives of $\widehat{\Omega}(Y^0, S, \Upsilon)$ are required [21] to transform according to

$$\left(\frac{\partial \widehat{\Omega}}{\partial S} \right)' - \Delta^2 \frac{\partial \widehat{\Omega}}{\partial S} = \frac{\partial \Delta}{\partial S} \left[\Delta \, z \left(\frac{\partial \widehat{\Omega}}{\partial z} - \frac{2 \Upsilon \, \gamma}{z} \right) - 2 \, \Upsilon^2 \, \alpha \, z^2 \, \frac{\partial \Delta}{\partial S} \right] \,,$$

$$\left(\frac{\partial \widehat{\Omega}}{\partial z} - \frac{2 \Upsilon \, \gamma}{z} \right)' = \Delta \left(\frac{\partial \widehat{\Omega}}{\partial z} - \frac{2 \Upsilon \, \gamma}{z} \right) - 4 \, \Upsilon^2 \, \alpha \, z \, \frac{\partial \Delta}{\partial S} \,, \tag{3.17}$$

where $'$ denotes the transformed quantity. Inserting the expression for $\widehat{\Omega}$ given in (3.15) into (3.17) results in

$$\left(\frac{\partial \Xi}{\partial S} \right)' = \Delta^2 \frac{\partial \Xi}{\partial S} + ic \, \Delta \, z \frac{\partial \Xi}{\partial z} \,,$$

$$\left(\frac{\partial \Xi}{\partial z} \right)' = \Delta \, \frac{\partial \Xi}{\partial z} \,, \tag{3.18}$$

where we used (3.10). Note that (3.18) is linear in Ξ. Expanding Ξ in powers of $\Upsilon \, z^2$ [21],

$$\Xi(z, S, \Upsilon) = \sum_{n=2}^{\infty} \Upsilon^n \, z^{2n} \, \omega^{(n+1)}(S) \,, \tag{3.19}$$

we find[5] that (3.19) solves (3.18), provided the coupling functions $\omega^{(n+1)}(S)$ in (3.19) are modular forms of weight $2n$. The solution (3.19) is therefore invariant under $\Gamma_0(2)_S$ transformations. The coupling functions $\omega^{(n+1)}(S)$ are expressed in terms of modular forms I_n introduced in [21], as follows.

Consider the combinations

$$I_1(S) = \frac{\partial \omega(S)}{\partial S} \,,$$

$$I_2(S) = \partial_S I_1 + \frac{1}{2\gamma} I_1^2 = \frac{\partial^2 \omega(S)}{\partial S^2} + \frac{1}{2\gamma} \left(\frac{\partial \omega(S)}{\partial S} \right)^2 \,. \tag{3.20}$$

$I_2(S)$ transforms as a modular form under $\Gamma_0(2)_S$ transformations, whereas $I_1(S)$ does not in view of (3.10). Therefore, and for latter use, we introduce the combination

$$\hat{I}_1(S, \bar{S}) = \frac{\partial \omega}{\partial S} - \frac{2\gamma}{(S + \bar{S})} \,, \tag{3.21}$$

which transforms covariantly (with weight 2) under $\Gamma_0(2)_S$ transformations by virtue of

$$\frac{1}{S + \bar{S}} \longrightarrow \frac{\Delta^2(S)}{S + \bar{S}} - ic \, \Delta(S) \,. \tag{3.22}$$

---

[4]The variable $z \equiv 1/Y^0$ should not be confused with the projective coordinates z^A defined in (3.1).

[5]Here we assume that we may differentiate (3.19) term by term, with respect to both z and S.

Next, define higher $I_n(S)$ by

$$I_{n+1}(S) = \mathcal{D}_S I_n(S), \quad n \geq 2.$$

(3.23)

Here, $\mathcal{D}_S$ denotes a holomorphic covariant derivative [55], an analogue of Serre derivative, which acts as follows on $\Gamma_0(2)_S$ modular forms $f_n(S)$ of weight $2n$,

$$\mathcal{D}_S f_n(S) = \left(\frac{\partial}{\partial S} + \frac{n}{\gamma} \frac{\partial \omega(S)}{\partial S} \right) f_n(S), \quad n \geq 2.$$

(3.24)

Under $\Gamma_0(2)_S$ transformations, the I_n (with $n \geq 2$) transform as modular forms of weight $2n$, i.e. $I_n(S) \to \Delta^{2n} I_n(S)$. We refer to appendix B for further properties of the I_n, in particular their relation with Eisenstein series of $\Gamma_0(2)$ [55].

The explicit form of $\Xi(z, S, \Upsilon)$, which was determined in [21], is in terms of a power series in $I_n(S)$ with $n \geq 2$,

$$\Xi(z, S, \Upsilon) = \sum_{n=2}^{\infty} \alpha_n \, \Upsilon^n \, z^{2n} \, I_n(S) + \sum_{m,n=2}^{\infty} \alpha_{m,n} \, \Upsilon^{m+n} \, z^{2(m+n)} \, I_m(S) \, I_n(S) + \dots \, . \quad (3.25)$$

In this expansion, each summand is invariant under $\Gamma_0(2)_S$ transformations. This expansion contains an infinite number of different sectors, characterized by different powers of I_n. The expansion coefficients in (3.25) can, in principle, be determined by following the rather laborious procedure described in [21], which consists in working at a generic point in moduli space, solving the associated conditions on Ω imposed by duality, and only then taking T and U to be large. In [21], only expressions for the coefficients α_n and $\alpha_{m,n}$ of the two first sectors were given: the α_n were determined fully, and partial expressions for the $\alpha_{m,n}$ were given. Below we write down explicit expressions for these coefficients (with $m, n \geq 2$):

$$\alpha_n = \frac{1}{n!} \left(\frac{\alpha}{\gamma} \right)^n,$$

(3.26)

$$\alpha_{m,n} = \beta_{m,n} \left(\frac{\alpha}{\gamma} \right)^{m+n},$$

$$\beta_{m,n} = -\frac{1}{2\gamma} \frac{1}{(m-1)!\,(n-1)!} \left[\frac{1}{m(m+1)} + \frac{1}{n(n+1)} - \frac{(m+n-2)!}{(m+n)!} + \frac{1}{2} \right].$$

The coefficients $\beta_{m,n}$ satisfy the following recursion relation (with $m, n \geq 2$),

$$(m+n)\,\beta_{m,n} = \beta_{m,n-1} + \beta_{m-1,n} - \frac{1}{2\gamma} \frac{1}{(m-1)!\,(n-1)!}$$
$$- \frac{1}{2\gamma} \frac{1}{(m-1)!} \, \delta_{n,2} - \frac{1}{2\gamma} \frac{1}{(n-1)!} \, \delta_{m,2}, \quad (3.27)$$

with $\beta_{1,n} = \beta_{m,1} \equiv 0$. Note that the terms in the bracket of $\beta_{m,n}$ are related to triangular numbers.

In this paper we will focus on the two first sectors displayed in (3.25), with the coefficients given by (3.26). Both series are convergent, as we will show in appendix D. The quantity $\Xi(z, S, \Upsilon)$ will play a role in the microstate counting proposal for large BPS black holes in section 4.

The expansion (3.25) in powers of I_n may look unfamiliar. In appendix D, we discuss another example of such an expansion, namely the expansion of $\ln \vartheta_2^8(S, z)$.

3.2 Properties of $\Xi(z, S, \Upsilon)$

The expansion (3.25) with coefficients (3.26) satisfies various interesting relations, which we now describe. First, we present a non-linear PDE that is satisfied by Ξ in (3.25). We also display a candidate for a non-linear PDE governing the all-order completion of (3.25). Next, we relate the monomial sector in (3.25) to (Serre-) Rankin-Cohen brackets at level 1 and comment on its relation with formal deformation. And finally, we deduce a Hamilton-Jacobi equation for $\widehat{\Omega}$ in (3.15), whose Hamiltonian describes a time-dependent deformation of a rational Calogero model.

3.2.1 Non-linear PDE

Let $S = -i\tau$, with τ taking values in the complex upper half plane $\mathcal{H}$, and $z \in \mathbb{C}$. Consider the following non-linear PDE for a complex function Ξ depending on S and z^2,

$$\nu \, D_S \Xi = \frac{\partial \Xi}{\partial u} + \frac{1}{2\gamma} u \left(\frac{\partial \Xi}{\partial u} \right)^2 + \frac{\nu^2}{\gamma} I_2 \, u^2 \, \frac{\partial \Xi}{\partial u} - \nu^2 I_2 \, u \,, \tag{3.28}$$

where we set $u \equiv z^2$ for ease of notation in this subsection, and where $\nu = \alpha \Upsilon / \gamma$. Here we defined

$$D_S \Xi \equiv \partial_S \Xi + \frac{1}{\gamma} I_1 \, u \, \partial_u \Xi \,. \tag{3.29}$$

I_1 and I_2 denote the combinations given in (3.20).

Proposition: the non-linear PDE (3.28), subject to $\Xi|_{u=0} = 0$ and $(\partial_u \Xi)|_{u=0} = 0$, admits a $\Gamma_0(2)_S$ invariant solution that is analytic in u in an open neighbourhood of $u = 0$. This solution is given by the series (3.25) with coefficients (3.26), up to terms that involve products of three I_n or higher.

Proof. We begin by showing that if a solution Ξ to (3.28) exists that satisfies $\Xi|_{u=0} = 0$ and $(\partial_u \Xi)|_{u=0} = 0$, and that is analytic in u in an open neighbourhood of $u = 0$, then it is unique. Such a solution will be given by the series $\Xi(u, S) = \sum_{n=2}^{\infty} u^n \, h_n(S)$, with $h_2(S) = \frac{\nu^2}{2} I_2(S)$. The latter follows immediately by inserting this series into (3.28) and assuming that one can differentiate it on a term by term basis. Now suppose that there are two such solutions, namely $\Xi_1(u, S) = \sum_{n=2}^{\infty} u^n \, h_n(S)$ and $\Xi_2(u, S) = \sum_{n=2}^{\infty} u^n \, g_n(S)$, such that $h_2(S) = g_2(S)$. Then, if the n-th coefficient functions are the same, i.e. $h_n(S) = g_n(S)$, also the $(n + 1)$-st coefficient functions will agree, i.e. $h_{n+1}(S) = g_{n+1}(S)$. This follows by direct inspection of the differential equation (3.28), which shows that the $(n + 1)$-st coefficient function is determined in terms of the lower coefficient functions. This also shows that the coefficient functions $h_n(S)$ are modular forms of weight $2n$ under $\Gamma_0(2)_S$ transformations, and hence $\Xi(u, S)$ is a $\Gamma_0(2)_S$ invariant solution.

Next, let us construct this $\Gamma_0(2)_S$ invariant solution $\Xi(u, S) = \sum_{n=2}^{\infty} u^n \, h_n(S)$. This solution will be given in terms of an expansion in powers of $I_n(S)$ (with $n \geq 2$). To distinguish between these various powers, we rescale Ξ and I_2 in (3.28) by a real parameter $\lambda \in \mathbb{R} \backslash \{0\}$, to obtain

$$\nu \, D_S \Xi = \frac{\partial \Xi}{\partial u} + \frac{\lambda}{2\gamma} u \left(\frac{\partial \Xi}{\partial u} \right)^2 + \lambda \frac{\nu^2}{\gamma} I_2 \, u^2 \, \frac{\partial \Xi}{\partial u} - \nu^2 I_2 \, u \,. \tag{3.30}$$

We then construct a solution to (3.30) order by order in λ. Since λ is a dummy variable, we set $\lambda = 1$ at the end, thereby arriving at a $\Gamma_0(2)_S$ invariant solution $\Xi(u, S)$ that is organized in powers of $I_n(S)$, as in (3.25).

To lowest order in λ, (3.30) reduces to

$$\nu \, D_S \Xi = \frac{\partial \Xi}{\partial u} - \nu^2 \, I_2 \, u \,. \tag{3.31}$$

We seek a solution to this differential equation of the form

$$\Xi(u, S) = \sum_{n=2}^{\infty} u^n \, f_n(S) \,. \tag{3.32}$$

Inserting (3.32) into (3.31), and assuming that we can differentiate (3.32) term by term[6], one immediately infers that $f_n(S) = \frac{\nu^n}{n!} \, I_n(S)$ for $n \geq 2$ by virtue of (3.24). This reproduces the first (monomial) sector in the expansion (3.25).

At the next order in λ, we return to (3.30) and substitute (3.32) into the term $(\partial \Xi / \partial u)^2$. At this order, (3.30) is solved by the second (binomial) sector in the series (3.25), with the coefficients satisfying the recursion relation (3.27), thus reproducing the result (3.26).

Setting $\lambda = 1$ we conclude that (3.25) solves the non-linear PDE (3.28), up to terms that involve products of three I_n or higher.

$\square$

The expansion Ξ in (3.25) receives corrections that are of higher order (higher than two) in the I_n. These will in turn lead to a modification of the PDE (3.28). The coefficients of these higher order terms were not determined in [21]. Inspection of (3.28) suggests the following all-order completion of (3.25) (under the assumption that the expressions for the coefficients $\alpha_{m,n}$ displayed in (3.26) are actually the exact expressions).

Conjecture: the non-linear PDE governing the all-order completion of (3.25) is given by

$$\nu \, D_S \left(\Xi / \gamma \right) = \frac{1}{u} \left(e^{u \partial (\Xi / \gamma) / \partial u} - 1 - \frac{\nu^2 \, u^2 \, I_2 / \gamma}{1 + u \frac{\partial (\Xi / \gamma)}{\partial u}} \right) \,. \tag{3.33}$$

However, at this stage, we cannot verify this, since we do not have at our disposal the coefficients of the higher-order terms.

Note that D_S acts as follows on a given summand of Ξ in (3.25). Consider the summand

$$\alpha_{n_1, n_2, \ldots, n_k} \, z^{2(n_1 + n_2 + \ldots n_k)} \, I_{n_1} \, I_{n_2} \ldots I_{n_k} \,, \tag{3.34}$$

with $\alpha_{n_1, n_2, \ldots, n_k}$ constant coefficients. Then,

$$D_S \left(\alpha_{n_1, n_2, \ldots, n_k} \, z^{2(n_1 + n_2 + \ldots n_k)} \, I_{n_1} \, I_{n_2} \ldots I_{n_k} \right) = \alpha_{n_1, n_2, \ldots, n_k} \, z^{2(n_1 + n_2 + \ldots n_k)} \, \mathcal{D}_S \left(I_{n_1} \, I_{n_2} \ldots I_{n_k} \right) \,, \tag{3.35}$$

where $\mathcal{D}_S$ denotes the covariant derivative introduced in (3.24).

---

[6]This will be shown to be the case in appendix D, at least so long as $\mathrm{Re} \, S$ is taken to be large.

3.2.2 I_n and Rankin-Cohen brackets

Second, let us focus on the terms linear in I_n in the expansion (3.25),

$$\Xi_1(z, S, \Upsilon) = \sum_{n=2}^{\infty} \frac{\nu^n}{n!} \, z^{2n} \, I_n(S) \,. \tag{3.36}$$

Note that the coefficients have a $1/n!$-suppression. This series is convergent, at least so long as $\mathrm{Re}\, S$ and $|Y^0|$ are taken to be large, as we show in appendix D.

Using the property $I_{n+1}(S) = \mathcal{D}_S I_n(S)$ for $n \geq 2$, together with $I_2(S) = \frac{1}{2}\pi\gamma\,\mathcal{D}_S\tilde{\mathcal{E}}_2(S)$, where $\tilde{\mathcal{E}}_2(S)$ denotes the basis vector of the vector space $\mathcal{M}_2(\Gamma_0(2))$, see (B.37), we obtain

$$\Xi_1(z, S, \Upsilon) = \frac{1}{2}\pi\gamma \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \nu^{n+1} z^{2(n+1)} \mathcal{D}_S^n \tilde{\mathcal{E}}_2(S) = \frac{1}{2}\pi\gamma \left(\frac{e^{\nu z^2 \mathcal{D}_S} - 1 - \nu z^2 \mathcal{D}_S}{\mathcal{D}_S} \right) \tilde{\mathcal{E}}_2(S), \tag{3.37}$$

where the operator on the right hand side is defined by the power series.

Next, let us add to $\Xi_1(z, S, \Upsilon)$ the term proportional to $I_1(S) = \partial\omega(S)/\partial S$ that appears in $\widehat{\Omega}$ given in (3.15), so that now we consider

$$\widehat{\Xi}_1(z, S, \Upsilon) = \sum_{n=1}^{\infty} \frac{\nu^n}{n!} \, z^{2n} \, I_n(S) \,. \tag{3.38}$$

As we review in appendix C, the I_n, with $n \geq 3$, can be expressed in terms of 1st Rankin-Cohen brackets for modular forms, while I_2 can be expressed in terms of the quasi-modular form I_1 by making use of 1st Rankin-Cohen brackets for quasi-modular forms of depth 1 [56],

$$I_2 = -\frac{1}{4g}[I_1, g]_1 \,, \tag{3.39}$$

where $g(S) = \vartheta_2^8(S)$ has weight 4. We thus have the following proposition.

Proposition: let $g(S) = \vartheta_2^8(S)$. Then, $\widehat{\Xi}_1(z, S, \Upsilon)$ can be expressed as

$$\widehat{\Xi}_1(z, S, \Upsilon) = \nu\, z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \left(\frac{\nu\, z^2}{4} \right)^n \left(\frac{1}{g}[\cdot, g]_1 \right)^n I_1(S) \,, \tag{3.40}$$

where

$$\left(\frac{1}{g}[\cdot, g]_1 \right)^n I_1 \equiv \frac{1}{g}\,[\frac{1}{g}[\dots[\frac{1}{g}[I_1, g]_1, \dots, g]_1, g]_1 \,, \tag{3.41}$$

with the understanding that this equals I_1 when $n = 0$. In the last step, the bracket $[\cdot, \cdot]_1$ denotes the 1st Rankin-Cohen bracket for quasi-modular forms of depth 1.

Proof. This follows immediately, using the results of appendix C.

$\square$

The expression (3.40) exhibits a formal similarity with the differential (C.14) of the exponential map $\exp : \mathfrak{g} \to G$ from the Lie algebra $\mathfrak{g}$ into the linear group G.

Now consider the nth Serre-Rankin-Cohen bracket (C.16) for $\Gamma_0(2)$,

$$\frac{1}{g} SRC_n(I_2, g)(S) = (-1)^n \begin{pmatrix} n+3 \\ n \end{pmatrix} I_{n+2}(S), \qquad (3.42)$$

where we used (B.34). Thus, we can write $\widehat{\Xi}_1(z, S, \Upsilon)$ as

$$\widehat{\Xi}_1(z, S, \Upsilon) = \nu z^2 I_1(S) + \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \nu^{n+2} z^{2(n+2)} I_{n+2}(S) \qquad (3.43)$$

$$= \nu z^2 I_1(S) + 6 \frac{\nu^2 z^4}{g} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(n+2)! (n+3)!} (\nu z^2)^n SRC_n(I_2, g)(S).$$

We relate this to the formal deformation Eholzer product (C.17) by means of the generalized hypergeometric function

$$_2F_2(a, b, c, d; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (d)_n} \frac{x^n}{n!}, \qquad (3.44)$$

where $(a)_n = a(a+1)\ldots(a+n-1)$. Using

$$_2F_2(1, 1, 3, 4; x) = 12 \sum_{n=0}^{\infty} \frac{n!}{(n+2)! (n+3)!} x^n, \qquad (3.45)$$

we obtain the formal expression

$$\widehat{\Xi}_1(z, S, \Upsilon) = \nu z^2 I_1(S) + \frac{\nu^2 z^4}{4\pi i g} \oint \frac{d\hbar}{\hbar} {}_2F_2(1, 1, 3, 4; -\frac{\nu z^2}{\hbar}) I_2 \# g, \qquad (3.46)$$

where $I_2 \# g$ is given as in (C.17), with $\hbar \in \mathbb{C}$ a deformation parameter, and the integration contour encloses the origin. It would be interesting to further study the relation of $\widehat{\Xi}_1(z, S, \Upsilon)$ with formal deformation.

3.2.3 Deformed Calogero model

We return to (3.17) and deduce a Hamilton-Jacobi equation from it, as follows.

Taking the square of the second equation in (3.17), and suitably combining it with the first equation in (3.17), we obtain

$$\left[8 \Upsilon^2 \alpha \frac{\partial \widehat{\Omega}}{\partial S} + \left(\frac{\partial \widehat{\Omega}}{\partial z} - \frac{2\Upsilon \gamma}{z} \right)^2 \right]' = \Delta^2 \left[8 \Upsilon^2 \alpha \frac{\partial \widehat{\Omega}}{\partial S} + \left(\frac{\partial \widehat{\Omega}}{\partial z} - \frac{2\Upsilon \gamma}{z} \right)^2 \right]. \qquad (3.47)$$

This implies that the combination $\left[8 \Upsilon^2 \alpha \, \partial\widehat{\Omega}/\partial S + (\partial\widehat{\Omega}/\partial z - 2\Upsilon \gamma/z)^2 \right]$ transforms as a modular form of weight 2 (with a trivial multiplier system) under $\Gamma_0(2)_S$ transformations. We denote this combination by $V(S, z)/z^2$, where $V(S, z)$ denotes a modular invariant function,

$$8 \Upsilon^2 \alpha \frac{\partial \widehat{\Omega}}{\partial S} = -\left[\left(\frac{\partial \widehat{\Omega}}{\partial z} - \frac{2\Upsilon \gamma}{z} \right)^2 - \frac{V(S, z)}{z^2} \right]. \qquad (3.48)$$

Then, inserting the expression (3.15) for $\widehat{\Omega}$ into (3.48) gives

$$\nu \left(\frac{\partial \Xi}{\partial S} + \frac{1}{2\gamma} I_1(S) \, z \, \frac{\partial \Xi}{\partial z} \right) = -\frac{1}{8\gamma} \left(\frac{\partial \Xi}{\partial z} \right)^2 + \frac{1}{2z} \frac{\partial \Xi}{\partial z} - \nu^2 \, I_2(S) \, z^2$$
$$- \left(\frac{4 \, (\gamma \, \Upsilon)^2 - V(S,z)}{8\gamma \, \Upsilon^2 \, z^2} \right) , \qquad (3.49)$$

where $\nu = \alpha \Upsilon / \gamma$, as before. By comparing this equation with (3.28), we infer

$$V(S,z) = 4 \, (\gamma \, \Upsilon)^2 + 2 \, \Upsilon^2 \, z^2 \left(\left(\frac{\partial \Xi}{\partial z} \right)^2 + 2\nu^2 \, I_2(S) \, z^3 \, \frac{\partial \Xi}{\partial z} \right) . \qquad (3.50)$$

Proposition: the partial differential equation (3.48) is a Hamilton-Jacobi equation with Hamilton's principal function $\mathcal{S}(t,z)$ given by

$$\mathcal{S}(t,z) = \widehat{\Omega}(t,z) - 2\Upsilon \, \gamma \, \ln z \qquad (3.51)$$

with $t = S/(4\Upsilon^2 \, \alpha)$. While $\mathcal{S}$ is not invariant under $\Gamma_0(2)$ transformations,

$$\mathcal{S}(t,z) \to \mathcal{S}(t,z) - \frac{z^2}{2\Delta} \, \frac{\partial \Delta}{\partial t} , \qquad (3.52)$$

the combination $\mathcal{S} - \mathcal{S}_1$, with $\mathcal{S}_1(t,z) = \frac{1}{2} \frac{z^2}{t+\bar{t}}$, is invariant.

Proof. Setting $t = S/(4\Upsilon^2 \, \alpha)$ we write (3.48) as

$$\frac{\partial(\widehat{\Omega} - 2\Upsilon \, \gamma \, \ln z)}{\partial t} = -\frac{1}{2} \left[\left(\frac{\partial(\widehat{\Omega} - 2\Upsilon \, \gamma \, \ln z)}{\partial z} \right)^2 - \frac{V(t,z)}{z^2} \right] . \qquad (3.53)$$

Using (3.51), we obtain

$$-\frac{\partial \mathcal{S}}{\partial t} = \mathcal{H} \left(\frac{\partial \mathcal{S}}{\partial z}, z, t \right) \qquad (3.54)$$

with

$$\mathcal{H}(p,z,t) = \frac{1}{2} \left(p^2 - \frac{V(t,z)}{z^2} \right) . \qquad (3.55)$$

Next, using (3.10) and (3.15), we infer the transformation behaviour (3.52). We can compensate for the term proportional to $\partial \Delta / \partial t$ on the right hand side of (3.52) by considering the $\Gamma_0(2)$ invariant combination $\mathcal{S} - \mathcal{S}_1$, where

$$\mathcal{S}_1 = \frac{1}{2} \frac{z^2}{t + \bar{t}} , \qquad (3.56)$$

which satisfies

$$-\frac{\partial \mathcal{S}_1}{\partial t} = \frac{1}{2} \left(\frac{\partial \mathcal{S}_1}{\partial z} \right)^2 . \qquad (3.57)$$

When t and z are real, $\mathcal{S}_1$ describes Hamilton's principal function for a free particle.

$\square$

Using (3.15), we note that the combination $\mathcal{S} - \mathcal{S}_1$, when expressed in terms of S and z, takes the form

$$\mathcal{S}(S,z) - \mathcal{S}_1(S,z) = \Upsilon \left[\omega(S) - 2\gamma \ln z + \nu \, z^2 \, \hat{I}_1(S,\bar{S}) + \Xi(z,S,\Upsilon) \right] , \qquad (3.58)$$

with $\hat{I}_1$ given in (3.21).

When truncating $V(S,z)$ in (3.50) to the constant term, $V(S,z) = 4\,(\gamma\,\Upsilon)^2$, the Hamiltonian (3.55) becomes $\mathcal{H}(p,z) = \frac{1}{2}(p^2 - g^2/z^2)$, with $g^2 = 4(\Upsilon\,\gamma)^2 > 0$. For real z, $\mathcal{H}(p,z)$ is the conformal mechanics Hamiltonian of [57] (with negative coupling constant, though), also related to the two-particle rational Calogero model [32]. When $V(S,z) = 4\,(\gamma\,\Upsilon)^2$, the PDE (3.49), subject to $\Xi|_{z=0} = 0$ and $(\partial_{z^2}\Xi)|_{z=0} = 0$, is solved by (3.36) in the approximation that we drop the term quadratic in $\partial\Xi/\partial z$.

On the other hand, the full $V(S,z)$ in (3.50) results in a time-dependent Hamiltonian (3.55), which can be viewed as a time-dependent deformation of the rational Calogero Hamiltonian by an infinite set of terms involving powers of $I_n(S)$, starting with terms quadratic in $I_n(S)$.

3.3 Charge bilinears in the STU model

In the STU model, BPS black holes may carry electric/magnetic charges (q_I, p^I), with $I = 0,1,2,3$. Under $\Gamma_0(2)_S$ they transform as follows [49],

$$
\begin{aligned}
p^0 &\to d\,p^0 + c\,p^1 , & q_0 &\to a\,q_0 - b\,q_1 , \\
p^1 &\to a\,p^1 + b\,p^0 , & q_1 &\to d\,q_1 - c\,q_0 , \\
p^2 &\to d\,p^2 - c\,q_3 , & q_2 &\to a\,q_2 - b\,p^3 , \\
p^3 &\to d\,p^3 - c\,q_2 , & q_3 &\to a\,q_3 - b\,p^2 .
\end{aligned}
\qquad (3.59)
$$

These charges can be assembled into three charge bilinears,

$$
\begin{aligned}
n &= -q_0 p^1 + q_2 q_3 , \\
m &= p^0 q_1 + p^2 p^3 , \\
l &= q_0 p^0 - q_1 p^1 + q_2 p^2 + q_3 p^3 .
\end{aligned}
\qquad (3.60)
$$

These bilinears transform as a triplet under $\Gamma_0(2)_S$ [49],

$$
\begin{pmatrix} n \\ m \\ l \end{pmatrix} \to
\begin{pmatrix} a^2 & b^2 & -ab \\ c^2 & d^2 & -cd \\ -2ac & -2bd & ad+bc \end{pmatrix}
\begin{pmatrix} n \\ m \\ l \end{pmatrix} ,
\qquad (3.61)
$$

and the $\Gamma_0(2)$ invariant norm of this vector is $4nm - l^2$.

3.4 Large single-center BPS black holes

Now we turn to the computation of the quantum entropy function (2.4) for large single-center BPS black holes in the STU model.

Definition: a large single-center BPS black hole is a dyonic spherically symmetric BPS black hole carrying electric/magnetic charges (q_I, p^I) such that the charge bilinears m, n and the charge combination $4nm - l^2$ are positive, which ensures that the black hole has a non-vanishing horizon area, proportional to $\sqrt{4nm - l^2}$, at the two-derivative level.

At the two-derivative level, the horizon area A_{BH} equals $A_{\text{BH}} = 4\pi(S + \bar{S})\, m$ [50], where S denotes the value at the horizon. Hence $S + \bar{S} > 0$ for a large BPS black hole, which implies that p^0 and p^1 cannot be simultaneously zero.[7]

The semi-classical macroscopic entropy $\mathcal{S}_{\text{BH}}$ of a BPS black hole equals $\mathcal{S}_{\text{BH}} = \pi H(\phi_*, p, q)$, with $H(\phi_*, p, q)$ given by (2.26). The quantum entropy function computes corrections to the semi-classical entropy.

To compute the quantum entropy function, we will work in a regime where T and U are large, so as to be able to use (3.15). We will expand $H(\phi, p, q)$ given in (2.19) around large values $\text{Re}\,T_0, \text{Re}\,U_0$ defined below in (3.67). These values, which depend on Y^0 and on S, are invariant under $\Gamma_0(2)_S$ transformations. When evaluated at the horizon of the BPS black hole, T_0 and U_0 become entirely expressed in terms of the charges carried by the black hole, and this implies that we will have to choose the charges of the black hole in such a way as to ensure that the horizon values of $\text{Re}\,T_0$ and $\text{Re}\,U_0$ are large.

We will now evaluate $H(\phi, p, q)$ on the attractor values ϕ_*^2, ϕ_*^3 that satisfy

$$F_a - \bar{F}_a = iq_a\,, \quad a = 2, 3\,. \tag{3.62}$$

We will denote the resulting expression by $H(\tau_1, \tau_2, p, q)$, which we subsequently expand around large values $\text{Re}\,T_0, \text{Re}\,U_0$. In doing so, we will keep all the charges, including p^0.

3.4.1 $H(\tau_1, \tau_2, p, q)$

We set $Y^2 = \frac{1}{2}(\phi^2 + ip^2)$ and $Y^3 = \frac{1}{2}(\phi^3 + ip^3)$, and we solve the attractor equations (3.62). We obtain

$$\phi_*^2 = \frac{2}{S + \bar{S}} \left(-q_3 - \frac{i}{2}(S - \bar{S})\, p^2 - 2i\, \Delta_U \right)\,,$$
$$\phi_*^3 = \frac{2}{S + \bar{S}} \left(-q_2 - \frac{i}{2}(S - \bar{S})\, p^3 - 2i\, \Delta_T \right)\,, \tag{3.63}$$

where

$$\Delta_T = \frac{1}{Y^0} \frac{\partial \Omega}{\partial T} - \text{c.c}\,,$$
$$\Delta_U = \frac{1}{Y^0} \frac{\partial \Omega}{\partial U} - \text{c.c}\,, \tag{3.64}$$

and we note that Δ_T and Δ_U depend on Y^0, S (and on their complex conjugates) as well as on ϕ_*^2, ϕ_*^3. This yields

$$T = -i\frac{Y_*^2}{Y^0} = -i\frac{(\phi_*^2 + ip^2)}{2\, Y^0} = T_0 + t\,, \quad U = -i\frac{Y_*^3}{Y^0} = -i\frac{(\phi_*^3 + ip^3)}{2\, Y^0} = U_0 + u\,, \tag{3.65}$$

---

[7]Imposing the magnetic attractor equation (2.2) gives $\text{Re}\,S = (p^1\, \phi^0 - p^0\, \phi^1)/|\phi^0 + ip^0|^2$.

where

$$T_0 = -\frac{i}{Y^0(S+\bar{S})}\left(-q_3 + i\bar{S}\,p^2\right),$$

$$U_0 = -\frac{i}{Y^0(S+\bar{S})}\left(-q_2 + i\bar{S}\,p^3\right),$$

$$t = -\frac{2}{Y^0(S+\bar{S})}\,\Delta_U\,,$$

$$u = -\frac{2}{Y^0(S+\bar{S})}\,\Delta_T\,. \tag{3.66}$$

We infer

$$T_0 + \bar{T}_0 = \frac{p^1 p^2 + p^0 q_3}{|Y^0|^2(S+\bar{S})}\,,$$

$$U_0 + \bar{U}_0 = \frac{p^1 p^3 + p^0 q_2}{|Y^0|^2(S+\bar{S})}\,. \tag{3.67}$$

Note that T_0 and U_0 are inert under $\Gamma_0(2)_S$ transformations, as can be checked by using the transformation rules (3.59) and (3.4); t and u, on the other hand, are not inert under $\Gamma_0(2)_S$ transformations.

Next, we evaluate $H(\phi,p,q)$ given in (2.19) at the values T,U in (3.65). To this end, we use the parametrization [48]

$$Y^0 = \frac{p^1 + i\bar{S}p^0}{S+\bar{S}}\,, \quad Y^1 = \frac{iSp^1 - |S|^2 p^0}{S+\bar{S}}\,. \tag{3.68}$$

We will also set

$$\tau = iS = \tau_1 + i\tau_2\,, \quad \bar{\tau} = -i\bar{S} = \tau_1 - i\tau_2 \tag{3.69}$$

in the following. Then, $H(\phi,p,q)$ becomes

$$H(\tau_1,\tau_2,p,q) = \frac{n + l\tau_1 + m\tau_1^2 + m\tau_2^2}{\tau_2} + 4\left(\Omega + \bar{\Omega}\right) + 4\frac{\Delta_T \Delta_U}{\tau_2}\,, \tag{3.70}$$

where we made use of the charge bilinears (3.60). This yields

$$\begin{aligned}
H(\tau_1,\tau_2,p,q) = {}& \frac{n + l\tau_1 + m\tau_1^2 + m\tau_2^2}{\tau_2} + 4\left(\Omega + \bar{\Omega}\right) \\
& + \frac{4}{\tau_2\,(Y^0)^2}\frac{\partial\Omega}{\partial T}\frac{\partial\Omega}{\partial U} + \frac{4}{\tau_2\,(\bar{Y}^0)^2}\frac{\partial\bar{\Omega}}{\partial\bar{T}}\frac{\partial\bar{\Omega}}{\partial\bar{U}} \\
& - \frac{4}{\tau_2\,|Y^0|^2}\left(\frac{\partial\Omega}{\partial T}\frac{\partial\bar{\Omega}}{\partial\bar{U}} + \frac{\partial\bar{\Omega}}{\partial\bar{T}}\frac{\partial\Omega}{\partial U}\right).
\end{aligned} \tag{3.71}$$

The terms on the right hand side are evaluated at ϕ_*^2, ϕ_*^3. The first term in this expression is invariant under $\Gamma_0(2)_S$. When extremized with respect to τ_1 and τ_2 (cf. (3.92)), it yields the entropy of a large BPS black hole at the two-derivative level. The terms proportional to $1/|Y^0|^2$ are also invariant under $\Gamma_0(2)_S$, since both $\partial\Omega/\partial T$ and $\partial\Omega/\partial U$ are invariant [49].

Now we expand $H(\tau_1, \tau_2, p, q)$ around large values of $\mathrm{Re}\, T_0, \mathrm{Re}\, U_0$. First, we expand

$$\omega(T) = \omega(T_0) - \frac{2}{Y^0(S + \bar{S})} \Delta_U \frac{\partial \omega}{\partial T}|_{T_0} + \dots,$$
$$\omega(U) = \omega(U_0) - \frac{2}{Y^0(S + \bar{S})} \Delta_T \frac{\partial \omega}{\partial U}|_{U_0} + \dots, \tag{3.72}$$

and drop terms that involve higher order derivatives of ω with respect to T and U in view of (3.11). Inserting this in the combination

$$\Omega + \frac{2}{(S + \bar{S})\,(Y^0)^2} \frac{\partial \Omega}{\partial T} \frac{\partial \Omega}{\partial U} \tag{3.73}$$

that appears in (3.71), we obtain, using (3.15),

$$\Omega + \frac{2}{(S + \bar{S})\,(Y^0)^2} \frac{\partial \Omega}{\partial T} \frac{\partial \Omega}{\partial U} = \Upsilon \left(\omega(S) + \omega(T_0) + \omega(U_0) \right) \tag{3.74}$$

$$+ \frac{\Upsilon^2}{(Y^0)^2} \frac{\alpha}{\gamma} \hat{I}_1(S, \bar{S}) + \Upsilon \, \Xi(Y^0, S, \Upsilon)$$

$$+ \frac{2\Upsilon\bar{\Upsilon}}{(S + \bar{S})\,|Y^0|^2} \left(\frac{\partial \omega}{\partial T} \frac{\partial \bar{\omega}}{\partial \bar{U}} + \frac{\partial \bar{\omega}}{\partial \bar{T}} \frac{\partial \omega}{\partial U} \right) |_{T_0, U_0},$$

where $\hat{I}_1(S, \bar{S})$ is given in (3.21).

Inserting the above into (3.71) gives

$$H(\tau_1, \tau_2, p, q) = \frac{n + l\tau_1 + m\tau_1^2 + m\tau_2^2}{\tau_2} + 4\Big(\Upsilon \left(\omega(T_0) + \omega(U_0) \right) + \text{c.c.} \Big)$$

$$+ 4\Big(\Upsilon \, \omega(\tau) + \text{c.c.} \Big)$$

$$+ 4\left(\frac{\Upsilon^2}{(Y^0)^2} \frac{\alpha}{\gamma} \hat{I}_1(\tau, \bar{\tau}) + \Upsilon \, \Xi(Y^0, \tau, \Upsilon) + \text{c.c.} \right)$$

$$+ \frac{8\,\alpha\,\Upsilon\bar{\Upsilon}}{\tau_2\,|Y^0|^2}, \tag{3.75}$$

with Y^0 expressed in terms of $\tau = iS$ and charges (p^0, p^1) as in (3.68), and where in the last line we replaced $\left(\frac{\partial \omega}{\partial T} \frac{\partial \bar{\omega}}{\partial \bar{U}} + \frac{\partial \bar{\omega}}{\partial \bar{T}} \frac{\partial \omega}{\partial U} \right) |_{T_0, U_0}$ by 2α in view of (3.13).

Summarizing, (3.75) gives the value of $H(\phi, p, q)$ evaluated at ϕ_*^2 and ϕ_*^3, in the approximation where the real part of T_0, U_0 is taken to be large, so that terms involving higher derivatives of ω with respect to T and U can be dropped.

In this approximation, all the terms in (3.75) are invariant under $\Gamma_0(2)_S$ transformations, except for the term in the second line, whose duality invariance can be repaired by

adding a term proportional to $\ln Y^0$ and its complex conjugate to $H(\tau_1, \tau_2, p, q)$,

$$
\begin{aligned}
H(\tau_1, \tau_2, p, q) + 8\gamma \Big(\Upsilon \ln Y^0 + \text{c.c.} \Big) = \; & \frac{n + l\tau_1 + m\tau_1^2 + m\tau_2^2}{\tau_2} \\
& + 4\Big(\Upsilon\left(\omega(T_0) + \omega(U_0)\right) + \text{c.c.} \Big) \\
& + 4\Big(\Upsilon\left(\omega(\tau) + 2\gamma \ln Y^0\right) + \text{c.c.} \Big) \\
& + 4\left(\frac{\Upsilon^2}{(Y^0)^2}\frac{\alpha}{\gamma}\hat{I}_1(\tau, \bar{\tau}) + \Upsilon\,\Xi(Y^0, \tau, \Upsilon) + \text{c.c.} \right) \\
& + \frac{8\,\alpha\,\Upsilon\bar{\Upsilon}}{\tau_2\,|Y^0|^2}\,.
\end{aligned}
\tag{3.76}
$$

We recall that for BPS black holes the horizon value of Υ is real and given by $\Upsilon = -64$, and that

$$
4\pi\,\Upsilon\,\gamma = 1\,.
\tag{3.77}
$$

The combination on the left hand side of (3.76) is thus $H(\tau_1, \tau_2, p, q) + (4/\pi)\ln|Y^0|$. This combination will play a role in the quantum entropy function below, cf. (3.100).

3.4.2 Evaluating $W(q, p)$ beyond saddle point approximation

In (2.24) we displayed the value of the quantum entropy function $W(q, p)$ in a saddle point approximation. Now, we proceed with the evaluation of $W(q, p)$ beyond the saddle point approximation. In doing so, we will impose approximations that we will clearly delineate in what follows.

We decompose

$$
Y^0 = \frac{1}{2}\left(\phi^0 + ip^0\right)\,, \quad Y^1 = \frac{1}{2}\left(\phi^1 + ip^1\right)\,.
\tag{3.78}
$$

Using (3.68) and (3.69) we infer

$$
\phi^0 = \frac{p^1 - \tau_1\,p^0}{\tau_2}\,, \quad \phi^1 = -p^0\,\tau_2 + \frac{\tau_1}{\tau_2}\left(p^1 - \tau_1\,p^0\right)\,,
\tag{3.79}
$$

to obtain

$$
d\phi^0 \wedge d\phi^1 = \frac{\left((\phi^0)^2 + (p^0)^2\right)}{\tau_2}\,d\tau_1 \wedge d\tau_2 = \frac{4|Y^0|^2}{\tau_2}\,d\tau_1 \wedge d\tau_2\,.
\tag{3.80}
$$

Then, the quantum entropy function reads (using the approximate measure factor (2.18) with $\chi = 0$)

$$
W(q, p) = 4 \int \frac{d\tau_1\,d\tau_2}{\tau_2}\,d\phi^a\,e^{\pi[4\,\text{Im}F(\phi + ip) - q\cdot\phi]}\,|Y^0|^4\,e^{-\mathcal{K}^{(0)}}\,, \quad a = 2, 3\,.
\tag{3.81}
$$

Next, we integrate out ϕ^2 and ϕ^3, by expanding the exponent $4\,\text{Im}F(\phi + ip) - q\cdot\phi$ around the attractor values ϕ_*^2 and ϕ_*^3 computed in (3.63), and retaining only quadratic fluctuations in ϕ^2 and ϕ^3. The associated quadratic form takes the form given in (2.21), with the indices I, J restricted to $I, J = 2, 3$. We approximate the resulting fluctuation determinant by

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replacing F by $F^{(0)}$, in which case it takes the value τ_2^2. Thus, in this approximation, the Gaussian integration[8] over fluctuations in ϕ^2, ϕ^3 yields a factor

$$1/\tau_2\,, \tag{3.82}$$

and we obtain the following approximate expression for the quantum entropy function,

$$W(q,p) = 4 \int \frac{d\tau_1\, d\tau_2}{\tau_2^2}\; e^{\pi\, H(\tau_1,\tau_2,p,q)}\, |Y^0|^4\, e^{-\mathcal{K}^{(0)}}\,, \tag{3.83}$$

with $H(\tau_1,\tau_2,p,q)$ given by (3.71). Here, $e^{-\mathcal{K}^{(0)}}$ is evaluated at the values T, U given in (3.65).

To proceed, we perform three more approximations. First, we approximate $e^{-\mathcal{K}^{(0)}}$ in (3.83) by replacing the values T, U by the values T_0, U_0 given in (3.67),

$$e^{-\mathcal{K}^{(0)}(\tau_1,\tau_2,T_0,U_0)} = \frac{(p^1 p^2 + p^0 q_3)(p^1 p^3 + p^0 q_2)}{2|Y^0|^2 \tau_2}\,. \tag{3.84}$$

Using

$$(p^1 p^2 + p^0 q_3)(p^1 p^3 + p^0 q_2) = (p^1)^2\, m + p^0 p^1\, l + (p^0)^2\, n\,, \tag{3.85}$$

where m, l, n denote the charge bilinears introduced in (3.60), we obtain

$$|Y^0|^4\, e^{-\mathcal{K}^{(0)}(\tau_1,\tau_2,T_0,U_0)} = |p^1 - \tau p^0|^2\, \frac{[(p^1)^2\, m + p^0 p^1\, l + (p^0)^2\, n]}{8\tau_2^3}\,, \tag{3.86}$$

and hence,

$$W(q,p) = \frac{1}{2} \int \frac{d\tau_1\, d\tau_2}{\tau_2^5}\; e^{\pi\, H(\tau_1,\tau_2,p,q)}\, |p^1 - \tau p^0|^2 \left((p^1)^2\, m + p^0 p^1\, l + (p^0)^2\, n \right). \tag{3.87}$$

Using

$$(p^1)^2\, m = |p^1 - \tau p^0|^2\, m + 2\tau_1\, p^0 p^1\, m - (p^0)^2(\tau_1^2 + \tau_2^2)\, m\,, \tag{3.88}$$

we obtain

$$(p^1)^2\, m + p^0 p^1\, l + (p^0)^2\, n = |p^1 - \tau p^0|^2\, m + p^0 p^1\, (l + 2\tau_1\, m) + (p^0)^2 \left(n - m(\tau_1^2 + \tau_2^2) \right). \tag{3.89}$$

Hence we write (3.87) as

$$W(q,p) = \frac{1}{2} \int \frac{d\tau_1\, d\tau_2}{\tau_2^5}\; e^{\pi\, H(\tau_1,\tau_2,p,q)}\, |p^1 - \tau p^0|^2$$
$$\left[|p^1 - \tau p^0|^2\, m + p^0 p^1\, (l + 2\tau_1\, m) + (p^0)^2 \left(n - m(\tau_1^2 + \tau_2^2) \right) \right]. \tag{3.90}$$

Now we note that the two combinations

$$l + 2\tau_1\, m\,, \quad n - m(\tau_1^2 + \tau_2^2) \tag{3.91}$$

---

[8]In doing so, we view (ϕ^2, ϕ^3) as local coordinates on $\mathbb{C}^2$, and we choose an appropriate path of integration to obtain a well defined Gaussian integral.

vanish at the extremum of the combination $(n + l\tau_1 + m\tau_1^2 + m\tau_2^2)/\tau_2$ that appears in the exponential of $H(\tau_1, \tau_2, p, q)$ in (3.90). This yields the attractor value for τ at the two-derivative level,

$$\tau_1^* = -\frac{l}{2m}, \quad \tau_2^* = \sqrt{\frac{4nm - l^2}{4m^2}} > 0\,. \tag{3.92}$$

Therefore, imposing the vanishing of (3.91) in (3.90) leads to the approximate result

$$W(q, p) = \frac{1}{2} \int \frac{d\tau_1 \, d\tau_2}{\tau_2^5} \, e^{\pi\, H(\tau_1, \tau_2, p, q)} |p^1 - \tau p^0|^4 \, m\,. \tag{3.93}$$

The third approximation consists in replacing $H(\tau_1, \tau_2, p, q)$ in (3.71) by (3.75). This is achieved by taking $\mathrm{Re}\, T_0, \mathrm{Re}\, U_0$ to be large. Since T_0, U_0 depend on τ, cf. (3.66), we replace τ by the saddle point values (3.92) in the expression for T_0, U_0. Thus, from now on, T_0, U_0 will refer to the attractor values of T, U at the two-derivative level, which are entirely expressed in terms of the charges of the BPS black hole. Then, requiring the real part of T_0, U_0 to be large, translates into a condition on the values of the charges.

Summarising, under these approximations, (3.93) becomes

$$W(q, p) = \frac{1}{2} e^{4\pi[\Upsilon(\omega(T_0) + \omega(U_0)) + c.c.]} \int \frac{d\tau_1 \, d\tau_2}{\tau_2^5} \, e^{\frac{\pi}{\tau_2}(n + l\tau_1 + m\tau_1^2 + m\tau_2^2) + \pi\Lambda} |p^1 - \tau p^0|^4 \, m\,, \tag{3.94}$$

where we introduced

$$\Lambda = 4\left(\Upsilon\, \omega(\tau) + \frac{\Upsilon^2}{(Y^0)^2} \frac{\alpha}{\gamma} \hat{I}_1(\tau, \bar{\tau}) + \Upsilon\, \Xi(Y^0, \tau, \Upsilon) + c.c.\right) + \frac{8\alpha\, \Upsilon\bar{\Upsilon}}{\tau_2 \, |Y^0|^2}\,, \tag{3.95}$$

and where we recall that Y^0 is expressed in terms of p^0, p^1 and τ through (3.68).

In what follows, we will use the approximate result (3.94) as a starting point for various considerations. One should keep in mind that there are subleading corrections to (3.94) that will not be considered in this paper.

The integrand in (3.94) is not invariant under $\Gamma_0(2)$-transformations of (τ_1, τ_2). To obtain an integrand that is invariant under $\Gamma_0(2)$-transformations, we perform a rewriting of (3.94) following an approach given in [36, 37]. This requires a certain assumption, as follows. We recall that the measure (2.18) that we used in (3.81) is an approximate measure. Following [36, 37], let us assume that there are subleading corrections to the measure such that m in (3.94) gets replaced by the combination $\left(m + \frac{1}{2}\frac{d\Lambda}{d\tau_2} + \frac{1}{2\pi}\frac{d\ln|p^1 - \tau p^0|^4}{d\tau_2}\right)$, in which case (3.94) becomes

$$W(q, p) = \frac{1}{2} e^{4\pi[\Upsilon(\omega(T_0) + \omega(U_0)) + c.c.]} \int \frac{d\tau_1 \, d\tau_2}{\tau_2^5} \, e^{\frac{\pi}{\tau_2}(n + l\tau_1 + m\tau_1^2 + m\tau_2^2) + \pi\Lambda} |p^1 - \tau p^0|^4$$

$$\left(m + \frac{1}{2}\frac{d\Lambda}{d\tau_2} + \frac{1}{2\pi}\frac{d\ln|p^1 - \tau p^0|^4}{d\tau_2}\right)\,. \tag{3.96}$$

Now, consider the integrand of (3.96) and compute

$$\frac{d}{d\tau_2}\left(\frac{1}{\tau_2^5}\,e^{\frac{\pi}{\tau_2}(n+l\tau_1+m\tau_1^2+m\tau_2^2)+\pi\Lambda}\,|p^1-\tau p^0|^4\right) \tag{3.97}$$

$$=\frac{\pi}{\tau_2^5}\left(2m+\frac{d\Lambda}{d\tau_2}+\frac{1}{\pi}\frac{d\ln|p^1-\tau p^0|^4}{d\tau_2}\right)e^{\frac{\pi}{\tau_2}(n+l\tau_1+m\tau_1^2+m\tau_2^2)+\pi\Lambda}\,|p^1-\tau p^0|^4$$

$$-\frac{1}{\tau_2^6}\left(5+\frac{\pi}{\tau_2}(n+l\tau_1+m\tau_1^2+m\tau_2^2)\right)e^{\frac{\pi}{\tau_2}(n+l\tau_1+m\tau_1^2+m\tau_2^2)+\pi\Lambda}\,|p^1-\tau p^0|^4\,.$$

Then, dropping an overall constant, we obtain for (3.96),

$$W(q,p)=\;e^{4\pi[\Upsilon(\omega(T_0)+\omega(U_0))+c.c.]} \tag{3.98}$$

$$\int\frac{d\tau_1\,d\tau_2}{\tau_2^6}\left(5+\frac{\pi}{\tau_2}(n+l\tau_1+m\tau_1^2+m\tau_2^2)\right)e^{\frac{\pi}{\tau_2}(n+l\tau_1+m\tau_1^2+m\tau_2^2)+\pi\Lambda}|p^1-\tau p^0|^4\,,$$

where we discarded the total derivative term in (3.97). We write (3.98) as

$$W(q,p)=\int\frac{d\tau_1\,d\tau_2}{\tau_2^2}\,e^{\Psi(\tau_1,\tau_2,p,q)}\,, \tag{3.99}$$

where

$$e^{\Psi(\tau_1,\tau_2,p,q)}=\left(5+\frac{\pi}{\tau_2}(n+l\tau_1+m\tau_1^2+m\tau_2^2)\right)e^{\pi H(\tau_1,\tau_2,p,q)+4\ln|Y^0(\tau_1,\tau_2,p^0,p^1)|}\,, \tag{3.100}$$

with the combination $\pi H(\tau_1,\tau_2,p,q)+4\ln|Y^0(\tau_1,\tau_2,p^0,p^1)|$ given in (3.76) (recall that $4\pi\Upsilon\gamma=1$). Note that Ψ is invariant under $\Gamma_0(2)_S$ transformations.

The purpose of the various manipulations described above was to bring $W(q,p)$ into the form (3.99), which is invariant under $\Gamma_0(2)_S$ transformations. This was achieved by subtracting a total derivative term from (3.96). The form of $W(q,p)$ given in (3.99) will provide the link with the microstate counting proposal that we will discuss in the next section. Thenceforth, we will refer to (3.99) as the quantum entropy function.

To fully specify $W(q,p)$, we also need to choose an integration contour $\mathcal{C}$ that passes through (τ_1^*,τ_2^*) given in (3.92). This requires extending $(\tau_1,\tau_2)\in\mathbb{R}^2$ to $(\tau_1,\tau_2)\in\mathbb{C}^2$. The contour $\mathcal{C}$ will be discussed in subsection 3.5, using a construction given in [38]. We thus have the following proposition.

Proposition: for large black holes, and for large attractor values T_0,U_0, the quantum entropy function $W(q,p)$ of the STU model is approximately given by

$$W(q,p)=\int_{\mathcal{C}}\frac{d\tau_1 d\tau_2}{\tau_2^2}\,e^{\Psi(\tau_1,\tau_2,p,q)}\,. \tag{3.101}$$

Finally, we note the following behaviour of the quantum entropy function (3.101):

Proposition: the quantum entropy function (3.101) is encoded in the function

$$\hat{F}(Y,\Upsilon)=F(Y,\Upsilon)+4i\,\Upsilon\gamma\ln Y^0\,, \tag{3.102}$$

where $F(Y, \Upsilon)$ denotes the Wilsonian function

$$
\begin{aligned}
F(Y, \Upsilon) &= F^{(0)}(Y) + 2i\Omega(Y, \Upsilon) \\
&= -\frac{Y^1 Y^2 Y^3}{Y^0} + 2i\Upsilon \left[\omega(S) + \omega(T) + \omega(U) + \frac{\Upsilon}{(Y^0)^2} \frac{\alpha}{\gamma} I_1(S) + \Xi(Y^0, S, \Upsilon) \right].
\end{aligned}
\tag{3.103}
$$

The function $\hat{F}$ is the one that was obtained recently in [21] using the duality symmetries of the model, in the limit of large T, U, see above (3.6).

Proof. It is straightforward to verify that for large values of T_0, U_0, the quantum entropy function (3.101) takes the form

$$
W(q, p) = \int_{\mathcal{C}} \frac{d\tau_1\, d\tau_2}{\tau_2^2} \left(5 + \frac{\pi}{\tau_2} (n + l\tau_1 + m\tau_1^2 + m\tau_2^2) \right) e^{\pi \left[4\mathrm{Im}\hat{F}(\phi+ip) - q\cdot\phi \right]}\big|_{\phi_*^2, \phi_*^3}, \quad (3.104)
$$

where ϕ_*^2, ϕ_*^3, given in (3.63), are evaluated at large attractor values of T_0, U_0 (cf. the discussion below (3.93)), and where Y^0 is given by (3.78) and (3.79).

$\square$

3.5 Choice of contour

The quantum entropy function (3.101) has a form that is suggestive of a Siegel modular form. Namely, following [58], we perform the change of variables

$$
\rho \equiv \tau_1 + i\tau_2, \quad \sigma \equiv -\tau_1 + i\tau_2, \tag{3.105}
$$

in which case

$$
\tau_1 = \frac{1}{2}(\rho - \sigma), \quad \tau_2 = -\frac{i}{2}(\rho + \sigma), \tag{3.106}
$$

and

$$
\frac{\pi}{\tau_2} \left(n + l\tau_1 + m\, \tau_1^2 + m\, \tau_2^2 \right) = -\frac{2\pi i}{\rho + \sigma} \left(-n - \frac{1}{2}l(\rho - \sigma) + m\,\rho\sigma \right). \tag{3.107}
$$

Next, introducing $k = 2$, and using $\Upsilon = -64$ as well as $\Upsilon\alpha/\gamma = \pi/4$, we write $W(q, p)$ in (3.101) as

$$
\begin{aligned}
&W(q, p)\, e^{\ln|\vartheta_2(T_0)|^8 \ln|\vartheta_2(U_0)|^8} = \int_{\mathcal{C}} \frac{d\rho\, d\sigma}{(\rho + \sigma)^2} \left(k + 3 - 2\pi i \left[\frac{\rho\sigma}{\rho + \sigma} m - \frac{1}{\rho + \sigma} n - \frac{1}{2}\frac{\rho - \sigma}{\rho + \sigma} l \right] \right) \\
&\exp\left\{ -2\pi i \left[\frac{\rho\sigma}{\rho + \sigma} m - \frac{1}{\rho + \sigma} n - \frac{1}{2}\frac{\rho - \sigma}{\rho + \sigma} l \right] - (k + 2)\ln(\rho + \sigma) + \pi\, \tilde{\Lambda} \right\},
\end{aligned}
\tag{3.108}
$$

up to an overall constant, where

$$
\pi\, \tilde{\Lambda} = -\frac{1}{2}\ln\vartheta_2^8(\rho) - \frac{1}{2}\ln\vartheta_2^8(\sigma) + 2\ln(p^1 - \rho\, p^0) + 2\ln(p^1 + \sigma\, p^0) + D(\rho, \sigma),
$$

$$
\begin{aligned}
D(\rho, \sigma) &= -256\pi \left(\frac{\pi}{4\,(Y^0)^2} \hat{I}_1(\rho, \sigma) + \frac{\pi}{4\,(\tilde{Y}^0)^2} \hat{I}_1(\sigma, \rho) + \Xi(Y^0, \rho) + \Xi(\tilde{Y}^0, \sigma) \right) \\
&\quad + \frac{i\pi}{(\rho + \sigma)\, Y^0(\rho, \sigma)\, \tilde{Y}^0(\rho, \sigma)}.
\end{aligned}
\tag{3.109}
$$

Here, $\Xi(Y^0, \rho)$ is given by (3.25) and (3.26), and

$$Y^0(\rho, \sigma) = i \frac{p^1 + \sigma\, p^0}{\rho + \sigma}\,,$$

$$\tilde{Y}^0(\rho, \sigma) = i \frac{p^1 - \rho\, p^0}{\rho + \sigma} = Y^0(\rho, \sigma) - ip^0\,,$$

$$\hat{I}_1(\rho, \sigma) = i \left(\frac{\partial \omega}{\partial \rho} - \frac{2\gamma}{\rho + \sigma} \right)\,. \tag{3.110}$$

The integrand in (3.108) exhibits a dependence on the parameter $k = 2$ (both in the exponent and in the measure) that is characteristic of a microstate counting formula based on a Siegel modular form Φ_2 of weight $k = 2$ [8], see (4.6) and (4.14) below. Thus, we expect the microstate counting formula for large BPS black holes to involve Φ_2. Therefore, we choose a contour $\mathcal{C}$ in (3.101) that captures the information about the locus of the second-order zero of the Siegel modular form which yields the leading microstate degeneracy. This contour was constructed in [38]. It is given by the image of a surface $\mathcal{B}'$ in the complex (τ_1, τ_2)-plane, as follows.

Consider Siegel's upper half plane $\mathcal{H}_2$ with local coordinates $\tilde{\rho} = \tilde{\rho}_1 + i\tilde{\rho}_2$, $\tilde{\sigma} = \tilde{\sigma}_1 + i\tilde{\sigma}_2$, $\tilde{v} = \tilde{v}_1 + i\tilde{v}_2$, where $\tilde{\rho}_2 > 0$, $\tilde{\sigma}_2 > 0$, $\tilde{\rho}_2\tilde{\sigma}_2 - \tilde{v}_2^2 > 0$, cf. (E.2). The surface $\mathcal{B}' \subset \mathcal{H}_2$ is obtained by intersecting $\tilde{\rho}_2 = \eta_1(\lambda)$, $\tilde{\sigma}_2 = \eta_2(\lambda)$, $\tilde{v}_2 = \eta_3(\lambda)$, with $\tilde{\rho}\tilde{\sigma} - \tilde{v}^2 + \tilde{v} = 0$. The latter describes the locus of the second-order zero of the Siegel modular form that is relevant for the microstate proposal. The intersection yields a family of curves given by [38],

$$\tilde{\rho}_2 = \eta_1(\lambda)\,, \quad \tilde{\sigma}_2 = \eta_2(\lambda)\,, \quad \tilde{v}_2 = \eta_3(\lambda)\,,$$

$$\tilde{\rho}_1 = -\left(\frac{\eta_1(\lambda)\, \tilde{\sigma}_1 - \eta_3(\lambda)\, (2\tilde{v}_1 - 1)}{\eta_2(\lambda)} \right)\,, \tag{3.111}$$

$$\frac{\eta_1(\lambda)}{\eta_2(\lambda)} (\tilde{\sigma}_1)^2 + \left(\tilde{v}_1 - \frac{1}{2} \right)^2 - 2 \frac{\eta_3(\lambda)}{\eta_2(\lambda)} \tilde{\sigma}_1 \left(\tilde{v}_1 - \frac{1}{2} \right) = \frac{1}{4} - \left(\eta_1(\lambda)\eta_2(\lambda) - \eta_3^2(\lambda) \right)\,.$$

The last equation describes an ellipse in the $(\tilde{\sigma}_1, \tilde{v}_1)$-plane so long as $0 \leq \eta_1(\lambda)\eta_2(\lambda) - \eta_3^2(\lambda) \leq 1/4$,

$$\frac{\left(\eta_1(\lambda)\eta_2(\lambda) - \eta_3^2(\lambda) \right)}{\eta_2^2(\lambda)} (\tilde{\sigma}_1)^2 + \left(\tilde{v}_1 - \frac{1}{2} - \frac{\eta_3(\lambda)}{\eta_2(\lambda)} \tilde{\sigma}_1 \right)^2 = \frac{1}{4} - \left(\eta_1(\lambda)\eta_2(\lambda) - \eta_3^2(\lambda) \right)\,. \tag{3.112}$$

The surface $\mathcal{B}'$ is obtained by picking

$$\eta_1(\lambda) = \lambda \frac{n}{\sqrt{4mn - l^2}}\,, \quad \eta_2(\lambda) = \lambda \frac{m}{\sqrt{4mn - l^2}}\,, \quad \eta_3(\lambda) = \lambda \frac{l}{2\sqrt{4mn - l^2}}\,, \tag{3.113}$$

and restricting λ to lie in the range $0 < \epsilon \leq \lambda \leq 1$, so that $0 < \eta_1(\lambda)\eta_2(\lambda) - \eta_3^2(\lambda) = \lambda^2/4 \leq 1/4$. The requirement $\epsilon > 0$ ensures the Siegel upper half plane conditions $\tilde{\rho}_2 > 0$, $\tilde{\sigma}_2 > 0$, $\tilde{\rho}_2\tilde{\sigma}_2 - \tilde{v}_2^2 > 0$. At $\lambda = 1$, the surface $\mathcal{B}'$ shrinks to a point,

$$\tilde{\sigma}_1 = 0\,, \quad \tilde{v}_1 = \frac{1}{2}\,, \quad \tilde{\rho}_1 = 0\,,$$

$$\tilde{\sigma}_2 = \frac{m}{\sqrt{4mn - l^2}}\,, \quad \tilde{v}_2 = \frac{l}{2\sqrt{4mn - l^2}}\,, \quad \tilde{\rho}_2 = \frac{n}{\sqrt{4mn - l^2}}\,, \tag{3.114}$$

while at $\lambda = \epsilon > 0$ the ellipse has maximal radii.

Next, introducing variables (ρ, σ, v) as [5]

$$\rho = \frac{\tilde{\rho}\,\tilde{\sigma} - \tilde{v}^2}{\tilde{\sigma}}\,, \quad \sigma = \frac{\tilde{\rho}\,\tilde{\sigma} - (\tilde{v}-1)^2}{\tilde{\sigma}}\,, \quad v = \frac{\tilde{\rho}\tilde{\sigma} - \tilde{v}^2 + \tilde{v}}{\tilde{\sigma}}\,, \tag{3.115}$$

it follows that on the locus $\tilde{\rho}\tilde{\sigma} - \tilde{v}^2 + \tilde{v} = 0$,

$$\rho = -\frac{\tilde{v}}{\tilde{\sigma}}\,, \quad \sigma = \frac{\tilde{v}-1}{\tilde{\sigma}}\,, \quad v = 0\,. \tag{3.116}$$

Then, using (3.106), we express (τ_1, τ_2) in terms of $(\tilde{\sigma}, \tilde{v})$ on the locus $v = 0$,

$$\tau_1 = \frac{1}{2}(\rho - \sigma) = -\frac{(\tilde{v} - \frac{1}{2})}{\tilde{\sigma}}\,, \quad \tau_2 = -\frac{i}{2}(\rho + \sigma) = \frac{i}{2\,\tilde{\sigma}}\,. \tag{3.117}$$

We parametrize the ellipse (3.112) in terms of an angle $0 \leq \theta < 2\pi$,

$$\tilde{\sigma}_1 = \frac{\sqrt{1-\lambda^2}\,\cos\theta}{2\tau_2^*}\,, \quad \tilde{v}_1 - \frac{1}{2} - \frac{l}{2m}\,\tilde{\sigma}_1 = \frac{\sqrt{1-\lambda^2}\,\sin\theta}{2}\,, \tag{3.118}$$

where τ_2^* denotes the attractor value given in (3.92). Then, using (3.117), we obtain

$$\tau_1 = \frac{1}{\lambda^2 + (1-\lambda^2)\cos^2\theta}\left[\tau_1^*\,\lambda^2 + \tau_1^*\,(1-\lambda^2)\cos^2\theta - \tau_2^*(1-\lambda^2)\sin\theta\cos\theta\right]$$

$$+ i\,\frac{\lambda\sqrt{1-\lambda^2}}{\lambda^2 + (1-\lambda^2)\cos^2\theta}\,\tau_2^*\sin\theta\,,$$

$$\tau_2 = \frac{\tau_2^*}{\lambda^2 + (1-\lambda^2)\cos^2\theta}\left[\lambda + i\,\sqrt{1-\lambda^2}\,\cos\theta\right]\,. \tag{3.119}$$

Since at $\lambda = 1$ (τ_1, τ_2) equals (τ_1^*, τ_2^*), the image of the surface $\mathcal{B}'$ in the complex (τ_1, τ_2)-plane is a surface that passes through the attractor point (3.92).

Next, for (ρ, σ) given in (3.116), we verify that $\mathrm{Im}\,\rho > 0$, $\mathrm{Im}\,\sigma > 0$. The condition $\mathrm{Im}\,\rho > 0$ gives

$$-\frac{1}{2} < \tilde{v}_1 - \frac{1}{2} - \frac{\eta_3(\lambda)}{\eta_2(\lambda)}\,\tilde{\sigma}_1\,, \tag{3.120}$$

while the condition $\mathrm{Im}\,\sigma > 0$ results in

$$\tilde{v}_1 - \frac{1}{2} - \frac{\eta_3(\lambda)}{\eta_2(\lambda)}\,\tilde{\sigma}_1 < \frac{1}{2}\,. \tag{3.121}$$

Inspection of (3.112) shows that both conditions are satisfied, for if $|\tilde{v}_1 - \frac{1}{2} - \frac{\eta_3(\lambda)}{\eta_2(\lambda)}\,\tilde{\sigma}_1| \geq 1/2$, (3.112) does not have a solution for $0 < \eta_1(\lambda)\eta_2(\lambda) - \eta_3^2(\lambda) \leq 1/4$. Then, defining $q = \exp[2\pi i\,\rho]$ and $\tilde{q} = \exp[2\pi i\,\sigma]$, we obtain $|q| < 1$, $|\tilde{q}| < 1$, which is a necessary condition for defining the modular forms $\vartheta_2^8(\rho)$, $\vartheta_2^8(\sigma)$, cf. (4.1). This gives another justification for having to require $\epsilon > 0$.

Let us return to the integrand in (3.108) and analyse the behaviour of the real part of the exponent. The three terms proportional to the charge bilinears, when evaluated on (3.116), yield the contribution $\pi\lambda[m(\tau_1^*)^2 + m(\tau_2^*)^2 + n + l\tau_1^*]/\tau_2^*$ [38], which is finite for $0 < \epsilon \leq \lambda \leq 1$. Now, consider the term

$$2\ln|Y^0\,\tilde{Y}^0| = 2\ln\left|\frac{p^1}{\tau_2} - p^0\,\frac{\tau_1}{\tau_2} + ip^0\right| + 2\ln\left|\frac{p^1}{\tau_2} - p^0\,\frac{\tau_1}{\tau_2} - ip^0\right| \tag{3.122}$$

contained in $\tilde\Lambda$. In the parametrization (3.119), this expression ceases to be defined when $\lambda^2 + (1 - \lambda^2)\cos^2\theta = 0$, which occurs when $\cos\theta = 0$ and $\lambda \to 0$. In this limit, τ_1/τ_2 remains finite, while $|\tau_2| \to \infty$, and hence $2\ln|Y^0 \tilde Y^0| \to -\infty$, which yields an exponentially damped contribution. Next, let us consider the contribution $D(\rho,\sigma)$ in (3.109), which was determined in the context of the Wilsonian effective action as power series in inverse powers of $(Y^0)^2$. This requires taking $|Y^0|$ to be large. To ensure that $|Y^0|$ remains large on the integration contour $\mathcal{C}$, we impose a charge dependent lower bound on ϵ, as follows. Computing $\mathrm{Re}\,Y^0(\rho,\sigma) = \mathrm{Re}\,\tilde Y^0(\rho,\sigma)$ on $\mathcal{C}$, we obtain

$$\mathrm{Re}\,Y^0(\rho,\sigma) = \tilde\sigma_2\, p^1 + \tilde v_2\, p^0\,. \tag{3.123}$$

At $\lambda = \epsilon$, we infer from (3.123) that $\mathrm{Re}\,Y^0 = \epsilon\left(m\,p^1 + \frac{1}{2}\,l\,p^0\right)/\sqrt{4mn - l^2}$. By taking $\mathcal{Q} \equiv |m\,p^1 + \frac{1}{2}\,l\,p^0|/\sqrt{4mn - l^2} \gg 1$ and choosing $\epsilon = 1/\sqrt{\mathcal{Q}}$ we can ensure that $|\mathrm{Re}\,Y^0| \gg 1$. Demanding $\mathcal{Q} \gg 1$ imposes a condition on the charges carried by the BPS black hole. Presumably, were we to have at our disposal the exact expression for the function H, there would be no need for such a charge dependent lower bound on ϵ. Noting that $|\rho + \sigma| = 2|\tau_2| = 2\tau_2^*/\sqrt{\lambda^2 + (1 - \lambda^2)\cos^2\theta} \geq 2\tau_2^* \gg 1$, we see that the first two terms and the last term in the expression for $D(\rho,\sigma)$ are well-behaved on $\mathcal{C}$. $\Xi(Y^0,\rho)$, on the other hand, is given by (3.25). As shown in appendix D, the series converges in an open neighbourhood of $(q,z) = (0,0)$, where $q = \exp[2\pi i\,\rho]$ and $z = 1/Y^0$. To ensure that we work in this neighbourhood, we impose a further condition on the charges, as follows. In the parametrization (3.119), we obtain

$$|q| = e^{-2\pi(\mathrm{Im}\,\tau_1 + \mathrm{Re}\,\tau_2)} = e^{-2\pi\tau_2^*\lambda\left(\sqrt{1-\lambda^2}\,\sin\theta + 1\right)/(\lambda^2 + (1-\lambda^2)\cos^2\theta)}\,. \tag{3.124}$$

To ensure that $|q| \ll 1$ when $\lambda = \epsilon$, we demand $\tau_2^*\epsilon \gg 1$, which results in $(\tau_2^*)^2 \gg \mathcal{Q} \gg 1$, which constitutes a further condition on the charges carried by the BPS black hole. In this way, we conclude that the real part of the exponent of the integrand in (3.108) is well-behaved in the range $\epsilon = 1/\sqrt{\mathcal{Q}} \leq \lambda \leq 1$.

Under $\Gamma_0(2)$-transformations, the critical point (τ_1^*, τ_2^*) gets mapped to a new critical point $(\tilde\tau_1^*, \tilde\tau_2^*)$, given in terms of transformed charge bilinears $\tilde m, \tilde n, \tilde l$. The transformed contour $\tilde{\mathcal{C}}$ will pass through $(\tilde\tau_1^*, \tilde\tau_2^*)$.

4 Microstate proposal

The quantum entropy function computes the macroscopic entropy of a BPS black hole. Ideally, we would like to reproduce it by state counting. Microstate counting formulae for BPS black holes in $N = 4, 8$ superstring theories [1, 3, 5, 9] suggest that the state counting will be based on modular objects. In the $N = 4$ context, these are Siegel modular forms, whose Fourier expansion yields integer coefficients that count microstates of $\frac{1}{4}$ BPS black holes. Encoding microstate degeneracies of BPS black holes in terms of modular forms is a powerful principle that we will also use in the context of the $N = 2$ STU model. This gives a microstate proposal that can be tested by a state counting process as and when it is divised.

As already mentioned in the previous subsection, the approximate result for the quantum entropy function $W(q,p)$ in (3.108) exhibits a dependence on the parameter $k = 2$ that is characteristic of a microstate counting formula based on a Siegel modular form Φ_2 of weight $k = 2$ [8]. Thus, we expect the microstate counting formula for large BPS black holes to involve Φ_2. On the other hand, the integrand in (3.108) also depends on Y^0, on $\tilde{Y}^0$ as well as on $\ln(p^1 - \rho \, p^0)$ and $\ln(p^1 + \sigma \, p^0)$, which means that the counting formula cannot solely be given in terms of Siegel modular form Φ_2. The microstate counting formula will, in particular, have to depend on the charges (p^0, p^1), which transform as a doublet under $\Gamma_0(2)_S$, cf. (3.59).

Let us first focus on the Siegel modular form Φ_2. For a given weight k with respect to a subgroup of the full modular group $\mathrm{Sp}(4, \mathbb{Z})$, there may exist one or more Siegel modular forms. We propose that the Siegel modular form $\Phi_2(\rho, \sigma, v)$ relevant for the microstate counting formula of large BPS black holes in the STU model is the Siegel modular form of weight $k = 2$ briefly discussed in [9] in the context of $N = 4$ BPS black holes, which in the limit $v \to 0$ behaves as

$$\Phi_2(\rho, \sigma, v) = 4\pi^2 \, 2^{-16} \, v^2 \, \vartheta_2^8(\rho) \, \vartheta_2^8(\sigma) + \mathcal{O}(v^4) \,. \tag{4.1}$$

We note that Φ_2 is related to the Siegel modular forms Φ_6 arising in the heterotic $\mathbb{Z}_2$ CHL orbifold model and Φ_{10} arising in the heterotic string theory on T^6 by $\Phi_2 = \Phi_6^2/\Phi_{10}$, up to a normalization constant. This Siegel modular form can be constructed as follows [5].

Proposition: there exists a Siegel modular form $\Phi_2(\rho, \sigma, v)$ of weight $k = 2$, symmetric in ρ and σ, with the property

$$\Phi_2(\rho, \sigma, v) = 4\pi v^2 \, 2^{-16} \, \vartheta_2^8(\rho) \, \vartheta_2^8(\sigma) + \mathcal{O}(v^4) \tag{4.2}$$

as $v \to 0$, that can be constructed by applying a Hecke lift to the Jacobi form

$$\phi_{2,1}(\rho, z) = \frac{\vartheta_1^2(\rho, z)}{\eta^6(\rho)} \, 2^{-8} \, \vartheta_2^8(\rho) \tag{4.3}$$

of weight $k = 2$ and index $m = 1$.

Proof. We refer to appendix G for the proof, which is based on the construction given in [5]. Our proof of the property (4.2) relies on the relation (G.22), which we prove by considering Hecke eigenforms.

Note that under $\Gamma_0(2)$ transformations, $\vartheta_1^2(\rho, z)/\eta^6(\rho)$ transforms as a modular form of weight -2, while $\vartheta_2^8(\rho)$ transforms with weight 4, so that the total weight is $k = 2$. Also note that none of these two factors has a non-trivial multiplier system. $\qed$

As mentioned above, the microstate counting formula for large BPS black holes cannot be solely based on the Siegel modular form Φ_2. It should be encoded in various modular objects, one of them being Φ_2. In the following, we make a proposal for reproducing $W(q,p) \, e^{\ln |\vartheta_2(T_0)|^8 + \ln |\vartheta_2(U_0)|^8}$ in terms of modular objects.

Proposition: the approximate expression for the quantum entropy function given in (3.108) is captured by the following integral in Siegel's upper half plane $\mathcal{H}_2$,

$$\int_{\mathcal{C}'} \frac{d\rho d\sigma dv}{(2v - \rho - \sigma)^5} \exp\left\{-2\pi i\left[\frac{v^2 - \rho\sigma}{2v - \rho - \sigma}m + \frac{1}{2v - \rho - \sigma}n + \frac{1}{2}\frac{\rho - \sigma}{2v - \rho - \sigma}l\right]\right\}$$
$$\frac{G(\rho, Y^0(\sigma, \rho))\, G(\sigma, \tilde{Y}^0(\rho, \sigma))\, \exp\left[\frac{i\pi}{(\rho+\sigma)\, Y^0(\rho,\sigma)\, \tilde{Y}^0(\rho,\sigma)}\right]}{\Phi_2(\rho, \sigma, v)}\,, \tag{4.4}$$

where

$$G(\rho, Y^0) = \exp\left[\left[\frac{1}{2}\ln\vartheta_2^8(\rho) + 2\ln(p^1 - \rho\, p^0)\right] - \frac{64\pi^2}{(Y^0)^2}\hat{I}_1(\rho, \sigma) - 256\pi\, \Xi(Y^0, \rho)\right],$$

$$G(\sigma, \tilde{Y}^0) = \exp\left[\left[\frac{1}{2}\ln\vartheta_2^8(\sigma) + 2\ln(p^1 + \sigma\, p^0)\right] - \frac{64\pi^2}{(\tilde{Y}^0)^2}\hat{I}_1(\sigma, \rho) - 256\pi\, \Xi(\tilde{Y}^0, \sigma)\right]. \tag{4.5}$$

Here, Y^0, $\tilde{Y}^0$ and $\hat{I}_1$ are given by (3.110), and Ξ is a solution to the non-linear PDE (3.28). The contour $\mathcal{C}'$ denotes a contour that encircles $v = 0$, and that in the (τ_1, τ_2)-plane is identified with the contour $\mathcal{C}$ discussed in subsection 3.5. Note that G is given in terms of three distinct building blocks.

Proof. Let us begin by considering the integral

$$\int_{\mathcal{C}'} \frac{d\rho d\sigma dv}{(2v - \rho - \sigma)^5} \exp\left\{-2\pi i\left[\frac{v^2 - \rho\sigma}{2v - \rho - \sigma}m + \frac{1}{2v - \rho - \sigma}n + \frac{1}{2}\frac{\rho - \sigma}{2v - \rho - \sigma}l\right]\right\}\frac{1}{\Phi_2(\rho, \sigma, v)}. \tag{4.6}$$

As shown in [5], the exponent in (4.6) as well as

$$\frac{d\rho d\sigma dv}{(2v - \rho - \sigma)^5\, \Phi_2(\rho, \sigma, v)} \tag{4.7}$$

are invariant under symplectic transformations acting on Siegel's upper half plane that belong to the subgroup $H \subset \mathrm{Sp}(4, \mathbb{Z})$, which consists of elements $h \in H$ given by

$$h = g_1(a, b, c, d)\, g_2\, g_1(a, -b, -c, d)\, (g_2)^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)\,, \tag{4.8}$$

with g_1 and g_2 given in (E.6) and (E.10), respectively. Indeed, using (E.4), one finds that

under H-transformations,

$$\rho \to \frac{\rho + ac(-\rho\sigma + v^2) + bc(\rho - \sigma) + bd}{\Delta},$$
$$\sigma \to \frac{\sigma + bc(\sigma - \rho) - bd + ac(\rho\sigma - v^2)}{\Delta},$$
$$v \to \frac{v}{\Delta},$$
$$2v - \rho - \sigma \to \frac{2v - \rho - \sigma}{\Delta},$$
$$\Phi_2(\rho, \sigma, v) \to \Delta^2\, \Phi_2(\rho, \sigma, v),$$
$$\Delta = c^2(-\rho\sigma + v^2) + cd(\rho - \sigma) + d^2. \tag{4.9}$$

The charge bilinears (m, n, l) transform as in (3.61). Using these transformation rules, one establishes that the exponent in (4.6) as well as (4.7) are invariant under H-transformations.

The integral (4.6) depends on a contour $\mathcal{C}'$, which transforms as follows under H. Under H, v gets mapped to v/Δ, see (4.9). Thus, a small contour around $v = 0$ gets mapped to a small contour around $v = 0$. On the other hand, ρ and σ get mapped to ρ' and σ' according to (4.9). As mentioned in subsection 3.5, this means that in the limit $v = 0$, the contour $\mathcal{C}$ passing through the attractor point (τ_1^*, τ_2^*) gets transformed into a new contour that passes through the transformed attractor point.

In the limit $v = 0$, the transformation rules (4.9) result in (using $ad - bc = 1$)

$$\Delta = (d + c\rho)(d - c\sigma) \tag{4.10}$$

and

$$\rho \to \frac{a\rho + b}{c\rho + d},$$
$$\sigma \to \frac{a\sigma - b}{-c\sigma + d},$$
$$\rho + \sigma \to \frac{\rho + \sigma}{\Delta}. \tag{4.11}$$

Supplementing this by (cf. (3.59))

$$p^1 \to a\, p^1 + b\, p^0,$$
$$p^0 \to d\, p^0 + c\, p^1, \tag{4.12}$$

we infer

$$p^1 - p^0\rho \to \frac{p^1 - p^0\rho}{c\rho + d},$$
$$p^1 + p^0\sigma \to \frac{p^1 + p^0\sigma}{-c\sigma + d},$$
$$Y^0(\rho, \sigma) \to (c\rho + d)\, Y^0(\rho, \sigma),$$
$$\tilde{Y}^0(\rho, \sigma) \to (-c\sigma + d)\, \tilde{Y}^0(\rho, \sigma). \tag{4.13}$$

It follows that each of the three building blocks that make up $G(\rho, Y^0(\sigma,\rho))$ and $G(\sigma, \tilde{Y}^0(\rho,\sigma))$ in (4.5) is invariant under H in the limit $v = 0$, and hence the whole integrand in (4.4) is invariant under H in this limit.

Why is the subgroup $H \subset \mathrm{Sp}(4, \mathbb{Z})$ of relevance? For once, it implements the $\Gamma_0(2)_S$-symmetry of the STU model, cf. (4.8). This is similar to the role played by H in the CHL $\mathbb{Z}_2$-orbifold counting formula discussed in [5]. In addition, in the limit $v = 0$, H-transformations result in a simultaneous transformation of $\rho, \sigma, Y^0, \tilde{Y}^0$ that is consistent with the attractor point identification $\rho = iS, \sigma = i\bar{S}, Y^0 = (p^1 + i\bar{S}p^0)/(S + \bar{S}), \tilde{Y}^0 = (p^1 - iSp^0)/(S + \bar{S})$.

Performing the contour integration over v in (4.6) using (4.2) results in (with $k = 2$)

$$\int_{\mathcal{C}} \frac{d\rho\, d\sigma}{(\rho+\sigma)^2} \left(k + 3 - 2\pi i \left[\frac{\rho\sigma}{\rho+\sigma}\, m - \frac{1}{\rho+\sigma}\, n - \frac{1}{2} \frac{\rho-\sigma}{\rho+\sigma}\, l \right] \right) \tag{4.14}$$
$$\exp\left\{ -2\pi i \left[\frac{\rho\sigma}{\rho+\sigma}\, m - \frac{1}{\rho+\sigma}\, n - \frac{1}{2} \frac{\rho-\sigma}{\rho+\sigma}\, l \right] - \ln \vartheta_2^8(\rho) - \ln \vartheta_2^8(\sigma) - (k+2)\ln(\rho+\sigma) \right\}.$$

Similarly, performing the contour integration over v in (4.4) yields the quantum entropy function integral $W(q,p)\, e^{\ln|\vartheta_2(T_0)|^8 + \ln|\vartheta_2(U_0)|^8}$ given in (3.108).

$\square$

Thus, our proposal for a microstate counting formula for large BPS black holes which reproduces the corresponding approximate quantum entropy function (3.108) is

$$d^{(p^0,p^1)}(m,n,l) = e^{-\ln|\vartheta_2(T_0)|^8 - \ln|\vartheta_2(U_0)|^8} \int_{\mathcal{C}'} \frac{d\rho\, d\sigma\, dv}{(2v - \rho - \sigma)^5}$$
$$\exp\left\{ -2\pi i \left[\frac{v^2 - \rho\sigma}{2v - \rho - \sigma}\, m + \frac{1}{2v - \rho - \sigma}\, n + \frac{1}{2} \frac{\rho - \sigma}{2v - \rho - \sigma}\, l \right] \right\}$$
$$\frac{G(\rho, Y^0(\sigma,\rho))\, G(\sigma, \tilde{Y}^0(\rho,\sigma))\, \exp\left[\frac{i\pi}{(\rho+\sigma)\, Y^0(\rho,\sigma)\, \tilde{Y}^0(\rho,\sigma)} \right]}{\Phi_2(\rho,\sigma,v)}. \tag{4.15}$$

Being approximate, this formula, which is based on modular objects, gives a non-integer value of $d^{(p^0,p^1)}(m,n,l)$. Note that we have attached the label (p^0, p^1) to the degeneracy $d^{(p^0,p^1)}(m,n,l)$, to indicate that it also depends on the $\Gamma_0(2)_S$ doublet (p^0, p^1).

5 Conclusions

We conclude with a brief summary and a few observations.

We computed the quantum entropy function (3.81) for large BPS black holes in the $N = 2$ model of Sen and Vafa, by integrating out the moduli ϕ^2 and ϕ^3 in Gaussian approximation, to arrive at the intermediate result (3.94), which we then converted into the integral (3.101) by adding a total derivative term (3.97). In doing so, we resorted to various approximations. The integrand in (3.101) is invariant under $\Gamma_0(2)$ transformations of (τ_1, τ_2). However, it also depends on T_0, U_0, and is not invariant under $\Gamma_0(2)$ transformations of T_0, U_0. This is due to the fact that when evaluating (3.81), we expanded around

background values T_0 and U_0 which we took to be large. There will then be subleading corrections in T_0, U_0 that will restore the invariance under $\Gamma_0(2)$ transformations of T_0, U_0.

Proceeding in the manner described above, namely integrating out the moduli $\{\phi^a\}_{a=2,3}$ and retaining the dependence on ϕ^0, ϕ^1, one obtains an integral which resembles, in part, the expression obtained by integrating the inverse of a Siegel modular form $\Phi_2(\rho, \sigma, v)$ of weight 2 along a closed contour surrounding $v = 0$, as in (4.6). However, the result of the quantum entropy function calculation also depends on the $\Gamma_0(2)_S$ doublet (p^0, p^1) through the dependence on Y^0, and hence, a microstate counting formula cannot solely be based on the inverse of a Siegel modular forms, since the evaluation of the latter gives a result that depends only on the three charge bilinears m, n, l, but not on the individual charges (p^0, p^1). Thus, the proposal for a microstate counting formula has to depend on an additional modular object $G(\rho, Y^0(\sigma, \rho))$, cf. (4.5).

We can generalize the above discussion to a certain class of $N = 2$ models with n_V vector multiplets, as follows. We take these models to have a holomorphic Wilsonian function $F(Y, \Upsilon) = F^{(0)}(Y) + 2i\Omega(Y, \Upsilon)$ with a heterotic type prepotential of the form

$$F^{(0)}(Y) = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0} + \dots, \quad a = 2, \dots, n_V. \tag{5.1}$$

The ellipsis in (5.1) stands for one-loop corrections that involve Y^0, Y^a, but not Y^1, since Y^1/Y^0 serves as the loop-counting parameter in heterotic string theory. Moreover, introducing $\tau = Y^1/Y^0$, we assume that the τ-dependence of the first gravitational coupling function $\omega^{(1)}$ in Ω is encoded in a modular form of a certain weight under (a subgroup of) SL(2, $\mathbb{Z}$). Suppressing the dependence on the other moduli, we set $4\pi\Upsilon\omega^{(1)}(\tau) = g(\tau)$. This modular form will then be related to the seed of an associated Siegel modular form, as we will discuss momentarily.

Setting $Y^I = \frac{1}{2}(\phi^I + ip^I)$, and using the approximate measure factor (2.18), the quantum entropy function for large BPS black holes in these models becomes

$$W(q, p) = \int \frac{d\tau_1 \, d\tau_2}{\tau_2} \, d\phi^a \, e^{\pi[4 \operatorname{Im} F(\phi + ip) - q \cdot \phi]} \, |Y^0|^{4 - \chi/12} \, e^{-\mathcal{K}^{(0)}}, \tag{5.2}$$

with $e^{-\mathcal{K}^{(0)}}$ given by (2.11), and with $\chi = 2(n_V - n_H + 1)$ determined in terms of the number of vector and hyper multiplets of the $N = 2$ model (n_V and n_H, respectively). Following the steps in subsection 3.4.2 and integrating out the $(n_V - 1)$ moduli ϕ^a in Gaussian approximation gives the approximate result

$$W(q, p) = \int \frac{d\tau_1 \, d\tau_2}{\tau_2^{(n_V+1)/2}} \, e^{\pi \, H(\tau_1, \tau_2, p, q)} \, |Y^0|^{4 - \chi/12} \, e^{-\mathcal{K}^{(0)}}, \tag{5.3}$$

with

$$|Y^0|^{4 - \chi/12} \, e^{-\mathcal{K}^{(0)}} = \frac{|p^1 - \tau p^0|^{2 - \chi/12}}{\tau_2^{3 - \chi/12}} \left((p^1)^2 \, m + p^0 p^1 \, l + (p^0)^2 \, n \right), \tag{5.4}$$

up to an overall constant. Here, $H(\tau_1, \tau_2, p, q)$ takes a form similar to (3.75),

$$\pi H(\tau_1, \tau_2, p, q) = \pi \frac{n + l\tau_1 + m\tau_1^2 + m\tau_2^2}{\tau_2} + \left(g(\tau) + \text{c.c.} \right) + h(Y^0, \bar{Y}^0, \tau_1, \tau_2), \tag{5.5}$$

where h encodes the dependence on $Y^0(\tau_1, \tau_2)$ due to presence of the higher gravitational coupling functions, and where we have suppressed the dependence on the other moduli.

Then, the analogue of the intermediate result (3.93) is

$$W(q,p) = \frac{1}{2} \int \frac{d\tau_1 \, d\tau_2}{\tau_2^{(n_V+7)/2-\chi/12}} \, e^{\pi \, H(\tau_1,\tau_2,p,q)} |p^1 - \tau p^0|^{4-\chi/12} \, m \,. \tag{5.6}$$

By adding an appropriate total derivative term to (5.6), the latter can be brought into a form analogous to (3.98),

$$W(q,p) = \int \frac{d\tau_1 \, d\tau_2}{\tau_2^{(n_V+9)/2-\chi/12}} \left(\frac{n_V + 7}{2} - \frac{\chi}{12} + \frac{\pi}{\tau_2}(n + l\tau_1 + m\tau_1^2 + m\tau_2^2) \right) e^{\pi \, H(\tau_1,\tau_2,p,q)}$$
$$|p^1 - \tau p^0|^{4-\chi/12} \,. \tag{5.7}$$

Introducing

$$k = \frac{n_V + 1}{2} - \frac{\chi}{12} = \frac{2(n_V + 1) + n_H}{6} > 0 \,, \tag{5.8}$$

we write (5.7) as

$$W(q,p) = \int_{\mathcal{C}} \frac{d\tau_1 \, d\tau_2}{\tau_2^2} \left(k + 3 + \frac{\pi}{\tau_2}(n + l\tau_1 + m\tau_1^2 + m\tau_2^2) \right)$$
$$e^{\pi \, H(\tau_1,\tau_2,p,q)+(2-\chi/24)\ln|p^1-\tau p^0|^2-(k+2)\ln \tau_2} \,, \tag{5.9}$$

with an appropriately chosen contour $\mathcal{C}$. Note the dependence of the measure on $k + 3$, which suggests a microstate counting formula based on a Siegel modular form $\Phi(\rho, \sigma, v)$ of weight k, with the property that as $v \to 0$,

$$\Phi_k(\rho, \sigma, v) \sim v^2 \, f_{k+2}(\rho) \, f_{k+2}(\sigma) \,, \tag{5.10}$$

where f_{k+2} is a modular form of weight $k + 2$ under (a subgroup of) $\mathrm{SL}(2, \mathbb{Z})$. Requiring k to be integer valued imposes a restriction on the allowed values of (n_V, n_H). We assume that this Siegel modular form can be constructed by applying a Hecke lift to a Jacobi form $\phi_{k,1}(\tau, z)$ of weight k and index 1 (cf. (G.2)),

$$\phi_{k,1}(\tau, z) = \frac{\vartheta_1^2(\tau, z)}{\eta^6(\tau)} \, f_{k+2}(\tau) \,. \tag{5.11}$$

$f_{k+2}(\tau)$ should be related to the modular form $g(\tau)$ that appears in the first gravitational coupling function $\omega^{(1)}$, as mentioned below (5.1), but need not coincide with it.

Evaluating the analogue of (4.6),

$$\int_{\mathcal{C}'} \frac{d\rho d\sigma dv}{(2v - \rho - \sigma)^{k+3}} \exp\left\{ -2\pi i \left[\frac{v^2 - \rho\sigma}{2v - \rho - \sigma} m + \frac{1}{2v - \rho - \sigma} n + \frac{1}{2} \frac{\rho - \sigma}{2v - \rho - \sigma} l \right] \right\} \frac{1}{\Phi_k(\rho, \sigma, v)} \tag{5.12}$$

by performing a contour integration over v around $v = 0$ gives [8]

$$\int_{\mathcal{C}} \frac{d\rho \, d\sigma}{(\rho + \sigma)^{k+4}} \left(k + 3 - 2\pi i \left[\frac{\rho\sigma}{\rho + \sigma} m - \frac{1}{\rho + \sigma} n - \frac{1}{2} \frac{\rho - \sigma}{\rho + \sigma} l \right] \right) \tag{5.13}$$
$$\exp\left\{ -2\pi i \left[\frac{\rho\sigma}{\rho + \sigma} m - \frac{1}{\rho + \sigma} n - \frac{1}{2} \frac{\rho - \sigma}{\rho + \sigma} l \right] - \ln f_{k+2}(\rho) - \ln f_{k+2}(\sigma) \right\} \,.$$

Fig (a).

Dyonic effective string web

Fig (b).

Effective string tubular web

Identically labelled end-regions are
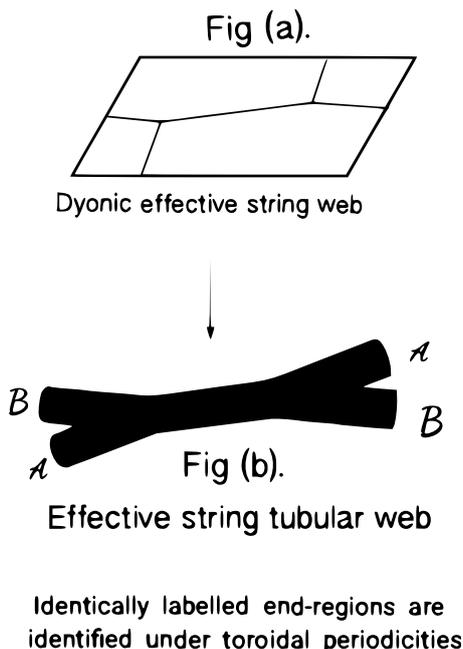identified under toroidal periodicities

Figure 1. String web picture.

Using (3.105) this reproduces part of (5.9). This then needs to be further supplemented
by modular objects that depend on (p^0, p^1) through $Y^0(\rho, \sigma)$, as in (4.15).

An example of a model in this class is the FHSV model, which has $n_V = 11, \chi = 0$,
$k = 6$ and modular subgroup $\Gamma_0(2)$. For this model, $g(\tau) = -\frac{1}{2}\ln\eta^{24}(2\tau)$, which we
write as

$$g(\tau) = -\ln(\vartheta_2^8(\tau)\mathcal{E}_4(\tau)) + \frac{1}{2}\ln\mathcal{E}_4(\tau), \tag{5.14}$$

up to a constant (here we used the relation given below (B.29)). This suggests to use
as seed (5.11) for the Hecke lift the cusp form $f_{k+2}(\tau) = \vartheta_2^8(\tau)\mathcal{E}_4(\tau)$, which has weight
$k + 2 = 8$, has trivial multiplier system and equals $[\eta(\tau)\eta(2\tau)]^8$, up to a normalization
constant (see below (G.2)). Since the vector space of cusp forms of weight 8 has dimension
one, $f_{k+2}(\tau)$ is a Hecke eigenform, and hence one deduces the property (5.10) in a manner
similar to the one given below (G.15). Using $g(\tau) = -\ln f_{k+2}(\tau) + \frac{1}{2}\ln\mathcal{E}_4(\tau)$, we note
that the transformation of the second term on the right hand side of $g(\tau)$ under $\Gamma_0(2)$-
transformations is precisely compensated by the term $2\ln(p^1 - \tau p^0)$ in (5.9), and hence,
the combination $(g(\tau) + \text{c.c.}) + 2\ln|p^1 - \tau p^0|^2 - 8\ln\tau_2$ in the exponent of (5.9) is invariant
under $\Gamma_0(2)$-transformations. We note that our proposal differs from the one made in [22].

We conclude by suggesting a string web picture of our counting proposal. We have
proposed an approximate degeneracy formula for $\frac{1}{2}$ BPS states that gravitate to form a
large black hole at strong 't Hooft coupling in terms of a contour integral of the inverse
of a Siegel modular form and of additional subleading contributions expressed as a series
in $1/(Y^0)^2$. A physical picture of this mathematical structure is suggested by the $N = 4$
picture discussed in [4, 11, 13], whereby a dyonic BPS black hole is viewed as a string web
wrapping a two-torus in type IIB string theory. In this picture, a dyonic black hole in

type IIB is described in terms of a web of effective strings. Viewing the six-dimensional compact manifold as a fibration over a two-torus, these effective strings wrap either one of the two cycles of the two-torus. Electric states correspond to wrappings on one of these cycles, while magnetic states correspond to wrappings on the other cycle. This effective description is valid at points in moduli space where the volume of the fibre is small compared to the volume of the two-torus. The electric-magnetic duality group $\Gamma_0(2)$ is viewed as a subgroup of the large diffeomorphism group of the two-torus. A computation of the Euclidean partition function of this string network requires time to be compactified, so that the arms of the torus are tube-like and the resultant partition function of the string web is a genus two partition function of an effective string theory that encodes BPS states of the original type II theory. In the present $N = 2$ case, topological string theory is a candidate for this effective string theory, since amplitude calculations in topological string theory result in corrections to the free energy of the type II BPS partition function. In this picture, the world sheet path integral will encode contributions from worldsheet instantons corresponding to higher genera string webs. These contributions should correspond to the series in $1/(Y^0)^2$ which appears in the proposed degeneracy formula, and which is absent in the $N = 4$ case as a result of the larger supersymmetry in the system.

Acknowledgments

A Modular forms for $\mathrm{SL}(2, \mathbb{Z})$

We review basic properties of modular forms for $\mathrm{SL}(2, \mathbb{Z})$, following [59].

Let $\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}\, \tau > 0\}$ denote the complex upper half plane, and let $\mathcal{H}^* = \mathcal{H} \cup \{\infty\}$.

Definition: let $k \in \mathbb{Z}$. A meromorphic function $f : \mathcal{H} \to \mathbb{C}$ is weakly modular of weight k if, $\forall \tau \in \mathcal{H}$,

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau)\,, \tag{A.1}$$

where

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d}\,, \tag{A.2}$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})\,, \tag{A.3}$$

where

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}. \tag{A.4}$$

Remark: $SL(2,\mathbb{Z})$ is generated by two elements,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \tau \mapsto \tau + 1\,, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \tau \mapsto -\frac{1}{\tau}\,. \tag{A.5}$$

Remark: consider an open subset $\mathcal{F} \subset \mathcal{H}$ such that no two distinct points of $\mathcal{F}$ are equivalent under the action of $SL(2,\mathbb{Z})$, and every $\tau \in \mathcal{H}$ is equivalent to a point in the closure $\mathcal{D} \equiv \overline{\mathcal{F}}$. Then, $\mathcal{D}$ is called a fundamental domain for $SL(2,\mathbb{Z})$,

$$\mathcal{D} = \{\tau \in \mathcal{H} \mid -\frac{1}{2} \leq \operatorname{Re}\tau \leq \frac{1}{2} \quad \text{and} \quad |\tau| \geq 1\}. \tag{A.6}$$

Remark: f is weakly modular of weight k if $f(\tau + 1) = f(\tau)$, $f(-1/\tau) = \tau^k f(\tau)$.

Definition: modular forms are weakly modular functions that are also holomorphic on $\mathcal{H}$ and at ∞, i.e. on $\mathcal{H}^*$.

Remark: to show that a weakly modular function f is holomorphic at ∞, it suffices to show that $f(\tau)$ is bounded at $\operatorname{Im}\tau \to \infty$, i.e. there exists $C \in \mathbb{R}$ such that $|f(\tau)| \leq C$ $\forall \tau$ with $\operatorname{Im}\tau \gg 1$.

Let $D = \{q \in \mathbb{C} : |q| < 1\}$ denote the open complex unit disc, and D' the punctured disc $D' = D - \{0\}$. Consider the map

$$\tau \mapsto e^{2\pi i \tau} = q\,, \tag{A.7}$$

which takes $\mathcal{H}^*$ to D. A modular form f is $\mathbb{Z}$-periodic, since $f(\tau + 1) = f(\tau)$. Then, f has the Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n\,, \quad q = e^{2\pi i \tau}\,. \tag{A.8}$$

Definition: the set of modular forms of weight $k \in \mathbb{Z}$ is denoted by $\mathcal{M}_k(SL(2,\mathbb{Z}))$.

Remark: $\mathcal{M}_k(SL(2,\mathbb{Z}))$ forms a finite-dimensional vector space over $\mathbb{C}$, and the direct sum

$$\mathcal{M}(SL(2,\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k(SL(2,\mathbb{Z})) \tag{A.9}$$

forms a graded ring.

Definition: a cusp form of weight $k \in \mathbb{Z}$ is a modular form of weight k whose Fourier expansion has a coefficient $a_0 = 0$, i.e.

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n\,, \quad q = e^{2\pi i \tau}\,. \tag{A.10}$$

The limit point ∞ represents the cusp of $SL(2,\mathbb{Z})$. The modular images of ∞ are the rational numbers $\gamma(\infty) = a/c \in \mathbb{Q}$.

Definition: the set of cusp forms of weight $k \in \mathbb{Z}$ is denoted by $\mathcal{S}_k(\mathrm{SL}(2,\mathbb{Z}))$.

Example: the discriminant function $\Delta(\tau) = \eta^{24}(\tau)$ is a cusp form with $\Delta \in \mathcal{S}_{12}(\mathrm{SL}(2,\mathbb{Z}))$. It has a simple zero at $q = 0$.

Example: let $k \in \mathbb{N}$ with $k \geq 2$. The Eisenstein series, defined by

$$G_{2k}(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 - \{(0,0)\}} \frac{1}{(m + n\,\tau)^{2k}}\,, \quad \tau \in \mathcal{H}\,, \tag{A.11}$$

is a modular form of weight $2k$. It has the following Fourier expansion,

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, q^n\,, \quad q = e^{2\pi i \tau}\,, \tag{A.12}$$

where the sum $\sigma_p(n) = \sum_{d|n} d^p$ is over positive divisors of n, and $\zeta(z)$ is Riemann's zeta function.

Setting $k = 1$ in (A.12) yields G_2, a quasi-modular form of weight 2 and depth $s = 1$, i.e. a holomorphic function $G_2 : \mathcal{H}^* \to \mathbb{C}$ that, compared to (A.1), transforms with an additional shift proportional to $c/(c\tau + d)$ under (A.2),

$$G_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2\, G_2(\tau) - 2\pi i\, c\,(c\tau + d)\,. \tag{A.13}$$

More generally, a quasi-modular function of weight k and depth s is defined as follows [56]:

Definition: let $k \in \mathbb{Z}$ and $s \in \mathbb{N}$. A holomorphic function $f : \mathcal{H}^* \to \mathbb{C}$ is a quasi-modular form of weight k and depth s if there exist holomorphic functions $Q_1(f), \ldots, Q_s(f)$ on $\mathcal{H}$ such that

$$(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau) + \sum_{i=1}^{s} Q_i(f)(\tau) \left(\frac{c}{c\tau + d}\right)^i \tag{A.14}$$

for all $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$, and such that $Q_s(f)$ is not identically zero.

The normalized Eisenstein series are $E_{2k} = G_{2k}/(2\zeta(2k))$. For $k \geq 2$, the normalized Eisenstein functions can also be defined by [60]

$$E_{2k}(\tau) = \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 - \{(0,0)\},\ \gcd(m,n)=1} \frac{1}{(m + n\,\tau)^{2k}} = \sum_{n>0, m \in \mathbb{Z},\ \gcd(m,n)=1} \frac{1}{(m + n\,\tau)^{2k}}\,. \tag{A.15}$$

Definition: a weakly holomorphic modular form of weight k [61] is a weakly modular function $f : \mathcal{H} \to \mathbb{C}$ that is holomorphic on $\mathcal{H}$, with a pole at $q = 0$. Its Fourier expansion is given by

$$f(\tau) = \sum_{n=-N}^{\infty} a_n\, q^n\,, \quad q = e^{2\pi i \tau}\,, \tag{A.16}$$

with $N \in \mathbb{N}$. Thus, $f(\tau)$ grow as q^{-N} at ∞.

Example: $Z(\tau) = 1/\Delta(\tau)$ is a weakly holomorphic modular form of weight -12 with a simple pole at $q = 0$. Hence

$$Z(\tau) = \sum_{n=-1}^{\infty} a_n q^n, \quad q = e^{2\pi i \tau}. \tag{A.17}$$

For large n, the Fourier coefficients a_n grow as

$$a_n \sim e^{4\pi\sqrt{n}}. \tag{A.18}$$

Thus, they exhibit exponential growth, as required for the microstate degeneracy of small BPS black holes in $N = 4$ string theories [62].

Depending on the nature of the modular form, the growth property of its Fourier coefficients can be markedly different [61, 63]:

Growth conditions:

1. $f \in \mathcal{S}_k(\mathrm{SL}(2,\mathbb{Z}))$: $a_n = \mathcal{O}(n^{k/2})$ as $n \to \infty$;

2. $f \in \mathcal{M}_k(\mathrm{SL}(2,\mathbb{Z}))$, $f \notin \mathcal{S}_k(\mathrm{SL}(2,\mathbb{Z}))$: $a_n = \mathcal{O}(n^{k-1})$ as $n \to \infty$;

3. f weakly holomorphic modular form of weight k: $a_n = \mathcal{O}(e^{C\sqrt{n}})$ as $n \to \infty$ for some $C > 0$.

B Congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$

We review basic properties of modular forms for congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$, following [59].

Definition: let N be a positive integer. The principal congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of level N is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}. \tag{B.1}$$

In particular, $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$.

Definition: a subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$ is a congruence subgroup if $\Gamma(N) \subset \Gamma$ for some $N \in \mathbb{Z}^+$. The least such N is called the level of Γ.

Example: let N be a positive integer. Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}, \tag{B.2}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}, \tag{B.3}$$

where $*$ can take any value in $\mathbb{Z}$. Hence, $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}(2,\mathbb{Z})$.

In particular, $\Gamma(2) \subset \Gamma_1(2) = \Gamma_0(2)$, since $a, d = 1 \mod 2$. This is the case of interest for this paper, $\Gamma = \Gamma_0(2)$.

Remark: $\Gamma_0(2)$ is generated by two elements,

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = S^2 \left(S\, T^{-2}\, S \right), \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{B.4}$$

where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{B.5}$$

Remark: each congruence subgroup Γ of $\mathrm{SL}(2, \mathbb{Z})$ contains a translation matrix of the form

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} : \tau \mapsto \tau + h \tag{B.6}$$

for some minimal $h \in \mathbb{Z}^+$. Hence, every function $f : \mathcal{H} \to \mathbb{C}$ that is weakly modular with respect to Γ is $h\mathbb{Z}$-periodic, and has a corresponding function $g : D' \to \mathbb{C}$ with $f(\tau) = g(q_h)$, where $q_h = e^{2\pi i \tau / h}$. Then, f is defined to be holomorphic at ∞ if g extends holomorphically to $q = 0$, in which case

$$f(\tau) = \sum_{n=0}^{\infty} a_n\, q_h^n, \quad q_h = e^{2\pi i \tau / h}. \tag{B.7}$$

To define modular forms for a congruence subgroup Γ, one adjoins not only ∞ to $\mathcal{H}$, but also the rational numbers $\mathbb{Q}$. Then, one identifies all elements in $\{\infty\} \cup \mathbb{Q}$ that are Γ-equivalent. When $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, all rational numbers are Γ-equivalent to ∞, i.e. $\gamma(-d/c) = \infty$, where $c, d \in \mathbb{Z}$ with $c \neq 0$. When $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ is a proper subgroup, fewer points are Γ-equivalent.

Definition: a Γ-equivalence class of points in $\{\infty\} \cup \mathbb{Q}$ is called a cusp of Γ.

Example: for $\Gamma = \Gamma_0(2)$ there are two cusps, 0 and ∞ [63]. The orbit of ∞ consists of the set of rational numbers of the form a/c with $a \neq 0$, $c = 0 \mod 2$ and $\gcd(a, c) = 1$. The orbit of 0 consists of the set of rational numbers of the form b/d with $\gcd(b, d) = 1$. Thus, every element of $\{\infty\} \cup \mathbb{Q}$ is in exactly one of the two orbits. Since there is no element in $\Gamma_0(2)$ that maps 0 to ∞, there are two cusps.

Proposition (fundamental domain): let Γ be a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$, and let R be a set of coset representatives for the quotient $\Gamma \backslash \mathrm{SL}(2, \mathbb{Z})$. Then, the set $\mathcal{D}_\Gamma = \cup_{\tilde{\gamma} \in R} \tilde{\gamma} \mathcal{D}$ is a fundamental domain for Γ [63]. Here, $\mathcal{D}$ denotes a fundamental domain for $\mathrm{SL}(2, \mathbb{Z})$.

Example: let $\Gamma = \Gamma_0(2)$. A system of coset representatives for $\Gamma \backslash \mathrm{SL}(2, \mathbb{Z})$ is [63]

$$R = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\} = \{\mathbb{I}, S, ST\}, \tag{B.8}$$

i.e. if $\alpha \in \mathrm{SL}(2, \mathbb{Z})$, then $\alpha = \gamma \tilde{\gamma}$ with $\gamma \in \Gamma$ and $\tilde{\gamma} \in R$. A fundamental domain for $\Gamma_0(2)$ is thus given by $\mathcal{D}_{\Gamma_0(2)} = \mathcal{D} \cup S(\mathcal{D}) \cup ST(\mathcal{D})$ [63, 64], see figure 2. Note that the two cusps, 0 and ∞, are in the closure of $\mathcal{D}_{\Gamma_0(2)}$ in the Riemann sphere, but not in $\mathcal{H}$ [63].
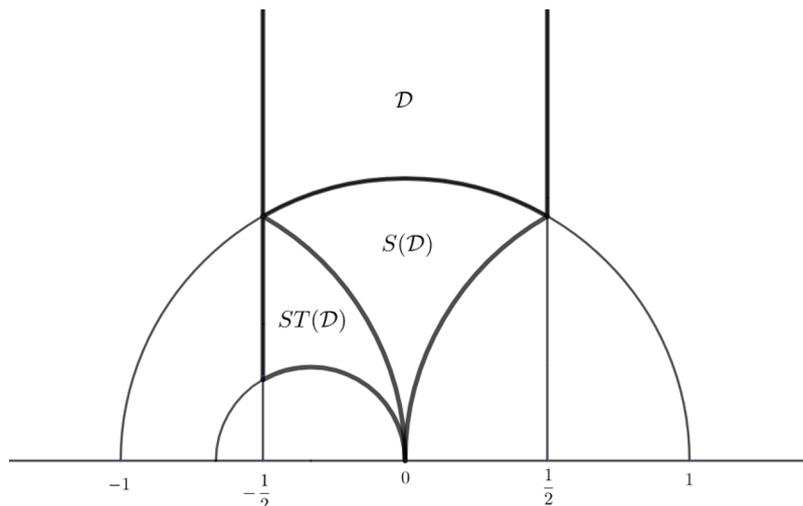
Figure 2. A fundamental domain for $\Gamma_0(2)$.

Definition: let Γ be a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, and let $k \in \mathbb{Z}$. A modular form of weight k for the subgroup Γ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ that is weakly modular of weight k for Γ and holomorphic at all cusps of Γ.

Definition: let Γ be a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, and let $k \in \mathbb{Z}$. Writing any element $s \in \{\infty\} \cup \mathbb{Q}$ as $s = \gamma(\infty)$ for some $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, holomorphy of f at s is defined in terms of holomorphy of $f[\gamma]_k$ at ∞ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. Here, $f[\gamma]_k : \mathcal{H} \to \mathbb{C}$ is defined by $(f[\gamma]_k)(\tau) = (c\tau + d)^{-k} f(\gamma(\tau))$. If $a_0 = 0$ in the Fourier expansion of $f[\gamma]_k$ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, then f is called a cusp form of weight k with respect to Γ.

Remark: for Γ, the finite-dimensional vector space over $\mathbb{C}$ of modular forms of weight k is denoted by $\mathcal{M}_k(\Gamma)$, and the vector subspace of cusp forms of weight k is denoted by $\mathcal{S}_k(\Gamma)$. The direct sums

$$\mathcal{M}(\Gamma) = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}_k(\Gamma), \quad \mathcal{S}(\Gamma) = \bigoplus_{k\in\mathbb{Z}} \mathcal{S}_k(\Gamma) \tag{B.9}$$

form a graded ring.

Example: an example of a modular form of weight 2 for $\Gamma_0(N)$ with $N \geq 2$ is $NG_2(N\tau) - G_2(\tau)$, where $G_2(\tau)$ denotes the Eisenstein series of weight 2 for $\mathrm{SL}(2,\mathbb{Z})$ introduced below (A.12).

Proposition: let $\Gamma = \Gamma_0(2)$. Let $k \in \mathbb{N}$, $k \geq 2$. Then, $\dim \mathcal{M}_k(\Gamma) = \left[\frac{k}{4}\right] + 1$ and $\dim \mathcal{S}_k(\Gamma) = \left[\frac{k}{4}\right] - 1$ for k even, while $\dim \mathcal{M}_k(\Gamma) = \dim \mathcal{S}_k(\Gamma) = 0$ for k odd [59]. Here, $[n]$ denotes the integer part of n.

Example: let $\Gamma = \Gamma_0(2)$. Then, $\dim \mathcal{M}_2(\Gamma) = 1$, $\dim \mathcal{M}_4(\Gamma) = 2$ and $\dim \mathcal{S}_4(\Gamma) = 0$. The vector space $\mathcal{M}_2(\Gamma)$ is generated by [55, 65]

$$\tilde{\mathcal{E}}_2(\tau) = \frac{1}{2}\left(3\,\mathcal{E}_2(\tau) - E_2(\tau)\right) = 2E_2(2\tau) - E_2(\tau). \tag{B.10}$$

Here, $E_2(\tau) = G_2(\tau)/(2\zeta(2))$ denotes the normalized quasi-modular form of weight 2 for $\mathrm{SL}(2,\mathbb{Z})$, while $\mathcal{E}_2(\tau)$ denotes one of the normalized Eisenstein series on $\Gamma_0(2)$, see (B.23).

Definition: let $k \geq 2$. For $\Gamma_0(2)$, define an Eisenstein series by [66]

$$G_{2k,N=2}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2,(m,n)=(1,0)\mod 2} \frac{1}{(m+n\,\tau)^{2k}},$$

$$= \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{\chi_2(m)}{(m+2\,n\,\tau)^{2k}}, \tag{B.11}$$

where

$$\chi_N(m) = \begin{cases} 1, & \text{if} \quad \gcd(m,N) = 1 \\ 0, & \text{if} \quad \gcd(m,N) > 1. \end{cases} \tag{B.12}$$

$\chi_N(m)$ is called the principal character $\mod N$.

Define the normalized Eisenstein series $\mathcal{E}_{2k}(\tau)$ by $G_{2k,N=2} = 2\zeta_2(2k)\,\mathcal{E}_{2k}$, where

$$\zeta_N(2k) = \sum_{r=1}^{\infty} \chi_N(r)\,r^{-2k}. \tag{B.13}$$

Remark: for $N=1$, $\zeta_1(2k) = \zeta(2k)$, while for $N=2$, $\zeta_2(2k) = (1-2^{-2k})\,\zeta(2k)$.

Proposition: let $k \geq 2$. Then, $G_{2k,N=2}(\tau)$ are modular forms of weight $2k$ for $\Gamma_0(2)$ [66].

Proposition: let $k \geq 2$. The normalized Eisenstein functions $\mathcal{E}_{2k}(\tau)$ are given by [67]

$$\mathcal{E}_{2k}(\tau) = \sum_{n>0,m\in\mathbb{Z},\,\gcd(m,2n)=1} \frac{1}{(m+2\,n\,\tau)^{2k}}. \tag{B.14}$$

Proof. Write [67]

$$G_{2k,N=2}(\tau) = \sum_{r=1}^{\infty} \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\},\,\gcd(m,n)=r} \frac{\chi_2(r)\,\chi_2(m/r)}{r^{2k}\,(m/r+2\,(n/r)\,\tau)^{2k}}$$

$$= \zeta_2(2k) \sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\},\,\gcd(c,d)=1} \frac{\chi_2(d)}{(d+2\,c\,\tau)^{2k}}$$

$$= \zeta_2(2k) \sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\},\,\gcd(2c,d)=1} \frac{1}{(d+2\,c\,\tau)^{2k}}. \tag{B.15}$$

$\square$

Proposition: let $k \geq 2$. Then, $\mathcal{E}_{2k}$ has the following expansion around the cusp ∞ [65],

$$\mathcal{E}_{2k}(\tau) = 1 + \frac{4k}{(1-2^{2k})B_{2k}} \sum_{n=1}^{\infty} \frac{(-1)^n\,n^{2k-1}\,q^n}{1-q^n}, \quad q = e^{2\pi i\tau}, \tag{B.16}$$

with B_k the k-th Bernoulli number. $\mathcal{E}_{2k}(\tau)$ can also be written as

$$\mathcal{E}_{2k}(\tau) = 1 + \frac{4k}{(1-2^{2k})B_{2k}} \sum_{n=1}^{\infty} \tilde{\sigma}_{2k-1}(n)\,q^n, \quad q = e^{2\pi i\tau}, \tag{B.17}$$

where the sum

$$\tilde{\sigma}_{2k-1}(n) = \sum_{d|n} (-1)^d \, d^{2k-1} \tag{B.18}$$

is over positive divisors of n.

Proof. Using absolute convergence, we write

$$
\begin{aligned}
G_{2k,N=2}(\tau) &= \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(m+2\,n\,\tau)^{2k}} - \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(2\,m+2\,n\,\tau)^{2k}} \\
&= G_{2k}(2\tau) - 2^{-2k}\, G_{2k}(\tau) \,,
\end{aligned}
\tag{B.19}
$$

where G_{2k} denote Eisenstein functions for $\mathrm{SL}(2,\mathbb{Z})$. Using (A.12) in the form

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(-1)^k(2\pi)^{2k}}{(2k-1)!} \sum_{m=1}^{\infty}\sum_{d=1}^{\infty} d^{2k-1} q^{dm}, \quad q = e^{2\pi i\tau}, \tag{B.20}$$

we obtain, using absolute convergence,

$$
\begin{aligned}
G_{2k,N=2}(\tau) &= 2\zeta_2(2k) + 2^{-2k}\frac{2(-1)^k(2\pi)^{2k}}{(2k-1)!} \left[2\sum_{m=1}^{\infty}\sum_{d=1}^{\infty}(2d)^{2k-1} q^{2dm} - \sum_{m=1}^{\infty}\sum_{d=1}^{\infty} d^{2k-1} q^{dm} \right] \\
&= 2\zeta_2(2k) + 2^{-2k}\frac{2(-1)^k(2\pi)^{2k}}{(2k-1)!} \sum_{m=1}^{\infty}\sum_{d=1}^{\infty}(-1)^d d^{2k-1} q^{dm} \\
&= 2\zeta_2(2k) + 2^{-2k}\frac{2(-1)^k(2\pi)^{2k}}{(2k-1)!} \sum_{d=1}^{\infty}(-1)^d d^{2k-1} \frac{q^d}{1-q^d} \\
&= 2\zeta_2(2k) \left[1 + \frac{4k}{(1-2^{2k})\,B_{2k}} \sum_{d=1}^{\infty}(-1)^d d^{2k-1} \frac{q^d}{1-q^d} \right] .
\end{aligned}
\tag{B.21}
$$

Expanding in a geometric series,

$$G_{2k,N=2}(\tau) = 2\zeta_2(2k) \left[1 + \frac{4k}{(1-2^{2k})\,B_{2k}} \sum_{d=1}^{\infty}\sum_{m=1}^{\infty}(-1)^d d^{2k-1} q^{dm} \right], \tag{B.22}$$

and interchanging the order of summations yields (B.17).

$\qquad\square$

Definition: setting $k=1$ in (B.16) defines the second Eisenstein series $\mathcal{E}_2$ [55, 65],

$$\mathcal{E}_2(\tau) = 1 - 8 \sum_{n=1}^{\infty} \frac{(-1)^n\, n\, q^n}{1-q^n}, \quad q = e^{2\pi i\tau}. \tag{B.23}$$

Proposition: $\mathcal{E}_2$ can be expressed as [55]

$$\mathcal{E}_2 = q\frac{d}{dq} \ln \vartheta_2^8, \tag{B.24}$$

where the theta function ϑ_2 has the product representation

$$\vartheta_2(\tau) = 2\,q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2\,, \quad q = e^{2\pi i\tau}\,, \tag{B.25}$$

valid in the open complex unit disc $D = \{q \in \mathbb{C} : |q| < 1\}$. Thus, $\mathcal{E}_2$ is a quasi-modular form for $\Gamma_0(2)$,

$$\mathcal{E}_2(\tau) = \frac{1}{\pi\gamma}\frac{\partial\omega(S)}{\partial S} = \frac{1}{\pi\gamma}\,I_1(S)\,, \quad S = -i\tau\,. \tag{B.26}$$

Proof. Using (B.25) and (B.23) as well as absolute convergence, we compute

$$q\frac{d}{dq}\ln\vartheta_2^8 - \mathcal{E}_2 = 16\left(\sum_{n=1}^{\infty}n\,\frac{q^n}{1+q^n} - \sum_{n=1}^{\infty}(2n-1)\,\frac{q^{2n-1}}{1-q^{2n-1}}\right)\,. \tag{B.27}$$

Using the relation

$$\frac{x}{1+x} = \frac{x}{1-x} - \frac{2x^2}{1-x^2}\,, \tag{B.28}$$

it follows that the right hand side of (B.27) vanishes. Since I_1 is quasi-modular, so is $\mathcal{E}_2$. $\qquad\square$

Remark: the vector space $\mathcal{M}_4(\Gamma)$, which has $\dim\mathcal{M}_4(\Gamma) = 2$, is generated by [65] $\tilde{\mathcal{E}}_2^2$ and $\mathcal{E}_4$. Therefore, $\vartheta_2^8(\tau)$, which has weight 4, is given by a linear combination of $\tilde{\mathcal{E}}_2^2$ and $\mathcal{E}_4$, namely [65]

$$2^{-8}\,\vartheta_2^8(\tau) = \frac{1}{64}\left(\tilde{\mathcal{E}}_2^2(\tau) - \mathcal{E}_4(\tau)\right) = \frac{1}{240}\left(E_4(\tau) - E_4(2\tau)\right)\,. \tag{B.29}$$

Note that $\vartheta_2^8(\tau)$ is not a cusp form of $\Gamma_0(2)$, and that $\mathcal{E}_4(\tau) = 16\,\eta^{12}(\tau)/\vartheta_2^4(\tau)$.

Remark: $\tilde{\mathcal{E}}_2(\tau)$ has the following q-expansion,

$$\tilde{\mathcal{E}}_2(\tau) = 1 + 24\sum_{n=1}^{\infty}\left(\sum_{d|n,\ d\,\text{odd}}d\right)q^n\,, \quad q = e^{2\pi i\tau}\,. \tag{B.30}$$

Proposition: the series $\mathcal{E}_2, \tilde{\mathcal{E}}_2, \mathcal{E}_4$ satisfy the differential equations [65]

$$\begin{aligned}
q\frac{d\mathcal{E}_2}{dq} &= \frac{1}{4}\left((\mathcal{E}_2)^2 - \mathcal{E}_4\right)\,, \\
q\frac{d\tilde{\mathcal{E}}_2}{dq} &= \frac{1}{2}\left(\tilde{\mathcal{E}}_2\,\mathcal{E}_2 - \mathcal{E}_4\right)\,, \\
q\frac{d\mathcal{E}_4}{dq} &= \mathcal{E}_2\,\mathcal{E}_4 - \tilde{\mathcal{E}}_2\,\mathcal{E}_4\,.
\end{aligned} \tag{B.31}$$

Definition: given a modular form $f(S)$ of $\Gamma_0(2)$ of weight $2k$, the operator [65]

$$f \to \mathcal{D}_S f = \partial_S f + \frac{k}{\gamma}\frac{\partial\omega(S)}{\partial S}\,f \tag{B.32}$$

maps $\mathcal{M}_{2k}(\Gamma_0(2))$ to $\mathcal{M}_{2k+2}(\Gamma_0(2))$. This operator can be written as

$$-\frac{1}{2\pi}\,f \to \left(q\frac{d}{dq} - \frac{k}{2}\,\mathcal{E}_2\right)f\,. \tag{B.33}$$

Using the above relations, one readily verifies that [55]

$$\mathcal{D}_S \vartheta_2^8(S) = 0 \,, \tag{B.34}$$

as well as

$$
\begin{aligned}
I_2(S) &= \frac{1}{2}\pi^2\,\gamma\,\mathcal{E}_4\,, \\
I_3(S) &= \pi^3\,\gamma\,\tilde{\mathcal{E}}_2\,\mathcal{E}_4\,, \\
I_4(S) &= \pi^4\,\gamma\left(\mathcal{E}_4^2 + 2(\tilde{\mathcal{E}}_2)^2\,\mathcal{E}_4\right)\,.
\end{aligned}
\tag{B.35}
$$

Acting with the operator (B.33) on $\tilde{\mathcal{E}}_2$ gives

$$-\frac{1}{2\pi}\mathcal{D}_S\tilde{\mathcal{E}}_2 = -\frac{1}{2}\,\mathcal{E}_4\,, \tag{B.36}$$

and hence we infer

$$I_2 = \frac{1}{2}\,\pi\,\gamma\,\mathcal{D}_S\tilde{\mathcal{E}}_2\,. \tag{B.37}$$

Proposition: for $k \geq 2$, the $I_k(S)$ are given by

$$I_k(S) = \gamma \sum_{m+2n=k,\,m\geq 0,n\geq 1} a_{m,n}\,(\tilde{\mathcal{E}}_2)^m\,(\mathcal{E}_4)^n\,, \tag{B.38}$$

where $a_{m,n}$ are real, positive constants.

Proof. The proof is by induction. The claim holds for $k = 2$. Assume that it holds for a k_0 with $k_0 > 2$. Then, consider operating with (B.33) on a summand $(\tilde{\mathcal{E}}_2)^m\,(\mathcal{E}_4)^n$ of weight $2k_0 = 2m + 4n$. Using the relations (B.31), one infers

$$\left(q\frac{d}{dq} - \frac{k}{2}\,\mathcal{E}_2\right)(\tilde{\mathcal{E}}_2)^m\,(\mathcal{E}_4)^n = -\frac{1}{2}\,m\,(\tilde{\mathcal{E}}_2)^{m-1}\,(\mathcal{E}_4)^{n+1} - n(\tilde{\mathcal{E}}_2)^{m+1}\,(\mathcal{E}_4)^n\,. \tag{B.39}$$

Note that the terms on the right hand side have weight $2k_0 + 2$, and that they have the same structure as in (B.38). Thus, when operating with (B.33) on I_{k_0}, the resulting sum is of the form (B.38) with m, n satisfying the relations $m \geq 0$, $n \geq 1$ and $m + 2n = k_0 + 1$. Multiplying (B.39) with (-2π), so as to obtain $\mathcal{D}_S$ on the left hand side, we infer that the coefficients $a_{m,n}$ of I_{k_0+1} are real and positive.

$\square$

Next, we discuss the growth properties of the Fourier coefficients of modular forms for $\Gamma_0(2)$.

Proposition: let $f \in \mathcal{S}_k(\Gamma_0(2))$, with $k \in \mathbb{N}$,

$$f(\tau) = \sum_{n=1}^{\infty} a_n\,q^n\,, \quad q = e^{2\pi i \tau}\,. \tag{B.40}$$

Then $|a_n| \leq C\,n^{k/2} \quad \forall n \in \mathbb{N}$ for some $C \in \mathbb{R}$ with $C > 0$ [59].

Proposition: let $\Gamma = \Gamma_0(2)$. Consider the Eisenstein series (B.17) with $k \geq 2$,

$$\mathcal{E}_{2k}(\tau) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i \tau}. \tag{B.41}$$

Then [60]

$$|a_n| \leq C n^{2k-1} \quad \forall n \in \mathbb{N} \tag{B.42}$$

for some $C \in \mathbb{R}$ with $C > 0$.

Proof. For $k \geq 2$, using [60]

$$|\tilde{\sigma}_{2k-1}(n)| \leq \sum_{d|n} d^{2k-1} \leq n^{2k-1} \zeta(2k-1), \tag{B.43}$$

we infer

$$|\mathcal{E}_{2k}(\tau)| \leq 1 + C \sum_{n=1}^{\infty} n^{2k-1} |q|^n, \tag{B.44}$$

with $C = |4k\, \zeta(2k-1)/(1 - 2^{2k})B_{2k}|$. Hence $|a_n| \leq C n^{2k-1}$ for $n \in \mathbb{N}$. $\qquad \square$

Proposition: consider the modular form $\tilde{\mathcal{E}}_2$ of weight 2 given in (B.30),

$$\tilde{\mathcal{E}}_2(\tau) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i \tau}. \tag{B.45}$$

Then

$$|a_n| \leq C n \quad \forall n \in \mathbb{N} \tag{B.46}$$

for some $C \in \mathbb{R}$ with $C > 0$.

Proof. We consider $\tilde{\mathcal{E}}_2^2$ and use (B.29),

$$\tilde{\mathcal{E}}_2^2(\tau) = \mathcal{E}_4(\tau) + \frac{8}{30} \left(E_4(\tau) - E_4(2\tau) \right). \tag{B.47}$$

The right hand side is a linear combination of modular forms of weight 4. Each of them has q-expansion coefficients a_n that exhibit the property $|a_n| \leq C n^3$, cf. (B.42). Hence, if we denote the q-expansion of $\tilde{\mathcal{E}}_2^2$ by $\tilde{\mathcal{E}}_2^2 = \sum_{N \geq 0} c_N q^N$, the coefficients c_N will also satisfy $|c_N| \leq D N^3$, for some constant $D > 0$. Now consider (B.45), and let us assume that its coefficients a_n satisfy the bound $|a_n| \leq A n^p$, with $p \in \mathbb{N}$, $A > 0$. Then $|\tilde{\mathcal{E}}_2|^2 \leq \left(1 + A \sum_{m \geq 1} m^p |q|^m\right) \left(1 + A \sum_{n \geq 1} n^p |q|^n\right)$. Using absolute convergence, we obtain, for $N > 0$, $|c_N| \leq \left(2A N^p + A^2 \sum_{m+n=N,\, m,n \geq 1} m^p n^p\right)$. We now place an upper bound on the sum, for a given N, as follows. Extremizing $m^p n^p = (N-n)^p n^p$ with respect to n gives $n_* = N/2$, in which case $m_*^p n_*^p = N^{2p}/2^{2p}$. Using that the number of partitions of N into two positive integers m and n with $m + n = N$ is $N - 1$, we obtain the bound $\sum_{m+n=N,\, m,n \geq 1} m^p n^p \leq m_*^p n_*^p (N - 1) \leq B N^{2p+1}$, for some constant $B > 0$. Hence, $|c_N| \leq D N^{2p+1}$, for some constant $D > 0$. On the other hand, we had already concluded that $|c_N| \leq D N^3$, so that $p = 1$.

$\qquad \square$

Proposition: let $\Gamma = \Gamma_0(2)$. Consider the modular form I_k of weight $2k$ given in (B.38), with $k \geq 2$. We express its q-expansion as

$$I_k(\tau) = \sum_{n=0}^{\infty} a_n \, q^n \,, \quad q = e^{2\pi i \tau} \,. \tag{B.48}$$

Then

$$|a_n| \leq C \, n^{2k-1} \quad \forall \, n \in \mathbb{N} \tag{B.49}$$

for some $C \in \mathbb{R}$ with $C > 0$.

Proof. The claim holds for $k = 2$, since $I_2 \propto \mathcal{E}_4$. Let us consider the case $k > 2$.

We begin by considering a summand $\mathcal{E}_4^2$ of weight 8 in (B.38). Then, using (B.42), we obtain $|\mathcal{E}_4^2| \leq \left(1 + a \sum_{m \geq 1} m^3 |q|^m\right)\left(1 + a \sum_{n \geq 1} n^3 |q|^n\right)$ for some constant $a > 0$. Denoting the q-expansion of $\mathcal{E}_4^2$ by $\mathcal{E}_4^2 = \sum_{N \geq 0} c_N q^N$, and following the same steps as in the proof given above, we obtain the bound $|c_N| \leq D \, N^{8-1}$, for some constant $D > 0$.

Next, consider a summand $\mathcal{E}_4^p$ of weight $4p$ in (B.38). We proceed by induction. Denoting the q-expansion of $\mathcal{E}_4^p$ by $\mathcal{E}_4^p = \sum_{n \geq 0} b(n) q^n$, and assuming $|b(n)| \leq b \, n^{4p-1}$, for some constant $b > 0$, we proceed to show that the expansion coefficients of $\mathcal{E}_4^{p+1}$ exhibit the growth property $|c_N| \leq D \, N^{4(p+1)-1}$. Namely, proceeding as before, we obtain $|\mathcal{E}_4 \, \mathcal{E}_4^p| \leq \left(1 + a \sum_{m \geq 1} m^3 |q|^m\right)\left(1 + b \sum_{n \geq 1} n^{4p-1} |q|^n\right)$ for some constants $a, b > 0$. Extremizing $(N - n)^3 \, n^{4p-1}$ with respect to n gives $n_* = N(4p - 1)/(4p + 2)$, in which case $(N - n)_*^3 \, n_*^{4p-1} = N^{4p+2} \, 3^3 (4p - 1)^{4p-1}/(4p + 2)^{4p+2}$. Hence, we obtain the bound $\sum_{m+n=N, \, m,n \geq 1} m^3 \, n^{4p-1} \leq m_*^3 \, n_*^{4p-1} \, (N - 1) \leq B \, N^{4p+3}$ for some constant $B > 0$. Thus, the expansion coefficients of $\mathcal{E}_4^{p+1}$ exhibit the growth property $|c_N| \leq D \, N^{4(p+1)-1}$, as we wanted to show. Note that this growth goes as N^{2k-1} with $2k = 4(p + 1)$.

Proceeding in a similar manner, one finds that the expansion coefficients of $\tilde{\mathcal{E}}_2 \, \mathcal{E}_4^p$ exhibit the growth property $|c_N| \leq D \, N^{4p+1}$. By induction, one then shows that the expansion coefficients of $\tilde{\mathcal{E}}_2^l \, \mathcal{E}_4^p$ exhibit the growth property $|c_N| \leq D \, N^{4p+2l-1}$. This growth goes as N^{2k-1} with $2k = 4p + 2l$.

Hence we conclude that expansion coefficients of I_k in (B.38), with $k \geq 2$, exhibit the growth property (B.49).

$\square$

C Rankin-Cohen brackets

Definition: let $n \in \mathbb{N}_0$. The nth Rankin-Cohen bracket is a bilinear, differential operator that acts on modular forms f, g of $\mathrm{SL}(2, \mathbb{Z})$, of weight $k \in \mathbb{N}$ and $l \in \mathbb{N}$, respectively, by [68]

$$[f, g]_n(\tau) = \frac{1}{(2\pi i)^n} \sum_{r=0}^{n} (-)^r \binom{k + n - 1}{n - r} \binom{l + n - 1}{r} f^{(r)}(\tau) \, g^{(n-r)}(\tau) \,, \tag{C.1}$$

where $f^{(r)}(\tau)$ denotes the rth derivative of f with respect to τ, and similarly for $g^{(n-r)}(\tau)$.

In the following, we drop the normalization factor $\frac{1}{(2\pi i)^n}$.

Proposition: the nth Rankin-Cohen bracket $[f, g]_n$ is a modular form for $\mathrm{SL}(2, \mathbb{Z})$, of weight $k + l + 2n$ [68].

Example: let $f(S)$ be a modular form for $\mathrm{SL}(2, \mathbb{Z})$ of weight k, and let $g(S) = \vartheta_2^8(S)$, which has weight $l = 4$. Then,

$$\frac{1}{g}[f, g]_1 = -4 \mathcal{D}_S f, \tag{C.2}$$

where (cf. (B.32))

$$\mathcal{D}_S f = \left(\partial_S - \frac{k}{4} \partial_S \ln g \right) f. \tag{C.3}$$

In particular, consider the case $f(S) = I_2(S)$, which we recall is proportional to the Eisenstein series $\mathcal{E}_4$ of weight 4, cf. (B.35). Then,

$$
\begin{aligned}
\frac{1}{g}[I_2, g]_1 &= -4 I_3, \\
\frac{1}{g}[I_2, g]_2 &= 10 \left(I_4 - \frac{4}{\gamma} I_2^2 \right), \\
\frac{1}{g}[I_2, g]_3 &= -20 \left(I_5 - \frac{16}{\gamma} I_2 I_3 \right).
\end{aligned} \tag{C.4}
$$

For $n > 2$ we obtain

$$I_n = -\frac{1}{4g}[I_{n-1}, g]_1 = \left(\frac{-1}{4} \right)^2 \frac{1}{g} \frac{1}{g}[\frac{1}{g}[I_{n-2}, g]_1, g]_1 = \left(\frac{-1}{4} \right)^{n-2} \frac{1}{g}[\frac{1}{g}[\ldots[\frac{1}{g}[I_2, g]_1, \ldots, g]_1, g]_1. \tag{C.5}$$

The above definition can be extended to include quasi-modular forms as well [56]. Here we focus on quasi-modular forms of weight k and depth $s = 1$ of $\mathrm{SL}(2, \mathbb{Z})$. An example thereof is provided by (A.13).

Definition: let f be a quasi-modular form of weight k and depth $s = 1$, and let g be a modular form of weight l, respectively, for $\mathrm{SL}(2, \mathbb{Z})$. Then, their nth Rankin-Cohen bracket is given by

$$[f, g]_n(\tau) = \frac{1}{(2\pi i)^n} \sum_{r=0}^{n} (-)^r \binom{k - s + n - 1}{n - r} \binom{l + n - 1}{r} f^{(r)}(\tau) g^{(n-r)}(\tau), \tag{C.6}$$

where $f^{(r)}(\tau)$ denotes the rth derivative of f with respect to τ, and similarly for $g^{(n-r)}(\tau)$.

In the following, we drop the normalization factor $\frac{1}{(2\pi i)^n}$.

Proposition: the nth Rankin-Cohen bracket $[f, g]_n$ is a quasi-modular form for $\mathrm{SL}(2, \mathbb{Z})$, of weight $k + l + 2n$ and depth $\leq s = 1$ [56].

Example: as an application, set

$$f(S) = I_1(S) = \frac{\partial \omega}{\partial S}, \quad g(S) = \vartheta_2^8(S), \tag{C.7}$$

with weights $k = 2$ and $l = 4$, respectively. The depth of f is $s = 1$. Then

$$\frac{1}{g}[f, g]_1 = -4 \left(\frac{\partial^2 \omega}{\partial S^2} + \frac{1}{2\gamma} \left(\frac{\partial \omega}{\partial S} \right)^2 \right) = -4 I_2(S), \tag{C.8}$$

which yields a modular form of weight 4 and depth $s = 0$.

Remark: the Rankin-Cohen bracket (C.1) endows the graded vector space M_* of modular forms of $\mathrm{SL}(2, \mathbb{Z})$, $M_* \equiv \oplus_{k \geq 0} \mathcal{M}_k(\mathrm{SL}(2, \mathbb{Z}))$, with an infinite set of bilinear operators [68],

$$[\cdot, \cdot]_n : M_* \times M_* \to M_{*+*+2n}. \tag{C.9}$$

The 0th bracket makes $(M_*, [\cdot, \cdot]_0)$ into a commutative and associative algebra. The 1st bracket, on the other hand, makes $(M_*, [\cdot, \cdot]_1)$ into a graded Lie algebra, since the 1st bracket, given by

$$[f, g]_1 = -[g, f]_1 = k\, f\, g' - l\, f'\, g, \tag{C.10}$$

satisfies the Jacobi identity

$$[[f, g]_1, h]_1 + [[g, h]_1, f]_1 + [[h, f]_1, g]_1 = 0, \quad \forall f, g, h \in M_*. \tag{C.11}$$

The space $(M_*, [\cdot, \cdot]_0, [\cdot, \cdot]_1)$ is a Poisson algebra in view of the Leibnitz rule relation

$$[[f, g]_0, h]_1 = [[f, h]_1, g]_0 + [[g, h]_1, f]_0, \quad \forall f, g, h \in M_*. \tag{C.12}$$

We note the formal analogy between Rankin-Cohen brackets $[\cdot, \cdot]_1$ and the adjoint representation of the Lie algebra $\mathfrak{g}$ of a linear group G. To this end, let us recall the differential of the exponential map $\exp : \mathfrak{g} \to G$ [69]. Define a curve $a(t) = \exp(Z(t)) \subset G$ with $Z(t) = X + t\,Y$ $(X, Y \in \mathfrak{g})$. Then

$$\exp(-X) \left(\frac{d}{dt} a(t) \right)_{|t=0} = \exp(-X) \left(\frac{d}{dt} \exp(X + t\,Y) \right)_{|t=0} = \frac{1 - \exp(-ad(X))}{ad(X)} Y \tag{C.13}$$

where

$$\frac{1 - \exp(-ad(X))}{ad(X)} \equiv \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (ad(X))^k, \tag{C.14}$$

and

$$ad(X) : \mathfrak{g} \to \mathfrak{g},$$
$$Y \mapsto [X, Y]. \tag{C.15}$$

Now consider replacing the derivatives with respect to τ in (C.1) by the covariant derivative $\mathcal{D}_\tau$ given in (B.32), to obtain the nth Serre-Rankin-Cohen bracket [35] for $\Gamma_0(2)$ (we drop the normalization factor $\frac{1}{(2\pi i)^n}$),

$$SRC_n(f, g)(\tau) = \sum_{r=0}^{n} (-)^r \binom{k+n-1}{n-r} \binom{l+n-1}{r} \mathcal{D}_\tau^r f(\tau)\, \mathcal{D}_\tau^{n-r} g(\tau). \tag{C.16}$$

Then, the sequence $(SRC_n)_{n \in \mathbb{Z}_{\geq 0}}$ of bilinear maps defines a formal deformation [35] of $M_* \equiv \oplus_{k \geq 0} \mathcal{M}_k(\Gamma_0(2))$ through the non-commutative Eholzer product #,

$$f \# g = \sum_{n \geq 0} SRC_n(f,g) \, \hbar^n \,, \tag{C.17}$$

which is associative [68], i.e. given any modular forms f, g, p (of weight $k, l, h \in \mathbb{N}$, respectively),

$$\sum_{r=0}^{n} SRC_{n-r} \left(SRC_r(f,g), p \right) = \sum_{r=0}^{n} SRC_{n-r} \left(f, SRC_r(g,p) \right), \quad \forall n \in \mathbb{N}_0 \,. \tag{C.18}$$

Note that the Eholzer product (C.17) is suggestive of a deformation quantization of the Poisson algebra (C.12).

D   Jacobi forms

In this section, we follow [61, 70].

Let $\mathcal{H}$ denote the complex upper half plane. Let $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$.

Definition: a Jacobi form of SL$(2, \mathbb{Z})$ is a holomorphic function $\phi_{k,m} : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ that transforms as follows under the modular group SL$(2, \mathbb{Z})$,

$$\phi_{k,m} \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^k \, e^{2\pi i \frac{mcz^2}{c\tau + d}} \, \phi_{k,m}(\tau, z) \,, \quad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \,, \tag{D.1}$$

and under translations of z by $\mathbb{Z}\tau + \mathbb{Z}$ as

$$\phi_{k,m}(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m (\lambda^2 \tau + 2\lambda z)} \, \phi_{k,m}(\tau, z) \,, \quad \forall \quad \lambda, \mu \in \mathbb{Z} \,. \tag{D.2}$$

Here, $k \in \mathbb{Z}$ is called the weight, and $m \in \mathbb{N}$ is called the index of the Jacobi form.
Remark: τ is called modular parameter, while z is called elliptic parameter.

Due to the periodicities $\phi_{k,m}(\tau + 1, z) = \phi_{k,m}(\tau, z)$ and $\phi_{k,m}(\tau, z + 1) = \phi_{k,m}(\tau, z)$, $\phi_{k,m}$ possesses a Fourier expansion,

$$\phi_{k,m}(\tau, z) = \sum_{n,r \in \mathbb{Z}} c(n,r) \, q^n \, y^r \,, \quad q = e^{2\pi i \tau} \,, \quad y = e^{2\pi i z} \,. \tag{D.3}$$

The transformation behaviour under elliptic transformations (D.2) implies that

$$c(n,r) = C(\Delta, r) \,, \quad \Delta \equiv 4nm - r^2 \,, \tag{D.4}$$

where $C(\Delta, r)$ depends only on $r \mod 2m$.

Definition: $\phi_{k,m}$ is called a holomorphic Jacobi form of weight k and index m if

$$c(n,r) = 0 \,, \quad \text{for } \Delta < 0 \,, \tag{D.5}$$

i.e.

$$\phi_{k,m}(\tau, z) = \sum_{n \geq 0, \, r \in \mathbb{Z}, \, 4nm \geq r^2} c(n,r) \, q^n \, y^r \,. \tag{D.6}$$

Definition: $\phi_{k,m}$ is called a cuspidal holomorphic Jacobi form if

$$c(n,r) = 0\,, \quad \text{for } \Delta \leq 0\,, \tag{D.7}$$

i.e.

$$\phi_{k,m}(\tau, z) = \sum_{n \geq 1,\, r \in \mathbb{Z},\, 4nm > r^2} c(n,r)\, q^n\, y^r\,. \tag{D.8}$$

Definition: $\phi_{k,m}$ is called a weak holomorphic Jacobi form if

$$c(n,r) = 0\,, \quad \text{for } n < 0\,, \tag{D.9}$$

i.e. if

$$\phi_{k,m}(\tau, z) = \sum_{n \geq 0,\, r \in \mathbb{Z}} c(n,r)\, q^n\, y^r\,. \tag{D.10}$$

Definition: $\phi_{k,m}$ is called a weakly holomorphic Jacobi form if

$$c(n,r) = 0\,, \quad \text{for } n < -n_0,\, , n_0 \in \mathbb{N}\,, \tag{D.11}$$

i.e. if

$$\phi_{k,m}(\tau, z) = \sum_{n \geq -n_0,\, r \in \mathbb{Z}} c(n,r)\, q^n\, y^r\,. \tag{D.12}$$

Depending on the nature of the Jacobi form, the coefficients $c(n,r)$ will have a certain growth property [61].

Remark: the above generalizes to Jacobi forms of subgroups $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ [70].

Example: consider $\vartheta_2^8(\tau, z)$, where $\vartheta_2(\tau, z)$ denotes the Jacobi theta function defined by

$$\vartheta_2(\tau, z) = q^{1/8} \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n(n+1)}\, e^{i\pi(2n+1)z}\,, \tag{D.13}$$

which has the following product representation, valid for $|q| < 1$,

$$\vartheta_2(\tau, z) = 2\, q^{1/8} \cos(\pi z) \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n y)\,(1 + q^n y^{-1})\,. \tag{D.14}$$

For fixed z, $\vartheta_2(\tau, z)$ has one branch point inside the unit circle of the complex q-plane, at $q = 0$, and a branch cut along the interval $(-1, 0)$ on the real axis. $\vartheta_2(\tau, z)$ has zeroes at $z = \frac{1}{2}(2m+1) + n\tau$ with $(m,n) \in \mathbb{Z}^2$.

$\vartheta_2^8(\tau, z)$, on the other hand, is analytic in both $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$, and is an example of a modular form of weight $k = 4$ and index $m = 4$ with trivial multiplyer system under $\Gamma_0(2)$-transformations.

Remark: ϑ_2 is even with respect to z,

$$\vartheta_2(\tau, -z) = \vartheta_2(\tau, z)\,, \tag{D.15}$$

and it solves the heat equation

$$\frac{\partial}{\partial S}\vartheta_2(S, z) = \frac{1}{4\pi}\frac{\partial^2}{\partial z^2}\vartheta_2(S, z)\,, \tag{D.16}$$

with periodic boundary conditions ($z \to z + 1$) imposed in the z direction. Here $\tau = iS$.

Also observe that when taking an odd number of derivatives with respect to z, we have

$$\frac{\partial^n}{\partial z^n}\vartheta_2(S,z)\Big|_{z=0} = 0 \quad \forall n \in 2\mathbb{N}+1 \tag{D.17}$$

due to (D.15).

Remark: consider $f(S,z) = \ln\vartheta_2^8(S,z)$. We fix S, and we view $\vartheta_2^8(S,z)$ as a function of z, which is non-vanishing in a small open neighbourhood of $z=0$. It is then possible to define a single-valued analytic branch of $\ln\vartheta_2^8(S,z)$ in that neighbourhood. We then Taylor expand around $z=0$, retaining only the first few terms for illustrative purposes,

$$f(S,z) = \ln\vartheta_2^8(S) + \frac{z^2}{2!}\partial_z^2 f|_{z=0} + \frac{z^4}{4!}\partial_z^4 f|_{z=0} + \frac{z^6}{6!}\partial_z^6 f|_{z=0} + \frac{z^8}{8!}\partial_z^8 f|_{z=0} + \mathcal{O}(z^{10})\,. \tag{D.18}$$

Using (D.15) and (D.17) to convert an even number of derivatives with respect to z into derivatives with respect to S, we obtain

$$f(S,z) = -\frac{2}{\gamma}\left(\omega(S) + 4\pi\frac{z^2}{2!}I_1(S) + (4\pi)^2\frac{z^4}{4!}I_2(S) + (4\pi)^3\frac{z^6}{6!}I_3(S)\right.$$
$$\left. + (4\pi)^4\frac{z^8}{8!}\left(I_4(S) + \frac{5}{\gamma}I_2^2(S)\right) + \mathcal{O}(z^8)\right)\,, \tag{D.19}$$

where

$$\omega(S) = -\frac{\gamma}{2}\ln\vartheta_2^8(S)\,, \quad \gamma = -\frac{1}{256\pi}\,, \tag{D.20}$$

and where the I_n are defined in (3.20) and in (3.23).

Thus, the expansion (D.19) is indicative of an expansion in powers of $I_n(S)$, as follows:

Proposition:

$$f(S,z) = -\frac{2}{\gamma}\left[\omega(S) + 4\pi\frac{z^2}{2!}I_1(S) + \sum_{n=2}^{\infty}(4\pi)^n\frac{z^{2n}}{(2n)!}I_n(S)\right.$$
$$\left. + \sum_{m,n\geq 2}(4\pi)^{m+n}f_{m,n}z^{2(m+n)}I_m(S)I_n(S) + \dots\right]\,, \tag{D.21}$$

where the dots stand for higher powers of I_n.

Proof. Let us first focus on the sector involving single powers of I_n only, and let us verify the expression for the monomials in I_n given in (D.21). Let us work in an open neighbourhood of $z=0$. First, observe that under $\Gamma_0(2)$-transformations (3.10), the first two terms in (D.21), $\omega(S) + 4\pi\frac{z^2}{2!}I_1(S)$, transform precisely as $f(S,z)$. Therefore, the Taylor series in (D.18) starting with the term z^4 has to be invariant under $\Gamma_0(2)$-transformations. Since each power of z transforms as $z \mapsto z/[\Delta(S)]$ under $\Gamma_0(2)$, it follows that each summand in (D.18) starting with the term z^4 must be separately invariant under $\Gamma_0(2)$, i. e. $\partial_z^{2n}f|_{z=0} \mapsto [\Delta(S)]^{2n}\partial_z^{2n}f|_{z=0}$ for $n \geq 2$. Using

$$\partial_z^{2n}f|_{z=0} = 8\frac{\partial_z^{2n}\vartheta_2(S,z)}{\vartheta_2(S,z)}|_{z=0} + \dots\,, \tag{D.22}$$

where the dots stand for products of terms involving a lower number of z-derivatives of $\vartheta_2(S,z)$, we infer, using (D.16),

$$\partial_z^{2n} f|_{z=0} = 8\,(4\pi)^n \, \frac{\partial_S^n \vartheta_2(S)}{\vartheta_2(S)} + \cdots = (4\pi)^n \, \partial_S^n \ln \vartheta_2^8(S) + \cdots = -\frac{2(4\pi)^n}{\gamma} \, I_n(S) + \dots \,. \quad \text{(D.23)}$$

Covariance under $\Gamma_0(2)$-transformations dictates that the terms involving a lower number of z-derivatives of $\vartheta_2(S,z)$ have to organize themselves into products of $I_k(S)$, so that the dots in (D.23) stand for higher powers of $I_k(S)$, such that under $\Gamma_0(2)$-transformations, $\partial_z^{2n} f|_{z=0} \mapsto [\Delta(S)]^{2n} \, \partial_z^{2n} f|_{z=0}$. Thus we establish (D.21).

$\square$

Another way of obtaining the same result is as follows. We note that $f(S,z) = \ln \vartheta_2^8(S,z)$, satisfies the non-linear PDE

$$32\pi \, \frac{\partial}{\partial S} f(S,z) = 8 \, \frac{\partial^2}{\partial z^2} f(S,z) + \left(\frac{\partial}{\partial z} f(S,z) \right)^2 , \quad \text{(D.24)}$$

as a consequence of (D.16), with boundary conditions $f(S,0) = \ln \vartheta_2^8(S)$, $\partial_z f|_{z=0} = 0$, $\partial_z^2 f|_{z=0} = -8\pi \, I_1(S)/\gamma$. Let us verify that (D.21) solves (D.24) when restricting to the monomial sector in (D.21). To distinguish between the various sectors, we rescale ω and each power of I_n in (D.21) by a real constant $\lambda \in \mathbb{R}$, in which case (D.21) becomes

$$f(S,z,\lambda) = -\frac{2}{\gamma} \left[\lambda\,\omega(S) + \lambda\,4\pi \, \frac{z^2}{2!} \, I_1(S) + \lambda \sum_{n=2}^{\infty} (4\pi)^n \, \frac{z^{2n}}{(2n)!} \, I_n(S) + \mathcal{O}(\lambda^2) \right]. \quad \text{(D.25)}$$

Inserting (D.25) into (D.24) (we verify below that we may differentiate the series (D.25) term by term) and working to first order in λ, we establish that (D.25) solves (D.24) at this order. Then, setting $\lambda = 1$ yields the expression for the monomials given in (D.21).

The uniqueness of solutions of the form (D.25) can be established as follows. Suppose that the differential equation (D.24) admits a solution $f(S,z)$ that is expressed as a power series expansions in z^2,

$$f(S,z) = \sum_{n \geq 0} f_n(S) z^{2n} , \quad \text{(D.26)}$$

satisfies the boundary conditions given above, and that can be differentiated term by term. Then, inspection of (D.24) shows that $f_2(S) = -4\pi^2 I_2(S)/(3\gamma)$, $f_3(S) = 2\pi \mathcal{D}_S f_2(S)/15$, and that the functions $f_{n+1}(S)$ with $n \geq 3$ are determined recursively by

$$(2n+2)(2n+1) \, f_{n+1}(S) = c_1 \mathcal{D}_S f_n(S) + c_2 \sum_{k+l=n+1;\,k,l>1} k\,l\,f_k(S)f_l(S) , \quad \text{(D.27)}$$

where c_1, c_2 are numerical constants. Hence, a solution of the form (D.26) satisfying the conditions mentioned above is uniquely specified.

Thus, we have the following corollary:

Corollary: let

$$g(S, z) \equiv \sum_{n=2}^{\infty} (4\pi)^n \frac{z^{2n}}{(2n)!} I_n(S) \,. \tag{D.28}$$

$g(S, z)$ is the unique $\Gamma_0(2)$-invariant solution, that is given by a Taylor series in even powers of z, to the PDE

$$4\pi \frac{\partial}{\partial S} g(S, z) = \frac{\partial^2}{\partial z^2} g(S, z) - \frac{2\pi}{\gamma} I_1(S) \, z \frac{\partial}{\partial z} g(S, z) - \frac{(4\pi)^2}{2} z^2 I_2(S) \,, \tag{D.29}$$

with boundary conditions $g|_{z=0} = \partial_z g|_{z=0} = \partial_z^2 g|_{z=0} = \partial_z^3 g|_{z=0} = 0$.

Proof. Let

$$g(S, z) \equiv \sum_{n=2}^{\infty} (4\pi)^n \frac{z^{2n}}{(2n)!} p_n(S) \,. \tag{D.30}$$

Imposing $\Gamma_0(2)$ invariance shows that $p_n(S)$ are modular forms for $\Gamma_0(2)$ of weight $2n$. Inserting (D.30) into (D.29) and differentiating term by term, which will be justified below, one infers $p_2(S) = I_2(S)$ and $p_{n+1}(S) = \mathcal{D}_S p_n(S)$ for $n \geq 2$. Hence, $p_n = I_n$.

$\square$

In the main part of the paper, we encountered a closely related series, namely (cf. (3.25))

$$H(S, z) \equiv \sum_{n=2}^{\infty} c^n \frac{z^{2n}}{n!} I_n(S) \,, \tag{D.31}$$

with $c \in \mathbb{C}$ a constant. Note that the suppression factor is now $1/n!$, instead of $1/(2n)!$ in (D.28). The PDE which $H(S, z)$ satisfies will thus be different from (D.29), cf. (3.28). For fixed S, this series is a power series in z, and hence both absolutely convergent in a open neighbourhood $D(0, R)$ of $z = 0$ as well as uniformly convergent on any compact subset contained in $D(0, R)$. The radius of convergence R will depend on S. We can infer a lower bound on R, as follows. First, using (B.25), we infer that for large $\operatorname{Re} S$ (i.e. $|q| \to 0$), $\omega(S)$ behaves as $\ln q \sim S$, and hence, $\partial \omega / \partial S$ is constant, up to exponentially suppressed corrections. Thus, at $q = 0$, I_k (with $k \geq 2$) is given by

$$I_k(q = 0) = \frac{(k-1)!}{2\gamma^{k-1}} \left(\frac{\partial \omega}{\partial S} \right)^k |_{q=0} \,. \tag{D.32}$$

On the other hand, evaluating (B.38) at $q = 0$, we infer

$$\sum_{m+2n=k, \, m \geq 0, n \geq 1} a_{m,n} = \frac{(k-1)!}{2\gamma^k} \left(\frac{\partial \omega}{\partial S} \right)^k |_{q=0} = \frac{(k-1)!}{2} \pi^k > 0 \,. \tag{D.33}$$

Using (B.30) and (B.44), we obtain (with $|q| < 1$)

$$|\tilde{\mathcal{E}}_2| \leq 1 + C_2 \sum_{n=1}^{\infty} n^2 |q|^n = 1 + C_2 \left(|q|\frac{d}{d|q|}\right)^2 \frac{|q|}{1-|q|} \equiv f_2(|q|), \quad C_2 = 24,$$

$$|\mathcal{E}_4| \leq 1 + C_4 \sum_{n=1}^{\infty} n^3 |q|^n = 1 + C_4 \left(|q|\frac{d}{d|q|}\right)^3 \frac{|q|}{1-|q|} \equiv f_4(|q|), \quad C_4 = 80\,\zeta(3),$$

$$(D.34)$$

and hence, using (B.38),

$$|I_k(S)| \leq |\gamma| \left|f_2^k(|q|)\right| \sum_{m+2n=k,\, m\geq 0, n\geq 1} a_{m,n} |h^n(|q|)|, \tag{D.35}$$

where we recall that $a_{m,n}$ are positive, and where

$$h(|q|) \equiv \frac{f_4(|q|)}{f_2^2(|q|)}. \tag{D.36}$$

If, for a fixed q, $|h(|q|)| \leq 1$, we may approximate (D.35) by

$$|I_k(S)| \leq |\gamma| \left|f_2^k(|q|)\right| \frac{(k-1)!}{2} \pi^k, \tag{D.37}$$

whereas if $|h(|q|)| > 1$, we have the bound $|h^n(|q|)| \leq |h(|q|)|^{[k/2]}$, and hence we approximate (D.35) by

$$|I_k(S)| \leq |\gamma| \left|f_2^k(|q|)\right| |h(|q|)|^{[k/2]} \frac{(k-1)!}{2} \pi^k. \tag{D.38}$$

Therefore, we obtain the following lower bound for the radius of convergence R,

$$\frac{1}{R} = \lim_{k\to\infty} (|c|^k |I_k(S)|/k!)^{1/k} \leq \pi |c| |f_2(q)| \begin{cases} 1, & |h(|q|)| \leq 1 \\ \sqrt{|h(|q|)|}, & |h(|q|)| > 1. \end{cases} \tag{D.39}$$

The functions $f_2(|q|)$ and $f_4(|q|)$ are monotonically increasing in the interval $0 \leq |q| < 1$, and they blow up at $|q| = 1$. The function $h(|q|)$ is bounded, $0 \leq h(|q|) < 1.5$. Thus, so long as $|q| \leq r < 1$, we have $R > 0$.

Next, let us consider fixing z in (D.31). Then, using (D.35), we infer

$$|H(S,z)| \leq \frac{1}{2}|\gamma| \sum_{k=2}^{\infty} \frac{|\pi\,c\,z^2\,f_2(|q|)|^k}{k} \begin{cases} 1, & |h(|q|)| \leq 1 \\ |h(|q|)|^{[k/2]}, & |h(|q|)| > 1. \end{cases} \tag{D.40}$$

The right hand side is a convergent series for $|q| < 1$, provided that $|\pi\,c\,z^2\,f_2(|q|)| < 1$ and $|\pi\,c\,z^2\,f_2(|q|)|\sqrt{|h(|q|)|} < 1$. Consider $|q| \leq r < 1$. Then, since $f_2(|q|)$ and $f_4(|q|)$ are monotonically increasing functions, we infer $|\pi\,c\,z^2\,f_2(|q|)| \leq |\pi\,c\,z^2\,f_2(r)| < 1$ and $|\pi\,c\,z^2\,f_2(|q|)|\sqrt{|h(|q|)|} \leq |\pi\,c\,z^2\,f_2(r)|\sqrt{|h(r)|} < 1$. Thus, by the Weierstrass M test, the function $H(S,z)$ converges both absolutely and uniformly for all $|q| \leq r < 1$, provided
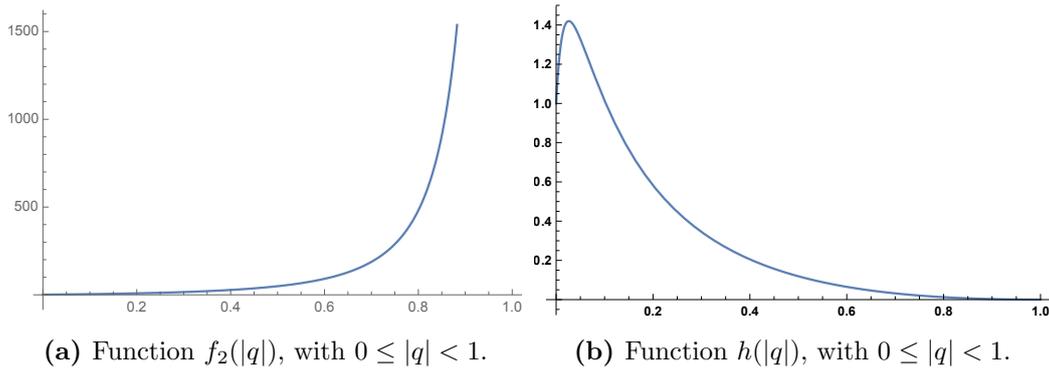
(a) Function $f_2(|q|)$, with $0 \le |q| < 1$. **(b)** Function $h(|q|)$, with $0 \le |q| < 1$.

Figure 3. Behaviour of the functions $f_2(|q|)$ and $h(|q|)$.

$|\pi\, c\, z^2| < 1/M$, with $M = \max\{f_2(r), f_2(r)\sqrt{|h(r)|}\}$. In particular, it converges absolutely and uniformly in an open neighbourhood of $(q, z) = (0, 0)$.

Next, let us consider the series of S-derivatives,

$$\sum_{n=2}^{\infty} c^n \frac{z^{2n}}{n!} \partial_S I_n(S) \,. \tag{D.41}$$

Proceeding as above we obtain the bound

$$|\partial_S I_k(S)| \le \pi^{k+1} |\gamma|\, k!\, v(|q|)\, |f_2(|q|)|^k \begin{cases} 1, & |h(|q|)| \le 1 \\ |h(|q|)|^{[k/2]}, & |h(|q|)| > 1 \,, \end{cases} \tag{D.42}$$

where the function $v(|q|)$ is given by

$$v(|q|) = 24 f_2^{-1}(|q|) \left(|q| \frac{d}{d|q|} \right)^3 \frac{|q|}{1 - |q|} + \frac{1}{2} C_4 f_4^{-1}(|q|) \left(|q| \frac{d}{d|q|} \right)^4 \frac{|q|}{1 - |q|} \,. \tag{D.43}$$

The function $v(|q|)$ is monotonically increasing in the interval $0 \le |q| < 1$ and blows up at $|q| = 1$. Hence we obtain the estimate

$$\sum_{k=2}^{\infty} |c^k \frac{z^{2k}}{k!} \partial_S I_k(S)| \le \pi |\gamma|\, v(|q|) \sum_{k=2}^{\infty} |\pi c\, z^2\, f_2(|q|)|^k \begin{cases} 1, & |h(|q|)| \le 1 \\ |h(|q|)|^{[k/2]}, & |h(|q|)| > 1 \,. \end{cases} \tag{D.44}$$

Thus, at fixed z, this series converges both absolutely and uniformly for all $|q| \le r < 1$, provided $|\pi\, c\, z^2| < 1/M$, with $M = \max\{f_2(r), f_2(r)\sqrt{|h(r)|}\}$, as above. In particular, it converges absolutely and uniformly in an open neighbourhood of $(q, z) = (0, 0)$.

Thus, at fixed z satisfying $|\pi\, c\, z^2| < 1/M$, both (D.31) and (D.41) converge absolutely and uniformly for all $|q| \le r < 1$, and hence the S-derivative of (D.31) equals (D.41).

In the main part of the paper, we also encountered the series (cf. (3.25))

$$\sum_{m,n \ge 2} c^{m+n} \beta_{m,n} z^{2(m+n)} I_m(S)\, I_n(S) \,, \tag{D.45}$$

with $c \in \mathbb{C}$ a constant. For large m, n, the coefficients behave as $\beta_{m,n} \propto 1/[(m-1)!\,(n-1)!]$, and the series (D.45) becomes

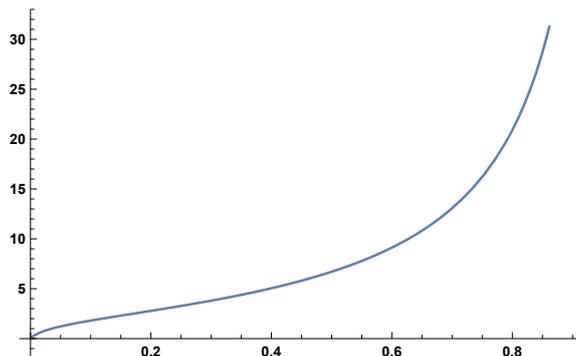$$\left(z^2 \partial_{z^2} H(S, z) \right)^2 \,, \tag{D.46}$$

Figure 4. Function $v(|q|)$, with $0 \le |q| < 1$.

with $H(S, z)$ given by (D.31). Thus, the series (D.45) has the same convergence properties as $H(S, z)$.

E   Siegel modular forms

In this section, we follow [60, 70] as well as appendix A of [71].

Definition: siegel's upper half plane $\mathcal{H}_2$ is defined by

$$\mathcal{H}_2 = \left\{ \Omega \in \mathrm{Mat}(2 \times 2, \mathbb{C}) : \Omega^T = \Omega, \ \mathrm{Im}\,\Omega > 0 \right\}, \tag{E.1}$$

where $\mathrm{Im}\,\Omega > 0$ means that $\mathrm{Im}\,\Omega$ is a positive definite matrix, i.e. $\mathcal{H}_2$ consists of complex symmetric matrices

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \tag{E.2}$$

with $\rho_2 > 0$, $\sigma_2 > 0$ and $\det(\mathrm{Im}\Omega) > 0$, i.e. $\rho_2\,\sigma_2 - v_2^2 > 0$, where $\rho = \rho_1 + i\rho_2$, $\sigma = \sigma_1 + i\sigma_2$, $v = v_1 + iv_2$.

The Siegel modular group $\mathrm{Sp}(4, \mathbb{Z})$ acts on $\mathcal{H}_2$ as follows. An element $M \in \mathrm{Sp}(4, \mathbb{Z})$,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{E.3}$$

can be written in terms of four real two-by-two blocks A, B, C, and D satisfying $A^T D - C^T B = D\,A^T - C\,B^T = \mathbb{I}_2$, $A^T C = C^T A$, $B^T D = D^T B$. Then, $M \in \mathrm{Sp}(4, \mathbb{Z})$ acts on $\mathcal{H}_2$ by

$$\Omega \mapsto \Omega' = (A\,\Omega + B)\,(C\,\Omega + D)^{-1}. \tag{E.4}$$

One then obtains $\Omega - \bar{\Omega} \mapsto (\Omega\,C^T + D^T)^{-1}(\Omega - \bar{\Omega})\,(C\,\bar{\Omega} + D)^{-1}$, which shows that $(\mathrm{Im}\,\Omega)'$ is positive definite.

A standard fundamental domain for the action of $\mathrm{Sp}(4, \mathbb{Z})$ on $\mathcal{H}_2$ is the set defined by [72]

$$-\frac{1}{2} \le \rho_1, \sigma_1, v_1 \le \frac{1}{2}, \quad 0 < 2v_2 \le \rho_2 \le \sigma_2, |\det(C\Omega + D)| \ge 1. \tag{E.5}$$

The condition $|\det(C\Omega + D)| \ge 1$ applies to all $\mathrm{Sp}(4, \mathbb{Z})$ transformations of Ω.

Next, we discuss the action of various subgroups of $\text{Sp}(4,\mathbb{Z})$ on $\mathcal{H}_2$:

1. $\text{SL}(2,\mathbb{Z})_\rho$ (which leaves $\sigma_2 - (v_2)^2/\rho_2$ invariant):

$$g_1(a,b,c,d) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\rho, v, \sigma) \mapsto \left(\frac{a\rho+b}{c\rho+d}, \frac{v}{c\rho+d}, \sigma - \frac{cv^2}{c\rho+d} \right),$$

(E.6)

where $a,b,c,d \in \mathbb{Z}$, $\quad ad - bc = 1$.

2. $\text{SL}(2,\mathbb{Z})_\sigma$ (which leaves $\rho_2 - (v_2)^2/\sigma_2$ invariant):

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad (\rho, v, \sigma) \mapsto \left(\rho - \frac{cv^2}{c\sigma+d}, \frac{v}{c\sigma+d}, \frac{a\sigma+b}{c\sigma+d} \right), \quad \text{(E.7)}$$

where $a,b,c,d \in \mathbb{Z}$, $\quad ad - bc = 1$.

3. Heis_ρ (which leaves $\Omega - \bar{\Omega}$ invariant)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\rho, v, \sigma) \mapsto (\rho, v + \lambda\rho + \mu, \sigma + 2\lambda v + \lambda^2 \rho + \lambda\mu + \kappa), \quad \text{(E.8)}$$

where $\lambda, \mu, \kappa \in \mathbb{Z}$.

4. Heis_σ (which leaves $\Omega - \bar{\Omega}$ invariant)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \lambda & \kappa & \mu \\ 0 & 1 & \mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 \end{pmatrix}, \quad (\rho, v, \sigma) \mapsto (\rho + 2\lambda v + \lambda^2 \sigma + \lambda\mu + \kappa, v + \lambda\sigma + \mu, \sigma), \quad \text{(E.9)}$$

where $\lambda, \mu, \kappa \in \mathbb{Z}$.

5. $\rho \leftrightarrow \sigma$, $\quad v \to -v$:

$$g_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad (\rho, v, \sigma) \mapsto (\sigma, -v, \rho). \quad \text{(E.10)}$$

In the main part of the paper, we are interested in the subgroup $H \subset \text{Sp}(4,\mathbb{Z})$ consisting of elements $h \in H$ with $h = g_1(a,b,c,d)\, g_2\, g_1(a,-b,-c,d)\, (g_2)^{-1}$ with the restriction $c = 0$

mod 2, $a, d = 1 \mod 2$,

$$h = g_1(a,b,c,d)\, g_2\, g_1(a,-b,-c,d)\, (g_2)^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2). \tag{E.11}$$

Definition: a Siegel modular form Φ_k of weight $k \in \mathbb{N}$ with respect to the full Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$ is a holomorphic function $\Phi_k : \mathcal{H}_2 \to \mathbb{C}$ that satisfies

$$\Phi_k((A\,\Omega + B)\,(C\,\Omega + D)^{-1}) = (\det(C\,\Omega + D))^k\, \Phi_k(\Omega), \quad \forall\, M \in \mathrm{Sp}(4,\mathbb{Z}). \tag{E.12}$$

Remark (Köcher's principle): a Siegel modular form Φ_k is bounded on any subset of $\mathcal{H}_2$ of the form $\{\Omega \in \mathcal{H}_2 | \mathrm{Im}(\Omega) \geq \mathrm{Im}(\Omega_0)\}$, for any $\mathrm{Im}(\Omega_0) > 0$.

Theorem: let Φ_k be a Siegel modular form of weight $k \in \mathbb{N}$. It has the Fourier development [70]

$$\Phi_k(\rho, \sigma, v) = \sum_{m \geq 0} \psi_{k,m}(\rho, v)\, e^{2\pi i m \sigma}. \tag{E.13}$$

For $m > 0$, the function $\psi_{k,m}$ is a holomorphic Jacobi form of weight $k \in \mathbb{N}$ and index m. For $m = 0$, the function $\psi_{k,0}$ transforms as a Jacobi form with $m = 0$. If the first coefficient $\psi_{k,0}$ is identically zero, the Siegel form is called Siegel cusp form.

Remark: the Fourier expansion of Φ_k is

$$\Phi_k(\rho, \sigma, v) = \sum_{n,m,r \in \mathbb{Z};\, n,m,4mn-r^2 \geq 0} A(n,m,r)\, e^{2\pi i (n\rho + m\sigma + rv)}. \tag{E.14}$$

It is well defined on $\mathcal{H}_2$.

Theorem: let $\psi_{k,1}$ be a holomorphic Jacobi form of weight k and index 1. Then the functions $T_m \psi_{k,1}$, with $m \geq 1$, defined in terms of the Hecke lift below, are the Fourier coefficients of a Siegel modular form of weight k [70].

Remark: these theorems also hold if $\mathrm{Sp}(4,\mathbb{Z})$ is replaced by a congruence subgroup $\Gamma \subset \mathrm{Sp}(4,\mathbb{Z})$. We shall be interested in the congruence subgroup $\Gamma = \Gamma_{2,0}(2)$ of $\mathrm{Sp}(4,\mathbb{Z})$, where one restricts to $C = 0 \mod 2$. In particular, we shall be interested in the subgroup H given by (E.11). Then, the functions $\psi_{k,m}$ in the Fourier development are Jacobi forms for $\Gamma_0(2) \subset \mathrm{SL}(2,\mathbb{Z})$.

F Hecke operators for SL(2, ℤ)

We follow [60]. Let $f \in \mathcal{M}_k(\mathrm{SL}(2,\mathbb{Z}))$ with Fourier expansion $f(\tau) = \sum_{n\geq 0} a_n q^n$. Let m be an integer with $m \geq 1$.

Definition: the m-th Hecke operator is a linear operator T_m that acts on modular forms of weight k by

$$
T_m : \mathcal{M}_k(\mathrm{SL}(2,\mathbb{Z})) \to \mathcal{M}_k(\mathrm{SL}(2,\mathbb{Z}))
$$
$$
f(\tau) \mapsto T_m f(\tau) = m^{k-1} \sum_{\alpha\delta=m,\ \alpha,\delta>0} \delta^{-k} \sum_{0\leq\beta<\delta} f\left(\frac{\alpha\tau+\beta}{\delta}\right), \quad \text{(F.1)}
$$

with $\alpha, \beta, \delta \in \mathbb{Z}$.

Proposition: [60]

$$
T_m f(\tau) = \sum_{n\geq 0} \left(\sum_{r|(m,n),\ r>0} r^{k-1}\, a_{mn/r^2} \right) q^n . \quad \text{(F.2)}
$$

Proof. Using $\sum_{0\leq\beta<\delta} e^{2\pi i\beta n/\delta} = \delta$ if $\delta|n$, and 0 otherwise, we get

$$
T_m f(\tau) = m^{k-1} \sum_{n\geq 0} \sum_{\alpha\delta=m,\ \alpha,\delta>0} \delta^{1-k}\, a_n\, q^{\alpha n/\delta} . \quad \text{(F.3)}
$$

Since $\delta|n$, setting $n' = \alpha n/\delta \in \mathbb{Z}$ and using $n' = \alpha n/\delta = \alpha^2 n/m \geq 0$, yields

$$
T_m f(\tau) = \sum_{n'\geq 0} \left(\sum_{\alpha|(m,n')} \alpha^{k-1}\, a_{mn'/\alpha^2} \right) q^{n'} , \quad \text{(F.4)}
$$

where we used $m/\alpha = \delta$, hence $\alpha|m$, and $n = mn'/\alpha^2 = (m/\alpha)(n'/\alpha)$, and hence $\alpha|n'$.

Note that $T_m f(\tau)$ is holomorphic at ∞. □

Now consider the case when $a_0 = 0$ and $a_1 = 1$. Suppose also that $f(\tau)$ is an eigenfunction of all the Hecke operators, i.e. $T_m f = \lambda_m f$, with eigenvalues λ_m. f is then called a Hecke eigenform. The first term of $\lambda_m f$ is $\lambda_m a_1 = \lambda_m$, whereas the first term of $T_m f$ (cf. (F.4)) is the term with $n' = 1$ given by a_m. Hence, $\lambda_m = a_m$. On the other hand, $T_m f = \lambda_m f$ implies

$$\sum_{\alpha | (m,n')} \alpha^{k-1} a_{mn'/\alpha^2} = \lambda_m a_{n'}, \tag{F.5}$$

and hence [60]

$$\sum_{\alpha | (m,n')} \alpha^{k-1} a_{mn'/\alpha^2} = a_m a_{n'}, \quad m, n' \geq 1. \tag{F.6}$$

Below we will encounter a similar relation when discussing the Hecke lift of a Jacobi form.

G Arithmetic lift of a particular Jacobi form for $\Gamma_0(2)$

In the following, we focus on a particular cuspidal holomorphic Jacobi form for $\Gamma_0(2)$ of weight $k = 2$ and index $m = 1$, and we discuss its arithmetic lift to a Siegel modular form Φ_2 of weight 2. The lift of this particular Jacobi form was briefly discussed in the context of black holes in [9], following the analysis given in [5]. Here, we first review this construction, and then we use the technique of Hecke eigenforms of the previous subsection to prove the decomposition (G.15) in the limit $v = 0$.

Jacobi forms $\phi_{2,1}$ of weight 2 and index $m = 1$ transform as follows under modular and elliptic transformations (cf. (D.1)),

$$\phi_{2,1}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^2 e^{2\pi i \frac{cz^2}{c\tau + d}} \phi_{2,1}(\tau, z),$$
$$\phi_{2,1}(\tau, z + \lambda\tau + \mu) = e^{-2\pi i (\lambda^2 \tau + 2\lambda z)} \phi_{2,1}(\tau, z) \quad \forall \quad \lambda, \mu \in \mathbb{Z}. \tag{G.1}$$

Here, we restrict to $\Gamma_0(2)$ transformations, i.e. $c = 0 \mod 2$, $a, d = 1 \mod 2$.

We will consider the following Jacobi form of $\Gamma_0(2)$ of weight $k = 2$ and index $m = 1$ (with trivial multiplyer system) as a seed for the arithmetic lift,

$$\phi_{2,1}(\tau, z) = \frac{\vartheta_1^2(\tau, z)}{\eta^6(\tau)} f(\tau), \quad f(\tau) = 2^{-8} \vartheta_2^8(\tau), \tag{G.2}$$

with $\vartheta_1(\tau, z)$ one of the Jacobi theta functions. Note that $f(\tau)$, which has weight 4, is not a cusp form of $\Gamma_0(2)$ (the first cusp form of $\Gamma_0(2)$ arises at weight 8; since $\dim \mathcal{S}_k(\Gamma_0(2)) = [\frac{k}{4}] - 1$, the vector space of cusp forms of weight 8 has dimension one, and is spanned by $\vartheta_2^8(\tau) \mathcal{E}_4(\tau)$ [65], which equals $[\eta(\tau)\eta(2\tau)]^8$, up to a normalization constant, as can be verified by using the relation given below (B.29) and $\vartheta_2^4(\tau) = 16\eta^8(2\tau)/\eta^4(\tau)$. The form $[\eta(\tau)\eta(2\tau)]^8$ is the standard cusp form of $\Gamma_0(2)$).

$\phi_{2,1}$ has a Fourier expansion in both τ and z given by

$$\phi_{2,1}(\tau, z) = \sum_{l,r \in \mathbb{Z}, \, l \geq 1, \, 4l > r^2} c(4l - r^2) \, e^{2\pi i l \tau} \, e^{2\pi i r z}, \tag{G.3}$$

with [5, 9]

$$c(\Delta) = (-1)^{\Delta} \sum_{s,n \in \mathbb{Z}, \, n \geq 1} f_n \, \delta_{4n+s^2-1,\Delta} \,, \qquad (G.4)$$

where f_n denote the Fourier coefficients of

$$\frac{f(\tau)}{\eta^6(\tau)} = \sum_{n \geq 1} f_n \, e^{2\pi i \tau (n - \frac{1}{4})} \,. \qquad (G.5)$$

Note the relations $4n + s^2 - 1 > 0$, $\Delta = 4l - r^2 > 0$, and hence $c(\Delta) = 0$ for $\Delta \leq 0$. Hence, $\phi_{2,1}$ is a cuspidal holomorphic Jacobi form of $\Gamma_0(2)$. Note that the coefficients f_n are integers, and so are the coefficients $c(\Delta)$.

For latter use, we note that in the limit $z \to 0$ we have $\vartheta_1(\tau, z) = 2\pi \, \eta^3(\tau) \, z + \mathcal{O}(z^3)$, and hence, we infer

$$\phi_{2,1}(\tau, z) = 4\pi^2 \, f(\tau) \, z^2 + \mathcal{O}(z^4) \,. \qquad (G.6)$$

Next, we construct Jacobi forms $\phi_{2,m}$ of $\Gamma_0(2)$ of weight $k = 2$ and index $m \in \mathbb{N}$ by means of the Hecke operator T_m, which maps $\phi_{2,1}$ in (G.3) to $\phi_{2,m}$. The action of the Hecke operator T_m on a Jacobi form $\phi_{k,1}$ is defined in a manner similar to (F.1) [70],

$$\phi_{2,m}(\tau, z) = T_m \phi_{2,1}(\tau, z) = m \sum_{\alpha\delta = m, \, \alpha,\delta > 0, \, \alpha = 1 \bmod 2} \delta^{-2} \sum_{0 \leq \beta < \delta} \phi_{2,1} \left(\frac{\alpha\tau + \beta}{\delta}, \frac{mz}{\delta} \right) \,. \quad (G.7)$$

One then verifies [5] that $\phi_{2,m}(\tau, z)$ transforms as a Jacobi form of weight $k = 2$ and index m under modular and elliptic transformations (D.1) and (D.2). $\phi_{2,m}(\tau, z)$ can be brought to the form

$$\phi_{2,m}(\tau, z) = \sum_{n,r \in \mathbb{Z}, \, n \geq 1, \, 4mn > r^2} a(n, m, r) \, e^{2\pi i n \tau} \, e^{2\pi i r z} \,, \qquad (G.8)$$

with Fourier expansion coefficients [5, 9]

$$a(n, m, r) = \sum_{\alpha \in \mathbb{Z}, \, \alpha > 0, \, \alpha = 1 \bmod 2, \, \alpha | (n,m,r)} \alpha \, c \left(\frac{4mn - r^2}{\alpha^2} \right) \,. \qquad (G.9)$$

Note that $a(n, m, r) = 0$ for $4mn - r^2 \leq 0$. The expression (G.9) is obtained in a manner analogous to (F.2). The Fourier coefficients $a(n, m, r)$ are integer valued.

Next, define the Siegel cusp form $\Phi_2(\rho, \sigma, v)$ by [9]

$$\Phi_2(\rho, \sigma, v) = \sum_{m \geq 1} \phi_{2,m}(\rho, v) \, e^{2\pi i m \sigma} = \sum_{n,m,r \in \mathbb{Z}, \, n,m \geq 1, \, 4mn - r^2 > 0} a(n, m, r) \, e^{2\pi i (n\rho + m\sigma + rv)} \,, \qquad (G.10)$$

where we have relabelled $\tau \to \rho, z \to v$. By construction, $\Phi_2(\rho, \sigma, v)$ is invariant under the exchange $\rho \leftrightarrow \sigma$, since the coefficients $a(n, m, r)$ are invariant under the exchange $n \leftrightarrow m$. $\Phi_2(\rho, \sigma, v)$ transforms as a modular form under a certain subgroup G of $\mathrm{Sp}(4, \mathbb{Z})$ [9]. Here, we will focus on the following subgroup H of G that consists of elements $h \in H$ given by (E.11). Under $H \subset \mathrm{Sp}(4, \mathbb{Z})$, Φ_2 transforms as [5]

$$\Phi_2((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^2 \, \Phi_2(\Omega) \,, \qquad (G.11)$$

with Ω given in (E.2). Introducing $d^3\Omega = d\rho \, d\sigma \, dv$, we note that

$$\frac{d^3\Omega}{(2v - \rho - \sigma)^5 \, \Phi_2} \tag{G.12}$$

is invariant under H-transformations. The combination (G.12) is one of the building blocks of the microstate proposal. Since $\Phi_2(\rho, \sigma, v)$ has integer valued Fourier coefficients (cf. (G.10)), so does $1/\Phi_2(\rho, \sigma, v)$ when expanded in powers of $e^{2\pi i (n\rho + m\sigma + rv)}$.

Finally, as $v \to 0$, we obtain from (G.10), using (G.7) and (G.6),

$$\Phi_2(\rho, \sigma, v) = v^2 \, F_2(\rho, \sigma) + \mathcal{O}(v^4) \,. \tag{G.13}$$

with

$$F_2(\rho, \sigma) = 4\pi^2 \sum_{m \geq 1} m^3 \, e^{2\pi i m\sigma} \sum_{\alpha\delta = m, \, \alpha,\delta > 0, \, \alpha = 1 \bmod 2} \delta^{-4} \sum_{0 \leq \beta < \delta} f\left(\frac{\alpha\rho + \beta}{\delta}\right) \,. \tag{G.14}$$

Although not apparent, $F_2(\rho, \sigma)$ is symmetric under the exchange $\rho \leftrightarrow \sigma$, and given by:

Proposition:

$$F_2(\rho, \sigma) = 4\pi^2 \, f(\rho) \, f(\sigma) \,. \tag{G.15}$$

Proof. Using the Fourier expansion of $f(\rho)$,

$$f(\rho) = \sum_{n \geq 1} a_n \, e^{2\pi i n\rho} \,, \tag{G.16}$$

and similarly to (F.2), we rewrite (G.14) into

$$F_2(\rho, \sigma) = 4\pi^2 \sum_{m \geq 1, n' \geq 1} e^{2\pi i m\sigma} \, e^{2\pi i n'\rho} \sum_{\alpha \in \mathbb{Z}, \, \alpha > 0, \, \alpha = 1 \bmod 2, \, \alpha | (m,n')} \alpha^3 \, a_{n'm/\alpha^2} \,. \tag{G.17}$$

Now we use that $f(\rho)$ can be expressed as in (B.29),

$$f(\rho) = 2^{-8} \, \vartheta_2^8(\rho) = \frac{1}{240} \left(E_4(\rho) - E_4(2\rho) \right) \,. \tag{G.18}$$

Inspection of (F.1), (G.14) and (G.17) shows that

$$T_m f(\rho) = \sum_{n' \geq 1} e^{2\pi i n'\rho} \sum_{\alpha \in \mathbb{Z}, \, \alpha > 0, \, \alpha = 1 \bmod 2, \, \alpha | (m,n')} \alpha^3 \, a_{n'm/\alpha^2} \,. \tag{G.19}$$

Now we use that $T_m f(\rho)$ is a modular form for $\Gamma_0(2)$ of weight 4. The vector space $\mathcal{M}_4(\Gamma_0(2))$ is two-dimensional and spanned by $\tilde{\mathcal{E}}_2^2$ and $\mathcal{E}_4$ [65]. Since (by inspection of (G.19)) $T_m f(\rho)$ does not have a constant term in its Fourier expansion, $T_m f$ must be proportional to the combination $\tilde{\mathcal{E}}_2^2 - \mathcal{E}_4$, i.e.

$$T_m f(\rho) = \lambda_m \left(E_4(\rho) - E_4(2\rho) \right) \,, \tag{G.20}$$

and hence f is a Hecke eigenform,

$$T_m f(\rho) = 240 \, \lambda_m f(\rho) \,, \forall \, m \geq 1 \,. \tag{G.21}$$

Using $a_1 = 1$ and repeating the argument that led to (F.6), we obtain

$$a_m \, a_{n'} = \sum_{\alpha \in \mathbb{Z}, \, \alpha > 0, \, \alpha = 1 \bmod 2, \, \alpha | (m, n')} \alpha^3 \, a_{n'm/\alpha^2} \,, \tag{G.22}$$

which ensures (G.15).

$\square$

We perform numerical checks on the validity of (G.22). We take $f(\rho) = \left(\frac{1}{2}\right)^8 \vartheta_2^8(\rho)$, and obtain for the first coefficients of the Fourier expansion,

$$f(\rho) = q + 8q^2 + 28q^3 + 64q^4 + 126q^5 + 224q^6 + 344q^7 + 512q^8 + 757q^9 + \ldots \tag{G.23}$$

We then test (G.22) by taking $m = n' = 1$ ($\alpha = 1 \to a_1^2 = a_1$), $m = 1, n' = 2$ ($\alpha = 1 \to a_1 a_2 = a_2$), $m = n' = 2$ ($\alpha = 1 \to a_2^2 = a_4$), $m = n' = 3$ ($\alpha = 1, 3 \to a_3^2 = a_9 + 27a_1$), to find perfect agreement.

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