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UC and BUC plane partitions

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Abstract This paper is concerned with the investigation of UC and BUC plane partitions based upon the fermion calculus approach. We construct generalized the vertex operators in terms of free charged fermions and neutral fermions and present the interlacing (strict) 2-partitions. Furthermore, it is showed that the generating functions of UC and BUC plane partitions can be written as product forms.

Mathematics Subject Classification 17B80 · 35Q55 · 37K10

Contents

1 Introduction

Free charged fermions and neutral fermions proposed by Kyoto school [1–5] play a crucial role in construction of τ-functions of integrable systems such as KP and BKP hierarchies. Tsuda [6] introduced the universal character (UC) hierarchy which is the generalization of KP hierarchy. Then Ogawa [7] constructed UC hierarchy of B-type (BUC hierarchy) which can be regarded as the extension of BKP hierarchy. The algebraic structures of UC and BUC hierarchies have been well discussed based upon the free fermions and neutral fermions [6–8]. By means of fermion calculus [5], the relations between vertex operators and KP plane partitions have been developed [9]. Fermionic approach is a extremely useful tool in exploring the structure and properties of integrable systems. Ünal [10,11] presented the τ-functions of the KP and BKP hierarchies as determinants and Pfaffians with charged free fermions and neutral free fermions.

Plane partitions are generated in crystal melting model [12,13] which have widely applications in various fields of mathematics and physics, such as statistical models, number theory and representation theory. The generating function of plane partitions describes the characteristics of plane partitions which has widely application in combinatorics [14,15], statistical mechanics [16,17] and integrable systems [18]. Okounkov et al. [19] analyzed generating function for plane partitions in terms of vertex operators expressed as exponentials of bilinear in fermions. Then the partition functions of the topological string theory have been developed by the fermion calculus approach [20]. Recently, Wang et al. [21,22] investigated 3-dimensional (3D) Boson representation of $W_{1+\infty}$ algebra and studied Littlewood-Richardson rule for 3-Jack polynomials by acting 3D Bosons on 3D Young diagrams (plane partitions). By using the fermion calculus approach, Foda et al. [9,23] established the product forms for the generating function of KP and BKP plane partitions based on the KP free charge fermions and BKP neutral fermions, respectively. It is also proved the generating function is a special τ-function of the 2D Toda lattice. The aim of this paper is to investigate the generating function of plane partitions for UC and BUC hierarchies.

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The paper is organized as follows. Section 2 provides a review of the fundamental facts of free fermions, plane partitions and generating functions. Section 3 is devoted to investigation of the UC plane partitions by fermion calculus approach. We introduce the interlacing partitions are presented with half-integers and construct interlacing 2-partitions, from which a product form of the generating function for UC plane partitions are derived. In Sect. 4, By introducing generating interlacing strict 2-partitions, we study the generating function for BUC plane partitions. The last section is conclusions and discussions.

2 Preliminaries

In this section, we mainly retrospect basic facts about free fermions, plane partitions and generating functions [5,6,23–26].

2.1 Charged fermions and UC hierarchy

ψ_m, ψ_m^*, ϕ_m and ϕ_m^* ($m \in \mathbb{Z}+\frac{1}{2}$) are charged fermions, the charge of the fermions is given by

Fermion	ψ_n	ψ_n^*	ϕ_n	ϕ_n^*
Charge	(1, 0)	(−1, 0)	(0, 1)	(0, −1)

Algebra $\mathcal{A}$ over $\mathbb{C}$ is generated by the commutation relations

$$[\psi_m,\psi_n]_+ = [\psi_m^*,\psi_n^*]_+ = 0, [\psi_m,\psi_n^*]_+ = \delta_{m+n,0},$$
$$[\phi_m,\phi_n]_+ = [\phi_m^*,\phi_n^*]_+ = 0, [\phi_m,\phi_n^*]_+ = \delta_{m+n,0},$$
$$[\psi_m,\phi_n] = [\psi_m,\phi_n^*] = [\psi_m^*,\phi_n] = [\psi_m^*,\phi_n^*] = 0, \tag{2.1}$$

and $\psi_n^2 = \psi_n^{*2} = \phi_n^2 = \phi_n^{*2} = 0$.

A Maya diagram is made up of black and white stones lined up along the real line, indexed by half-integers. It is required that far away to the right (when $n \gg 0$) all the stones are black, whereas far away to the left (when $n \ll 0$), they are all white. By writing $\alpha_j \in \mathbb{Z}+\frac{1}{2}$ for the position of the black stone, we can describe a Maya diagram as an increasing sequence of half-integers

$$\boldsymbol{\alpha}=\{\alpha_j\}_{j\geq 1}=(\alpha_1,\alpha_2,\alpha_3,\ldots) \quad \text{with } \alpha_1<\alpha_2<\alpha_3<\cdots, \tag{2.2}$$

that satisfies the following conditions

$$\text{(i) } \alpha_j<\alpha_{j+1} \quad \text{for all } j\geq 1,$$
$$\text{(ii) } \alpha_{j+1}=\alpha_j+1 \quad \text{for all sufficiently large } j. \tag{2.3}$$

The right state corresponding to the Maya diagram α is defined as

$$|\boldsymbol{\alpha}\rangle = |\alpha_1,\alpha_2,\alpha_3\cdots\rangle. \tag{2.4}$$

A left action of the fermions is given by the following rules

$$\psi_n|\boldsymbol{\alpha}\rangle = \begin{cases} (-1)^{i-1}|\ldots,\alpha_{i-1},\alpha_{i+1},\ldots\rangle, & \\ \quad \text{if } \alpha_i=-n \text{ for some } i, & \\ 0, & \text{otherwise,}\end{cases} \tag{2.5}$$

$$\psi_n^*|\boldsymbol{\alpha}\rangle = \begin{cases} (-1)^{i}|\ldots,\alpha_{i},n,\alpha_{i+1},\ldots\rangle, & \\ \quad \text{if } \alpha_i<n<\alpha_{i+1} \text{ for some } i, & \\ 0, & \text{otherwise.}\end{cases} \tag{2.6}$$

In particular,

$$\psi_n|\boldsymbol{\alpha}\rangle = |\alpha_2,\alpha_3,\ldots\rangle \quad \text{for } \alpha_1=-n, \tag{2.7}$$
$$\psi_n^*|\boldsymbol{\alpha}\rangle = |n,\alpha_1,\alpha_2,\ldots\rangle \quad \text{for } n<\alpha_1. \tag{2.8}$$

Similarly, a Maya diagram can also be represented as

$$\boldsymbol{\alpha} = \{\alpha'_j\}_{j\geq 1} = (\ldots,\alpha'_3,\alpha'_2,\alpha'_1) \quad \text{with } \cdots<\alpha'_3<\alpha'_2 <\alpha'_1, \tag{2.9}$$

where $\alpha'_j \in \mathbb{Z}+\frac{1}{2}$ denotes the position of the white stone and $\alpha'_{j+1}=\alpha'_j-1$ for all sufficiently large j. The left state corresponding to the Maya diagram α is denoted as

$$\langle\boldsymbol{\alpha}| = \langle\ldots,\alpha'_3,\alpha'_2,\alpha'_1|. \tag{2.10}$$

A right action of the fermions is given by

$$\langle\boldsymbol{\alpha}|\psi_n = \begin{cases} (-1)^{i}\langle\ldots,\alpha'_{i+1},n,\alpha'_i,\ldots|, & \text{if } \alpha'_{i+1}<n<\alpha'_i \\ \quad \text{for some } i, & \\ 0, & \text{otherwise,}\end{cases} \tag{2.11}$$

$$\langle\boldsymbol{\alpha}|\psi_n^* = \begin{cases} (-1)^{i-1}\langle\ldots,\alpha'_{i+1},\alpha'_{i-1},\ldots|, & \text{if } n=-\alpha'_i \\ \quad \text{for some } i, & \\ 0, & \text{otherwise,}\end{cases} \tag{2.12}$$

while ϕ and ϕ^* have respectively the same action as ψ and ψ^*, except replacing ψ with ϕ. Particularly, the vacuum state $|\text{vac}\rangle$ and the dual vacuum state $\langle\text{vac}|$ are defined as

$$|\text{vac}\rangle = |\frac{1}{2},\frac{3}{2},\ldots\rangle \quad \text{and} \quad \langle\text{vac}| = \langle\ldots,-\frac{3}{2},-\frac{1}{2}|, \tag{2.13}$$

which satisfy

$$\begin{aligned}\psi_n|\text{vac}\rangle &= \psi_n^*|\text{vac}\rangle \\ &= \phi_n|\text{vac}\rangle = \phi_n^*|\text{vac}\rangle = 0 \quad \text{for } n > 0, \\ \langle\text{vac}|\psi_n &= \langle\text{vac}|\psi_n^* \\ &= \langle\text{vac}|\phi_n = \langle\text{vac}|\phi_n^* = 0 \quad \text{for } n < 0. \end{aligned} \tag{2.14}$$

The charged fermionic Fock space $\mathcal{F}$ and the dual Fock space $\mathcal{F}^*$ are generated by

$$\begin{aligned}\mathcal{F} &\overset{\text{def}}{=} \mathcal{A} \cdot |\text{vac}\rangle = \{a|\text{vac}\rangle \mid a \in \mathcal{A}\}, \\ \mathcal{F}^* &\overset{\text{def}}{=} \langle\text{vac}| \cdot \mathcal{A} = \{\langle\text{vac}|a \mid a \in \mathcal{A}\}, \end{aligned} \tag{2.15}$$

where

$$a = \psi_{m_1} \cdots \psi_{m_r} \psi_{n_1}^* \cdots \psi_{n_s}^* \phi_{\tilde{m}_1'} \cdots \phi_{\tilde{m}_i'} \phi_{\tilde{n}_1'}^* \cdots \phi_{\tilde{n}_j'}^*. \tag{2.16}$$

The vector subspace of $\mathcal{F}$ with charge (l_1, l_2) is written as $\mathcal{F}_{l_1,l_2}$. Consider a pairing $\langle\ \ \rangle : \mathcal{F}^* \times \mathcal{F} \to \mathbb{C}$ denoted by

$$(\langle\text{vac}|a, b|\text{vac}\rangle) \longmapsto \langle\text{vac}|a \cdot b|\text{vac}\rangle = \langle ab\rangle, \tag{2.17}$$

where $\langle\ \ \rangle$ is called the vacuum expectation value. The following properties hold

$$\langle\text{vac}|vac\rangle = 1, \quad \langle\psi_m\psi_n^*\rangle = \langle\phi_m\phi_n^*\rangle = \begin{cases}\delta_{m+n,0} & (m > 0), \\ 0 & (\text{otherwise}).\end{cases} \tag{2.18}$$

The UC hierarchy is a system satisfying the following bilinear identity

$$\sum_{j\in\mathbb{Z}+1/2} \psi_{-j}|u\rangle \otimes \psi_j^*|u\rangle = \sum_{j\in\mathbb{Z}+1/2} \phi_{-j}|u\rangle \otimes \phi_j^*|u\rangle = 0, \tag{2.19}$$

where $|u\rangle \in \mathcal{F}_{0,0}$ has charge $(0, 0)$.

Define the colon operator $:\ \ :$ as

$$:\psi_m\psi_n^*: = \psi_m\psi_n^* - \langle\psi_m\psi_n^*\rangle, \quad :\phi_m\phi_n^*: = \phi_m\phi_n^* - \langle\phi_m\phi_n^*\rangle. \tag{2.20}$$

Consider the operators H_n and $\tilde{H}_n$ $(n \in \mathbb{Z})$,

$$H_n = \sum_{j\in\mathbb{Z}+1/2} :\psi_{-j}\psi_{j+n}^*:, \quad \tilde{H}_n = \sum_{j\in\mathbb{Z}+1/2} :\phi_{-j}\phi_{j+n}^*:. \tag{2.21}$$

Then the following properties hold

$$\begin{aligned}&[H_n, \psi_m] = \psi_{m+n}, \quad [H_n, \psi_m^*] = -\psi_{m+n}^*, \\ &[H_m, H_n] = m\delta_{m+n,0}, [\tilde{H}_n, \phi_m] = \phi_{m+n}, \\ &[\tilde{H}_n, \phi_m^*] = -\phi_{m+n}^*, \quad [\tilde{H}_m, \tilde{H}_n] = m\delta_{m+n,0}. \end{aligned} \tag{2.22}$$

Noting

$$H_n|\text{vac}\rangle = \tilde{H}_n|\text{vac}\rangle = 0 \quad \text{if } n > 0. \tag{2.23}$$

The operators called Hamiltonian are defined as

$$\begin{aligned}H_\pm(\mathbf{x}, \mathbf{y}; \partial_\mathbf{x}, \partial_\mathbf{y}) = \sum_{n\in\pm\mathbb{N}} \Bigg\{ &\left(x_n - \frac{1}{n}\frac{\partial}{\partial y_n}\right) H_n \\ &+ \left(y_n - \frac{1}{n}\frac{\partial}{\partial x_n}\right)\tilde{H}_n \Bigg\}, \end{aligned} \tag{2.24}$$

along with the generating functions of charged fermions

$$\begin{aligned}\psi(k) &= \sum_{n\in\mathbb{Z}+1/2} \psi_n k^{-n-1/2}, \quad \psi^*(k) = \sum_{n\in\mathbb{Z}+1/2} \psi_n^* k^{-n-1/2}, \\ \phi(k) &= \sum_{n\in\mathbb{Z}+1/2} \phi_n k^{-n-1/2}, \quad \phi^*(k) = \sum_{n\in\mathbb{Z}+1/2} \phi_n^* k^{-n-1/2}. \end{aligned} \tag{2.25}$$

For convenience, $H_\pm(\mathbf{x}, \mathbf{y}; \partial_\mathbf{x}, \partial_\mathbf{y})$ is represented as $H_\pm(\mathbf{x}, \mathbf{y})$.

Proposition 2.1 *The commutative relations between the Hamiltonian $H_\pm(\mathbf{x}, \mathbf{y})$ and generating functions of charged fermions are as follows*

$$\begin{aligned}&[H_\pm(\mathbf{x}, \mathbf{y}), \psi(k)] = \xi_\pm(\mathbf{x} - \tilde{\partial}_\mathbf{y}, k)\psi(k), \\ &[H_\pm(\mathbf{x}, \mathbf{y}), \psi(k)^*] = -\xi_\pm(\mathbf{x} - \tilde{\partial}_\mathbf{y}, k)\psi^*(k), \\ &[H_\pm(\mathbf{x}, \mathbf{y}), \phi(k)] = \xi_\pm(\mathbf{y} - \tilde{\partial}_\mathbf{x}, k)\phi(k), \\ &[H_\pm(\mathbf{x}, \mathbf{y}), \phi(k)^*] = -\xi_\pm(\mathbf{y} - \tilde{\partial}_\mathbf{x}, k)\phi^*(k), \end{aligned} \tag{2.26}$$

where

$$\begin{aligned}&\xi_\pm(x, k) = \sum_{n\in\pm\mathbf{N}} x_n k^n, \\ &\tilde{\partial}_\mathbf{y} = \left(\frac{\partial}{\partial y_1}, \frac{1}{2}\frac{\partial}{\partial y_2}, \frac{1}{3}\frac{\partial}{\partial y_3}, \ldots\right). \end{aligned} \tag{2.27}$$

Proof By means of Eqs. (2.22) and (2.24), we obtain

$$\begin{aligned}&[H_\pm(\mathbf{x}, \mathbf{y}), \psi(k)] \\ &= \sum_{\substack{n\in\mathbb{N} \\ m\in\mathbb{Z}+1/2}} \left(x_{\pm n} - \frac{1}{\pm n}\frac{\partial}{\partial y_{\pm n}}\right) \cdot \left[H_{\pm n}, \psi_m\right] \cdot k^{-m-\frac{1}{2}} \\ &= \sum_{n\in\pm\mathbb{N}} \left(x_n - \frac{1}{n}\frac{\partial}{\partial y_n}\right) k^n \sum_{m'\in\mathbb{Z}+1/2} \psi_{m'} k^{-m'-1/2} \\ &= \xi_\pm(\mathbf{x} - \tilde{\partial}_\mathbf{y}, k)\psi(k). \end{aligned} \tag{2.28}$$

The other formulas can be proved in the same way. □

Lemma 2.2 *The following equations hold*

$$
\begin{aligned}
e^{H_\pm(\mathbf{x},\mathbf{y})}\psi(k)e^{-H_\pm(\mathbf{x},\mathbf{y})} &= e^{\xi_\pm(\mathbf{x}-\tilde{\partial}_\mathbf{y},k)}\psi(k),\\
e^{H_\pm(\mathbf{x},\mathbf{y})}\psi^*(k)e^{-H_\pm(\mathbf{x},\mathbf{y})} &= e^{-\xi_\pm(\mathbf{x}-\tilde{\partial}_\mathbf{y},k)}\psi^*(k),\\
e^{H_\pm(\mathbf{x},\mathbf{y})}\phi(k)e^{-H_\pm(\mathbf{x},\mathbf{y})} &= e^{\xi_\pm(\mathbf{y}-\tilde{\partial}_\mathbf{x},k)}\phi(k),\\
e^{H_\pm(\mathbf{x},\mathbf{y})}\phi^*(k)e^{-H_\pm(\mathbf{x},\mathbf{y})} &= e^{-\xi_\pm(\mathbf{y}-\tilde{\partial}_\mathbf{x},k)}\phi^*(k).
\end{aligned}
\tag{2.29}
$$

Proof From the Eq. (2.26) , it follows that

$$
\begin{aligned}
&e^{H_\pm(\mathbf{x},\mathbf{y})}\psi(k)e^{-H_\pm(\mathbf{x},\mathbf{y})}\\
&\quad= \psi(k) + [H_\pm(\mathbf{x},\mathbf{y}),\psi(k)]\\
&\qquad+\frac{1}{2!}[H_\pm(\mathbf{x},\mathbf{y}),[H_\pm(\mathbf{x},\mathbf{y}),\psi(k)]]+\cdots\cdots\\
&\quad= \psi(k)+\xi_\pm(\mathbf{x}-\tilde{\partial}_\mathbf{y},k)\psi(k)+\frac{1}{2!}\xi_\pm^2(\mathbf{x}-\tilde{\partial}_\mathbf{y},k)\psi(k)+\cdots\\
&\quad= e^{\xi_\pm(\mathbf{x}-\tilde{\partial}_\mathbf{y},k)}\psi(k).
\end{aligned}
\tag{2.30}
$$

Using the similar procedure, we can prove other equations. □

Remark 2.3 Under the reduction $\phi_m = \phi_m^* = 0$, Eq. (2.19) leads to bilinear identity of KP hierarchy. The Eqs. (2.1)–(2.29) leads to definitions and properties in KP hierarchy.

2.2 Neutral fermions and BUC hierarchy

In this section, we introduce neutral fermions ϕ_n and $\bar{\phi}_m$ $(n, m \in \mathbf{Z})$, which are generators of the algebra $\widetilde{\mathcal{A}}$ over $\mathbb{C}$ and satisfy

$$
\begin{aligned}
&[\phi_m,\phi_n]_+ = [\bar{\phi}_m,\bar{\phi}_n]_+ = (-1)^m\delta_{m+n,0},\quad [\phi_m,\bar{\phi}_n]=0,\\
&\phi_0^2=\bar{\phi}_0^2=\frac{1}{2}.
\end{aligned}
\tag{2.31}
$$

The neutral fermionic Fock space $\widetilde{\mathcal{F}}$ and the dual Fock space $\widetilde{\mathcal{F}}^*$ can be defined as

$$
\begin{aligned}
\widetilde{\mathcal{F}} &\overset{\text{def}}{=} \widetilde{\mathcal{A}}\cdot|0\rangle = \{a|0\rangle | a\in\widetilde{\mathcal{A}}\},\\
\widetilde{\mathcal{F}}^* &\overset{\text{def}}{=} \langle 0|\cdot\widetilde{\mathcal{A}} = \{\langle 0|a \mid a\in\widetilde{\mathcal{A}}\},
\end{aligned}
\tag{2.32}
$$

where the vacuum state $|0\rangle$ and the dual vacuum state $\langle 0|$ are denoted by

$$
\begin{aligned}
&\phi_n|0\rangle = \bar{\phi}_n|0\rangle = 0 \quad \text{for } n<0,\\
&\langle 0|\phi_n = \langle 0|\bar{\phi}_n = 0 \quad \text{for } n>0.
\end{aligned}
\tag{2.33}
$$

Introduce the operators H'_m and $\bar{H}'_m$ $(m \in \mathbf{N}_{\text{odd}})$

$$
H'_m = \frac{1}{2}\sum_{j\in\mathbf{Z}}(-1)^{j+1}\phi_j\phi_{-j-m},
$$

$$
\bar{H}'_m = \frac{1}{2}\sum_{j\in\mathbf{Z}}(-1)^{j+1}\bar{\phi}_j\bar{\phi}_{-j-m}. \tag{2.34}
$$

Note that

$$
\begin{aligned}
&[H'_m,\phi_n]=\phi_{n-m},\quad [\bar{H}'_m,\bar{\phi}_n]=\bar{\phi}_{n-m},\\
&[H'_m,H'_n]=[\bar{H}'_m,\bar{H}'_n]=\frac{m}{2}\delta_{m+n,0}.
\end{aligned}
\tag{2.35}
$$

In particular,

$$
H'_m|0\rangle = \bar{H}'_m|0\rangle = 0 \quad \text{if } m>0. \tag{2.36}
$$

The Hamiltonian is written as

$$
\begin{aligned}
H'_\pm(\mathbf{x},\mathbf{y}) = \sum_{l\in\pm\mathbf{N}_{\text{odd}}}\Bigg\{&\left(x_l-\frac{2}{l}\frac{\partial}{\partial y_l}\right)H'_l\\
&+\left(y_l-\frac{2}{l}\frac{\partial}{\partial x_l}\right)\bar{H}'_l\Bigg\}.
\end{aligned}
\tag{2.37}
$$

It should be noted that BUC hierarchy satisfies the bilinear identity, which is given in [7].

Lemma 2.4 *For the generating functions of neutral fermions,*

$$
\phi(k)=\sum_{n\in\mathbf{Z}}\phi_n k^n,\quad \bar{\phi}(k)=\sum_{n\in\mathbf{Z}}\bar{\phi}_n k^n, \tag{2.38}
$$

we have

$$
\begin{aligned}
e^{H'_\pm(\mathbf{x},\mathbf{y})}\phi(k)e^{-H'_\pm(\mathbf{x},\mathbf{y})} &= e^{\zeta_\pm(\mathbf{x}-2\tilde{\partial}'_\mathbf{y},k)}\phi(k),\\
e^{H'_\pm(\mathbf{x},\mathbf{y})}\bar{\phi}(k)e^{-H'_\pm(\mathbf{x},\mathbf{y})} &= e^{\zeta_\pm(\mathbf{y}-2\tilde{\partial}'_\mathbf{x},k)}\bar{\phi}(k),
\end{aligned}
\tag{2.39}
$$

where

$$
\zeta_\pm(\mathbf{x},k)=\sum_{n\in\pm\mathbf{N}_{\text{odd}}}x_n k^n,\quad \tilde{\partial}'_\mathbf{y}=\left(\frac{\partial}{\partial y_1},\frac{1}{3}\frac{\partial}{\partial y_3},\frac{1}{5}\frac{\partial}{\partial y_5},\ldots\right). \tag{2.40}
$$

Proof From Eqs. (2.35) and (2.37), we obtain

$$
\begin{aligned}
[H'_\pm(\mathbf{x},\mathbf{y}),\phi(k)] &= \sum_{l\in\pm\mathbf{N}_{\text{odd}}}\left(x_l-\frac{2}{l}\frac{\partial}{\partial y_l}\right)k^l\phi(k)\\
&= \zeta_\pm(\mathbf{x}-2\tilde{\partial}'_\mathbf{y},k)\phi(k),\\
[H'_\pm(\mathbf{x},\mathbf{y}),\bar{\phi}(k)] &= \sum_{l\in\pm\mathbf{N}_{\text{odd}}}\left(y_l-\frac{2}{l}\frac{\partial}{\partial x_l}\right)k^l\bar{\phi}(k)\\
&= \zeta_\pm(\mathbf{y}-2\tilde{\partial}'_\mathbf{x},k)\bar{\phi}(k).
\end{aligned}
\tag{2.41}
$$

Therefore, we have

$$e^{H'_{\pm}(\mathbf{x},\mathbf{y})}\psi(k)e^{-H'_{\pm}(\mathbf{x},\mathbf{y})}=e^{\zeta_{\pm}(\mathbf{x}-\tilde{\partial}'_{\mathbf{y}},k)}\psi(k). \tag{2.42}$$

The proof of the other formula is quite similar, so is omitted. □

Consider the neutral fermion vertex operators

$$\Upsilon_+(\mathbf{z})=e^{H'_+(\mathbf{z})}=\exp\left(\sum_{n\in+\mathbb{N}_{\text{odd}}}\frac{2}{n}z^{-n}H'_n\right),$$
$$\Upsilon_-(\mathbf{z})=e^{-H'_-(\mathbf{z})}=\exp\left(\sum_{n\in+\mathbb{N}_{\text{odd}}}\frac{2}{n}z^{n}H'_{-n}\right), \tag{2.43}$$

where

$$H'_+(\mathbf{z})=\sum_{n\in+\mathbb{N}_{\text{odd}}}\frac{2}{n}z^{-n}H'_n,$$
$$H'_-(\mathbf{z})=-\sum_{n\in+\mathbb{N}_{\text{odd}}}\frac{2}{n}z^{n}H'_{-n}. \tag{2.44}$$

2.3 Plane partitions

A partition (strict partition) is a non-increasing (strictly decreasing) sequence consisting of non-negative integers, denoted as $\alpha=(\alpha_1,\alpha_2,\ldots)$, with weights $|\alpha|=\sum_{i\geq 1}\alpha_i$. Define a partition $\alpha'=(\alpha'_1,\alpha'_2,\ldots)$, which is obtained by taking the transpose of α. Suppose that there are r nodes on the main diagonal of partitions α and set $t_i=\alpha_i-i$, $p_i=\alpha'_i-i$ for $1\leq i\leq r$, we have $p_1>p_2>\cdots>p_r\geq 0$, $t_1>t_2>\cdots>t_r\geq 0$. The partition α can be also expressed as

$$\alpha=(t_1,t_2,\ldots,t_r|p_1,p_2,\ldots,p_r). \tag{2.45}$$

A hook refers to the set of boxes

$$h(p,t|j)=\left\{\bigcup_{k=0}^{p}(j+k,j)\right\}\cup\left\{\bigcup_{l=0}^{t}(j,j+l)\right\},\quad p\geqslant 0,\ t\geqslant 0. \tag{2.46}$$

The partition α can be denoted by the hook as

$$\alpha=\bigcup_{j=1}^{r}h(p_j,t_j|j), \tag{2.47}$$

where $r\geq 1$, $p_1>\cdots>p_r\geqslant 0$ and $t_1>\cdots>t_r\geqslant 0$.

Example 2.5 The partition $\alpha=(4,2,0|3,1,0)$ in Fig. 1 can be constructed by a set of hooks, where $h(3,4|1)=\left\{\bigcup_{k=0}^{3}(1+k,1)\right\}\cup\left\{\bigcup_{l=0}^{4}(1,1+l)\right\}$, $h(1,2|2)=\{(2,2),(3,2),(2,3),(2,4)\}$ and $h(0,0|3)=\{(3,3)\}$.

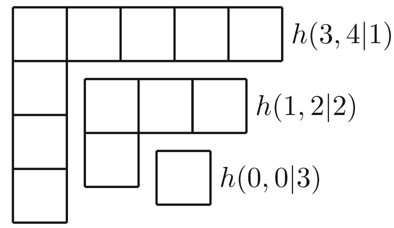

Fig. 1 Partition $\alpha=(4,2,0|3,1,0)$

For the partitions $\alpha=(\alpha_1,\alpha_2,\ldots)$ and $\beta=(\beta_1,\beta_2,\ldots)$, we say that β interlaces α and write $\alpha\succ\beta$, which is defined by the following relation

$$\alpha\succ\beta\quad\Longleftrightarrow\quad\alpha_1\geq\beta_1\geq\alpha_2\geq\beta_2\geq\cdots, \tag{2.48}$$

where $\alpha_1\geq\alpha_2\geq\cdots\geq 0$ and $\beta_1\geq\beta_2\geq\cdots\geq 0$. Consider the set

$$\mathcal{D}_\alpha=\left\{\bigcup_{j=1}^{r}h(p'_j,t'_j|j)\;\middle|\;p_j\geqslant p'_j\geqslant p_j-1,\ t_j\geqslant t'_j\geqslant t_{j+1}+1,\ \forall 1\leqslant j\leqslant r\right\}, \tag{2.49}$$

where $t_{r+1}\equiv 0$ and $h(-1,t'_r|r)\equiv\emptyset$. All partitions that intersect α and β are contained in $\mathcal{D}_\alpha$.

Let a strict partitions $\tilde{\alpha}=(m_1,m_2,\ldots,m_{2r})$, the right state and left state corresponding to $\tilde{\alpha}$ can be written as

$$|\tilde{\alpha}\rangle:=(-1)^r\omega\phi_{m_1}\cdots\phi_{m_{2r}}|0\rangle=(-1)^r\omega\overleftarrow{\prod_{j=1}^{2r}}\phi_{m_j}|0\rangle,$$
$$\langle\tilde{\alpha}|:=(-1)^{r+|\tilde{\alpha}|}\omega\langle 0|\phi_{-m_{2r}}\cdots\phi_{-m_1}$$
$$=(-1)^{r+|\tilde{\alpha}|}\omega\langle 0|\overrightarrow{\prod_{j=1}^{2r}}\phi_{-m_j}, \tag{2.50}$$

where

$$m_1>\cdots>m_{2r}\geqslant 0,\quad \omega:=\begin{cases}1, & m_{2r}\geqslant 1,\\ \sqrt{2}, & m_{2r}=0.\end{cases} \tag{2.51}$$

Lemma 2.6 [9] *Setting $\tilde{\alpha}=(m_1,\ldots,m_{2r})$, from Eqs. (2.43) and (2.50), the following relations hold*

$$\langle\tilde{\beta}|\Upsilon_+(\mathbf{z})|\tilde{\alpha}\rangle = \begin{cases} 2^{n(\tilde{\beta}|\tilde{\alpha})} z^{|\tilde{\beta}|-|\tilde{\alpha}|}, & \tilde{\beta} \prec \tilde{\alpha} \textit{ and } n(\tilde{\beta}) = n(\tilde{\alpha}), \\ (-1)^{n(\tilde{\alpha})} 2^{n(\tilde{\beta}|\tilde{\alpha})+\frac{1}{2}} z^{|\tilde{\beta}|-|\tilde{\alpha}|}, & \tilde{\beta} \prec \tilde{\alpha} \textit{ and } n(\tilde{\beta}) = n(\tilde{\alpha}) - 1, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.52}$$

$$\langle\tilde{\alpha}|\Upsilon_-(\mathbf{z})\tilde{\beta}\rangle = \begin{cases} 2^{n(\tilde{\beta}|\tilde{\alpha})} z^{|\tilde{\alpha}|-|\tilde{\beta}|}, & \tilde{\beta} \prec \tilde{\alpha} \textit{ and } n(\tilde{\beta}) = n(\tilde{\alpha}), \\ (-1)^{n(\tilde{\alpha})} 2^{n(\tilde{\beta}|\tilde{\alpha})+\frac{1}{2}} z^{|\tilde{\alpha}|-|\tilde{\beta}|}, & \tilde{\beta} \prec \tilde{\alpha} \textit{ and } n(\tilde{\beta}) = n(\tilde{\alpha}) - 1, \\ 0, & \textit{otherwise}. \end{cases} \tag{2.53}$$

A plane partition π is a set of non-negative integers π_{ij} which satisfies

$$\pi_{ij} \geq \pi_{(i+1)j}, \quad \pi_{ij} \geq \pi_{i(j+1)}, \\ \lim_{i\to\infty} \pi_{ij} = \lim_{j\to\infty} \pi_{ij} = 0, \quad \text{for } i, j \geq 1. \tag{2.54}$$

Each plane partition can be represented as a composition of specific partitions, denoted as $(\ldots, \pi_{-1}, \pi_0, \pi_1, \ldots)$. Indicate π_i as

$$\pi_i = \begin{cases} (\pi_{1(i+1)}, \pi_{2(i+2)}, \pi_{3(i+3)}, \ldots) & \text{for } i \geq 0, \\ (\pi_{(-i+1)1}, \pi_{(-i+2)2}, \pi_{(-i+3)3}, \ldots) & \text{for } i \leq -1, \end{cases} \tag{2.55}$$

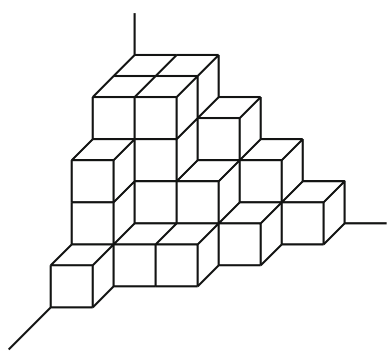

Fig. 2 A 3-dimensional view of a plane partition π. The value of π_{ij} denotes the number of boxes stacked at the location

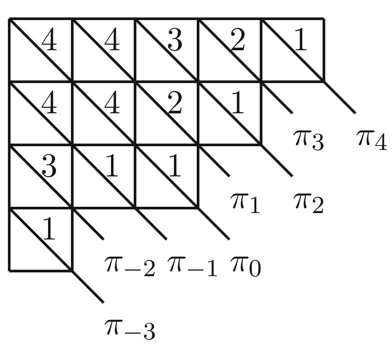

Fig. 3 A 2-dimensional view of the plane partition in Fig. 2. The sequence of values covered in the slice is the corresponding partition. In particular, $\pi_0 = (4, 4, 1)$ and $|\pi| = 31$

then the plane partition π satisfies

$$\emptyset = \pi_{-M} \prec \cdots \prec \pi_{-2} \prec \pi_{-1} \prec \pi_0 \succ \pi_1 \succ \pi_2 \succ \cdots \succ \\ \pi_N = \emptyset, \tag{2.56}$$

for sufficiently large $M, N \in \mathbb{N}$ and the weight $|\pi| = \sum_{i=-M}^{N} |\pi_i|$.

A strict plane partition $\tilde{\pi}$ satisfies

$$\tilde{\pi}_{ij} \geq \tilde{\pi}_{(i+1)j}, \quad \tilde{\pi}_{ij} \geq \tilde{\pi}_{i(j+1)}, \quad \tilde{\pi}_{ij} > \tilde{\pi}_{(i+1)(j+1)}, \\ \lim_{i\to\infty} \tilde{\pi}_{ij} = \lim_{j\to\infty} \tilde{\pi}_{ij} = 0, \tag{2.57}$$

for all integers $i, j \geq 1$.

For a strict plane partition $\tilde{\pi}$, we refer to the set of all connected boxes as paths, which are connected horizontal plateaux in the 3-dimensional view. $p(\tilde{\pi})$ denotes the number of paths possessed by $\tilde{\pi}$. For strict plane partitions $\tilde{\pi}_i$ and $\tilde{\pi}_j$, $n(\tilde{\pi}_i)$ represents the number of nonzero elements in $\tilde{\pi}_i$ and $n(\tilde{\pi}_i|\tilde{\pi}_j)$ represents the number of non-zero elements in $\tilde{\pi}_i$ but not in $\tilde{\pi}_j$.

The generating function for plane partitions is given by

$$\sum_{\substack{\pi \text{ is a} \\ \text{plane partition}}} q^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n}\right)^n. \tag{2.58}$$

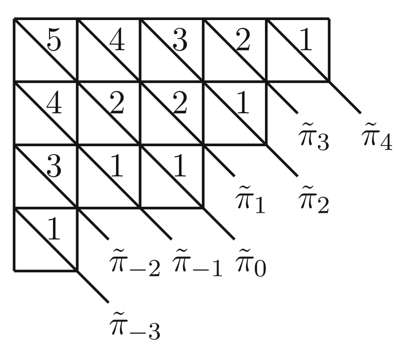

Fig. 4 A 2-dimensional view of a strict plane partition $\tilde{\pi}$. The sequence of values covered in the slice are strictly decreasing. The difference between Fig. 4 and Fig. 3 is the main diagonal

The generating function for strict plane partitions can be expressed as

$$\sum_{\substack{\pi \text{ is a strict} \\ \text{plane partition}}} 2^{p(\pi)} q^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^n. \tag{2.59}$$

3 UC plane partitions

In this section, we construct generalized charged fermion vertex operators and investigate interlacing 2-partitions. By means of fermion calculus approach, the generating function for UC plane partitions has been developed.

3.1 Generalized charged fermion vertex operators

Introduce

$$x_n - \frac{1}{n}\frac{\partial}{\partial y_n} = -\frac{z^{-n}}{n}, \quad y_n - \frac{1}{n}\frac{\partial}{\partial x_n} = -\frac{v^{-n}}{n}, \quad \forall n \in \pm\mathbb{N}, \tag{3.1}$$

where z and v are indeterminate. The Hamiltonian $H_{\pm}(\mathbf{x}, \mathbf{y})$ can be rewritten as

$$H_+(\mathbf{z}, \mathbf{v}) = \sum_{n=1}^{\infty} \left(-\frac{z^{-n}}{n} H_n - \frac{v^{-n}}{n} \tilde{H}_n \right),$$
$$H_-(\mathbf{z}, \mathbf{v}) = \sum_{n=1}^{\infty} \left(\frac{z^{n}}{n} H_{-n} + \frac{v^{n}}{n} \tilde{H}_{-n} \right). \tag{3.2}$$

Let us define the generalized charged fermion vertex operators

$$\Gamma_+(\mathbf{z}, \mathbf{v}) = e^{H_+(\mathbf{z},\mathbf{v})} = \exp\left(-\sum_{n=1}^{\infty} \left(\frac{z^{-n}}{n} H_n + \frac{v^{-n}}{n} \tilde{H}_n \right) \right),$$
$$\Gamma_-(\mathbf{z}, \mathbf{v}) = e^{-H_-(\mathbf{z},\mathbf{v})} = \exp\left(-\sum_{n=1}^{\infty} \left(\frac{z^{n}}{n} H_{-n} + \frac{v^{n}}{n} \tilde{H}_{-n} \right) \right). \tag{3.3}$$

It is easy to derive

$$\Gamma_+(\mathbf{z}, \mathbf{v})|\text{vac}\rangle = |\text{vac}\rangle, \quad \langle\text{vac}|\Gamma_-(\mathbf{z}, \mathbf{v}) = \langle\text{vac}|. \tag{3.4}$$

Taking $\xi_{\pm}(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}, k) = \xi_{\pm}(\mathbf{z}, k)$ and $\xi_{\pm}(\mathbf{y} - \tilde{\partial}_{\mathbf{x}}, k) = \xi_{\pm}(\mathbf{v}, k)$, we have

$$\xi_{\pm}(\mathbf{x} - \tilde{\partial}_{\mathbf{y}}, k) = \xi_{\pm}(\mathbf{z}, k) = \mp \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{k}{z} \right)^{\pm n}$$
$$= \pm \ln \left(1 - \left(\frac{k}{z} \right)^{\pm 1} \right),$$

$$\xi_{\pm}(\mathbf{y} - \tilde{\partial}_{\mathbf{x}}, k) = \xi_{\pm}(\mathbf{v}, k) = \mp \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{k}{v} \right)^{\pm n}$$
$$= \pm \ln \left(1 - \left(\frac{k}{v} \right)^{\pm 1} \right). \tag{3.5}$$

Proposition 3.1 *The vertex operators $\Gamma_+(\mathbf{z}, \mathbf{v})$ and $\Gamma_-(\mathbf{z}, \mathbf{v})$ satisfy the following relations*

$$\Gamma_+(\mathbf{z}, \mathbf{v})\psi_{n'}\Gamma_+^{-1}(\mathbf{z}, \mathbf{v}) = \psi_{n'} - \frac{1}{z}\psi_{(n'+1)},$$
$$\Gamma_+(\mathbf{z}, \mathbf{v})\psi^*_{n'}\Gamma_+^{-1}(\mathbf{z}, \mathbf{v}) = \sum_{m=0}^{\infty} \frac{\psi^*_{(n'+m)}}{z^m},$$
$$\Gamma_-^{-1}(\mathbf{z}, \mathbf{v})\psi_{n'}\Gamma_-(\mathbf{z}, \mathbf{v}) = \sum_{m=0}^{\infty} z^m \psi_{(n'-m)},$$
$$\Gamma_-^{-1}(\mathbf{z}, \mathbf{v})\psi^*_{n'}\Gamma_-(\mathbf{z}, \mathbf{v}) = \psi^*_{n'} - z\psi^*_{(n'+1)},$$
$$\Gamma_+(\mathbf{z}, \mathbf{v})\phi_{n'}\Gamma_+^{-1}(\mathbf{z}, \mathbf{v}) = \phi_{n'} - \frac{1}{v}\phi_{(n'+1)},$$
$$\Gamma_+(\mathbf{z}, \mathbf{v})\phi^*_{n'}\Gamma_+^{-1}(\mathbf{z}, \mathbf{v}) = \sum_{m=0}^{\infty} \frac{\phi^*_{(n'+m)}}{v^m},$$
$$\Gamma_-^{-1}(\mathbf{z}, \mathbf{v})\phi_{n'}\Gamma_-(\mathbf{z}, \mathbf{v}) = \sum_{m=0}^{\infty} v^m \phi_{(n'-m)},$$
$$\Gamma_-^{-1}(\mathbf{z}, \mathbf{v})\phi^*_{n'}\Gamma_-(\mathbf{z}, \mathbf{v}) = \psi^*_{n'} - v\phi^*_{(n'+1)}. \tag{3.6}$$

Proof We only prove the first formula of Eq. (3.6), other formulas can be proved similarly. By means of Eqs. (2.26), (2.29) and (3.5), we get

$$\Gamma_+(\mathbf{z}, \mathbf{v})\psi(k)\Gamma_+^{-1}(\mathbf{z}, \mathbf{v})$$
$$= e^{H_+(\mathbf{z},\mathbf{v})}\psi(k)e^{-H_+(\mathbf{z},\mathbf{v})} = \left(1 - \frac{k}{z}\right)\psi(k). \tag{3.7}$$

It follows from Eq. (2.25) that

$$\sum_{n' \in \mathbb{Z}+1/2} \Gamma_+(\mathbf{z}, \mathbf{v})\psi_{n'}k^{-n'-\frac{1}{2}}\Gamma_+^{-1}(\mathbf{z}, \mathbf{v})$$
$$= \sum_{n \in \mathbb{Z}+1/2} \left(\psi_n k^{-n-\frac{1}{2}} - \frac{\psi_n}{z} k^{-n+\frac{1}{2}} \right). \tag{3.8}$$

Comparing the orders of k on both sides yields

$$\Gamma_+(\mathbf{z}, \mathbf{v})\psi_{n'}\Gamma_+^{-1}(\mathbf{z}, \mathbf{v}) = \psi_{n'} - \frac{1}{z}\psi_{(n'+1)}. \tag{3.9}$$

□

The operators $H_+(\mathbf{z}, \mathbf{v})$ and $H_-(\mathbf{z}', \mathbf{v}')$ satisfy

$$[H_+(\mathbf{z}, \mathbf{v}), -H_-(\mathbf{z}', \mathbf{v}')] = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{mn} z^{-m}(z')^n [H_m, H_{-n}]$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}v^{-m}(v')^{n}[\tilde{H}_m,\tilde{H}_{-n}]$$
$$=\ln\left(1-\frac{z'}{z}\right)^{-1}+\ln\left(1-\frac{v'}{v}\right)^{-1}.$$

Then

$$\Gamma_+(\mathbf{z},\mathbf{v})\Gamma_-(\mathbf{z}',\mathbf{v}')=e^{[H_+(\mathbf{z},\mathbf{v}),-H_-(\mathbf{z}',\mathbf{v}')]}\Gamma_-(\mathbf{z}',\mathbf{v}')\Gamma_+(\mathbf{z},\mathbf{v})$$
$$=\left(1-\frac{z'}{z}\right)^{-1}\left(1-\frac{v'}{v}\right)^{-1}\Gamma_-(\mathbf{z}',\mathbf{v}')\Gamma_+(\mathbf{z},\mathbf{v}).\quad(3.10)$$

Remark 3.2 The vertex operators $\Gamma_+(\mathbf{z},\mathbf{v})$ and $\Gamma_-(\mathbf{z},\mathbf{v})$ are reduced to the charged fermion vertex operators $\Gamma_+(\mathbf{z})$ and $\Gamma_-(\mathbf{z})$ by deleting the variables ϕ_m,ϕ_m^* and v, respectively. Then Eqs. (3.4)–(3.10) lead to the properties for KP hierarchy.

3.2 Generating interlacing 2-partitions

If the Maya diagram has charge 0, there is a one-to-one correspondence between the Maya diagram and the partition. The right state corresponding to partitions α and β in space $\mathcal{F}_{0,0}$ can be represented as

$$|\alpha,\beta\rangle := (-1)^{\kappa+\tilde{\kappa}}\psi_{m_1}\cdots\psi_{m_r}\psi^*_{n_1}\cdots\psi^*_{n_r}\phi_{\tilde{m}_1}\cdots\phi_{\tilde{m}_s}\phi^*_{\tilde{n}_1},\ldots\phi^*_{\tilde{n}_s}|\mathrm{vac}\rangle,$$
$$=(-1)^{\kappa+\tilde{\kappa}}\prod_{j=1}^{r}\psi_{\tilde{m}_j}\prod_{k=1}^{r}\psi^*_{\tilde{n}_k}\prod_{\tilde{j}=1}^{s}\phi_{\tilde{m}_{\tilde{j}}}\prod_{\tilde{k}=1}^{s}\phi^*_{\tilde{n}_{\tilde{k}}}|\mathrm{vac}\rangle,\quad(3.11)$$

where $\kappa=\sum_{k=1}^{r}\left[(m_k+\frac{1}{2})+k\right]$, $\tilde{\kappa}=\sum_{k=1}^{s}\left[(\tilde{m}_k+\frac{1}{2})+k\right]$, $m_1<\cdots<m_r<0, n_1<\cdots<n_r<0$, $\tilde{m}_1<\cdots<\tilde{m}_s<0$ and $\tilde{n}_1<\cdots<\tilde{m}_s<0$. The left state has a similar representation in the charge (0, 0) sector of the dual Fock space $\mathcal{F}^*$,

$$\langle\alpha,\beta| := (-1)^{\kappa'+\tilde{\kappa}'}\langle\mathrm{vac}|\phi_{\tilde{m}_s}\cdots\phi_{\tilde{m}_1}\phi^*_{\tilde{n}_s}\cdots\phi^*_{\tilde{n}_1}\psi_{n_r}\cdots\psi_{n_1}\psi^*_{m_r}\cdots\psi^*_{m_1}$$
$$=(-1)^{\kappa'+\tilde{\kappa}'}\langle\mathrm{vac}|\prod_{\tilde{j}=1}^{s}\phi_{\tilde{m}_{\tilde{j}}}\prod_{\tilde{k}=1}^{s}\phi^*_{\tilde{n}_{\tilde{k}}}\prod_{j=1}^{r}\psi_{n_j}\prod_{k=1}^{r}\psi^*_{m_k},\quad(3.12)$$

where $\kappa'=\sum_{k=1}^{r}\left[(m_k-\frac{1}{2})+k\right]$, $\tilde{\kappa}'=\sum_{k=1}^{s}\left[(\tilde{m}_k-\frac{1}{2})+k\right]$, $0<m_1<\cdots<m_r, 0<n_1<\cdots<n_r, 0<\tilde{m}_1<\cdots<\tilde{m}_s$ and $0<\tilde{n}_1<\cdots<\tilde{n}_s$.

Define 2-partition χ and write $(\chi)=(\alpha,\beta)$, which represents a pair of partitions α and β. Then we have $|\chi\rangle=|\alpha,\beta\rangle$, $\langle\chi|=\langle\alpha,\beta|$ and the weight $|\chi|=|\alpha|+|\beta|$. Let 2-partitions $(\chi)=(\alpha,\beta)$ and $(\bar{\chi})=(\bar{\alpha},\bar{\beta})$, we say that $(\bar{\chi})$ interlaces (χ), and write $(\chi)\succ(\bar{\chi})$,

$$(\chi)\succ(\bar{\chi})\iff\alpha\succ\bar{\alpha}\ \text{ and }\ \beta\succ\bar{\beta}.\quad(3.13)$$

In particular, if $\beta=\emptyset$, 2-partition χ is reduced to the partition α. Equations (3.11) and (3.12) lead to

$$|\alpha\rangle := (-1)^{\kappa}\psi_{m_1}\cdots\psi_{m_r}\psi^*_{n_1}\cdots\psi^*_{n_r}|\mathrm{vac}\rangle$$
$$=(-1)^{\kappa}\prod_{j=1}^{r}\psi_{m_j}\prod_{k=1}^{r}\psi^*_{n_k}|\mathrm{vac}\rangle,$$
$$\langle\alpha| := (-1)^{\kappa'}\langle\mathrm{vac}|\psi_{n_r}\cdots\psi_{n_1}\psi^*_{m_r}\cdots\psi^*_{m_1}$$
$$=(-1)^{\kappa'}\langle\mathrm{vac}|\prod_{j=1}^{r}\psi_{n_j}\prod_{k=1}^{r}\psi^*_{m_k}.\quad(3.14)$$

Definition 3.3 An 'UC plane partition' is defined as $(\ldots,\chi_{-1},\chi_0,\chi_1,\ldots)$, which denotes a pair of plane partitions and satisfies

$$\emptyset=\chi_{-M}\prec\cdots\prec\chi_{-2}\prec\chi_{-1}\prec\chi_0\succ\chi_1\succ\chi_2\succ\cdots\succ\chi_N=\emptyset,\quad(3.15)$$

where the weight of the UC plane partition is the sum of the weights of these 2-partitions.

Example 3.4 The UC plane partition $(\chi_{-3},\chi_{-2},\chi_{-1},\chi_0,\chi_1,\chi_2,\chi_3,\chi_4)$ in Fig. 5 represents a pair of plane partitions $\pi=(\alpha_{-3},\alpha_{-2},\alpha_{-1},\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ and $\tilde{\pi}=(\beta_{-3},\beta_{-2},\beta_{-1},\beta_0,\beta_1,\beta_2,\beta_3,\beta_4)$, where $(\chi_i)=(\alpha_i,\beta_i)$ and the weight is $\sum_{i=-3}^{4}|\alpha_i|+|\beta_i|=61$.

Lemma 3.5 *Let $|\alpha\rangle$ and $\langle\alpha|$ be states corresponding to the partition α in the Fock space $\mathcal{F}$ and the dual Fock space $\mathcal{F}^*$, which are described in Eq. (3.14). Then we have*

$$\langle\beta|\Gamma_+(\mathbf{z})|\alpha\rangle=\begin{cases}z^{|\beta|-|\alpha|}, & \beta\prec\alpha,\\ 0, & otherwise,\end{cases}\quad(3.16)$$

$$\langle\alpha|\Gamma_-(\mathbf{z})|\beta\rangle=\begin{cases}z^{|\alpha|-|\beta|}, & \beta\prec\alpha,\\ 0, & otherwise.\end{cases}\quad(3.17)$$

Proof Set $n_{(r+1)}\equiv\frac{1}{2}$. From Eqs. (3.6) and (3.14), one obtains

$$\Gamma_+(\mathbf{z})|\alpha\rangle=(-1)^{\kappa}\prod_{j=1}^{r}\left(\Gamma_+(\mathbf{z})\psi_{m_j}\Gamma_+^{-1}(\mathbf{z})\right)$$
$$\times\prod_{k=1}^{r}\left(\Gamma_+(\mathbf{z})\psi^*_{n_k}\Gamma_+^{-1}(\mathbf{z})\right)\Gamma_+(\mathbf{z})|\mathrm{vac}\rangle$$

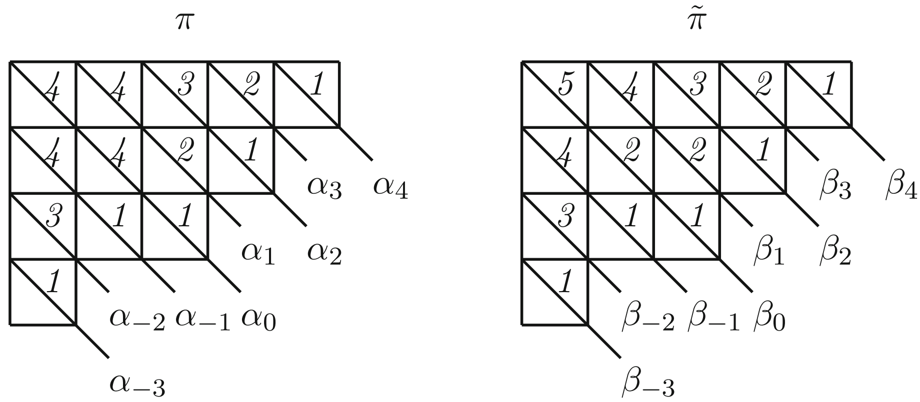

Fig. 5 The UC plane partition $(\chi_{-3}, \chi_{-2}, \chi_{-1}, \chi_0, \chi_1, \chi_2, \chi_3, \chi_4)$

$$= (-1)^{\kappa} \prod_{j=1}^{r} \left(\psi_{m_j} - \frac{1}{z} \psi_{(m_j+1)} \right) \cdot T|\text{vac}\rangle, \tag{3.18}$$

where

$$\begin{aligned} T|\text{vac}\rangle &= \prod_{k=1}^{r} \left(\sum_{i=0}^{\infty} \frac{1}{z^i} \psi^*_{(n_k+i)} \right) |\text{vac}\rangle \\ &= \left(\sum_{i=0}^{\infty} \frac{1}{z^i} \psi^*_{(n_1+i)} \right) \left(\sum_{i=0}^{\infty} \frac{1}{z^i} \psi^*_{(n_2+i)} \right) \\ &\quad \cdots \left(\sum_{i=0}^{\infty} \frac{1}{z^i} \psi^*_{(n_r+i)} \right) |\text{vac}\rangle. \end{aligned} \tag{3.19}$$

The following equation holds

$$\begin{aligned} \sum_{i=0}^{\infty} \frac{1}{z^i} \psi^*_{(n_k+i)} &= \sum_{i=0}^{-n_k+n_{(k+1)}-1} \frac{1}{z^i} \psi^*_{(n_k+i)} \\ &\quad + \frac{1}{z^{n_{k+1}-n_k}} \sum_{i=0}^{\infty} \frac{1}{z^i} \psi^*_{(n_{(k+1)}+i)}, \quad 1 \le k \le r-1. \end{aligned} \tag{3.20}$$

Using the commutation relations (2.1), we have

$$\left(\sum_{i=0}^{\infty} \frac{1}{z^i} \psi^*_{(n+i)} \right) \left(\sum_{i=0}^{\infty} \frac{1}{z^i} \psi^*_{(n+i)} \right) = 0. \tag{3.21}$$

From Eqs. (2.14), (3.20) and (3.21), we obtain

$$\begin{aligned} T|\text{vac}\rangle &= \left(\sum_{i=0}^{-n_1+n_2-1} \frac{1}{z^i} \psi^*_{(n_1+i)} \right) \left(\sum_{i=0}^{-n_2+n_3-1} \frac{1}{z^i} \psi^*_{(n_2+i)} \right) \\ &\quad \cdots \left(\sum_{i=0}^{-n_r+\frac{1}{2}-1} \frac{1}{z^i} \psi^*_{(n_r+i)} \right) |\text{vac}\rangle. \end{aligned} \tag{3.22}$$

Therefore

$$\begin{aligned} \Gamma_+(\mathbf{z})|\alpha\rangle &= (-1)^{\kappa} \prod_{j=1}^{r} \left(\psi_{m_j} - \frac{1}{z} \psi_{(m_j+1)} \right) \\ &\quad \times \prod_{k=1}^{r} \left(\sum_{i=0}^{-n_k+n_{(k+1)}-1} \frac{1}{z^i} \psi^*_{(n_k+i)} \right) |\text{vac}\rangle. \end{aligned} \tag{3.23}$$

Set

$$\begin{aligned} & p_j = -m_j - \frac{1}{2}, \quad t_j = -n_j - \frac{1}{2}, \quad \forall 1 \leqslant j \leqslant r, \\ & t_{(r+1)} = -n_{(r+1)} - \frac{1}{2} \equiv -1, \\ & h\left(-1, -n'_r - \frac{1}{2}|r\right) \equiv \emptyset, \quad -m_j \ge -m'_j \ge -m_j - 1, \\ & -n_j \ge -n'_j \ge -n_{j+1} + 1. \end{aligned} \tag{3.24}$$

It can be clearly found that the terms of the expansion of the Eq. (3.23) contain all of the partitions in $\mathcal{D}_\alpha$, accompanied by the weighting factor z. Each weighted partition can be expressed as

$$\prod_{j=1}^{r} z^{m_j - m'_j} z^{n_j - n'_j} \bigcup_{k=1}^{r} h\left(-m'_k - \frac{1}{2}, -n'_k - \frac{1}{2} |k \right). \tag{3.25}$$

The powers of z can be written as

$$\sum_{j=1}^{r} (m_j + n_j - m'_j - n'_j) = |\beta| - |\alpha|, \tag{3.26}$$

where

$$\begin{aligned} \alpha &= \bigcup_{j=1}^{r} h\left(-m_j - \frac{1}{2}, -n_j - \frac{1}{2} |j \right), \\ \beta &= \bigcup_{j=1}^{r} h\left(-m'_j - \frac{1}{2}, -n'_j - \frac{1}{2} |j \right). \end{aligned} \tag{3.27}$$

From Eqs. (3.24)–(3.27), the Eq. (3.23) can be rewritten as

$$\Gamma_+(\mathbf{z})|\alpha\rangle = \sum_{\beta \prec \alpha} z^{|\beta| - |\alpha|} |\beta\rangle. \tag{3.28}$$

A similar proof for the left state yields

$$\langle\alpha|\Gamma_-(\mathbf{z}) = (-1)^{\kappa}\langle\text{vac}|\prod_{j=1}^{r}\left(\sum_{i=0}^{-n_{(j-1)}+n_j-1} z^i\psi_{(n_j-i)}\right)\times\prod_{k=1}^{r}(\psi^*_{m_k} - z\psi^*_{(m_k-1)}). \tag{3.29}$$

For $\forall 1 \leqslant j \leqslant r$, let

$$\begin{aligned} &p_j = m_{(r+1)-j} - \frac{1}{2}, \quad t_j = n_{(r+1)-j} - \frac{1}{2}, \\ &t_{r+1} = n_0 - \frac{1}{2} \equiv -1, \quad h\left(-1, n'_1 - \frac{1}{2}|r\right) \equiv \emptyset, \\ &m_{(r+1)-j} \geq m'_{(r+1)-j} \geq m_{(r+1)-j} - 1, \\ &n_{(r+1)-j} \geq n'_{(r+1)-j} \geq n_{(r+1)-j} + 1. \end{aligned} \tag{3.30}$$

Then one obtains

$$\langle\alpha|\Gamma_-(\mathbf{z}) = \sum_{\beta\prec\alpha} z^{|\alpha|-|\beta|}\langle\beta|. \tag{3.31}$$

□

Lemma 3.6 *Let the states corresponding to the* 2*-partition* $(\chi) = (\alpha, \beta)$ *be* $|\chi\rangle = |\alpha, \beta\rangle$ *and* $\langle\chi| = \langle\alpha, \beta|$*. The following relations hold*

$$\langle\chi'|\Gamma_+(\mathbf{z}, \mathbf{v})|\chi\rangle = \langle\alpha', \beta'|\Gamma_+(\mathbf{z}, \mathbf{v})|\alpha, \beta\rangle = \begin{cases} z^{|\alpha'|-|\alpha|}v^{|\beta'|-|\beta|}, & \alpha' \prec \alpha, \beta' \prec \beta, \\ 0, & otherwise, \end{cases} \tag{3.32}$$

$$\langle\chi|\Gamma_-(\mathbf{z}, \mathbf{v})|\chi'\rangle = \langle\alpha, \beta|\Gamma_-(\mathbf{z}, \mathbf{v})|\alpha', \beta'\rangle = \begin{cases} z^{|\alpha|-|\alpha'|}v^{|\beta|-|\beta'|}, & \alpha' \prec \alpha, \beta' \prec \beta, \\ 0, & otherwise. \end{cases} \tag{3.33}$$

Proof By means of the Eq. (3.11)

$$\begin{aligned} \Gamma_+(\mathbf{z}, \mathbf{v})|\alpha, \beta\rangle = &(-1)^{\kappa+\tilde{\kappa}}\prod_{j=1}^{r}(\Gamma_+(\mathbf{z}, \mathbf{v})\psi_{m_j}\Gamma_+^{-1}(\mathbf{z}, \mathbf{v})) \\ &\times\prod_{k=1}^{r}(\Gamma_+(\mathbf{z}, \mathbf{v})\psi^*_{n_k}\Gamma_+^{-1}(\mathbf{z}, \mathbf{v})) \\ &\times\prod_{\tilde{j}=1}^{s}(\Gamma_+(\mathbf{z}, \mathbf{v})\phi_{\tilde{m}_{\tilde{j}}}\Gamma_+^{-1}(\mathbf{z}, \mathbf{v})) \\ &\times\prod_{\tilde{k}=1}^{s}(\Gamma_+(\mathbf{z}, \mathbf{v})\phi^*_{\tilde{n}_{\tilde{k}}}\Gamma_+^{-1}(\mathbf{z}, \mathbf{v}))\Gamma_+(\mathbf{z}, \mathbf{v})|\text{vac}\rangle \\ = &(-1)^{\kappa}\prod_{j=1}^{r}\left(\psi_{m_j} - \frac{1}{z}\psi_{(m_j+1)}\right) \\ &\times\prod_{k=1}^{r}\left(\sum_{i=0}^{\infty}\frac{1}{z^i}\psi^*_{(n_k+i)}\right)\cdot T_1|\text{vac}\rangle, \end{aligned} \tag{3.34}$$

where

$$T_1 = (-1)^{\tilde{\kappa}_1}\prod_{j=1}^{r}\left(\phi_{m_j} - \frac{1}{v}\phi_{(m_j+1)}\right)\times\prod_{k=1}^{r}\left(\sum_{i=0}^{\infty}\frac{1}{v^i}\phi^*_{(n_k+i)}\right)|\text{vac}\rangle. \tag{3.35}$$

By using the Eq. (3.16), we have

$$T_1 = \sum_{\beta'\prec\beta} v^{|\beta'|-|\beta|}|\beta'\rangle. \tag{3.36}$$

Setting $|\beta'\rangle = (-1)^{\tilde{\kappa}_1}\phi_{\tilde{m}'_1}\cdots\phi_{\tilde{m}'_s}\phi^*_{\tilde{n}'_1}\cdots\phi^*_{\tilde{n}'_s}|\text{vac}\rangle$, T_1 is rewritten as

$$T_1 = \sum_{\beta'\prec\beta} v^{|\beta'|-|\beta|}(-1)^{\tilde{\kappa}_1}\phi_{\tilde{m}'_1}\cdots\phi_{\tilde{m}'_s}\phi^*_{\tilde{n}'_1}\cdots\phi^*_{\tilde{n}'_s}|\text{vac}\rangle. \tag{3.37}$$

According to the commutation relations (2.1) one obtains

$$\begin{aligned} \Gamma_+(\mathbf{z}, \mathbf{v})|\alpha, \beta\rangle = &\sum_{\beta'\prec\beta} v^{|\beta'|-|\beta|}(-1)^{\tilde{\kappa}_1}\phi_{\tilde{m}'_1} \\ &\cdots\phi_{\tilde{m}'_s}\phi^*_{\tilde{n}'_1}\cdots\phi^*_{\tilde{n}'_s}(-1)^{\kappa}\prod_{j=1}^{r}\left(\psi_{m_j}\right. \\ &\left.-\frac{1}{z}\psi_{(m_j+1)}\right)\prod_{k=1}^{r}\left(\sum_{i=0}^{\infty}\frac{1}{z^i}\psi^*_{(n_k+i)}\right)|\text{vac}\rangle \\ = &\sum_{\beta'\prec\beta} v^{|\beta'|-|\beta|}(-1)^{\tilde{\kappa}_1}\phi_{\tilde{m}'_1}\cdots\phi_{\tilde{m}'_s}\phi^*_{\tilde{n}'_1}\cdots\phi^*_{\tilde{n}'_s}\cdot T_2|\text{vac}\rangle, \end{aligned} \tag{3.38}$$

where

$$T_2 = (-1)^{\kappa}\prod_{j=1}^{r}\left(\psi_{m_j} - \frac{1}{z}\psi_{(m_j+1)}\right)\times\prod_{k=1}^{r}\left(\sum_{i=0}^{\infty}\frac{1}{z^i}\psi^*_{(n_k+i)}\right)|\text{vac}\rangle = \sum_{\alpha'\prec\alpha} z^{|\alpha'|-|\alpha|}|\alpha'\rangle. \tag{3.39}$$

Taking $|\alpha'\rangle = (-1)^{\kappa_1}\psi_{m'_1}\cdots\psi_{m'_r}\psi^*_{n'_1}\cdots\psi^*_{n'_r}|\text{vac}\rangle$, combining the Eq. (3.11) gets

$$\begin{aligned} &\Gamma_+(\mathbf{z}, \mathbf{v})|\alpha, \beta\rangle \\ &= \sum_{\beta'\prec\beta} v^{|\beta'|-|\beta|}\sum_{\alpha'\prec\alpha} z^{|\alpha'|-|\alpha|}(-1)^{\tilde{\kappa}_1+\kappa_1}\phi_{\tilde{m}'_1} \\ &\quad\cdots\phi_{\tilde{m}'_s}\phi^*_{\tilde{n}'_1}\cdots\phi^*_{\tilde{n}'_s}\psi_{m'_1}\cdots\psi_{m'_r}\psi^*_{n'_1}\cdots\psi^*_{n'_r}|\text{vac}\rangle \\ &= \sum_{\substack{\alpha'\prec\alpha \\ \beta'\prec\beta}} z^{|\alpha'|-|\alpha|}v^{|\beta'|-|\beta|}|\alpha', \beta'\rangle. \end{aligned} \tag{3.40}$$

Using the similar approach yields

$$\langle\alpha,\beta|\Gamma_-(\mathbf{z},\mathbf{v}) = \sum_{\substack{\alpha'\prec\alpha\\ \beta'\prec\beta}} z^{|\alpha|-|\alpha'|}v^{|\beta|-|\beta'|}\langle\alpha',\beta'|. \tag{3.41}$$

□

Setting $\beta=\emptyset$, Eqs. (3.40) and (3.41) are reduced to

$$\Gamma_+(\mathbf{z},\mathbf{v})|\alpha\rangle = \sum_{\alpha'\prec\alpha} z^{|\alpha'|-|\alpha|}|\alpha'\rangle,$$
$$\langle\alpha|\Gamma_-(\mathbf{z},\mathbf{v}) = \sum_{\alpha'\prec\alpha} z^{|\alpha|-|\alpha'|}\langle\alpha'|. \tag{3.42}$$

The case of $\alpha=\emptyset$ is similar to the above.

3.3 Generating function for UC plane partitions

Consider the correlation function

$$S_A(p,q) = \langle\text{vac}|\overrightarrow{\prod_{i=1}^{\infty}}\Gamma_+(p^{\frac{-2i+1}{2}},q^{\frac{-2i+1}{2}}) \times\overleftarrow{\prod_{k=1}^{\infty}}\Gamma_-(p^{\frac{2k-1}{2}},q^{\frac{2k-1}{2}})|\text{vac}\rangle, \tag{3.43}$$

where p and q are indeterminate. Set 2-partition $(\chi)=(\alpha,\beta)$, and insert $\sum_\chi|\chi\rangle\langle\chi|$ in the middle of a pair of multiplicative vertex operators. It follows that

$$\begin{aligned}S_A(p,q) &= \sum_{\substack{\chi\text{ is a}\\ \text{2-partition}}}\langle\text{vac}|\overrightarrow{\prod_{i=1}^{\infty}}\Gamma_+(p^{\frac{-2i+1}{2}},q^{\frac{-2i+1}{2}})|\chi\rangle \times\langle\chi|\overleftarrow{\prod_{k=1}^{\infty}}\Gamma_-(p^{\frac{2k-1}{2}},q^{\frac{2k-1}{2}})|\text{vac}\rangle\\ &= \sum_{\substack{\alpha\text{ and }\beta\\ \text{are partitions}}}\langle\text{vac}|\overrightarrow{\prod_{i=1}^{\infty}}\Gamma_+(p^{\frac{-2i+1}{2}},q^{\frac{-2i+1}{2}})|\alpha,\beta\rangle \times\langle\alpha,\beta|\overleftarrow{\prod_{k=1}^{\infty}}\Gamma_-(p^{\frac{2k-1}{2}},q^{\frac{2k-1}{2}})|\text{vac}\rangle.\end{aligned} \tag{3.44}$$

By means of Eqs. (3.40)–(3.42), the generated weights are of the form

$$\begin{aligned}&\overrightarrow{\prod_{i=1}^{M}}\langle\alpha'_{-i},\beta'_{-i}|\Gamma_+(p^{\frac{-2i+1}{2}},q^{\frac{-2i+1}{2}})|\alpha'_{-i+1},\beta'_{-i+1}\rangle\\ &\times\overleftarrow{\prod_{k=1}^{N}}\langle\alpha'_{k-1},\beta'_{k-1}|\Gamma_-(p^{\frac{2k-1}{2}},q^{\frac{2k-1}{2}})|\alpha'_k,\beta'_k\rangle\\ &= \prod_{j=-M}^{N} p^{|\alpha'_j|}q^{|\beta'_j|}.\end{aligned} \tag{3.45}$$

Set $(\chi'_j)=(\alpha'_j,\beta'_j)$ and $\alpha'_{-M_1}=\alpha'_{N_1}=\beta'_{-M_2}=\beta'_{N_2}=\emptyset$, then we have

$$\begin{aligned}\emptyset &= (\chi'_{-M})\prec\cdots\prec(\chi'_{-2})\prec(\chi'_{-1})\\ &\prec(\chi'_0)\succ(\chi'_1)\succ(\chi'_2)\succ\cdots\succ(\chi'_N)=\emptyset,\end{aligned} \tag{3.46}$$

where $M=\max\{M_1,M_2\}$, $N=\max\{N_1,N_2\}$ and $-M\le j\le N$. Note that the plane partition π consists of α'_j and the plane partition π' consists of β'_j. Hence interlacing relation above indicates

$$\begin{aligned}&\emptyset=\alpha'_{-M}=\cdots=\alpha'_{-M_1}\prec\cdots\prec\alpha'_{-2}\prec\alpha'_{-1}\prec\alpha'_0\succ\alpha'_1\succ\alpha'_2\\ &\succ\cdots\succ\alpha'_{N_1}=\cdots=\alpha'_N=\emptyset,\\ &\emptyset=\beta'_{-M}=\cdots=\beta'_{-M_2}\prec\cdots\prec\beta'_{-2}\prec\beta'_{-1}\prec\beta'_0\succ\beta'_1\\ &\succ\beta'_2\succ\cdots\succ\beta'_{N_2}=\cdots=\beta'_N=\emptyset.\end{aligned} \tag{3.47}$$

The Eq. (3.45) can be rewritten as

$$\prod_{j=-M}^{N} p^{|\alpha'_j|}q^{|\beta'_j|} = \prod_{i=-M_1}^{N_1} p^{|\alpha'_i|}\prod_{k=-M_2}^{N_2} q^{|\beta'_k|}. \tag{3.48}$$

Then we derive the generating function for UC plane partitions

$$S_A(p,q) = \sum_{\substack{\pi\text{ and }\pi'\text{ are}\\ \text{plane partitions}}} p^{|\pi|}q^{|\pi'|}. \tag{3.49}$$

On the other hand, by using the Eqs. (3.4) and (3.10), we can express the generating function for UC plane partitions as the product of the generalized MacMahon's formula

$$\begin{aligned}S_A(p,q) &= \prod_{n_1=1}^{\infty}\frac{1}{1-p^{n_1}}\prod_{m_1=1}^{\infty}\frac{1}{1-q^{m_1}}\langle\text{vac}|\\ &\times\overrightarrow{\prod_{j=2}^{\infty}}\Gamma_+(p^{\frac{-2j+1}{2}},q^{\frac{-2j+1}{2}})\\ &\times\overleftarrow{\prod_{k=1}^{\infty}}\Gamma_-(p^{\frac{2k-1}{2}},q^{\frac{2k-1}{2}})|\text{vac}\rangle\\ &= \cdots = \prod_{n=1}^{\infty}\left(\frac{1}{1-p^n}\right)^n\prod_{m=1}^{\infty}\left(\frac{1}{1-q^m}\right)^m.\end{aligned} \tag{3.50}$$

4 BUC plane partitions

In this section, the BUC plane partitions will be developed. By using the fermion calculus method, we construct generalized neutral fermion vertex operators. Based upon interlacing strict 2-partitions derived by the vertex operator, we investigate the properties of the generating function for BUC plane partitions.

4.1 Generalized neutral fermion vertex operators

Set

$$x_n - \frac{2}{n}\frac{\partial}{\partial y_n} = \frac{2}{n}z^{-n}, \quad y_n - \frac{2}{n}\frac{\partial}{\partial x_n} = \frac{2}{n}v^{-n}, \quad \forall n \in \mathbb{Z}_{\text{odd}}, \tag{4.1}$$

where z and v are indeterminate. Replacing the variables above, we obtain

$$H'_+(\mathbf{z},\mathbf{v}) = \sum_{n\in+\mathbb{N}_{\text{odd}}} \left(\frac{2}{n}z^{-n}H'_n + \frac{2}{n}v^{-n}\bar{H}'_n\right),$$
$$H'_-(\mathbf{z},\mathbf{v}) = -\sum_{n\in+\mathbb{N}_{\text{odd}}} \left(\frac{2}{n}z^{n}H'_{-n} + \frac{2}{n}v^{n}\bar{H}'_{-n}\right). \tag{4.2}$$

Meanwhile the generalized neutral fermion vertex operators $\Upsilon_+(\mathbf{z},\mathbf{v})$ and $\Upsilon_-(\mathbf{z},\mathbf{v})$ are defined as

$$\Upsilon_+(\mathbf{z},\mathbf{v}) = e^{H'_+(\mathbf{z},\mathbf{v})} = \exp\left(\sum_{n\in+\mathbb{N}_{\text{odd}}} \left(\frac{2}{n}z^{-n}H'_n + \frac{2}{n}v^{-n}\bar{H}'_n\right)\right),$$
$$\Upsilon_-(\mathbf{z},\mathbf{v}) = e^{-H'_-(\mathbf{z},\mathbf{v})} = \exp\left(\sum_{n\in+\mathbb{N}_{\text{odd}}} \left(\frac{2}{n}z^{n}H'_{-n} + \frac{2}{n}v^{n}\bar{H}'_{-n}\right)\right). \tag{4.3}$$

In particular,

$$\Upsilon_+(\mathbf{z},\mathbf{v})|0\rangle = |0\rangle, \quad \langle 0|\Upsilon_-(\mathbf{z},\mathbf{v}) = \langle 0|. \tag{4.4}$$

Taking the transformation of $\zeta_\pm(\mathbf{x} - 2\tilde{\partial}'_{\mathbf{y}}, k)$ and $\zeta_\pm(\mathbf{y} - 2\tilde{\partial}'_{\mathbf{x}}, k)$, we have

$$\zeta_\pm(\mathbf{x} - 2\tilde{\partial}'_{\mathbf{y}}, k) = \zeta_\pm(\mathbf{z}, k) = \pm\sum_{m\in\mathbb{N}_{\text{odd}}} \frac{2}{m}\left(\frac{k}{z}\right)^{\pm m} = \ln\left(\pm\frac{z+k}{z-k}\right)^{\pm 1},$$
$$\zeta_\pm(\mathbf{y} - 2\tilde{\partial}'_{\mathbf{x}}, k) = \zeta_\pm(\mathbf{v}, k) = \pm\sum_{m\in\mathbb{N}_{\text{odd}}} \frac{2}{m}\left(\frac{k}{v}\right)^{\pm m} = \ln\left(\pm\frac{v+k}{v-k}\right)^{\pm 1}. \tag{4.5}$$

Proposition 4.1 *The following equations hold*

$$\Upsilon_+(\mathbf{z},\mathbf{v})\phi_j\Upsilon_+(-\mathbf{z},-\mathbf{v}) = \phi_j + 2\sum_{n=1}^{\infty}\frac{1}{z^n}\phi_{j-n},$$
$$\Upsilon_-(-\mathbf{z},-\mathbf{v})\phi_j\Upsilon_-(\mathbf{z},\mathbf{v}) = \phi_j + 2\sum_{n=1}^{\infty}(-z)^n\phi_{j+n},$$
$$\Upsilon_+(\mathbf{z},\mathbf{v})\bar{\phi}_j\Upsilon_+(-\mathbf{z},-\mathbf{v}) = \bar{\phi}_j + 2\sum_{n=1}^{\infty}\frac{1}{z^n}\bar{\phi}_{j-n},$$
$$\Upsilon_-(-\mathbf{z},-\mathbf{v})\bar{\phi}_j\Upsilon_-(\mathbf{z},\mathbf{v}) = \bar{\phi}_j + 2\sum_{n=1}^{\infty}(-z)^n\bar{\phi}_{j+n}. \tag{4.6}$$

Proof From Eqs. (2.39), (2.41) and (4.5), it is clear that

$$\Upsilon_+(\mathbf{z},\mathbf{v})\phi(k)\Upsilon_+(-\mathbf{z},-\mathbf{v}) = \phi(k)\left(\frac{z+k}{z-k}\right). \tag{4.7}$$

Substituting Eq. (2.38) into the above equation and comparing the orders of k, one obtains

$$\Upsilon_+(\mathbf{z},\mathbf{v})\phi_j\Upsilon_+(-\mathbf{z},-\mathbf{v}) = \phi_j + 2\sum_{n=1}^{\infty}\frac{1}{z^n}\phi_{j-n}. \tag{4.8}$$

Other equations can be proved with the same method. □

By means of Eqs. (2.35) and (4.2), we have

$$[H'_+(\mathbf{z},\mathbf{v}), -H'_-(\mathbf{z}',\mathbf{v}')] = \ln\left(\frac{z+z'}{z-z'}\right) + \ln\left(\frac{v+v'}{v-v'}\right). \tag{4.9}$$

It follows that

$$\Upsilon_+(\mathbf{z},\mathbf{v})\Upsilon_-(\mathbf{z}',\mathbf{v}') = e^{[H'_+(\mathbf{z},\mathbf{v}), -H'_-(\mathbf{z}',\mathbf{v}')]}e^{-H'_-(\mathbf{z}',\mathbf{v}')}e^{H'_+(\mathbf{z},\mathbf{v})} = \left(\frac{z+z'}{z-z'}\right)\left(\frac{v+v'}{v-v'}\right) \times \Upsilon_-(\mathbf{z}',\mathbf{v}')\Upsilon_+(\mathbf{z},\mathbf{v}). \tag{4.10}$$

4.2 Generating interlacing strict 2-partitions

Let strict partitions $\tilde{\alpha} = (m_1, m_2, \ldots, m_{2r})$ and $\tilde{\beta} = (n_1, n_2, \ldots, n_{2s})$. In the Fock space $\widetilde{\mathcal{F}}$ and the dual Fock space $\widetilde{\mathcal{F}}^*$, the states corresponding to $\tilde{\alpha}$ and $\tilde{\beta}$ can be described as

$$|\tilde{\alpha},\tilde{\beta}\rangle := (-1)^{r+s}\omega\omega'\phi_{m_1}\cdots\phi_{m_{2r}}\bar{\phi}_{n_1}\cdots\bar{\phi}_{n_{2s}}|0\rangle = (-1)^{r+s}\omega\omega'\prod_{j=1}^{2r}\phi_{m_j}\prod_{j=1}^{2s}\bar{\phi}_{n_j}|0\rangle,$$
$$\langle\tilde{\alpha},\tilde{\beta}| := (-1)^{r+s+|\tilde{\beta}|+|\tilde{\alpha}|}\omega\omega'\langle 0|\phi_{-n_{2s}}\cdots\phi_{-n_1}\phi_{-m_{2r}} \times \cdots\phi_{-m_1}$$

$$= (-1)^{r+s+|\tilde{\beta}|+|\tilde{\alpha}|}\omega\omega'\langle 0|\prod_{j=1}^{2s}\bar{\phi}_{-n_j}\prod_{j=1}^{2r}\phi_{-m_j}, \tag{4.11}$$

where $m_1 > \cdots > m_{2r} \geqslant 0$, $n_1 > \cdots > n_{2r} \geqslant 0$ and

$$\omega := \begin{cases} 1, & m_{2r} \geqslant 1, \\ \sqrt{2}, & m_{2r} = 0. \end{cases} \quad \omega' := \begin{cases} 1, & n_{2s} \geqslant 1, \\ \sqrt{2}, & n_{2s} = 0. \end{cases} \tag{4.12}$$

Denote strict 2-partition $\tilde{\chi}$ as $(\tilde{\chi}) = (\tilde{\alpha}, \tilde{\beta})$, which possesses the same properties as 2-partition. Note that if $\beta = \emptyset$, strict 2-partition $(\tilde{\chi}) = (\tilde{\alpha}, \tilde{\beta})$ is equivalent to the strict partition $\tilde{\alpha}$. Eq. (4.11) leads to

$$|\tilde{\alpha}\rangle := (-1)^r\omega\phi_{m_1}\cdots\phi_{m_{2r}}|0\rangle = (-1)^r\omega\prod_{j=1}^{2r}\phi_{m_j}|0\rangle,$$
$$\langle\tilde{\alpha}| := (-1)^{r+|\tilde{\alpha}|}\omega\langle 0|\phi_{-m_{2r}}\cdots\phi_{-m_1}$$
$$= (-1)^{r+|\tilde{\alpha}|}\omega\langle 0|\prod_{j=1}^{2r}\phi_{-m_j}. \tag{4.13}$$

Definition 4.2 Define the 'BUC plane partition' as $(\ldots, \tilde{\chi}_{-1}, \tilde{\chi}_0, \tilde{\chi}_1, \ldots)$ which represents a pair of BKP plane partitions $\tilde{\pi}$ and $\tilde{\pi}'$, where $\tilde{\pi} = (\ldots, \tilde{\alpha}_{-1}, \tilde{\alpha}_0, \tilde{\alpha}_1, \ldots)$, $\tilde{\pi}' = (\ldots, \tilde{\beta}_{-1}, \tilde{\beta}_0, \tilde{\beta}_1, \ldots)$ and $(\tilde{\chi}_k) = (\tilde{\alpha}_k, \tilde{\beta}_k)$.

Lemma 4.3 *Let the states corresponding to the strict 2-partition* $(\tilde{\chi}) = (\tilde{\alpha}, \tilde{\beta})$ *be* $|\tilde{\chi}\rangle = |\tilde{\alpha}, \tilde{\beta}\rangle$ *and* $\langle\tilde{\chi}| = \langle\tilde{\alpha}, \tilde{\beta}|$. *Then*

$$\langle\tilde{\chi}'|\Upsilon_+(\mathbf{z},\mathbf{v})|\tilde{\chi}\rangle = \begin{cases} 2^{n(\tilde{\alpha}'|\tilde{\alpha})}z^{|\tilde{\alpha}'|-|\tilde{\alpha}|}2^{n(\tilde{\beta}'|\tilde{\beta})}z^{|\tilde{\beta}'|-|\tilde{\beta}|}, & if\ A\ holds, \\ 2^{n(\tilde{\alpha}'|\tilde{\alpha})}z^{|\tilde{\alpha}'|-|\tilde{\alpha}|}(-1)^{n(\tilde{\beta})}2^{n(\tilde{\beta}'|\tilde{\beta})+\frac{1}{2}}z^{|\tilde{\beta}'|-|\tilde{\beta}|}, & if\ B\ holds, \\ (-1)^{n(\tilde{\alpha})}2^{n(\tilde{\alpha}'|\tilde{\alpha})+\frac{1}{2}}z^{|\tilde{\alpha}'|-|\tilde{\alpha}|}2^{n(\tilde{\beta}'|\tilde{\beta})}z^{|\tilde{\beta}'|-|\tilde{\beta}|}, & if\ B\ holds, \\ (-1)^{n(\tilde{\alpha})+n(\tilde{\beta})}2^{n(\tilde{\alpha}'|\tilde{\alpha})+n(\tilde{\beta}'|\tilde{\beta})+1}z^{|\tilde{\alpha}'|+|\tilde{\beta}'|-|\tilde{\alpha}|-|\tilde{\beta}|}, & if\ D\ holds, \\ 0, & otherwise, \end{cases} \tag{4.14}$$

$$\langle\tilde{\chi}|\Upsilon_-(\mathbf{z},\mathbf{v})|\tilde{\chi}'\rangle = \begin{cases} 2^{n(\tilde{\alpha}'|\tilde{\alpha})}z^{|\tilde{\alpha}|-|\tilde{\alpha}'|}2^{n(\tilde{\beta}'|\tilde{\beta})}z^{|\tilde{\beta}|-|\tilde{\beta}'|}, & if\ A\ holds, \\ 2^{n(\tilde{\alpha}'|\tilde{\alpha})}z^{|\tilde{\alpha}|-|\tilde{\alpha}'|}(-1)^{n(\tilde{\beta})}2^{n(\tilde{\beta}'|\tilde{\beta})+\frac{1}{2}}z^{|\tilde{\beta}|-|\tilde{\beta}'|}, & if\ B\ holds, \\ (-1)^{n(\tilde{\alpha})}2^{n(\tilde{\alpha}'|\tilde{\alpha})+\frac{1}{2}}z^{|\tilde{\alpha}|-|\tilde{\alpha}'|}2^{n(\tilde{\beta}'|\tilde{\beta})}z^{|\tilde{\beta}|-|\tilde{\beta}'|}, & if\ C\ holds, \\ (-1)^{n(\tilde{\alpha})+n(\tilde{\beta})}2^{n(\tilde{\alpha}'|\tilde{\alpha})+n(\tilde{\beta}'|\tilde{\beta})+1}z^{|\tilde{\alpha}|+|\tilde{\beta}|-|\tilde{\alpha}'|-|\tilde{\beta}'|}, & if\ D\ holds, \\ 0, & otherwise, \end{cases} \tag{4.15}$$

where $(\tilde{\chi}') = (\tilde{\alpha}', \tilde{\beta}')$ *and*

A *stands for* $\tilde{\alpha}' \prec \tilde{\alpha},\ \tilde{\beta}' \prec \tilde{\beta},\ n(\tilde{\alpha}') = n(\tilde{\alpha})$ *and* $n(\tilde{\beta}') = n(\tilde{\beta})$,

B *stands for* $\tilde{\alpha}' \prec \tilde{\alpha},\ \tilde{\beta}' \prec \tilde{\beta},\ n(\tilde{\alpha}') = n(\tilde{\alpha})$ *and* $n(\tilde{\beta}') = n(\tilde{\beta}) - 1$,

C *stands for* $\tilde{\alpha}' \prec \tilde{\alpha},\ \tilde{\beta}' \prec \tilde{\beta},\ n(\tilde{\alpha}') = n(\tilde{\alpha}) - 1$ *and* $n(\tilde{\beta}') = n(\tilde{\beta})$,

D *stands for* $\tilde{\alpha}' \prec \tilde{\alpha},\ \tilde{\beta}' \prec \tilde{\beta},\ n(\tilde{\alpha}') = n(\tilde{\alpha}) - 1$ *and* $n(\tilde{\beta}') = n(\tilde{\beta}) - 1$. (4.16)

Proof By means of Eqs. (4.6) and (4.11), one obtains

$$\Upsilon_+(\mathbf{z},\mathbf{v})|\tilde{\alpha},\tilde{\beta}\rangle$$
$$= (-1)^{r+s}\omega\omega'\prod_{j=1}^{2r}\left(\Upsilon_+(\mathbf{z},\mathbf{v})\phi_{m_j}\Upsilon_+(-\mathbf{z},-\mathbf{v})\right)$$
$$\times\prod_{k=1}^{2s}\left(\Upsilon_+(\mathbf{z},\mathbf{v})\bar{\phi}_{n_k}\Upsilon_+(-\mathbf{z},-\mathbf{v})\right)\Upsilon_+(\mathbf{z},\mathbf{v})|0\rangle$$
$$= (-1)^r\omega\prod_{j=1}^{2r}\left(\phi_{m_j} + 2\sum_{i=1}^{\infty}\frac{1}{z^i}\phi_{(m_j-i)}\right)(-1)^s\omega'$$
$$\times\prod_{k=1}^{2s}\left(\bar{\phi}_{n_k} + 2\sum_{i=1}^{\infty}\frac{1}{v^i}\bar{\phi}_{(n_k-i)}\right)|0\rangle = \tilde{T}_1\cdot\tilde{T}_2|0\rangle, \tag{4.17}$$

where

$$\tilde{T}_1 = (-1)^r\omega\prod_{j=1}^{2r}\left(\phi_{m_j} + 2\sum_{i=1}^{\infty}\frac{1}{z^i}\phi_{(m_j-i)}\right),$$
$$\tilde{T}_2 = (-1)^s\omega'\prod_{k=1}^{2s}\left(\bar{\phi}_{n_k} + 2\sum_{i=1}^{\infty}\frac{1}{v^i}\bar{\phi}_{(n_k-i)}\right). \tag{4.18}$$

Substituting Eq. (2.52) into Eq. (4.18), we have

$$\tilde{T}_2|0\rangle = \sum_{\substack{\tilde{\beta}'\prec\tilde{\beta} \\ n(\tilde{\beta}')=n(\tilde{\beta})}} 2^{n(\tilde{\beta}'|\tilde{\beta})}z^{|\tilde{\beta}'|-|\tilde{\beta}|}|\tilde{\beta}'\rangle + (-1)^{n(\tilde{\beta})}\sqrt{2}$$
$$\sum_{\substack{\tilde{\beta}'\prec\tilde{\beta} \\ n(\tilde{\beta}')=n(\tilde{\beta})-1}} 2^{n(\tilde{\beta}'|\tilde{\beta})}z^{|\tilde{\beta}'|-|\tilde{\beta}|}|\tilde{\beta}'\rangle. \tag{4.19}$$

Since the assumed state is not involved in the subsequent calculations, we let

$$|\tilde{\beta}'\rangle = (-1)^s \omega_1' \bar{\phi}_{\tilde{n}_1} \cdots \bar{\phi}_{\tilde{n}_{2s}} |0\rangle = (-1)^s \omega_1' \prod_{j=1}^{2s} \bar{\phi}_{\tilde{n}_j} |0\rangle. \tag{4.20}$$

From the commutation relations (2.31), the Eq. (4.17) can be rewritten as

$$\begin{aligned}\Upsilon_+(\mathbf{z},\mathbf{v})|\alpha_1,\alpha_2\rangle &= \sum_{\substack{\tilde{\beta}'\prec\tilde{\beta}\\ n(\tilde{\beta}')=n(\tilde{\beta})}} 2^{n(\tilde{\beta}'|\tilde{\beta})} z^{|\tilde{\beta}'|-|\tilde{\beta}|} (-1)^s \omega_1' \bar{\phi}_{\tilde{n}_1} \cdots \bar{\phi}_{\tilde{n}_{2s}} \tilde{T}_1 |0\rangle \\ &+ (-1)^{n(\tilde{\beta})}\sqrt{2} \sum_{\substack{\tilde{\beta}'\prec\tilde{\beta}\\ n(\tilde{\beta}')=n(\tilde{\beta})-1}} 2^{n(\tilde{\beta}'|\tilde{\beta})} z^{|\tilde{\beta}'|-|\tilde{\beta}|} (-1)^s \omega_1' \bar{\phi}_{\tilde{n}_1} \\ &\cdots \bar{\phi}_{\tilde{n}_{2s}} \tilde{T}_1 |0\rangle. \end{aligned} \tag{4.21}$$

It follows from Eq. (2.52) that

$$\begin{aligned}\tilde{T}_1|0\rangle = &\sum_{\substack{\tilde{\alpha}'\prec\tilde{\alpha}\\ n(\tilde{\alpha}')=n(\tilde{\alpha})}} 2^{n(\tilde{\alpha}'|\tilde{\alpha})} z^{|\tilde{\alpha}'|-|\tilde{\alpha}|} |\tilde{\alpha}'\rangle \\ &+ (-1)^{n(\tilde{\alpha})}\sqrt{2} \sum_{\substack{\tilde{\alpha}'\prec\tilde{\alpha}\\ n(\tilde{\alpha}')=n(\tilde{\alpha})-1}} 2^{n(\tilde{\alpha}'|\tilde{\alpha})} z^{|\tilde{\alpha}'|-|\tilde{\alpha}|} |\tilde{\alpha}'\rangle. \end{aligned} \tag{4.22}$$

Setting

$$|\tilde{\alpha}'\rangle = (-1)^r \omega_1 \phi_{\tilde{m}_1} \cdots \phi_{\tilde{m}_{2r}} |0\rangle = (-1)^r \omega_1 \prod_{j=1}^{2r} \phi_{\tilde{m}_j} |0\rangle. \tag{4.23}$$

Applying the above results to Eq. (4.21) yields

$$\begin{aligned}\Upsilon_+(\mathbf{z},\mathbf{v})|\tilde{\alpha},\tilde{\beta}\rangle = &\sum_{\substack{\tilde{\alpha}'\prec\tilde{\alpha}\\ n(\tilde{\alpha}')=n(\tilde{\alpha})}} \\ \times &\sum_{\substack{\tilde{\beta}'\prec\tilde{\beta}\\ n(\tilde{\beta}')=n(\tilde{\beta})}} 2^{n(\tilde{\alpha}'|\tilde{\alpha})} z^{|\tilde{\alpha}'|-|\tilde{\alpha}|} 2^{n(\tilde{\beta}'|\tilde{\beta})} z^{|\tilde{\beta}'|-|\tilde{\beta}|} |\tilde{\alpha}',\tilde{\beta}'\rangle \\ &+ (-1)^{n(\tilde{\beta})}\sqrt{2} \sum_{\substack{\tilde{\alpha}'\prec\tilde{\alpha}\\ n(\tilde{\alpha}')=n(\tilde{\alpha})}} \\ \times &\sum_{\substack{\tilde{\beta}'\prec\tilde{\beta}\\ n(\tilde{\beta}')=n(\tilde{\beta})-1}} 2^{n(\tilde{\alpha}'|\tilde{\alpha})} z^{|\tilde{\alpha}'|-|\tilde{\alpha}|} 2^{n(\tilde{\beta}'|\tilde{\beta})} z^{|\tilde{\beta}'|-|\tilde{\beta}|} |\tilde{\alpha}',\tilde{\beta}'\rangle \\ &+ (-1)^{n(\tilde{\alpha})}\sqrt{2} \sum_{\substack{\tilde{\alpha}'\prec\tilde{\alpha}\\ n(\tilde{\alpha}')=n(\tilde{\alpha})-1}} \\ \times &\sum_{\substack{\tilde{\beta}'\prec\tilde{\beta}\\ n(\tilde{\beta}')=n(\tilde{\beta})}} 2^{n(\tilde{\alpha}'|\tilde{\alpha})} z^{|\tilde{\alpha}'|-|\tilde{\alpha}|} 2^{n(\tilde{\beta}'|\tilde{\beta})} z^{|\tilde{\beta}'|-|\tilde{\beta}|} |\tilde{\alpha}',\tilde{\beta}'\rangle \\ &+ (-1)^{n(\tilde{\alpha})+n(\tilde{\beta})} 2 \sum_{\substack{\tilde{\alpha}'\prec\tilde{\alpha}\\ n(\tilde{\alpha}')=n(\tilde{\alpha})-1}} \\ \times &\sum_{\substack{\tilde{\beta}'\prec\tilde{\beta}\\ n(\tilde{\beta}')=n(\tilde{\beta})-1}} 2^{n(\tilde{\alpha}'|\tilde{\alpha})} z^{|\tilde{\alpha}'|-|\tilde{\alpha}|} 2^{n(\tilde{\beta}'|\tilde{\beta})} z^{|\tilde{\beta}'|-|\tilde{\beta}|} |\tilde{\alpha}',\tilde{\beta}'\rangle. \end{aligned} \tag{4.24}$$

Similarly, it is show that

$$\begin{aligned}\langle\tilde{\alpha},\tilde{\beta}|\Upsilon_-(\mathbf{z},\mathbf{v}) = &\sum_{\substack{\tilde{\alpha}'\prec\tilde{\alpha}\\ n(\tilde{\alpha}')=n(\tilde{\alpha})}} \\ \times &\sum_{\substack{\tilde{\beta}'\prec\tilde{\beta}\\ n(\tilde{\beta}')=n(\tilde{\beta})}} 2^{n(\tilde{\alpha}'|\tilde{\alpha})} z^{|\tilde{\alpha}|-|\tilde{\alpha}'|} 2^{n(\tilde{\beta}'|\tilde{\beta})} z^{|\tilde{\beta}|-|\tilde{\beta}'|} \langle\tilde{\alpha}',\tilde{\beta}'| \\ &+ (-1)^{n(\tilde{\beta})}\sqrt{2} \sum_{\substack{\tilde{\alpha}'\prec\tilde{\alpha}\\ n(\tilde{\alpha}')=n(\tilde{\alpha})}} \\ \times &\sum_{\substack{\tilde{\beta}'\prec\tilde{\beta}\\ n(\tilde{\beta}')=n(\tilde{\beta})-1}} 2^{n(\tilde{\alpha}'|\tilde{\alpha})} z^{|\tilde{\alpha}|-|\tilde{\alpha}'|} 2^{n(\tilde{\beta}'|\tilde{\beta})} z^{|\tilde{\beta}|-|\tilde{\beta}'|} \langle\tilde{\alpha}',\tilde{\beta}'| \\ &+ (-1)^{n(\tilde{\alpha})}\sqrt{2} \sum_{\substack{\tilde{\alpha}'\prec\tilde{\alpha}\\ n(\tilde{\alpha}')=n(\tilde{\alpha})-1}} \\ \times &\sum_{\substack{\tilde{\beta}'\prec\tilde{\beta}\\ n(\tilde{\beta}')=n(\tilde{\beta})}} 2^{n(\tilde{\alpha}'|\tilde{\alpha})} z^{|\tilde{\alpha}|-|\tilde{\alpha}'|} 2^{n(\tilde{\beta}'|\tilde{\beta})} z^{|\tilde{\beta}|-|\tilde{\beta}'|} \langle\tilde{\alpha}',\tilde{\beta}'| \\ &+ (-1)^{n(\tilde{\alpha})+n(\tilde{\beta})} 2 \sum_{\substack{\tilde{\alpha}'\prec\tilde{\alpha}\\ n(\tilde{\alpha}')=n(\tilde{\alpha})-1}} \\ \times &\sum_{\substack{\tilde{\beta}'\prec\tilde{\beta}\\ n(\tilde{\beta}')=n(\tilde{\beta})-1}} 2^{n(\tilde{\alpha}'|\tilde{\alpha})} z^{|\tilde{\alpha}|-|\tilde{\alpha}'|} 2^{n(\tilde{\beta}'|\tilde{\beta})} z^{|\tilde{\beta}|-|\tilde{\beta}'|} \langle\tilde{\alpha}',\tilde{\beta}'|. \end{aligned} \tag{4.25}$$

□

In particular, if $\tilde{\beta} = \emptyset$, Eqs. (4.24) and (4.25) are respectively transformed into

$$\begin{aligned}\Upsilon_+(\mathbf{z},\mathbf{v})|\tilde{\alpha}\rangle = &\sum_{\substack{\tilde{\alpha}'\prec\tilde{\alpha}\\ n(\tilde{\alpha}')=n(\tilde{\alpha})}} 2^{n(\tilde{\alpha}'|\tilde{\alpha})} z^{|\tilde{\alpha}'|-|\tilde{\alpha}|} |\tilde{\alpha}'\rangle \\ &+ (-1)^{n(\tilde{\alpha})}\sqrt{2} \sum_{\substack{\tilde{\alpha}'\prec\tilde{\alpha}\\ n(\tilde{\alpha}')=n(\tilde{\alpha})-1}} 2^{n(\tilde{\alpha}'|\tilde{\alpha})} z^{|\tilde{\alpha}'|-|\tilde{\alpha}|} |\tilde{\alpha}'\rangle, \\ \langle\tilde{\alpha}|\Upsilon_-(\mathbf{z},\mathbf{v}) = &\sum_{\substack{\tilde{\alpha}'\prec\tilde{\alpha}\\ n(\tilde{\alpha}')=n(\tilde{\alpha})}} 2^{n(\tilde{\alpha}'|\tilde{\alpha})} z^{|\tilde{\alpha}|-|\tilde{\alpha}'|} \langle\tilde{\alpha}'| \end{aligned}$$

$$+(-1)^{n(\tilde{\alpha})}\sqrt{2}\sum_{n(\tilde{\alpha}')=n(\tilde{\alpha})-1}2^{n(\tilde{\alpha}'|\tilde{\alpha})}z^{|\tilde{\alpha}|-|\tilde{\alpha}'|}\langle\tilde{\alpha}'|. \tag{4.26}$$

A similar conclusion can be obtained for $\tilde{\alpha}=\emptyset$.

4.3 Generating function for BUC plane partitions

Define correlation function $S_B(t,q)$ as

$$S_B(t,q)=\langle 0|\overrightarrow{\prod_{i=1}^{\infty}}\Upsilon_+(t^{\frac{-2i+1}{2}},q^{\frac{-2i+1}{2}})\times\overleftarrow{\prod_{k=1}^{\infty}}\Upsilon_-(t^{\frac{2k-1}{2}},q^{\frac{2k-1}{2}})|0\rangle, \tag{4.27}$$

which provides a generating function for BUC plane partitions, where t and q are indeterminate.

Proposition 4.4 *For a strict plane partition $\tilde{\pi}$, we have*

$$2^{n(\tilde{\pi}_0)}\prod_{i=1}^{M}2^{n(\tilde{\pi}_{-i}|\tilde{\pi}_{-i+1})}\prod_{j=1}^{N}2^{n(\tilde{\pi}_j|\tilde{\pi}_{j-1})}=2^{p(\tilde{\pi})}. \tag{4.28}$$

Proof Let us use the example of strict plane partition in Fig. 4 to explain this formula. From Fig. 4, it is clear that

$$\begin{aligned}&\tilde{\pi}^0=(5,2,1),\quad \tilde{\pi}^{-1}=(4,1),\quad \tilde{\pi}^{-2}=(3)\\ &\quad\tilde{\pi}^{-3}=(1),\quad \tilde{\pi}^{-4}=\emptyset,\\ &\tilde{\pi}^1=(4,2),\quad \tilde{\pi}^2=(3,1),\\ &\quad\tilde{\pi}^3=(2),\quad \tilde{\pi}^4=(1),\quad \tilde{\pi}^5=\emptyset,\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}&2^{n(\tilde{\pi}^1|\tilde{\pi}^0)}=1,\quad 2^{n(\tilde{\pi}^2|\tilde{\pi}^1)}=2,\quad 2^{n(\tilde{\pi}^3|\tilde{\pi}^2)}=1,\\ &2^{n(\tilde{\pi}^4|\tilde{\pi}^3)}=1,\quad 2^{n(\tilde{\pi}^5|\tilde{\pi}^4)}=0,\quad 2^{n(\tilde{\pi}^{-1}|\tilde{\pi}^0)}=1,\\ &2^{n(\tilde{\pi}^{-2}|\tilde{\pi}^{-1})}=1,\quad 2^{n(\tilde{\pi}^{-3}|\tilde{\pi}^{-2})}=1,\quad 2^{n(\tilde{\pi}^{-4}|\tilde{\pi}^{-3})}=0.\end{aligned} \tag{4.30}$$

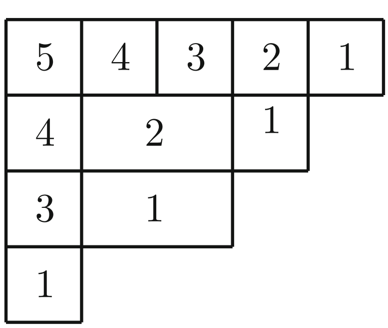

Fig. 6 All the paths in Fig. 4

From the above relations, it is showed that diagonal slices not being intersected receive a factor of 2, otherwise zero. Multiplying $n(\tilde{\pi}^0)=3$ as the power of 2 and $p(\tilde{\pi})=11$ in Fig. 6, we have

$$2^{n(\tilde{\pi}^0)}\prod_{i=1}^{4}2^{n(\tilde{\pi}_{-i}|\tilde{\pi}_{-i+1})}\prod_{j=1}^{5}2^{n(\tilde{\pi}_j|\tilde{\pi}_{j-1})}=2^{11}=2^{p(\tilde{\pi})}. \tag{4.31}$$

It follows that this method extends to arbitrary strict plane partitions. □

Setting a strict 2-partition $(\tilde{\chi})=(\tilde{\alpha},\tilde{\beta})$ and inserting $\sum_{\tilde{\chi}}|\tilde{\chi}\rangle\langle\tilde{\chi}|$ to the above equation yields

$$\begin{aligned}S_B(t,q)=&\sum_{\substack{\tilde{\chi}\text{ is a strict}\\ \text{2-partition}}}\langle 0|\overrightarrow{\prod_{i=1}^{\infty}}\Upsilon_+(t^{\frac{-2i+1}{2}},q^{\frac{-2i+1}{2}})|\tilde{\chi}\rangle\langle\tilde{\chi}|\\ &\times\overleftarrow{\prod_{k=1}^{\infty}}\Upsilon_-(t^{\frac{2k-1}{2}},q^{\frac{2k-1}{2}})|0\rangle\\ =&\sum_{\substack{\tilde{\alpha}\text{ and }\tilde{\beta}\text{ are}\\ \text{strict partitions}}}\langle 0|\overrightarrow{\prod_{i=1}^{\infty}}\Upsilon_+(t^{\frac{-2i+1}{2}},q^{\frac{-2i+1}{2}})|\tilde{\alpha},\tilde{\beta}\rangle\langle\tilde{\alpha},\tilde{\beta}|\\ &\times\overleftarrow{\prod_{k=1}^{\infty}}\Upsilon_-(t^{\frac{2k-1}{2}},q^{\frac{2k-1}{2}})|0\rangle.\end{aligned} \tag{4.32}$$

Equations (4.24)–(4.26) show that the generated weights are given by

$$\overrightarrow{\prod_{i=1}^{M}}\langle\tilde{\alpha}'_{-i},\tilde{\beta}'_{-i}|\Upsilon_+(t^{\frac{-2i+1}{2}},q^{\frac{-2i+1}{2}})|\tilde{\alpha}'_{-i+1},\tilde{\beta}'_{-i+1}\rangle\times\overleftarrow{\prod_{k=1}^{N}}\langle\tilde{\alpha}'_{k-1}\tilde{\beta}'_{k-1}|\Upsilon_-(t^{\frac{-2j+1}{2}},q^{\frac{-2j+1}{2}})|\tilde{\alpha}'_k,\tilde{\beta}'_k\rangle. \tag{4.33}$$

Let $(\tilde{\chi}'_j)=(\tilde{\alpha}'_j,\tilde{\beta}'_j)$, $\tilde{\alpha}'_{-M_1}=\tilde{\alpha}'_{N_1}=\tilde{\beta}'_{-M_2}=\tilde{\beta}'_{N_2}=\emptyset$, $M=\max\{M_1,M_2\}$, $N=\max\{N_1,N_2\}$ and $-M\leq j\leq N$. Note that the plane partition $\tilde{\pi}$ is made up of $\tilde{\alpha}'_j$ and the plane partition $\tilde{\pi}'$ is made up of $\tilde{\beta}'_j$. Combining Proposition 4.4, the Eq. (4.33) can be represented as

$$\begin{aligned}&2^{n(\tilde{\alpha}'_0)}\prod_{i_1=1}^{M_1}2^{n(\tilde{\alpha}'_{-i_1}|\tilde{\alpha}'_{-i_1+1})}\prod_{k_1=1}^{N_1}2^{n(\tilde{\alpha}_{k_1}|\tilde{\alpha}'_{k_1-1})}\\ &\times\overrightarrow{\prod_{i_1=1}^{M_1}}t^{(\frac{-2i_1+1}{2})(|\tilde{\alpha}'_{-i_1}|-|\tilde{\alpha}'_{-i_1+1}|)}\overleftarrow{\prod_{k_1=1}^{N_1}}t^{(\frac{-2k_1+1}{2})(|\tilde{\alpha}'_{k_1-1}|-|\tilde{\alpha}'_{k_1}|)}\\ &2^{n(\tilde{\beta}'_0)}\prod_{i_2=1}^{M_2}2^{n(\tilde{\beta}'_{-i_2}|\tilde{\beta}'_{-i_2+1})}\prod_{k_2=1}^{N_2}2^{n(\tilde{\beta}_{k_2}|\tilde{\beta}'_{k_2-1})}\end{aligned}$$

$$\times \prod_{i_2=1}^{M_2} q^{(\frac{-2i_2+1}{2})(|\tilde{\beta}'_{-i_2}|-|\tilde{\beta}'_{-i_2+1}|)} \prod_{k_2=1}^{N_2} q^{(\frac{-2k_2+1}{2})(|\tilde{\beta}'_{k_2-1}|-|\tilde{\beta}'_{k_2}|)}$$
$$= 2^{p(\tilde{\pi})+p(\tilde{\pi}')} \prod_{j=-M}^{N} t^{|\tilde{\alpha}'_j|} q^{|\tilde{\beta}'_j|}. \tag{4.34}$$

It shows that all strict 2-partitions $(\tilde{\chi}'_j) = (\tilde{\alpha}'_j, \tilde{\beta}'_j)$ satisfy

$$\emptyset = (\tilde{\chi}'_{-M}) \prec \cdots \prec (\tilde{\chi}'_{-2}) \prec (\tilde{\chi}'_{-1}) \prec (\tilde{\chi}'_0) \succ (\tilde{\chi}'_1) \succ (\tilde{\chi}'_2) \succ \cdots \succ (\tilde{\chi}'_N) = \emptyset, \tag{4.35}$$

which is equivalent to

$$\begin{aligned}\emptyset &= \tilde{\alpha}'_{-M} = \cdots = \tilde{\alpha}'_{-M_1} \prec \cdots \prec \tilde{\alpha}'_{-2} \prec \tilde{\alpha}'_{-1} \prec \tilde{\alpha}'_0 \succ \tilde{\alpha}'_1 \\ &\succ \tilde{\alpha}'_2 \succ \cdots \succ \tilde{\alpha}'_{N_1} = \cdots = \tilde{\alpha}'_N = \emptyset, \\ \emptyset &= \tilde{\beta}'_{-M} = \cdots = \tilde{\beta}'_{-M_1} \prec \cdots \prec \tilde{\beta}'_{-2} \prec \tilde{\beta}'_{-1} \\ &\prec \tilde{\beta}'_0 \succ \tilde{\beta}'_1 \succ \tilde{\beta}'_2 \succ \cdots \succ \tilde{\beta}'_{N_2} = \cdots = \tilde{\beta}'_N = \emptyset.\end{aligned} \tag{4.36}$$

Then the Eq. (4.34) can be rewritten as

$$2^{p(\tilde{\pi})+p(\tilde{\pi}')} \prod_{j=-M}^{N} t^{|\tilde{\alpha}'_j|} q^{|\tilde{\beta}'_j|} = 2^{p(\tilde{\pi})+p(\tilde{\pi}')} \times \prod_{i=-M_1}^{N_1} t^{|\tilde{\alpha}'_i|} \prod_{k=-M_2}^{N_2} q^{|\tilde{\beta}'_k|}. \tag{4.37}$$

It follows that

$$S_B(t,q) = \sum_{\substack{\tilde{\pi} \text{ and } \tilde{\pi}' \text{ are} \\ \text{strict plane partitions}}} 2^{p(\tilde{\pi})+p(\tilde{\pi}')} t^{|\tilde{\pi}|} q^{|\tilde{\pi}'|}. \tag{4.38}$$

In addition, by means of the Eq. (4.10), the generating function for BUC plane partitions can be represented as

$$\begin{aligned}S_B(p,q) &= \prod_{n_1=1}^{\infty} \left(\frac{1+t^{n_1}}{1-t^{n_1}}\right) \prod_{m_1=1}^{\infty} \left(\frac{1+q^{m_1}}{1-q^{m_1}}\right) \\ &\times \langle 0| \prod_{j=2}^{\infty} \Upsilon_+(p^{\frac{-2j+1}{2}}, q^{\frac{-2j+1}{2}}) \prod_{k=1}^{\infty} \Upsilon_-(p^{\frac{2k-1}{2}}, q^{\frac{2k-1}{2}})|0\rangle \\ &= \cdots = \prod_{n=1}^{\infty} \left(\frac{1+t^n}{1-t^n}\right)^n \prod_{m=1}^{\infty} \left(\frac{1+q^m}{1-q^m}\right)^m.\end{aligned} \tag{4.39}$$

Equation (4.39) can be regarded as the extension of the shifted MacMahon's formula [26].

5 Conclusions and discussions

In this paper, by means of constructing the generalized fermion vertex operators and interlacing (strict) 2-partitions, we have discussed generating functions for UC and BUC plane partitions which can be written as product forms. It should be pointed out that the fermion calculus approach play a vital role in establishing generating functions of plane partitions. How to use this method to look for the structure and properties of plane partitions in other integrable systems, such as symplectic universal character (SUC) hierarchy and the orthogonal universal character (OUC) should be an interesting question, which will be studied in the near future.

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Code availability My manuscript has no associated code/software. [Author's comment: Code/Software sharing not applicable to this article as no code/software was generated or analysed during the current study.]

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