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Fermion hierarchies in $SU(5)$ grand unification from Γ'_6 modular flavor symmetry

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ABSTRACT: We construct an $SU(5)$ grand unified model in which the hierarchies of the quark and lepton masses and mixing are explained by the Γ'_6 modular flavor symmetry. The hierarchies are realized by the Froggatt-Nielsen-like mechanism due to the residual Z_6^T symmetry, approximately unbroken at $\tau \sim i\infty$. We argue that the $\Gamma_6^{(l)}$ symmetry is the minimal possibility to realize the up-type quark mass hierarchies, since the Yukawa matrix is symmetric. We find a combination of the representations and modular weights and then show numerical values of $\mathcal{O}(1)$ coefficients for the realistic fermion hierarchies.

KEYWORDS: Theories of Flavour, Grand Unification

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Contents

1	Introduction	1
2	Fermion hierarchies in SU(5) GUT	2
2.1	SU(5) GUT	2
2.2	Modular flavor symmetry	3
2.3	Assignments of representations	5
3	Model	6
3.1	Yukawa couplings	6
3.2	Assigning modular weights	7
3.3	Benchmark points	8
4	Neutrino sector	10
5	Summary	12
A	Γ'_6 modular symmetry	13
A.1	Group theory of Γ'_6	13
A.2	Modular forms	15

1 Introduction

The Grand Unified Theory (GUT) is an attractive framework to understand the gauge structure of the Standard Model (SM) [1]. In the SU(5) GUT, the quarks and leptons in a generation are unified into $\bar{5}$ and 10 representations. As a result, for instance, the exactly opposite charges of the electron and the proton are manifestly explained. The SU(5) GUT predicts the unification of the three gauge coupling constants which is consistent with the Minimal Supersymmetric SM (MSSM) [2–9]. Due to the unification of the quarks and leptons, the Yukawa couplings are also constrained to be consistent with the SU(5) gauge symmetry, as we shall study in this paper.

Higher dimensional theory such as superstring theory is an interesting candidate for the underlying theory beyond the SM. Such a theory includes a compact space, whose geometrical aspects such as a size and shape are characterized by moduli. Yukawa couplings as well as higher dimensional couplings depend on moduli, and hence they have a geometrical symmetry, the so-called modular symmetry. The moduli dependencies of the Yukawa couplings were studied in magnetized D-brane models [10–16] and heterotic orbifold models [17–22]. For a single modulus τ , the modular symmetry corresponds to $\Gamma = \text{SL}(2, \mathbb{Z})$. Since Yukawa couplings are functions of the modulus, they transform non-trivially under Γ . Thus, the modular symmetry may play an important role in four-dimensional low-energy effective field theory.¹

¹For instance, the three-generation models based on the magnetized extra dimension invariant under the modular flavor symmetry were studied in refs. [16, 23].

The modular flavor symmetry provides an interesting possibility to understand the flavor structures of the three generations of quarks and leptons in the SM [24]. Under the modular symmetry, Yukawa coupling constants are treated as the so-called modular forms, holomorphic functions of modulus τ . The finite modular groups Γ_N , $N \in \mathbb{N}$, are generalization of the non-Abelian discrete flavor symmetries [25], as well studied in the literature [24, 26–44]. In fact, we can find isomorphisms $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$, and so on. There have been many attempts to understand the flavor structures of the SM by the finite modular flavor symmetries [45–83], especially under the GUTs [84–92].

There are the hierarchies in the masses of the SM fermions and in the Cabibbo-Kobayashi-Maskawa (CKM) matrix. The residual \mathbb{Z}_N symmetry of the modular symmetry induces hierarchies of Yukawa couplings if the modulus τ has a value near a fixed point. For instance, if $\tau \sim i\infty$, the \mathbb{Z}_N symmetry remains in the modular Γ_N symmetry, and then the Yukawa couplings are suppressed by powers of $q_N := e^{-2\pi\text{Im}\tau/N} \ll 1$ depending on the charge under the \mathbb{Z}_N symmetry. This is a realization of the Froggatt-Nielson (FN) mechanism, where the powers of the flavon field are replaced by those of q_N . Recently, this idea is applied for the quark sector based on the modular A_4 [93, 94], $A_4 \times A_4 \times A_4$ [95], S_4 [96] and Γ_6 [97] symmetries, and the lepton sector is also studied in refs. [98, 99].² In ref. [101], the present authors constructed a model for both quark and lepton sectors based on $\Gamma'_4 \simeq S'_4$ symmetry, where Γ'_N is the double covering of Γ_N symmetry.

In this work, we construct a model to explain the hierarchies of the quarks and leptons in the SU(5) GUT. It will be turned out that the residual symmetry should be \mathbb{Z}_N with $N \geq 5$, to realize the hierarchy of the up-type quark masses because the up Yukawa matrix is a symmetric matrix. Since the $\Gamma_5^{(')}$ symmetry does not have proper representation to realize the hierarchy [28, 80] as explained later, we shall consider the Γ'_6 symmetry as the minimal possibility. We consider Γ'_6 , double covering of Γ_6 , to construct a model with non-singlet representations. We also study the neutrino sector assuming the type-I seesaw mechanism by introducing three SU(5) singlets.

This paper is organized as follows. In section 2, we briefly review the flavor structures of the SU(5) GUT and the modular flavor symmetry, and then discuss how to explain the hierarchies of the masses and mixing. The explicit model is studied and our results of numerical calculations for the charged fermions are shown in section 3, and then the neutrino sector is discussed in section 4. Section 5 concludes. The details of the modular Γ'_6 symmetry are shown in appendix A.

2 Fermion hierarchies in SU(5) GUT

2.1 SU(5) GUT

The superpotential at the dimension-4 is given by

$$W_4 = \frac{1}{2} y_{ij} \epsilon_{ABCDE} 10_i^{AB} 10_j^{CD} H^E + h_{ij} 10_i^{AB} \bar{5}_{jA} \bar{H}_B, \quad (2.1)$$

²In ref. [100], the lepton mass hierarchies are explained by assuming the smallness of the modular form ~ 0.07 at weight 2 in the S_3 modular symmetry.

where $A, B, \dots = 1, 2, 3, 4, 5$ are the $SU(5)$ indices, and $i, j = 1, 2, 3$ are the flavor indices. The MSSM fields are in the $SU(5)$ multiplets as

$$\bar{5}_a = \bar{d}_a, \quad \bar{5}_{3+\alpha} = \epsilon_{\alpha\beta} \ell^\beta, \quad 10^{a,3+\alpha} = Q^{a\alpha}, \quad 10^{ab} = \epsilon^{abc} \bar{u}_c, \quad 10^{3+\alpha,3+\beta} = \epsilon^{\alpha\beta} \bar{e}, \quad (2.2)$$

where $a, b, c = 1, 2, 3$ and $\alpha, \beta = 1, 2$ are the $SU(3)_C$ and $SU(2)_L$ indices, respectively. In these expressions, ϵ 's with the indices are the complete anti-symmetric tensors. The Higgs doublets are included in the (anti-)fundamental representations H (\bar{H}) as

$$H^{3+\alpha} = \epsilon^{\alpha\beta} H_{u\beta}, \quad \bar{H}_{3+\alpha} = H_{d\alpha}. \quad (2.3)$$

We assume that the triplets are so heavy that it will not induce too fast proton decay by a certain mechanism [102–105].³ We also assume a discrete R symmetry, such as the R-parity, to forbid the dimension-4 operator $10\bar{5}\bar{5}$ [107, 108]. Since the Yukawa matrices of the down quarks and charged leptons are given by the common Yukawa matrix h_{ij} , the dimension-4 Yukawa couplings in eq. (2.1) can not explain the realistic fermion masses.

In this work, we assume that the $SU(5)$ gauge symmetry is broken by an adjoint field Σ_A^B whose VEV is given by

$$\langle \Sigma \rangle = v_\Sigma \times \text{diag}(2, 2, 2, -3, -3), \quad (2.4)$$

so that the GUT symmetry is broken down to the SM gauge symmetry. We shall consider the dimension-5 interaction involving the adjoint field,

$$W_5 \ni \frac{1}{\Lambda} C_{ij} 10_i^{AB} \Sigma_A^C \bar{5}_{jC} \bar{H}_B, \quad (2.5)$$

which splits the Yukawa couplings of the down quarks and the charged leptons. Here Λ is a cutoff scale. The dimension-5 operators inserting the adjoint field to other places will not change the flavor structure, and thus we do not write them explicitly.

The Yukawa couplings for the MSSM fields are given by

$$W = \bar{u} Y_u Q H_u + \bar{d} Y_d H_d + \bar{e} Y_e \ell H_d, \quad (2.6)$$

where the Yukawa matrices are given by

$$[Y_u]_{ij} = y_{ij}, \quad [Y_d]_{ij} = h_{ij} + 2R_\Sigma C_{ij}, \quad [Y_e]_{ij} = h_{ij} - 3R_\Sigma C_{ij}, \quad (2.7)$$

where $R_\Sigma := v_\Sigma/\Lambda$. Thus the $\mathcal{O}(R_\Sigma)$ splitting of the down-type Yukawa couplings is induced via the VEV of the adjoint scalar field v_Σ .

2.2 Modular flavor symmetry

We consider the so-called principal congruence subgroups $\Gamma(N)$, $N \in \mathbb{N}$, defined as

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}, \quad (2.8)$$

³As far as the triplet Higgses are at the GUT scale, the proton lifetime would be long enough if soft SUSY breaking parameters are larger than $\mathcal{O}(100)$ TeV [106].

where $\Gamma := \text{SL}(2, \mathbb{Z}) = \Gamma(1)$ is the special linear group of 2×2 matrices of integers. The modulus τ is transformed by the group Γ as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}. \tag{2.9}$$

The finite modular group Γ'_N is defined as a quotient group $\Gamma'_N := \Gamma/\Gamma(N)$, which can be generated by the generators,

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.10}$$

One can consider the quotient group $\Gamma_N := \bar{\Gamma}/\Gamma(N)$, where $\bar{\Gamma} := \Gamma/\mathbb{Z}_2^R$ with \mathbb{Z}_2^R being the \mathbb{Z}_2 symmetry generated by R . Γ'_N is the double covering of the group Γ_N . For $N < 6$, the generators of Γ'_N satisfy

$$S^2 = R, \quad (ST)^3 = R^2 = T^N = 1, \quad TR = RT, \tag{2.11}$$

and those for Γ_N are given by taking $R = 1$. At $N = 6$, the generators should also satisfy

$$ST^2ST^3ST^4ST^3 = 1, \tag{2.12}$$

in addition to those in eq. (2.11). Under Γ'_N , a modular form, a holomorphic function of τ , $Y_r^{(k)}$ with representation r and modular weight k transforms as

$$Y_r^{(k)} \rightarrow (c\tau + d)^k \rho(r) Y_r^{(k)}(\tau), \tag{2.13}$$

where $\rho(r)$ is the representation matrix of r . We assume that the chiral superfield Φ with representation r_Φ and weight $-k_\Phi$ transform as

$$\Phi \rightarrow (c\tau + d)^{-k_\Phi} \rho(r_\Phi) \Phi. \tag{2.14}$$

If the modulus is stabilized near a fixed point $\tau \sim i$, $w := e^{2\pi i/3}$ or $i\infty$, there remains a residual symmetry \mathbb{Z}_4^S , \mathbb{Z}_3^{ST} or \mathbb{Z}_N^T , respectively. Here, the superscript represents a generator associated with the residual symmetry.

The modular invariant Kähler potential for the kinetic term is given by

$$K \ni \frac{\Phi^\dagger \Phi}{(-i\tau + i\tau^*)^{k_\Phi}}, \tag{2.15}$$

and hence the canonically normalized chiral superfield is given by

$$\hat{\Phi} = \left(\sqrt{2\text{Im}\tau}\right)^{-k_\Phi} \Phi. \tag{2.16}$$

The normalization factors from $\sqrt{2\text{Im}\tau}$ will be important for $\tau \sim i\infty$, as we shall consider in this work.

2.3 Assignments of representations

The hierarchical Yukawa matrices can be realized by the FN mechanism with a \mathbb{Z}_N symmetry [109, 110] if the modulus field is stabilized near a fixed point. We shall consider the FN-like mechanism by \mathbb{Z}_N^T symmetry, and define $\epsilon := \sqrt{3}e^{2\pi i\tau/6}$ in eq. (A.24) throughout this paper. See appendix A for more details. Since the up-type Yukawa matrix is symmetric in the SU(5) GUT, the texture of Y_u is, in general, given by

$$Y_u \sim \begin{pmatrix} \epsilon^{2n} & \epsilon^{n+m} & \epsilon^n \\ \epsilon^{n+m} & \epsilon^{2m} & \epsilon^m \\ \epsilon^n & \epsilon^m & 1 \end{pmatrix}, \quad (2.17)$$

where $\epsilon \ll 1$ and $n, m \in \mathbb{N}$. The singular values are read as $(\epsilon^{2n}, \epsilon^{2m}, 1)$, and hence the minimal possibility to obtain the hierarchal up quark masses is $(m, n) = (1, 2)$, so that the singular values are given by $(\epsilon^4, \epsilon^2, 1)$. Thus $N > 4$ is required to realize the hierarchy by a \mathbb{Z}_N symmetry.

The $\Gamma_6^{(l)}$ symmetry is the minimal possibility for the up quark mass hierarchy. $N > 4$ can be realized only from the residual T symmetry in the modular $\Gamma_N^{(l)}$ symmetry with $N > 4$, where \mathbb{Z}_N^T symmetry remains unbroken at $\tau \sim i\infty$. In the case of $N = 5$, however, there are no such representations whose T -charges are $(0, -1, -2) \equiv (0, 4, 3) \pmod{N = 5}$ [28, 80]. Therefore, $N = 6$ is the minimal possibility to realize the hierarchal up Yukawa couplings.

As shown in detail in appendix A, the irreducible representations of Γ_6 are [111]

$$1_b^s, 2_b, 3^s, 6, \quad (2.18)$$

where $s = 0, 1$ and $b = 0, 1, 2$. Since Γ_6 is isomorphic to $\Gamma_2 \times \Gamma_3 \simeq S_3 \times A_4$, upper (lower) indices can be considered as S_3 (A_4) indices. For the double covering Γ_6' , there are additional representations,

$$2_b^s, 4_b. \quad (2.19)$$

The charges under the residual \mathbb{Z}_6^T symmetry of the representations less than three dimensions are given by

$$T(1_b^s) = 3s + 2b, \quad T(2_b) = \begin{pmatrix} 2b \\ 2b + 3 \end{pmatrix}, \quad T(3^s) = \begin{pmatrix} 3s \\ 3s + 2 \\ 3s + 4 \end{pmatrix}, \quad T(2_b^s) = \begin{pmatrix} 3s + 2b \\ 3s + 2b + 2 \end{pmatrix}, \quad (2.20)$$

modulo 6.

At $N = 6$, the texture can be realized for,

$$10_{i=1,2,3} = 1_1^0 \oplus 2_2^1 =: 10_1 \oplus (10_2, 10_3). \quad (2.21)$$

The \mathbb{Z}_6^T -charges are $(2, 1, 3)$. Since the doublet representation 2_2^1 exists only for Γ_6' , we should consider the double covering of Γ_6 . We can replace 2_2^1 to $1_2^1 \oplus 1_0^1$ which exist in Γ_6 ,

but we do not consider this case because the model is less predictive and may need larger modular weights to have certain modular forms [97].

Since the doublet lepton ℓ_i is in $\bar{5}_i$, the hierarchical structures of the neutrino masses and the PMNS matrix are determined by the representation of $\bar{5}_i$. As $\epsilon \sim (m_u/m_t)^{1/4} \sim \mathcal{O}(0.05)$ for the charged fermion hierarchies, ℓ_i should have a common \mathbb{Z}_6^T charge, so that there is no hierarchy in the neutrino sector. Thus we assign

$$\bar{5}_{i=1,2,3} = 1_b^s \oplus 1_b^s \oplus 1_b^s =: \bar{5}_1 \oplus \bar{5}_2 \oplus \bar{5}_3, \tag{2.22}$$

with $s = 0, 1$ and $b = 0, 1, 2$. The values of (s, b) will be determined to have the modular forms for a given modular weights. In this case, the singular values of the down quarks and charged leptons are given by $\epsilon^q(\epsilon^2, \epsilon, 1)$, where $q = 0, 1, 2, 3$ depends on (s, b) . The textures of the CKM and PMNS matrices are given by

$$V_{\text{CKM}} \sim \begin{pmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & 1 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}, \quad V_{\text{PMNS}} \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{2.23}$$

and there is no hierarchy in the neutrino masses originated from ϵ . We discuss the neutrino sector explicitly in section 4. We also note that the textures will also be modified by powers of $\sqrt{2\text{Im}\tau}$ as in eq. (2.16) depending on modular weights, as will be discussed in the next section in which we will take $\sqrt{2\text{Im}\tau} \sim 2.5$.

3 Model

3.1 Yukawa couplings

We denote the modular weights of $10_{1,2}$ and $\bar{5}_{1,2,3}$ as

$$k_{10} = (k_{10}^1, k_{10}^2), \quad k_5 = (k_5^1, k_5^2, k_5^3). \tag{3.1}$$

The modular invariant superpotential of the Yukawa couplings are given by

$$W = \alpha_1 \left(Y_{1_1^0}^{(2k_{10}^1)} 10_1 10_1 \right) H + \alpha_2 \left(Y_{2_1^1}^{(k_{10}^1+k_{10}^2)} 10_1 10_2 \right)_1 H + \alpha_3 \left(Y_{3^0}^{(2k_{10}^2)} \overline{10}_2 10_2 \right)_1 H \tag{3.2}$$

$$+ \sum_{i=1,2,3} \left[\left(\beta_{i1} + \frac{\Sigma}{\Lambda} \gamma_{i1} \right) \left(Y_{1_{2-b}^s}^{(k_{10}^1+k_5^i)} \bar{5}_i 10_1 \right) + \left(\beta_{i2} + \frac{\Sigma}{\Lambda} \gamma_{i2} \right) \left(Y_{2_{-b}^{s+1}}^{(k_{10}^2+k_5^i)} \bar{5}_i 10_2 \right) \right]_1 \bar{H},$$

where the contractions of the SU(5) indices are implicit. It is assumed that Σ , H and \bar{H} are trivial singlets with modular weight $k = 0$. The symbol $(\dots)_1$ indicates that the trivial singlet combinations inside the parenthesis. Here α_i , β_{1i} and β_{2i} are $\mathcal{O}(1)$ coefficients. The

Yukawa matrices for the MSSM quarks and leptons are read as

$$Y_u = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{6}\alpha_1 Y_{10}^{(2k_{10}^1)} & \sqrt{3}\alpha_2 \left[Y_{22}^{(k_{10}^1+k_{10}^2)} \right]_2 & -\sqrt{3}\alpha_2 \left[Y_{22}^{(k_{10}^1+k_{10}^2)} \right]_1 \\ \sqrt{3}\alpha_2 \left[Y_{22}^{(k_{10}^2+k_{10}^1)} \right]_2 & -\sqrt{2}\alpha_3 \left[Y_{30}^{(2k_{10}^2)} \right]_3 & \alpha_3 \left[Y_{30}^{(2k_{10}^2)} \right]_2 \\ -\sqrt{3}\alpha_2 \left[Y_{22}^{(k_{10}^2+k_{10}^1)} \right]_1 & \alpha_3 \left[Y_{30}^{(2k_{10}^2)} \right]_2 & \sqrt{2}\alpha_3 \left[Y_{30}^{(2k_{10}^2)} \right]_1 \end{pmatrix}, \quad (3.3)$$

$$Y_d = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\beta_{11}^d Y_{12-b}^{(k_5^1+k_{10}^1)} & \beta_{12}^d \left[Y_{2s-b}^{(k_5^1+k_{10}^2)} \right]_2 & -\beta_{12}^d \left[Y_{2s-b}^{(k_5^1+k_{10}^2)} \right]_1 \\ \sqrt{2}\beta_{21}^d Y_{12-b}^{(k_5^2+k_{10}^1)} & \beta_{22}^d \left[Y_{2s+1}^{(k_5^2+k_{10}^2)} \right]_2 & -\beta_{22}^d \left[Y_{2s+1}^{(k_5^2+k_{10}^2)} \right]_1 \\ \sqrt{2}\beta_{31}^d Y_{12-b}^{(k_5^3+k_{10}^1)} & \beta_{32}^d \left[Y_{2s+1}^{(k_5^3+k_{10}^2)} \right]_2 & -\beta_{32}^d \left[Y_{2s+1}^{(k_5^3+k_{10}^2)} \right]_1 \end{pmatrix}, \quad (3.4)$$

$$Y_e = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\beta_{11}^e Y_{12-b}^{(k_5^1+k_{10}^1)} & \sqrt{2}\beta_{21}^e Y_{12-b}^{(k_5^2+k_{10}^1)} & \sqrt{2}\beta_{31}^e Y_{12-b}^{(k_5^3+k_{10}^1)} \\ \beta_{12}^e \left[Y_{2s+1}^{(k_5^1+k_{10}^2)} \right]_2 & \beta_{22}^e \left[Y_{2s+1}^{(k_5^2+k_{10}^2)} \right]_2 & \beta_{32}^e \left[Y_{2s+1}^{(k_5^3+k_{10}^2)} \right]_2 \\ -\beta_{12}^e \left[Y_{2s+1}^{(k_5^1+k_{10}^2)} \right]_1 & -\beta_{22}^e \left[Y_{2s+1}^{(k_5^2+k_{10}^2)} \right]_1 & -\beta_{32}^e \left[Y_{2s+1}^{(k_5^3+k_{10}^2)} \right]_1 \end{pmatrix}, \quad (3.5)$$

where

$$\beta_{ij}^d := \beta_{ij} + 2R_\Sigma \gamma_{ij}, \quad \beta_{ij}^e := \beta_{ij} - 3R_\Sigma \gamma_{ij}. \quad (3.6)$$

Here, $[Y_r^{(k)}]_i$ denotes the i -th component of the modular form $Y_r^{(k)}$. As shown in eq. (2.7), $Y_d = Y_e^T$ up to the splitting at $\mathcal{O}(R_\Sigma)$ because $\bar{d} \in \bar{5}$ and $Q \in 10$ while $\bar{e} \in 10$ and $\ell \in \bar{5}$. Note that the number of coefficients may change depending on the number of modular forms for a given modular weight.

3.2 Assigning modular weights

The minimal choice to obtain the realistic Yukawa matrices is

$$k_{10} = (2, 3), \quad k_5 = (0, 0, 2), \quad (3.7)$$

and we take $(s, b) = (1, 0)$ for $\bar{5}_i = 1_s^i$, so that the Yukawa matrices are given by

$$Y_u = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{6}\alpha_1 Y_{10}^{(4)} & \sqrt{3}\alpha_2 \left[Y_{22}^{(5)} \right]_2 & -\sqrt{3}\alpha_2 \left[Y_{22}^{(5)} \right]_1 \\ \sqrt{3}\alpha_2 \left[Y_{22}^{(5)} \right]_2 & -\sqrt{2}\alpha_3^i \left[Y_{30,i}^{(6)} \right]_3 & \alpha_3^i \left[Y_{30,i}^{(6)} \right]_2 \\ -\sqrt{3}\alpha_2 \left[Y_{22}^{(5)} \right]_1 & \alpha_3^i \left[Y_{30,i}^{(6)} \right]_2 & \sqrt{2}\alpha_3^i \left[Y_{30,i}^{(6)} \right]_1 \end{pmatrix}, \quad (3.8)$$

$$Y_d = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\beta_{11}^d Y_{12}^{(2)} & \beta_{12}^d \left[Y_{20}^{(3)} \right]_2 & -\beta_{12}^d \left[Y_{20}^{(3)} \right]_1 \\ \sqrt{2}\beta_{21}^d Y_{12}^{(2)} & \beta_{22}^d \left[Y_{20}^{(3)} \right]_2 & -\beta_{22}^d \left[Y_{20}^{(3)} \right]_1 \\ 0 & \beta_{32}^d \left[Y_{20}^{(5)} \right]_2 & -\beta_{32}^d \left[Y_{20}^{(5)} \right]_1 \end{pmatrix}, \quad (3.9)$$

$$Y_e = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\beta_{11}^e Y_{12}^{(2)} & \sqrt{2}\beta_{21}^e Y_{12}^{(2)} & 0 \\ \beta_{12}^e \left[Y_{20}^{(3)} \right]_2 & \beta_{22}^e \left[Y_{20}^{(3)} \right]_2 & \beta_{32}^e \left[Y_{20}^{(5)} \right]_2 \\ -\beta_{12}^e \left[Y_{20}^{(3)} \right]_1 & -\beta_{22}^e \left[Y_{20}^{(3)} \right]_1 & -\beta_{32}^e \left[Y_{20}^{(5)} \right]_1 \end{pmatrix}.$$

There are two modular forms $Y_{30,i}^{(6)}$, $i = 1, 2$, and there is no $1_{\frac{1}{2}}$ at $k = 4$. The Yukawa matrices are rescaled by the canonical normalization of the kinetic terms as in eq. (2.15). Since the top Yukawa coupling is $\mathcal{O}(1)$, we fix the overall factor of the Yukawa couplings, so that the factor $(\sqrt{2\text{Im}\tau})^6$ is compensated. Note that the normalization of the modular form is not fixed from the bottom-up approach.⁴ We absorb this effect into the coefficients, rather than the modular forms, by defining

$$\hat{\alpha}_i := (2\text{Im}\tau)^3 \alpha_i, \quad \hat{\beta}_{ij} := (2\text{Im}\tau)^3 \beta_{ij}, \quad \hat{\gamma}_{ij} := (2\text{Im}\tau)^3 \gamma_{ij}. \quad (3.10)$$

With the assignments of the modular weights, the texture of the Yukawa matrices are given by

$$Y_u \sim \begin{pmatrix} \eta\epsilon^2 & \eta^{1/2}\epsilon^3 & \eta^{1/2}\epsilon \\ \eta^{1/2}\epsilon^3 & \epsilon^4 & \epsilon^2 \\ \eta^{1/2}\epsilon & \epsilon^2 & 1 \end{pmatrix}, \quad Y_d \sim Y_e^T \sim \begin{pmatrix} \eta^2\epsilon & \eta^{3/2}\epsilon^2 & \eta^{3/2} \\ \eta^2\epsilon & \eta^{3/2}\epsilon^2 & \eta^{3/2} \\ 0 & \eta^{1/2}\epsilon^2 & \eta^{1/2} \end{pmatrix}, \quad (3.11)$$

where $\eta := 1/(2\text{Im}\tau)$ is the factor from the canonical normalization. The hierarchical structures of the quark masses and the CKM matrix is given by

$$\begin{aligned} (m_u, m_c, m_d, m_s, m_b)/m_t &\sim (\epsilon^4, \eta\epsilon^2, \eta^{3/2}\epsilon^2, \eta^2\epsilon, \eta^{1/2}) \\ &\sim (2 \times 10^{-5}, 7 \times 10^{-4}, 3 \times 10^{-4}, 2 \times 10^{-3}, 0.4), \\ (s_{12}^Q, s_{23}^Q, s_{13}^Q) &\sim (\epsilon/\eta^{1/2}, \eta^{1/2}\epsilon, \epsilon^2) \sim (0.2, 0.03, 0.005), \end{aligned} \quad (3.12)$$

where $\epsilon = 0.067$ and $\eta = 0.16$ are used for the numerical estimations. Here, s_{ij}^Q is the mixing angles in the standard parametrization of the CKM matrix. The mass hierarchies of the charged leptons are the same as those for the down-type quarks. These values well fit to the data shown in table 1 at the benchmark points discussed in the next section, except for y_u and y_e which are about 10 larger than the experimental values. These differences will be explained by the numerical factors in the modular forms and the $\mathcal{O}(1)$ coefficients. It is interesting that the CKM angles in our models fit to the data after taking account the powers of η from the canonical normalization, so that $s_{23}^Q/s_{12}^Q \sim \eta \sim 0.2$.

3.3 Benchmark points

We find the numerical values of the parameters by numerical optimization. We restrict the parameter space to be

$$\tan \bar{\beta} \in (5, 60), \quad \bar{\eta}_b \in (-0.6, 0.6), \quad |\hat{\gamma}_{ij}| \in (-1, 1). \quad (3.13)$$

Here, $\tan \bar{\beta}$ includes the threshold correction to the tau lepton, and $\bar{\eta}_b$ is that for the bottom quark as defined in ref. [112]. In this work, we treat $\bar{\eta}_b$ as a parameter which will be determined from the soft SUSY breaking parameters, see for threshold corrections refs. [113, 114]. The threshold corrections to the light flavors, $\bar{\eta}_q$ and $\bar{\eta}_\ell$, are assumed to be

⁴We also assume that this normalization factor is universal for all of the matter fields $\bar{5}_i$ and 10_i , and hence do not change the flavor structure.

obs.	value	center	error	obs.	value	center	error
$y_u/10^{-6}$	2.73	2.74	0.85	$y_u/10^{-6}$	2.89	2.74	0.85
$y_c/10^{-3}$	1.391	1.421	0.050	$y_c/10^{-3}$	1.450	1.421	0.050
y_t	0.5041	0.5033	0.0050	y_t	0.5008	0.5029	0.0050
$y_d/10^{-5}$	5.94	5.90	0.65	$y_d/10^{-5}$	7.12	5.87	0.65
$y_s/10^{-3}$	1.145	1.167	0.063	$y_s/10^{-3}$	1.053	1.161	0.063
$y_b/10^{-2}$	5.375	5.388	0.054	$y_b/10^{-2}$	3.977	3.983	0.040
$y_e/10^{-5}$	2.389	2.389	0.014	$y_e/10^{-5}$	2.377	2.378	0.014
$y_\mu/10^{-3}$	5.049	5.043	0.030	$y_\mu/10^{-3}$	5.000	5.019	0.030
$y_\tau/10^{-2}$	8.611	8.622	0.086	$y_\tau/10^{-2}$	8.674	8.582	0.086
s_{12}^Q	0.22540	0.22541	0.00072	s_{12}^Q	0.22558	0.22541	0.00072
$s_{23}^Q/10^{-2}$	4.770	4.769	0.076	$s_{23}^Q/10^{-2}$	6.47	6.42	0.10
$s_{13}^Q/10^{-3}$	4.13	4.15	0.15	$s_{13}^Q/10^{-3}$	5.61	5.58	0.20
δ_{CKM}	1.2270	1.2080	0.0540	δ_{CKM}	1.2335	1.2080	0.0540

Table 1. The values of the Yukawa couplings of the SM fermions and the mixing angles in the CKM matrix at the BP1 and BP2 in the left and right panels, respectively. The second column is the value at the benchmark points, and the third (fourth) column is the experimental value (and its error). The experimental values are given at the GUT scale calculated in ref. [112].

zero for simplicity. We take $R_\Sigma = 0.1$ and restrict $|\hat{\gamma}_{ij}| < 1$, so that the contribution from the dimension-5 operator is sub-dominant.

We found the following two benchmark points. At the first benchmark point (BP1), the inputs are given by $\tan \bar{\beta} = 11.4643$, $\bar{\eta}_b = 0.187818$, $\tau = 0.0592 + 3.1033i$,

$$\begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\alpha}_3^1 \\ \hat{\alpha}_3^2 \end{pmatrix} = \begin{pmatrix} 2.1054 \\ -1.9005 \\ -1.5069 \\ 1.7427 \end{pmatrix}, \quad \begin{pmatrix} \hat{\beta}_{11} \\ \hat{\beta}_{12} \\ \hat{\beta}_{21} \\ \hat{\beta}_{22} \\ \hat{\beta}_{32} \end{pmatrix} = \begin{pmatrix} 2.2077e^{0.1766i} \\ 1.8263 \\ -0.6111 \\ 0.2838 \\ 0.1922 \end{pmatrix}, \quad \begin{pmatrix} \hat{\gamma}_{11} \\ \hat{\gamma}_{12} \\ \hat{\gamma}_{21} \\ \hat{\gamma}_{22} \\ \hat{\gamma}_{32} \end{pmatrix} = \begin{pmatrix} 0.2568 \\ 0.8436 \\ -0.9999 \\ 0.2433 \\ -0.8884 \end{pmatrix}, \quad (3.14)$$

and at the second point (BP2), $\tan \bar{\beta} = 11.4303$, $\bar{\eta}_b = 0.598145$, $\tau = 0.0661 + 3.0791i$,

$$\begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\alpha}_3^1 \\ \hat{\alpha}_3^2 \end{pmatrix} = \begin{pmatrix} 1.5437 \\ -1.4035 \\ -1.4993 \\ 1.3019 \end{pmatrix}, \quad \begin{pmatrix} \hat{\beta}_{11} \\ \hat{\beta}_{12} \\ \hat{\beta}_{21} \\ \hat{\beta}_{22} \\ \hat{\beta}_{32} \end{pmatrix} = \begin{pmatrix} 1.7296e^{0.2134i} \\ 1.2887 \\ -0.5323 \\ 0.2484 \\ 0.2092 \end{pmatrix}, \quad \begin{pmatrix} \hat{\gamma}_{11} \\ \hat{\gamma}_{12} \\ \hat{\gamma}_{21} \\ \hat{\gamma}_{22} \\ \hat{\gamma}_{32} \end{pmatrix} = \begin{pmatrix} 0.9997 \\ 0.7726 \\ -0.9332 \\ 0.3163 \\ -0.9619 \end{pmatrix}. \quad (3.15)$$

These points realize the quark and charged lepton masses, and the CKM angles as shown in table 1. At both points, all of the observables are within 2σ range, and the largest discrepancy is 0.61σ (1.93σ) at the BP1 (BP2) for the charm (down) quark mass. The values of the parameters are similar at both points, but $\bar{\eta}_b$ is relatively small (large) at the BP1

(BP2). The absolute values of the coefficients are in the range of $[0.19, 2, 2]$ and $[0.20, 1.7]$ at BP1 and BP2, respectively. Thus, the $\mathcal{O}(1)$ coefficients can explain the hierarchies with the good accuracy.

4 Neutrino sector

We shall consider the neutrino sector in this section. We assume that the neutrinos are Majorana, so the neutrino masses are given by the Weinberg operator $W \ni (\ell H_u)^2$ at low-energies. With our choice of the representations and the weights of $\ell \in \bar{5}$, the masses of the neutrinos and the mixing angles in the PMNS matrix are predicted to be

$$(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) \sim (\eta^2, \eta^2, 1) \sim (0.03, 0.03, 1), \quad (s_{12}, s_{23}, s_{13}) \sim (1, \eta, \eta) \sim (1, 0.2, 0.2), \quad (4.1)$$

and thus the neutrino observables will have the texture

$$R_{32}^{21} := \frac{m_{\nu_2}^2 - m_{\nu_1}^2}{m_{\nu_3}^2 - m_{\nu_1}^2} \sim \eta^4 \sim 0.0007, \quad (s_{12}^2, s_{23}^2, s_{13}^2) \sim (1, \eta^2, \eta^2) \sim (1, 0.03, 0.03), \quad (4.2)$$

independently to the UV completion of the Weinberg operator. Hence, the angle s_{13}^2 is naturally explained, while the ratio of the mass squared differences R_{32}^{21} and s_{23}^2 are predicted to be about an order of magnitude smaller than the observed values. We will discuss how these discrepancies are explained in an explicit model based on the type-I seesaw mechanism.

For illustration, we assume the type-I seesaw mechanism to realize the tiny neutrino masses by introducing the three generations of singlets N_i . The superpotential is given by

$$W = \frac{1}{2} N^T M_N N + N^T Y_n \bar{5}_A H^A \ni \frac{1}{2} N^T M_N N + N^T Y_n \ell H_u. \quad (4.3)$$

We choose the representations and modular weights of the right-handed neutrinos as

$$N = 1_0^1 \oplus 2_0 =: N_1 \oplus N_2, \quad k_N = (0, 2). \quad (4.4)$$

The modular invariant superpotential is given by

$$W = \frac{M_0}{2} \left[A_1 N_1 N_1 + 2A_2 \left(Y_{2_0}^{(2)} N_1 N_2 \right)_1 + A_3 Y_{1_0}^{(4)} (N_2 N_2)_1 + A_4 \left(Y_{2_0}^{(4)} (N_2 N_2)_{2_0} \right)_1 \right] \quad (4.5)$$

$$+ \sum_{i=1,2} \left[B_{1i} (N_1 \bar{5}_i)_1 + B_{2i} \left(Y_{2_0}^{(2)} N_2 \bar{5}_i \right)_1 \right] H + B_{23} \left(Y_{2_0}^{(4)} N_2 \bar{5}_3 \right)_1 H,$$

where A_i and B_{ij} are the $\mathcal{O}(1)$ coefficients, and M_0 is the overall scale of the Majorana mass term. $(N_2 N_2)_{2_0}$ takes the combination of the representation 2_0 . Here, we take $Y_{1_0}^{(0)} = 1$. There is no 1_0^0 at $k = 2$. The Majorana mass matrix M_N and the neutrino Yukawa matrix

Y_n are given by

$$M_N = \frac{M_0}{2} \begin{pmatrix} 2A_1 & -\sqrt{2}A_2 [Y_{20}^{(2)}]_2 & \sqrt{2}A_2 [Y_{20}^{(2)}]_1 \\ -\sqrt{2}A_2 [Y_{20}^{(2)}]_2 & \sqrt{2}A_3 Y_{10}^{(4)} - A_4 [Y_{20}^{(4)}]_1 & A_4 [Y_{20}^{(4)}]_2 \\ \sqrt{2}A_2 [Y_{20}^{(2)}]_1 & A_4 [Y_{20}^{(4)}]_2 & \sqrt{2}A_3 Y_{10}^{(4)} + A_4 [Y_{20}^{(4)}]_1 \end{pmatrix}, \quad (4.6)$$

$$Y_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}B_{11} & \sqrt{2}B_{12} & 0 \\ -B_{21} [Y_{20}^{(2)}]_2 & -B_{22} [Y_{20}^{(2)}]_2 & -B_{23} [Y_{20}^{(4)}]_2 \\ B_{21} [Y_{20}^{(2)}]_1 & B_{22} [Y_{20}^{(2)}]_1 & B_{23} [Y_{20}^{(4)}]_1 \end{pmatrix}.$$

The neutrino mass matrix is given by

$$M_\nu = v_u^2 \hat{Y}_n^T \hat{M}_N^{-1} \hat{Y}_n, \quad (4.7)$$

where \hat{Y}_n and \hat{M}_N are after the canonical normalization. For the neutrino observables, the elements suppressed by $\epsilon \sim \mathcal{O}(0.05)$ are irrelevant, and hence the matrices are approximately given by

$$\frac{\hat{M}_N}{M_0(2\text{Im}\tau)^2} \sim \begin{pmatrix} A_1\eta^2 & 0 & -\frac{A_2}{2}\eta \\ 0 & \frac{A_3}{\sqrt{6}} + \frac{A_4}{4\sqrt{2}} & 0 \\ -\frac{A_2}{2}\eta & 0 & \frac{A_3}{\sqrt{6}} - \frac{A_4}{4\sqrt{2}} \end{pmatrix}, \quad \frac{\hat{Y}_n}{(2\text{Im}\tau)^2} \sim \begin{pmatrix} B_{11}\eta^2 & B_{12}\eta^2 & 0 \\ 0 & 0 & 0 \\ -\frac{B_{21}}{2}\eta & -\frac{B_{22}}{2}\eta & -\frac{B_{23}}{4} \end{pmatrix}. \quad (4.8)$$

Since the neutrino mass matrix M_ν is rank-2, the lightest neutrino mass appears only at $\mathcal{O}(\epsilon^3)$.⁵ The ratio of the heavier two neutrinos are given by

$$\sqrt{R_{32}^{21}} \simeq \frac{m_{\nu_2}}{m_{\nu_3}} \sim \frac{16(B_{11}^2 + B_{22}^2)}{B_{23}^2 A_1} \left(\frac{A_3}{\sqrt{6}} - \frac{A_4}{4\sqrt{2}} - \frac{A_2^2}{4A_1} \right) \eta^2, \quad (4.9)$$

at the leading order in η . Because of the hierarchy in the neutrino masses, the model predicts the normal ordering [115]. The ratio is enhanced by $(B_{11}^2 + B_{22}^2)/(B_{23}/4)^2 \sim \mathcal{O}(10)$ coming from the Dirac Yukawa matrix Y_n for all $A, B = \mathcal{O}(1)$, and thus the ratio of the mass squared difference R_{23}^{12} is enhanced by $\mathcal{O}(100)$, consistent with the observed value ~ 0.03 . While the mild discrepancy of s_{23}^2 will be explained simply by $\mathcal{O}(5)$ ratios of the coefficients.

We find the values to explain the neutrino observables at the benchmark points. At the BP1, the fitted values are $M_0 = 1.2819 \times 10^{16}$ GeV,

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 1.8038 \\ -1.3098 \\ 1.0235 \\ -4.0367 \end{pmatrix}, \quad \begin{pmatrix} B_{11} \\ B_{12} \\ B_{21} \\ B_{22} \\ B_{23} \end{pmatrix} = \begin{pmatrix} -2.4174 \\ 1.0236 \\ 4.0214 \\ 1.0236 \\ 2.6574 \end{pmatrix}, \quad (4.10)$$

⁵If N is assigned to be a triplet, the neutrino mass will be rank-1 for $\epsilon \rightarrow 0$, and thus the observed pattern is more difficult to be realized.

obs.	value	center	error	obs.	value	center	error
$R_{32}^{21}/10^{-2}$	3.070	3.070	0.084	$R_{32}^{21}/10^{-2}$	3.069	3.070	0.084
s_{12}^2	0.307	0.307	0.013	s_{12}^2	0.307	0.307	0.013
s_{23}^2	0.544	0.546	0.021	s_{23}^2	0.546	0.546	0.021
$s_{13}^2/10^{-2}$	2.200	2.200	0.070	$s_{13}^2/10^{-2}$	2.200	2.200	0.070
$\delta_{\text{PMNS}}/10^0$	-3.02	-2.01	0.63	$\delta_{\text{PMNS}}/10^0$	-3.00	-2.01	0.63

Table 2. The values of the neutrino observables at the benchmark points.

and at BP2, $M_0 = 1.7775 \times 10^{15}$ GeV,

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 1.8984 \\ -0.6240 \\ -0.4992 \\ 1.1528 \end{pmatrix}, \quad \begin{pmatrix} B_{11} \\ B_{12} \\ B_{21} \\ B_{22} \\ B_{23} \end{pmatrix} = \begin{pmatrix} 0.8983 \\ -0.7126 \\ 1.8984 \\ 0.6480 \\ -0.3344 \end{pmatrix}. \quad (4.11)$$

The values of the neutrino observables are shown in table 2. At these points, the CP phase in the PMNS matrix is originated from that in the charged lepton Yukawa matrix, which is common to the CKM matrix. We see that the values of the coefficients are close to 1 and the ratio of the coefficients are at most 3.9 (5.7) at the BP1 (BP2).

5 Summary

In this work, we build a model to realize the fermion hierarchy in SU(5) GUT utilizing the modular Γ'_6 symmetry. The residual symmetry associated with the generator T , namely \mathbb{Z}_6^T symmetry, controls the powers of the small parameter $e^{-2\pi\text{Im}\tau/6}$. We argue that the Γ'_6 is the minimal possibility to realize the hierarchical masses of the up-type quarks, because the Yukawa matrix for the up-type quarks is symmetric in the SU(5) GUT. We assign the representations of 10 as $1_1^0 \oplus 2_2^0$ and $\bar{5}$ as $1_b^s \oplus 1_b^s \oplus 1_b^s$. We have to consider the double covering Γ'_6 to have the representations 2_b^s . The CKM matrix is hierarchical because the T charges of $Q \in 10$ are different, while the PMNS matrix is not hierarchical because the those of $\ell \in \bar{5}$ are the same. The representations of r and (s, b) are chosen such that the realistic Yukawa matrices are realized for the modular weights smaller than 6. We assigned the modular weights, so that certain modular forms exists and the Yukawa matrices are rank 3 up to $\mathcal{O}(\epsilon^6)$. In the model, we show that the $\mathcal{O}(1)$ coefficients without hierarchies explain the observed quark and lepton masses and the CKM elements.

We assume the type-I seesaw mechanism to explain the tiny neutrino masses. The singlet right-handed neutrinos are assigned to be $1_0^1 \oplus 2_0$ and their modular weights are chosen to be $0 \oplus 2$. With these assignments, one of the three neutrinos is suppressed by $\mathcal{O}(\epsilon^3)$ compared with the other two heavier neutrinos. The mass ratio of the heavier two neutrinos is not suppressed by ϵ , but is suppressed by $(\text{Im}\tau/2)^2 \sim \mathcal{O}(10)$ due to the canonical normalization, see eq. (4.9). We showed that the $\mathcal{O}(1)$ coefficients can explain the neutrino observables in the SU(5) GUT model.

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A Γ'_6 modular symmetry

A.1 Group theory of Γ'_6

The algebra of Γ'_6 is given by [111]

$$S^2 = R, \quad TR = RT, \quad R^2 = T^6 = (ST)^3 = ST^2ST^3ST^4ST^3 = 1. \quad (\text{A.1})$$

That for $\Gamma_6 = \Gamma'_6/\mathbb{Z}_2^R$ is given by taking $R = 1$. The irreducible representations of Γ'_6 are given by

$$1_b^s, 2_b, 3^s, 6, \quad \text{and} \quad 2_b^s, 4_b, \quad (\text{A.2})$$

where $s = 0, 1$ and $b = 0, 1, 2$ correspond to the S_3 and A'_4 indices respectively, since Γ'_6 is isomorphic to $S_3 \times A'_4$. The latter two representations exist only in Γ'_6 . The representation matrices are given by

$$\rho_S(1_b^s) = (-1)^s, \quad \rho_T(1_b^s) = (-1)^s w^b, \quad (\text{A.3})$$

for the singlet 1_b^s ,

$$\rho_S(2_b) = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \quad \rho_T(2_b) = w^b \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{A.4})$$

for the doublet 2_b ,

$$\rho_S(3^s) = (-1)^s \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \rho_T(3^s) = (-1)^s \begin{pmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^2 \end{pmatrix}, \quad (\text{A.5})$$

for the triplet 3^s and

$$\rho_S(6) = \frac{1}{2} \begin{pmatrix} -\rho_S(3^0) & \sqrt{3}\rho_S(3^0) \\ \sqrt{3}\rho_S(3^0) & \rho_S(3^0) \end{pmatrix}, \quad \rho_T(6) = \text{diag} \left(1, w, w^2, -1, -w, -w^2 \right), \quad (\text{A.6})$$

for the sextet 6, where $w := e^{2\pi i/3}$. The representation matrices of 2_b^s and 4_b are respectively given by

$$\rho_S(2_b^s) = (-1)^s \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}, \quad \rho_T(2_b^s) = (-1)^s w^b \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix}, \quad (\text{A.7})$$

and

$$\rho_S(4_b) = \frac{1}{2} \begin{pmatrix} -\rho_S(2_b^0) & \sqrt{3}\rho_S(2_b^0) \\ \sqrt{3}\rho_S(2_b^0) & \rho_S(2_b^0) \end{pmatrix}, \quad \rho_T(4_b) = \begin{pmatrix} \rho_T(2_b^0) & 0 \\ 0 & -\rho_T(2_b^0) \end{pmatrix}. \quad (\text{A.8})$$

Note that the definition of 2_b^s is different from that in ref. [111], where $\rho_T(2_b^s) \propto w^{b+1}$.

The direct product of the singlets is given by

$$1_b^s \otimes 1_c^t = 1_{b+c}^{s+t}. \quad (\text{A.9})$$

Here and hereafter, $s + t$ ($b + c$) should be understood as modulo 2 (3). The products involving the singlet are given by

$$\begin{aligned} 1_b^s(\alpha) \otimes 2_c(\beta) &= P_2^s \begin{pmatrix} \alpha\beta_1 \\ \alpha\beta_2 \end{pmatrix}_{2b+c}, \\ 1_b^s(\alpha) \otimes 3^s(\beta) &= P_3^b \begin{pmatrix} \alpha\beta_1 \\ \alpha\beta_2 \\ \alpha\beta_3 \end{pmatrix}_{3r+s}, \\ 1_b^s(\alpha) \otimes 6(\beta) &= P_{sb} \begin{pmatrix} \alpha\beta_1 \\ \alpha\beta_2 \\ \alpha\beta_3 \\ \alpha\beta_4 \\ \alpha\beta_5 \\ \alpha\beta_6 \end{pmatrix}_6, \end{aligned}$$

and

$$1_b^s \otimes 2_c^t = 2_{b+c}^{s+t}, \quad 1_b^s(\alpha) \otimes 4_c(\beta) = P_4^s \begin{pmatrix} \alpha\beta_1 \\ \alpha\beta_2 \\ \alpha\beta_3 \\ \alpha\beta_4 \end{pmatrix}_{4b+c}, \quad (\text{A.10})$$

where

$$P_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P_3 := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_{sb} := \begin{pmatrix} 0 & \mathbf{1}_3 \\ -\mathbf{1}_3 & 0 \end{pmatrix}^s \begin{pmatrix} P_3 & 0 \\ 0 & P_3 \end{pmatrix}^b, \quad P_4 := \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}. \quad (\text{A.11})$$

The products of the doublets are given by

$$\begin{aligned} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}_{2b} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}_{2c} &= \frac{1}{\sqrt{2}} (\alpha_1\beta_1 + \alpha_2\beta_2)_{1_{b+c}^0} \oplus \frac{1}{\sqrt{2}} (\alpha_1\beta_2 - \alpha_2\beta_1)_{1_{b+c}^1} \\ &\quad \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_2\beta_2 - \alpha_1\beta_1 \\ \alpha_1\beta_2 + \alpha_2\beta_1 \end{pmatrix}_{2b+c}, \\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}_{2_b^s} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}_{2_c} &= P_4^s \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_2\beta_1 \\ \alpha_1\beta_2 \\ \alpha_2\beta_2 \end{pmatrix}_{4b+c}, \\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}_{2_b^s} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}_{2_c^t} &= \frac{1}{\sqrt{2}} (\alpha_1\beta_2 - \alpha_2\beta_1)_{1_{b+c+1}^{s+t}} \oplus P_3^{b+c} \frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{2}\alpha_1\beta_1 \\ \alpha_1\beta_2 + \alpha_2\beta_1 \\ \sqrt{2}\alpha_2\beta_2 \end{pmatrix}_{3^{s+t}}. \end{aligned} \quad (\text{A.12})$$

The products of the doublet and triplet are

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}_{2_b} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}_{3^s} = P_{sb} \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \alpha_1\beta_3 \\ \alpha_2\beta_1 \\ \alpha_2\beta_2 \\ \alpha_2\beta_3 \end{pmatrix}_6, \tag{A.13}$$

$$\begin{aligned} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}_{2_b^s} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}_{3^t} &= \frac{1}{\sqrt{3}} \begin{pmatrix} \alpha_1\beta_1 + \sqrt{2}\alpha_2\beta_3 \\ \sqrt{2}\alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix}_{2_b^{s+t}} \oplus \frac{1}{\sqrt{3}} \begin{pmatrix} \alpha_1\beta_2 + \sqrt{2}\alpha_2\beta_1 \\ \sqrt{2}\alpha_1\beta_3 - \alpha_2\beta_2 \end{pmatrix}_{2_{b+1}^{s+t}} \\ &\oplus \frac{1}{\sqrt{3}} \begin{pmatrix} \alpha_1\beta_3 + \sqrt{2}\alpha_2\beta_2 \\ \sqrt{2}\alpha_1\beta_1 - \alpha_2\beta_3 \end{pmatrix}_{2_{b+2}^{s+t}}, \end{aligned}$$

and the product of the triplets is given by

$$\begin{aligned} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}_{3^s} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}_{3^t} &= \frac{1}{\sqrt{3}} (\alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2)_{1_0^{s+t}} \oplus \frac{1}{\sqrt{3}} (\alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1)_{1_1^{s+t}} \tag{A.14} \\ &\oplus \frac{1}{\sqrt{3}} (\alpha_2\beta_2 + \alpha_3\beta_1 + \alpha_1\beta_3)_{1_2^{s+t}} \oplus \frac{1}{\sqrt{6}} \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \\ 2\alpha_2\beta_2 - \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}_{3_S^{s+b}} \\ &\oplus \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}_{3_A^{s+b}}. \end{aligned}$$

The product rules involving the higher dimensional representations are shown in ref. [111].

A.2 Modular forms

The modular forms at $k = 1$ are given by [111]

$$\begin{aligned} Y_{2_0^0}^{(1)} &= \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{2_0^0} = \begin{pmatrix} 3e_1 + e_2 \\ 3\sqrt{2}e_1 \end{pmatrix}, \\ Y_{4_2}^{(1)} &= \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix}_{4_2} = \begin{pmatrix} 3\sqrt{2}e_3 \\ -3e_3 - e_5 \\ \sqrt{6}e_3 - \sqrt{6}e_6 \\ -\sqrt{3}e_3 + 1/\sqrt{3}e_4 - 1/\sqrt{3}e_5 + \sqrt{3}e_6 \end{pmatrix}, \tag{A.15} \end{aligned}$$

where the functions are defined as

$$\begin{aligned} e_1(\tau) &:= \frac{\eta(3\tau)^3}{\eta(\tau)}, & e_2(\tau) &:= \frac{\eta(\tau/3)^3}{\eta(\tau)}, & e_3(\tau) &:= \frac{\eta(6\tau)^3}{\eta(2\tau)}, \\ e_4(\tau) &:= \frac{\eta(\tau/6)^3}{\eta(\tau/2)}, & e_5(\tau) &:= \frac{\eta(2\tau/3)^3}{\eta(2\tau)}, & e_6(\tau) &:= \frac{\eta(3\tau/2)^3}{\eta(\tau/2)}. \end{aligned} \tag{A.16}$$

The modular forms with higher weights can be constructed by taking the direct products.

The modular forms at weight $k = 2$ are

$$\begin{aligned}
 Y_{1\frac{1}{2}}^{(2)} &= (Y_3Y_6 - Y_4Y_5)_{1\frac{1}{2}}, & Y_{2_0}^{(2)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} Y_1Y_4 - Y_2Y_3 \\ Y_1Y_6 - Y_2Y_5 \end{pmatrix}, & (A.17) \\
 Y_{3_0}^{(2)} &= \begin{pmatrix} -Y_1^2 \\ \sqrt{2}Y_1Y_2 \\ Y_2^2 \end{pmatrix}, & Y_6^{(2)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} Y_1Y_4 + Y_2Y_3 \\ \sqrt{2}Y_2Y_4 \\ -\sqrt{2}Y_1Y_3 \\ Y_1Y_6 + Y_2Y_5 \\ \sqrt{2}Y_2Y_6 \\ -\sqrt{2}Y_1Y_5 \end{pmatrix}.
 \end{aligned}$$

At weight $k = 3$,

$$\begin{aligned}
 Y_{2_0}^{(3)} &= \frac{1}{\sqrt{3}} \begin{pmatrix} -Y_1^3 + \sqrt{2}Y_2^3 \\ 3Y_1^2Y_2 \end{pmatrix}, & Y_{2_2}^{(3)} &= \frac{1}{\sqrt{3}} \begin{pmatrix} 3Y_1Y_2^2 \\ -\sqrt{2}Y_1^3 - Y_2^3 \end{pmatrix}, & Y_{2\frac{1}{2}}^{(3)} &= (Y_3Y_6 - Y_4Y_5) \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \\
 Y_{4_0}^{(3)} &= \sqrt{\frac{2}{3}} \begin{pmatrix} Y_1Y_2Y_3 - Y_1^2Y_4 \\ Y_2^2Y_3 - Y_1Y_2Y_4 \\ Y_1Y_2Y_5 - Y_1^2Y_6 \\ Y_2^2Y_5 - Y_1Y_2Y_6 \end{pmatrix}, & Y_{4_1}^{(3)} &= \frac{1}{\sqrt{3}} \begin{pmatrix} Y_2^2Y_3 + 2Y_1Y_2Y_4 \\ -Y_2^2Y_4 - \sqrt{2}Y_1^2Y_3 \\ Y_2^2Y_5 + 2Y_1Y_2Y_6 \\ -Y_2^2Y_6 - \sqrt{2}Y_1^2Y_5 \end{pmatrix}, & (A.18) \\
 Y_{4_2}^{(3)} &= \frac{1}{\sqrt{3}} \begin{pmatrix} -Y_1^2Y_3 + \sqrt{2}Y_2^2Y_4 \\ Y_1^2Y_4 + 2Y_1Y_2Y_3 \\ -Y_1^2Y_5 + \sqrt{2}Y_2^2Y_6 \\ Y_1^2Y_6 + 2Y_1Y_2Y_5 \end{pmatrix},
 \end{aligned}$$

and at weight $k = 4$,

$$\begin{aligned}
 Y_{1_0}^{(4)} &= \frac{1}{\sqrt{3}} Y_1 (Y_1^3 + 2\sqrt{2}Y_2^3), & Y_{1_1}^{(4)} &= (Y_3Y_6 - Y_4Y_5)^2, & (A.19) \\
 Y_{2_0}^{(4)} &= \frac{1}{2\sqrt{2}} \left((Y_1Y_6 - Y_2Y_5)^2 - (Y_1Y_4 - Y_2Y_3)^2 \right), & Y_{2_2}^{(4)} &= \frac{1}{\sqrt{2}} (Y_3Y_6 - Y_4Y_5) \begin{pmatrix} Y_1Y_6 - Y_2Y_5 \\ Y_2Y_3 - Y_1Y_4 \end{pmatrix}, \\
 Y_{3_0}^{(4)} &= \sqrt{\frac{2}{3}} \begin{pmatrix} Y_1 (Y_1^3 - \sqrt{2}Y_2^3) \\ Y_2 (Y_2^3 + \sqrt{2}Y_1^3) \\ 3Y_1^2Y_2^2 \end{pmatrix}, & Y_{3_1}^{(4)} &= (Y_3Y_6 - Y_4Y_5) \begin{pmatrix} \sqrt{2}Y_1Y_2 \\ Y_2^2 \\ -Y_1^2 \end{pmatrix}, \\
 Y_{6,1}^{1(4)} &= \frac{1}{\sqrt{2}} (Y_3Y_6 - Y_4Y_5) \begin{pmatrix} \sqrt{2}Y_2Y_6 \\ -\sqrt{2}Y_1Y_5 \\ Y_1Y_6 + Y_2Y_5 \\ -\sqrt{2}Y_2Y_4 \\ \sqrt{2}Y_1Y_3 \\ -Y_1Y_4 - Y_2Y_3 \end{pmatrix}, & Y_{6,2}^{2(4)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} -(Y_1Y_4 - Y_2Y_3)Y_1^2 \\ \sqrt{2}(Y_1Y_4 - Y_2Y_3)Y_1Y_2 \\ (Y_1Y_4 - Y_2Y_3)Y_2^2 \\ -(Y_1Y_6 - Y_2Y_5)Y_1^2 \\ \sqrt{2}(Y_1Y_6 - Y_2Y_5)Y_1Y_2 \\ (Y_1Y_6 - Y_2Y_5)Y_2^2 \end{pmatrix}.
 \end{aligned}$$

At $k = 5$,

$$\begin{aligned}
 Y_{20}^{(5)} &= \frac{1}{\sqrt{3}} Y_1 \begin{pmatrix} Y_1^3 + 2\sqrt{2}Y_2^3 \\ Y_2 \end{pmatrix}, & Y_{20}^{(5)} &= (Y_3Y_6 - Y_4Y_5)^2 \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, & (A.20) \\
 Y_{22}^{(5)} &= \frac{1}{3} \begin{pmatrix} -5Y_1^3Y_2^2 - \sqrt{2}Y_2^5 \\ 5Y_1^2Y_2^3 - \sqrt{2}Y_1^5 \end{pmatrix}, & Y_{21}^{(5)} &= \frac{1}{\sqrt{3}} (Y_4Y_5 - Y_3Y_6) \begin{pmatrix} -3Y_1Y_2^2 \\ \sqrt{2}Y_1^3 + Y_2^3 \end{pmatrix}, \\
 Y_{22}^{(5)} &= \frac{1}{\sqrt{3}} (Y_3Y_6 - Y_4Y_5) \begin{pmatrix} -Y_1^3 + \sqrt{2}Y_2^3 \\ 3Y_1^2Y_2 \end{pmatrix}, \\
 Y_{40,1}^{(5)} &= \frac{1}{\sqrt{3}} (Y_3Y_6 - Y_4Y_5) \begin{pmatrix} Y_2^2Y_5 + 2Y_1Y_2Y_6 \\ -Y_2^2Y_6 - \sqrt{2}Y_1^2Y_5 \\ -Y_2^2Y_3 - 2Y_1Y_2Y_4 \\ Y_2^2Y_4 + \sqrt{2}Y_1^2Y_3 \end{pmatrix}, & Y_{40,2}^{(5)} &= (Y_3Y_6 - Y_4Y_5)^2 \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix}, \\
 Y_{41}^{(5)} &= \frac{1}{\sqrt{3}} (Y_3Y_6 - Y_4Y_5) \begin{pmatrix} -Y_1^2Y_5 + \sqrt{2}Y_2^2Y_6 \\ Y_1^2Y_6 + 2Y_1Y_2Y_5 \\ Y_1^2Y_3 - \sqrt{2}Y_2^2Y_4 \\ -Y_1^2Y_4 - 2Y_1Y_2Y_3 \end{pmatrix}, \\
 Y_{42,1}^{(5)} &= \sqrt{\frac{2}{3}} (Y_3Y_6 - Y_4Y_5) \begin{pmatrix} Y_1Y_2Y_5 - Y_1^2Y_6 \\ Y_2^2Y_5 - Y_1Y_2Y_6 \\ -Y_1Y_2Y_3 + Y_1^2Y_4 \\ -Y_2^2Y_3 + Y_1Y_2Y_4 \end{pmatrix}, & Y_{42,2}^{(5)} &= \frac{1}{\sqrt{3}} Y_1 (Y_1^3 + 2\sqrt{2}Y_2^3) \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix}.
 \end{aligned}$$

At weight $k = 6$,

$$\begin{aligned}
 Y_{10}^{(6)} &= \frac{1}{4\sqrt{2}} (Y_1Y_4 - Y_2Y_3) \left\{ 3(Y_1Y_6 - Y_2Y_5)^2 - (Y_1Y_4 - Y_2Y_3)^2 \right\}, & (A.21) \\
 Y_{11}^{(6)} &= (Y_3Y_6 - Y_4Y_5)^3, & Y_{12}^{(6)} &= \frac{1}{\sqrt{3}} Y_1 (Y_1^3 + 2\sqrt{2}Y_2^3) (Y_3Y_6 - Y_4Y_5), \\
 Y_{20}^{(6)} &= \frac{1}{\sqrt{6}} Y_1 \begin{pmatrix} Y_1^3 + 2\sqrt{2}Y_2^3 \\ Y_1Y_6 - Y_2Y_5 \end{pmatrix}, \\
 Y_{21}^{(6)} &= \frac{1}{\sqrt{2}} (Y_3Y_6 - Y_4Y_5)^2 \begin{pmatrix} Y_2Y_3 - Y_1Y_4 \\ Y_2Y_5 - Y_1Y_6 \end{pmatrix}, \\
 Y_{22}^{(6)} &= \frac{1}{2\sqrt{2}} (Y_3Y_6 - Y_4Y_5) \begin{pmatrix} 2(Y_1Y_4 - Y_2Y_3)(Y_1Y_6 - Y_2Y_5) \\ (Y_1Y_4 - Y_2Y_3)^2 - (Y_1Y_6 - Y_2Y_5)^2 \end{pmatrix}, \\
 Y_{30,1}^{(6)} &= \frac{1}{\sqrt{3}} Y_1 (Y_1^3 + 2\sqrt{2}Y_2^3) \begin{pmatrix} -Y_1^2 \\ \sqrt{2}Y_1Y_2 \\ Y_2^2 \end{pmatrix}, & Y_{30,2}^{(6)} &= \frac{1}{\sqrt{3}} \begin{pmatrix} Y_2^6 - 2\sqrt{2}Y_1^3Y_2^3 \\ 2\sqrt{2}Y_1^5Y_2 - Y_1^2Y_2^4 \\ -4Y_1^4Y_2^2 + \sqrt{2}Y_1Y_2^5 \end{pmatrix}, \\
 Y_{31}^{(6)} &= \sqrt{\frac{2}{3}} (Y_3Y_6 - Y_4Y_5) \begin{pmatrix} Y_2 (Y_2^3 + \sqrt{2}Y_1^3) \\ 3Y_1^2Y_2^2 \\ Y_1 (Y_1^3 - \sqrt{2}Y_2^3) \end{pmatrix}, \\
 Y_{6,1}^{(6)} &= Y_{1\frac{1}{2}}^{(2)} \otimes Y_{6,1}^{(4)}, & Y_{6,2}^{(6)} &= Y_{1\frac{1}{2}}^{(2)} \otimes Y_{6,2}^{(4)}, & Y_{6,3}^{(6)} &= Y_{20}^{(2)} \otimes Y_{30}^{(4)}. & (A.22)
 \end{aligned}$$

The representations existing for $k \leq 6$ are summarized in table 3.

weight	representations
$k = 1$	$2_0^0, 4_2$
$k = 2$	$1_2^1, 2_0, 3^0, 6$
$k = 3$	$2_0^0, 2_2^0, 2_2^1, 4_0, 4_1, 4_2$
$k = 4$	$1_0^0, 1_1^0, 2_0, 2_2, 3^0, 3^1, 6, 6$
$k = 5$	$2_0^0, 2_1^0, 2_2^0, 2_1^1, 2_2^1, 4_0, 4_0, 4_1, 4_2, 4_2$
$k = 6$	$1_0^0, 1_0^1, 1_2^1, 2_0, 2_1, 2_2, 3^0, 3^0, 3^1, 6, 6, 6$

Table 3. The representations of modular forms at $k \leq 6$.

At $\tau \sim i\infty$, it is convenient to explained the modular forms by $q := e^{2\pi i\tau}$ whose absolute values is small. The q -expansions of the functions e_i are given by

$$\begin{aligned}
 e_1 &= q^{1/3} + q^{4/3}, & e_2 &= 1 - 3q^{1/3} + 6q - 3q^{4/3}, & (A.23) \\
 e_3 &= q^{2/3}, & e_4 &= 1 - 3q^{1/6} + 6q^{1/2} - 3q^{2/3} - 6q^{7/6} + 6q^{3/2}, \\
 e_5 &= 1 - 3q^{2/3}, & e_6 &= q^{1/6} + q^{2/3} + 2q^{7/6},
 \end{aligned}$$

where $\mathcal{O}(q^2)$ is neglected. Defining

$$\epsilon := \sqrt{3}q^{1/6}, \tag{A.24}$$

the modular forms up to $\mathcal{O}(\epsilon^5)$ are given by

$$Y_{2_0^0}^{(1)} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \simeq \begin{pmatrix} 1 \\ \sqrt{2}\epsilon^2 \end{pmatrix}, \quad Y_{4_2}^{(1)} = \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} \sim \frac{1}{3} \begin{pmatrix} \sqrt{2}\epsilon^4 \\ -3 \\ -3\sqrt{2}\epsilon \\ 2\epsilon^3 \end{pmatrix}, \tag{A.25}$$

at $k = 1$. For reference, we show the expansions of the modular forms of up to the three dimensional representations for $k \leq 6$. At $k = 2$,

$$Y_{1_2^1}^{(2)} \sim -\sqrt{2}\epsilon, \quad Y_{2_0^0}^{(2)} \sim \frac{1}{3\sqrt{2}} \begin{pmatrix} -3 \\ 8\epsilon^3 \end{pmatrix}, \quad Y_{3_0^0}^{(2)} = \begin{pmatrix} -1 \\ 2\epsilon^2 \\ 2\epsilon^4 \end{pmatrix}. \tag{A.26}$$

At $k = 3$,

$$Y_{2_0^0}^{(3)} \sim \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 3\sqrt{2}\epsilon^2 \end{pmatrix}, \quad Y_{2_2^0}^{(3)} \sim \frac{1}{\sqrt{3}} \begin{pmatrix} 6\epsilon^4 \\ -\sqrt{2} \end{pmatrix}, \quad Y_{2_2^1}^{(3)} \sim -\sqrt{2} \begin{pmatrix} \epsilon \\ \sqrt{2}\epsilon^3 \end{pmatrix}. \tag{A.27}$$

At $k = 4$,

$$\begin{aligned}
 Y_{1_0^0}^{(4)} &\sim \frac{1}{\sqrt{3}}, \quad Y_{1_1^0}^{(4)} \sim 2\epsilon^2, \quad Y_{2_0^0}^{(4)} \sim -\frac{1}{6\sqrt{2}} \begin{pmatrix} 3 \\ 16\epsilon^3 \end{pmatrix}, \quad Y_{2_2^0}^{(4)} \sim -\frac{1}{3} \begin{pmatrix} 8\epsilon^4 \\ 3\epsilon \end{pmatrix}, & (A.28) \\
 Y_{3_0^0}^{(4)} &\sim \sqrt{\frac{2}{3}} \begin{pmatrix} 1 \\ 2\epsilon^2 \\ 6\epsilon^4 \end{pmatrix}, \quad Y_{3_1^1}^{(4)} \sim \sqrt{2} \begin{pmatrix} -2\epsilon^3 \\ -2\epsilon^5 \\ \epsilon \end{pmatrix}.
 \end{aligned}$$

At $k = 5$,

$$\begin{aligned}
 Y_{2_0^0}^{(5)} &\sim \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \sqrt{2}\epsilon^2 \end{pmatrix}, & Y_{2_1^0}^{(5)} &\sim \begin{pmatrix} 2\epsilon^2 \\ 2\sqrt{2}\epsilon^4 \end{pmatrix}, & Y_{2_2^0}^{(5)} &\sim -\frac{1}{3} \begin{pmatrix} 10\epsilon^4 \\ \sqrt{2} \end{pmatrix}, \\
 Y_{2_1^1}^{(5)} &\sim \frac{1}{\sqrt{3}} \begin{pmatrix} -6\sqrt{2}\epsilon^5 \\ 2\epsilon \end{pmatrix}, & Y_{2_2^1}^{(5)} &\sim \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2}\epsilon \\ -6\epsilon^3 \end{pmatrix}.
 \end{aligned}
 \tag{A.29}$$

At $k = 6$,

$$\begin{aligned}
 Y_{1_0^0}^{(6)} &\sim \frac{1}{4\sqrt{2}}, & Y_{1_1^0}^{(6)} &\sim -2\sqrt{2}\epsilon^3, & Y_{1_2^0}^{(6)} &\sim -\sqrt{\frac{2}{3}}\epsilon, \\
 Y_{2_0^0}^{(6)} &\sim \frac{1}{3\sqrt{6}} \begin{pmatrix} -3 \\ 8\epsilon^3 \end{pmatrix}, & Y_{2_1^0}^{(6)} &\sim \frac{1}{3} \begin{pmatrix} 3\sqrt{2}\epsilon^2 \\ -8\sqrt{2}\epsilon^5 \end{pmatrix}, & Y_{2_2^0}^{(6)} &\sim \frac{1}{6} \begin{pmatrix} 16\epsilon^4 \\ -3\epsilon \end{pmatrix}, \\
 Y_{3_0,1}^{(6)} &\sim \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 2\epsilon^2 \\ 2\epsilon^4 \end{pmatrix}, & Y_{3_0,2}^{(6)} &\sim \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 4\epsilon^2 \\ -8\epsilon^4 \end{pmatrix}, & Y_{3_1^0}^{(6)} &\sim \frac{1}{\sqrt{3}} \begin{pmatrix} -4\epsilon^3 \\ -12\epsilon^5 \\ -2\epsilon \end{pmatrix}.
 \end{aligned}
 \tag{A.30}$$

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