Symbology for elliptic multiple polylogarithms and the symbol prime

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ABSTRACT: Elliptic multiple polylogarithms occur in Feynman integrals and in particular in scattering amplitudes. They can be characterized by their symbol, a tensor product in the so-called symbol letters. In contrast to the non-elliptic case, the elliptic letters themselves satisfy highly non-trivial identities, which we discuss in this paper. Moreover, we introduce the symbol prime, an analog of the symbol for elliptic symbol letters, which makes these identities manifest. We demonstrate its use in two explicit examples at two-loop order: the unequal-mass sunrise integral in two dimensions and the ten-point double-box integral in four dimensions. Finally, we also report the result of the polylogarithmic nine-point double-box integral, which arises as the soft limit of the ten-point integral.

Contents

1 Introduction

Scattering amplitudes as well as further quantities in Quantum Field Theory contain a rich mathematical structure, whose understanding has frequently expanded our calculational reach — benefiting both phenomenological tests of the Standard Model of Particle Physics at the LHC as well as more formal studies.

At one-loop order, and in certain cases also at higher loop orders, the functions that occur in Feynman integrals and thus in Quantum Field Theory are multiple polylogarithms (MPLs) [1–6], which are by now very well understood. Increasing the loop order, the next class of functions we encounter are elliptic multiple polylogarithms (eMPLs), on which there has been much recent progress [7–34], in particular by studying the two-loop massive sunrise integral in two dimensions [7, 10–14, 16, 20, 35–41] (see figure 1a). More recently, also more complicated Feynman integrals are starting to be understood, in particular the two-loop ten-point double-box integral with massless internal propagators in four dimensions [23, 33] (see figure 1b). Beyond eMPLs, also integrals over more complicated geometries than elliptic curves occur [42–60]; an understanding of the corresponding spaces of functions is still in its infancy. For a recent review on functions in scattering amplitudes beyond MPLs, see [61].

Our good understanding of MPLs is to a large extend due to the Hopf algebra structure underlying these functions [62–66], and in particular the symbol [63]. The symbol map associates to each MPL f a simple tensor product, $\mathcal{S}(f) = \sum \log(\phi_{i_1}) \otimes \cdots \otimes \log(\phi_{i_k})$.[1] The entries in this tensor product, called symbol letters, are logarithms of rational or algebraic functions ϕ_i of the kinematic invariants. Since a tensor product is easy to manipulate, and the identities of the symbol letters, $\log(a) + \log(b) = \log(ab)$, are well understood, the symbol provides a powerful way of finding identities between MPLs and for simplifying expressions. The symbol moreover manifests physical properties of the corresponding function. For example, the first entry of the symbol describes the discontinuities, which are heavily restricted in particular in massless theories resulting in so-called first-entry conditions [67]. Moreover, discontinuities in overlapping channels are forbidden by the so-called Steinmann conditions [68, 69], restricting which symbol letter in the second entry can follow a particular letter in the first entry. The symbol has made possible enormous progress for quantities consisting of MPLs, both in relation to phenomenology and more formal studies, including in particular powerful bootstrap techniques [70–99].

While the symbol for eMPLs was defined in [25, 100], it has so far not been put to much use, and is still much less understood than its analog for MPLs. One reason is that the symbol letters $\Omega^{(i)}$ for eMPLs are themselves elliptic functions of the kinematic invariants. In particular, $\Omega^{(-1)} = -2\pi i \tau$ occurs as a letter, where $\tau = \omega_2/\omega_1$ is the ratio of the two periods of the elliptic curve. The letters $\Omega^{(0)}$ satisfy simple identities as the consequence of the group law on the elliptic curve. In [33], some identities for the elliptic letters $\Omega^{(i)}$ with $i = 1, 2$ were observed numerically in the study of the symbol of the ten-point elliptic double-box integral. Using these identities, it was found that the elliptic letters in the first two entries simplify to logarithms, manifesting the same first-entry condition as for polylogarithmic amplitudes as well as the Steinmann conditions. Moreover, the last entries were found to be given by simple elliptic integrals $\Omega^{(0)}$, with $\Omega^{(2)}$ only occurring in the third entry preceding the modular parameter τ in the last entry.

In this paper, we show how the identities observed in [33] for $\Omega^{(1)}$ are a consequence of Abel's theorem [101]. Moreover, we demonstrate that the identities observed in [33] for $\Omega^{(2)}$ are a consequence of the elliptic Bloch relation [102, 103] for the elliptic dilogarithm, which generalizes the Bloch relation for the classic dilogarithm and which have also been studied in the context of finding identities between elliptic multiple polylogarithms [104, 105]. While the identities for $\Omega^{(1)}$ can be reduced to three-term identities similar to $\log(a) + \log(b) = \log(ab)$ in the case of the logarithm, the elliptic Bloch relation, and thus the identities for $\Omega^{(2)}$, are five-term identities similar to the Bloch relation for the classical dilogarithm, which are made manifest only by the symbol. Thus, we introduce a symbol prime $\mathcal{S}'$ for the symbol letters $\Omega^{(2)}$ (and similarly for $\Omega^{(n>2)}$) in analogy to the symbol for MPLs and eMPLs, which makes the identities due to the elliptic Bloch relation manifest.

In general, eMPLs transform under modular transformations of τ in a complicated way [30, 32], and results given in terms of eMPLs are not manifestly double periodic. However, in the examples we studied, we find that the symbol prime makes both double

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[1]Note that in contrast to much of the literature on MPLs, we are not suppressing the log in the notation.

(a)   (b)
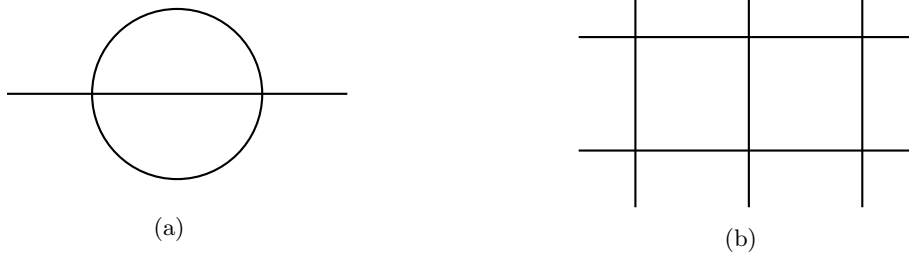
Figure 1. The sunrise integral in two dimensions with unequal internal masses (a) as well as the ten-point double-box integral in four dimensions with massless internal propagators (b).

periodicity as well as a simple behavior under modular transformations manifest. Finally, it makes also part of the integrability conditions manifest, which result from the requirement that partial derivatives commute.

To illustrate the use of the symbol for eMPLs, the application of the identities of elliptic letters as well as the symbol prime, we study two concrete examples. The first example is the two-loop sunrise integral in two dimensions with all internal masses being unequal (see figure 1a). The second example is the ten-point two-loop double-box integral in four dimensions with massless internal propagators (see figure 1b). In addition to the aforementioned properties and techniques, we also demonstrate how the (elliptic) symbol reduces to polylogarithmic symbol in kinematic limits where the elliptic curve degenerates.

The remainder of this paper is organized as follows. We review elliptic multiple poly-logarithms in section 2. In section 3, we derive identities for elliptic symbol letters — based on Abel's addition theorem for $\Omega^{(1)}$'s and by introducing the symbol prime map for $\Omega^{(2)}$'s. We illustrate the use of these techniques for the unequal-mass sunrise integral in section 4 and for the ten-point double-box integral in section 5. In particular, we provide analytic results for the non-elliptic nine-point double-box integral and its symbol, which result from taking the soft limit of the ten-point double-box integral. We conclude with a discussion and an outlook on open questions in section 6. In appendix A, we review the calcula-tion of the unequal-mass sunrise integral via Feynman parameters using our conventions and notation. The details of simplifying the symbols for the sunrise integral, as well as the expressions of the functions and symbols for the ten-point elliptic integral and its soft limit, are included as supplementary material attached to this paper (`sunrise_symbol.nb`, `doublebox_omega2` and `doublebox_soft`).[2]

2 Review of elliptic multiple polylogarithms

Let us first review several elementary facts about elliptic multiple polylogarithms; see [19, 20, 25, 26] for further details. We follow the notations and conventions of [33], which differ slightly from those of [19, 20, 25, 26].

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[2]In this article, we only provide the expression for the ten-point double box in the normalization by the period $-\omega_2$ since the corresponding expressions in the normalization by the period ω_1 can be found in the ancillary files of [33].

2.1 Elliptic multiple polylogarithms on the elliptic curve

In this paper, the elliptic curves $\mathcal{C}$ are described by monic quartic polynomials:

$$y^2 = P_4(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \,. \tag{2.1}$$

Such elliptic curves can be birationally mapped to Weierstrass form

$$Y^2 = 4X^3 - g_2 X - g_3 \tag{2.2}$$

using the rational point at $(x, y) = (+\infty, +\infty)$, where (X, Y) are related to (x, y) by

$$\begin{aligned}
X &= \frac{1}{12}\left(a_2 + 3a_3 x + 6x^2 + 6y\right), \\
Y &= \frac{1}{4}\left(a_1 + 2a_2 x + 3a_3 x^2 + 4x^3 + a_3 y + 4xy\right).
\end{aligned} \tag{2.3}$$

On the curve $\mathcal{C}$, we can introduce elliptic multiple polylogarithms E_4, which are recursively defined as [19][3]

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right) = \int_0^x \mathrm{d}x'\, \psi_{n_1}(c_1, x')\, \mathrm{E}_4\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix}; x'\right) \tag{2.4}$$

with $\mathrm{E}_4(; x) = 1$, where

$$\begin{aligned}
\psi_0(0, x) &= \frac{1}{y}, & \psi_{-1}(\infty, x) &= \frac{x}{y}, \\
\psi_1(c, x) &= \frac{1}{x - c}, & \psi_{-1}(c, x) &= \frac{y_c}{y(x - c)},
\end{aligned} \tag{2.5}$$

with $y_c = y|_{x=c}$. The definitions of $\psi_n(c, x)$ for $n = \pm 2, \pm 3, \ldots$ can be found in [19]; the kernels (2.5) are sufficient for the purpose of this paper, though.

The class of elliptic multiple polylogarithms $\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right)$ contains in particular all non-elliptic (Goncharov) multiple polylogarithms, defined by

$$G(c_1, \ldots, c_n; x) = \int_0^x \frac{\mathrm{d}x'}{x' - c_1} G(c_2, \ldots, c_n; x') \tag{2.6}$$

with $G(; x) = 1$, since by definition $\mathrm{E}_4\left(\begin{smallmatrix} 1 & \cdots & 1 \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right) \equiv G(c_1, \ldots, c_k; x)$.

In general, any integral of the form

$$\int \frac{\mathrm{d}x}{y}\, \mathcal{G}(x, y), \tag{2.7}$$

where $\mathcal{G}$ is a polylogarithm whose letters are rational functions of x and y, can be converted to E_4 functions with only the four kinds of integration kernels defined in (2.5).[4] In particular, this is the case for the (unequal-mass) sunrise integral and the ten-point double-box integral, which we will study as examples in sections 4–5.

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[3]The subscript "4" indicates that the elliptic curve is given by a quartic polynomial. Analogous functions for a cubic polynomial were also defined in [19].

[4]Since $\mathcal{G}$ is a polylogarithm, the integration kernels have only simple poles, in addition to being rational in x and y. While all integration kernels ψ_n have only simple poles, only $\psi_{-1,0,+1}$ are rational functions of x, y.

2.2 From the elliptic curve to the torus

The functions E_4 are defined on the elliptic curve and directly arise from the Feynman-parameter representation of Feynman integrals. However, the purity of some elliptic Feynman integrals, such as integrals of the form (2.7), is not visible when expressed in terms of E_4 functions since taking the total derivative of a E_4 function does not necessarily decrease its length [25].[5] On the other hand, iterated integrals defined on the torus, such as the $\tilde{\Gamma}$ functions we will review below, are manifestly pure and hence allow a symbol map defined via the total derivative.

To connect both sides, we first need a bijection between the elliptic curve $\mathcal{C}$ and the torus $\mathbb{C}/\Lambda$, where Λ is the lattice generated by the periods ω_1 and ω_2 of the elliptic curve. For an elliptic curve of the form (2.1), one can find such a map through the birationally equivalent curve in the Weierstrass normal form: first solve (x, y) in terms of (X, Y) from (2.3), then replace X and Y with the Weierstrass elliptic function $\wp(z)$ and its derivative $\wp'(z)$, respectively. This gives

$$z \mapsto (x, y) = (\kappa(z), \kappa'(z)),\tag{2.8}$$

where

$$\kappa(z) = \frac{6a_1 - a_2a_3 + 12a_3\wp(z) - 24\wp'(z)}{3a_3^2 - 8a_2 - 48\wp(z)}.\tag{2.9}$$

It is easy to see that $\kappa(0) = \infty$, and hence all lattice points are mapped to the infinity point in the (x, y)-space. Furthermore, each point c in the x-space corresponds to two points $(c, \pm y_c)$ on the elliptic curve $\mathcal{C}$ and hence to two images on the torus $\mathbb{C}/\Lambda$, which we denote by $z_c^{\pm}$; these two images satisfy

$$z_c^+ + z_c^- = z_\infty^- + z_\infty^+ \equiv z_\infty^- \mod \Lambda,\tag{2.10}$$

since the corresponding points $(X_c^{\pm}, Y_c^{\pm})$ and

$$(X_\infty^-, Y_\infty^-) = \left(\tfrac{1}{48}\left(3a_3^2 - 8a_2\right), \tfrac{1}{32}\left(-8a_1 + 4a_2a_3 - a_3^3\right)\right)\tag{2.11}$$

are on the same line in (X, Y)-space.

The inverse map from the torus to the elliptic curve is simply given by the Abel-Jacobi map. We assume that the four roots of $y^2(x)$ come in complex conjugate pairs as shown in figure 2.[6] Then, the torus image z_c^+ for any real c is given by

$$z_c^+ = \int_{-\infty}^c \frac{\mathrm{d}x}{y}.\tag{2.12}$$

Hence, z_∞^+ is one period of the torus, and we choose it to be ω_2. The image z_c^- can be obtained by (2.10) together with $z_\infty^- = \int_{\gamma_-} \mathrm{d}x/y$, and the other period is $\omega_1 = \int_{\gamma_1} \mathrm{d}x/y$; see figure 2 for the definitions of the integration contours γ_- and γ_1. Due to the distribution of roots, ω_2 and $i\omega_1$ are *positive* reals.

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[5]This can be seen concretely by how the integration kernels $\psi_{-1}(\infty, x)\mathrm{d}x$ and $\psi_{-1}(c, x)\mathrm{d}x$ are related to the kernels of pure functions given below in (2.16).

[6]For a discussion of other possible distributions of roots, see [19].
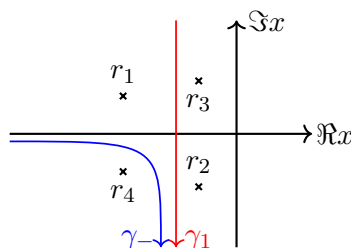
Figure 2. The distribution of the four roots of $y^2(x)$ and the two integration contours γ_1 and γ_- defining the period ω_1 and z_∞^-. The contour γ_2 which defines the period ω_2 runs along the real axis.

2.3 Elliptic multiple polylogarithms on the torus

Due to the equivalence between the elliptic curve and the torus, another way to define elliptic multiple polylogarithms is via iterated integrals on a torus. Such iterated integrals can be formulated in several ways. In this paper, we use the so-called $\tilde{\Gamma}$ functions [19, 25], which are defined as[7]

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ w_1 & \cdots & w_k \end{smallmatrix}; w|\tau\right) = \int_0^w \mathrm{d}w'\, g^{(n_1)}\left(w'-w_1, \tau\right) \widetilde{\Gamma}\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ w_2 & \cdots & w_k \end{smallmatrix}; w'|\tau\right) \tag{2.13}$$

with $\tilde{\Gamma}(; w|\tau) = 1$; we will frequently suppress the dependence on τ for ease of notation. Such an iterated integral is said to have length k and weight $\sum_k n_k$, and in contrast to the case of MPLs both quantities are not necessarily equal. The integration kernels $g^{(n)}(w, \tau)$ are generated by the *Eisenstein-Kronecker series*

$$\frac{\partial_w \theta_1(0|\tau)\theta_1(w+\alpha|\tau)}{\theta_1(w|\tau)\theta_1(\alpha|\tau)} = \sum_{n\geq 0} \alpha^{n-1} g^{(n)}(w, \tau)\,, \tag{2.14}$$

where $\theta_1(w|\tau)$ is the odd Jacobi theta function. All the integration kernels $g^{(n)}$ except $g^{(0)} = 1$ are quasi double periodic,

$$g^{(n)}(w+1) = g^{(n)}(w)\,, \qquad g^{(n)}(w+\tau) = \sum_{j=0}^n \frac{(-2\pi i)^j}{j!} g^{(n-j)}(w)\,, \tag{2.15}$$

but meromorphic with only logarithmic poles at most [19, 25].

Note that the functions $\tilde{\Gamma}$ and the integration kernels $g^{(n)}$ are defined on the torus with one period rescaled to be 1, and that the torus with periods (ω_1, ω_2) has the two possible rescalings $[1 : \tau = \omega_2/\omega_1]$ and $[1 : \tau' = -\omega_1/\omega_2]$, which are related by the modular S-transformation $\tau \to -\tau^{-1}$. We denote the images of c on $[1 : \tau = \omega_2/\omega_1]$ as $w_c^\pm = z_c^\pm/\omega_1$ and the images on $[1 : \tau' = -\omega_1/\omega_2]$ as $\xi_c^\pm = z_c^\pm/(-\omega_2)$. The two are related by $\xi_c^\pm = \tau' w_c^\pm$. In what follows, most of the results are written in terms of w-coordinates, but the corresponding results with w replaced by ξ also hold unless otherwise indicated.

The integration kernels ψ_n can be identified as combinations of $g^{(j)}$'s by matching poles on both sides. On the torus $[1 : \tau = \omega_2/\omega_1]$, one can easily find the following relations

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[7]Another variant of such iterated integrals extensively used in one-loop string amplitudes are the so-called Γ functions, whose integration kernels $f^{(n)}$ are double periodic but *not* meromorphic; see e.g. [106].

between $g^{(j)}$'s and ψ_n's,

$$\psi_1(c,x)\,\mathrm{d}x = \Big(g^{(1)}\left(w-w_c^+\right) + g^{(1)}\left(w-w_c^-\right)$$
$$- g^{(1)}\left(w-w_\infty^+\right) - g^{(1)}\left(w-w_\infty^-\right)\Big)\mathrm{d}w\,, \tag{2.16a}$$

$$\psi_{-1}(c,x)\,\mathrm{d}x = \Big(g^{(1)}\left(w-w_c^+\right) - g^{(1)}\left(w-w_c^-\right) + g^{(1)}\left(w_c^+\right) - g^{(1)}\left(w_c^-\right)\Big)\mathrm{d}w\,, \tag{2.16b}$$

$$\psi_{-1}(\infty,x)\,\mathrm{d}x = \Big(g^{(1)}\left(w-w_\infty^-\right) - g^{(1)}\left(w\right) + g^{(1)}\left(w_\infty^-\right) - \omega_1 a_3/4\Big)\mathrm{d}w\,, \tag{2.16c}$$

$$\psi_0(0,x)\mathrm{d}x = \omega_1\mathrm{d}w\,. \tag{2.16d}$$

On the torus $[1:\tau']$, the corresponding relations can be obtained by replacing $w\to\xi$ and $\omega_1\to-\omega_2$ in (2.16).

Sometimes, it is more convenient to combine $\tilde{\Gamma}$ functions into the so-called $\mathcal{E}_4$ functions [26], especially if the $\tilde{\Gamma}$ functions originally arose from an expression of E_4 functions. The elliptic multiple polylogarithms $\mathcal{E}_4$ are defined in complete analogy to (2.4):

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x\right) = \int_0^x \mathrm{d}x'\,\Psi_{n_1}(c_1,x')\mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; x'\right) \tag{2.17}$$

with $\mathcal{E}_4(;x)=1$,

$$\Psi_{\pm(n>0)}(c,x)\mathrm{d}x = \Big(g^{(n)}(w-w_c^+) \pm g^{(n)}(w-w_c^-)$$
$$- \delta_{\pm n,1}\big(g^{(1)}(w-w_\infty^+) + g^{(1)}(w-w_\infty^-)\big)\Big)\mathrm{d}w \tag{2.18}$$

and $\Psi_0(x)\mathrm{d}x = \mathrm{d}w$, as well as analogous expressions in terms of ξ. The weight of a function $\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x\right)$ is defined as $\sum_i |n_i|$.

2.4 Symbol

By construction, the total derivative of $\tilde{\Gamma}$ admits a recursive structure [25],

$$\mathrm{d}\tilde{\Gamma}(A_1,\dots,A_k;w)$$
$$= \sum_{p=1}^{k-1}(-1)^{n_{p+1}}\tilde{\Gamma}(A_1,\dots,A_{p-1},\vec{0},A_{p+2},\dots,A_k;w)\times\omega^{(n_p+n_{p+1})}(w_{p+1,p}) \tag{2.19}$$
$$+ \sum_{p=1}^{k}\sum_{r=0}^{n_p+1}\Bigg[\binom{n_{p-1}+r-1}{n_{p-1}-1}\tilde{\Gamma}(A_1,\dots,A_{p-1}^{[r]},A_{p+1},\dots,A_k;w)\times\omega^{(n_p-r)}(w_{p-1,p})$$
$$- \binom{n_{p+1}+r-1}{n_{p+1}-1}\tilde{\Gamma}(A_1,\dots,A_{p-1},A_{p+1}^{[r]},\dots,A_k;w)\times\omega^{(n_p-r)}(w_{p+1,p})\Bigg]\,,$$

where $\vec{0}\equiv\big(\begin{smallmatrix}0\\0\end{smallmatrix}\big)$, $w_{i,j}\equiv w_i-w_j$, $(w_0,w_{k+1})\equiv(w,0)$, $(n_0,n_{k+1})\equiv(0,0)$ as well as

$$A_i^{[r]}\equiv\big(\begin{smallmatrix} n_i+r \\ w_i \end{smallmatrix}\big)\,, \qquad A_i^{[0]}\equiv A_i\,. \tag{2.20}$$

The forms $\omega^{(j)}(w)$ are exact, and we can thus write them as

$$\omega^{(j)}(w,\tau) = (2\pi i)^{j-1}\mathrm{d}\Omega^{(j)}(w,\tau)\,, \tag{2.21}$$

with

$$\Omega^{(-1)}(w,\tau) = -2\pi i \tau, \qquad \Omega^{(0)}(w,\tau) = 2\pi i w, \qquad \Omega^{(1)}(w,\tau) = \log \frac{\theta_1(w|\tau)}{\eta(\tau)},$$

$$\Omega^{(\mathrm{odd}\ j>1)}(w,\tau) = -\frac{2j\zeta_{j+1}\tau}{(2\pi i)^j} + \frac{1}{(j-1)!} \sum_{n=1}^{\infty} n^{j-1} \log\left(\left(1 - \mathrm{e}^{2\pi i(n\tau - w)}\right)\left(1 - \mathrm{e}^{2\pi i(n\tau + w)}\right)\right),$$

$$\Omega^{(\mathrm{even}\ j)}(w,\tau) = -\frac{2\zeta_j w}{(2\pi i)^{j-1}} + \frac{1}{(j-1)!} \sum_{n=1}^{\infty} n^{j-1} \log \frac{1 - \mathrm{e}^{2\pi i(n\tau + w)}}{1 - \mathrm{e}^{2\pi i(n\tau - w)}}, \tag{2.22}$$

where $\eta(\tau)$ is the Dedekind eta function and $\zeta_j = \sum_{n \in \mathbb{Z}_+} n^{-j}$ are the Riemann zeta values.[8] The functions $\Omega^{(j)}$ satisfy

$$g^{(j)}(w,\tau) = (2\pi i)^{j-1}\partial_w \Omega^{(j)}(w,\tau) = \frac{(2\pi i)^{j-1}}{j-1}\partial_\tau \Omega^{(j-1)}(w,\tau). \tag{2.23}$$

The sum representation (2.22) can be derived using the sum representation of the $g^{(n)}$ functions given in [25]. In particular,

$$(2\pi i)^{1-n}\widetilde{\Gamma}\left(\begin{smallmatrix} n \\ 0 \end{smallmatrix}; w\right) = \Omega^{(n)}(w) - \Omega^{(n)}(0), \tag{2.24}$$

where $\Omega^{(n)}(0)$ vanishes for even n and is the primitive of the Eisenstein series for odd n; see [25]. We will see below that the functions $\Omega^{(j)}$ play the role of elliptic symbol letters. As can be seen from (2.22), $\Omega^{(1)}$ has a logarithmic singularity at all lattice points, while $\Omega^{(j>1)}$ has a logarithmic singularity at all lattice points except for the origin [25].

For a function $\widetilde{\Gamma}_k^{(n)}$ of weight n and length k, we can define $\underline{\widetilde{\Gamma}}_k^{(n)} = (2\pi i)^{k-n}\widetilde{\Gamma}_k^{(n)}$. Schematically, the differential of $\underline{\widetilde{\Gamma}}_k^{(n)}$ then takes the form

$$\mathrm{d}\underline{\widetilde{\Gamma}}_k^{(n)} = \sum_i \underline{\widetilde{\Gamma}}_{k-1}^{(n-j_i)}\mathrm{d}\Omega^{(j_i)}(w_i). \tag{2.25}$$

Thus, it is natural to define the symbol of $\underline{\widetilde{\Gamma}}_k^{(n)}$ as

$$\mathcal{S}\left(\underline{\widetilde{\Gamma}}_k^{(n)}\right) = \sum_i \mathcal{S}\left(\underline{\widetilde{\Gamma}}_{k-1}^{(n-j_i)}\right) \otimes \Omega^{(j_i)}(w_i). \tag{2.26}$$

Note that in contrast to [25] we have included additional factors of $(2\pi i)$ in the definition of the elliptic letters $\Omega^{(n)}$ and consider the symbol of $\underline{\widetilde{\Gamma}}_k^{(n)}$ rather than $\widetilde{\Gamma}_k^{(n)}$.[9] This is such that the elliptic letters and symbols degenerate to logarithms and polylogarithmic symbols without additional factors of $(2\pi i)$ in the limit where the elliptic curve degenerates, see sections 4–5.

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[8]Recall that $\zeta_{2n} = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!}$ with B_{2n} being the $(2n)^{\mathrm{th}}$ Bernoulli number, such that the first terms in (2.22) can equivalently be written in terms of Bernoulli numbers.

[9]In [25], there is also a projection operator π_k in the definition of the symbol for $\tilde{\Gamma}$ functions due to the fact that some eMPLs of weight 0 evaluate to rational numbers, such as $\widetilde{\Gamma}\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}; 1\right) = 1$. Here we exclude it by introducing these $2\pi i$ factors.

2.5 Shuffle regularization

Let us close this section by remarking on shuffle regularization. One can easily see that $\widetilde{\Gamma}\left(\begin{smallmatrix}1\\0\end{smallmatrix}; z\right) = \Omega^{(1)}(z) - \Omega^{(1)}(0)$ is divergent since $\Omega^{(1)}(0)$ is singular according to (2.22). The shuffle regularization used in [9, 25] takes $\Omega^{(1)}(0) \equiv 2 \log \eta(\tau)$. However, that regularization leads to an issue if we start with an integral of the form (2.7) since it is inconsistent with the usual shuffle regularization for polylogarithms, $G(0; x) \equiv \log x$. To see this, we apply (2.16) to $G(0; 1) = \log 1 = 0$ and find

$$0 \overset{?}{=} \widetilde{\Gamma}\left(\begin{smallmatrix}1\\0\end{smallmatrix}; w_1^+ - w_0^+\right) + \widetilde{\Gamma}\left(\begin{smallmatrix}1\\w_0^- - w_0^+\end{smallmatrix}; w_1^+ - w_0^+\right) - \widetilde{\Gamma}\left(\begin{smallmatrix}1\\w_\infty^+ - w_0^+\end{smallmatrix}; w_1^+ - w_0^+\right) - \widetilde{\Gamma}\left(\begin{smallmatrix}1\\w_\infty^- - w_0^+\end{smallmatrix}; w_1^+ - w_0^+\right),$$
(2.27)

which is in general not true if we use $\Omega^{(1)}(0) \equiv 2 \log \eta(\tau)$. To reconcile both sides, we expand $\widetilde{\Gamma}\left(\begin{smallmatrix}1\\w'\end{smallmatrix}; w\right) = \Omega^{(1)}(w - w') - \Omega^{(1)}(-w')$ to arrive at the following regularization for elliptic multiple polylogarithms:

$$\begin{aligned}
\Omega^{(1)}(0) &\equiv \Omega^{(1)}\left(w_0^+ - w_\infty^-\right) + \Omega^{(1)}\left(w_0^+ - w_\infty^+\right) - \Omega^{(1)}\left(w_0^+ - w_0^-\right) \\
&\quad + \Omega^{(1)}\left(w_1^+ - w_0^-\right) + \Omega^{(1)}\left(w_1^+ - w_0^+\right) - \Omega^{(1)}\left(w_1^+ - w_\infty^-\right) - \Omega^{(1)}\left(w_1^+ - w_\infty^+\right) \\
&= 2 \log \eta(\tau) + \log \frac{2\pi i}{\omega_1} - \log y_0 = \frac{1}{12} \log \Delta - \log y_0,
\end{aligned}$$
(2.28)

where $\Delta = g_2^3 - 27 g_3^2$ is the discriminant of the elliptic curve. The second equality will be explained in the next section and the third equality shows that this regularization is actually independent of the way we rescale the torus.

3 Identities of elliptic symbol letters and the symbol prime

We have briefly reviewed several elementary facts about elliptic multiple polylogarithms, and we saw that the symbol letters of elliptic multiple polylogarithms are the functions $\Omega^{(n)}(w, \tau)$. These functions stand in the way of analyzing the elliptic symbols. For one thing, the relations among $\Omega^{(n)}$'s are much more complicated than the manipulation rules $\log a + \log b = \log ab$ for the symbol letters of multiple polylogarithms. For another, they depend on the kinematics in a rather indirect way — their arguments w and τ are (ratios) of elliptic integrals involving kinematics.

In this section, we investigate the identities of the elliptic letter $\Omega^{(n)}(w, \tau)$. The most trivial identities these letters satisfy are the following:

$$\text{Parity}: \qquad \Omega^{(n)}(-w) = (-1)^{n+1} \Omega^{(n)}(w), \tag{3.1}$$

$$\text{Quasi periodicity}: \qquad \Omega^{(n)}(w + \tau) = \sum_{j=0}^{n+1} \frac{(-1)^j}{j!} \Omega^{(n-j)}(w). \tag{3.2}$$

They immediately follow from (2.22). Our investigation of more non-trivial identities will be focussed on the cases $n = 0, 1, 2$ since — for the two examples considered in this paper, namely the sunrise integral and the double-box integral — the identities among $\Omega^{(n \leq 2)}$ are sufficient to simplify the symbols after using (3.1) and (3.2). We comment on a generalization to identities among $\Omega^{(n > 2)}$ at the end of this section.

Let us start with the slightly trivial identity

$$\log \frac{c-b}{c-a} = \sum_{\sigma \in \pm} \left(\Omega^{(1)} \left(w_c^\sigma - w_b^+ \right) - \Omega^{(1)} \left(w_c^\sigma - w_a^+ \right) \right.$$
$$\left. - \Omega^{(1)} \left(w_\infty^\sigma - w_b^+ \right) + \Omega^{(1)} (w_\infty^\sigma - w_a^+) \right) , \qquad (3.3)$$

which is a simple consequence of applying (2.16a) to $\int_a^b \psi_1(c, x) \mathrm{d}x = \log \frac{c-b}{c-a}$. The identity (3.3) has two important special cases. One is obtained by taking $a \to \infty$, giving

$$\sum_{\sigma \in \pm} \Omega^{(1)} (w_c^\sigma - w_b^+) = \Omega^{(1)} (w_\infty^- - w_\infty^+) + \sum_{\sigma \in \pm} \left[\Omega^{(1)} (w_c^\sigma - w_\infty^+) + \Omega^{(1)} (w_b^\sigma - w_\infty^+) \right]$$
$$- \log \left(\frac{2\pi i}{\omega_1} \right) - 2 \log \eta(\tau) + \log(c-b) ; \qquad (3.4)$$

the other one is obtained by further taking $b \to c$, which yields

$$\Omega^{(1)} (w_c^- - w_c^+) = 2 \left(\Omega^{(1)} \left(w_c^- - w_\infty^+ \right) + \Omega^{(1)} \left(w_c^+ - w_\infty^+ \right) - 2 \log \eta(\tau) - \log \frac{2\pi i}{\omega_1} \right)$$
$$- \Omega^{(1)} \left(w_\infty^- - w_\infty^+ \right) - \log y_c . \qquad (3.5)$$

Now one can easily see that the second equality in (2.28) is the consequence of applying (3.4) and (3.5). The two special cases are particularly useful since the letters $\Omega^{(1)}$ on their right-hand sides always involve $w_\infty^\pm$ and hence can serve as a basis.

3.1 Abel's addition theorem

Surprisingly, a very classical and powerful theorem, Abel's addition theorem [101], yields other identities for $\Omega^{(0)}$ and $\Omega^{(1)}$.[10]

Let us first spell out this theorem: let $\mathcal{C}$ and $\mathcal{C}'$ be curves given by two polynomial equations

$$\mathcal{C}: \quad F(x, y) = 0 , \qquad (3.6)$$
$$\mathcal{C}': \quad Q(x, y) = 0 , \qquad (3.7)$$

where $\mathcal{C}$ is viewed as a *fixed* curve and $\mathcal{C}'$ as a *variable* curve with coefficients collectively denoted as $\{b_i\}$. Suppose that these two curves intersect at n points (x_1, y_1), ..., (x_n, y_n). Let $R(x, y)$ be a rational function defined on $\mathcal{C}$. Then the following holds.

Theorem (Abel)**.** *The integral*

$$I(\{b_i\}) = \sum_{i=1}^n \int_{x_*}^{x_i} R(x, y) \, \mathrm{d}x , \qquad (3.8)$$

where x_* is an arbitrary base point, contains at most rational functions and logarithms of $\{b_i\}$.

This theorem can be proven by showing that $\partial_{b_\nu} I$ is always a rational function of $\{b_i\}$ for all b_ν.

---

[10]See e.g. [107] for a textbook treatment of Abel's addition theorem and [108, 109] for previous applications to Feynman integrals.

If a symbol letter $\phi(u)$ can be expressed as $\int^u R(x,y)\mathrm{d}x$, one can try to find an addition formula for $\phi(u)$ through Abel's addition theorem. Of all applications of this theorem, we are most interested in the cases that $\mathcal{C}'$ has only *two* degrees of freedom and intersects $\mathcal{C}$ at *three* points. For this case, Abel's addition theorem gives

$$\phi(u) + \phi(v) = \phi\big(T(u,v)\big) + \cdots, \tag{3.9}$$

where $T(u,v)$ is an algebraic function of u and v and '$\cdots$' denotes simpler objects, like logarithms.

An example is the addition formula for logarithms, $\log(x_1) + \log(x_2) = \log(x_1 x_2)$,[11] which is given by choosing

$$\mathcal{C}: \quad y = \frac{1}{x}, \tag{3.10}$$

$$\mathcal{C}': \quad y = x^2 + b_1 x + b_2. \tag{3.11}$$

These two curves intersect at the three points x_1, x_2, x_3 that solve

$$x^3 + b_1 x^2 + b_2 x - 1 = 0, \tag{3.12}$$

and hence satisfy $x_1 x_2 x_3 = 1$. Now consider

$$I = \sum_{i=1}^{3} \int_1^{x_i} \frac{\mathrm{d}x}{x}, \tag{3.13}$$

which satisfies

$$\partial_{b_j} I = \sum_{i=1}^{3} \frac{1}{x_i} \frac{\partial x_i}{\partial b_j} = \frac{1}{x_1 x_2 x_3} \frac{\partial}{\partial b_j} x_1 x_2 x_3 = 0, \tag{3.14}$$

since $x_1 x_2 x_3 = 1$. Thus, I is a constant. To fix this constant, we can pick $b_1 = -3, b_2 = 3$, such that $x_1 = x_2 = x_3 = 1$, yielding $I = 0$. Again using $x_1 x_2 x_3 = 1$, we thus have

$$0 = I = \log(x_1) + \log(x_2) + \log(x_3) = \log(x_1) + \log(x_2) - \log(x_1 x_2), \tag{3.15}$$

as claimed.

For the case we are most interested in, the fixed curve $\mathcal{C}$ is given by (2.1), and we find that a convenient choice for $\mathcal{C}'$ is

$$y = -x^2 + b_1 x + b_2. \tag{3.16}$$

One can easily check that these two curves intersects at three point at most. Suppose that two intersection points are $(x_1, y_1 = \sqrt{P_4(x_1)})$ and $(x_2, y_2 = \sqrt{P_4(x_2)})$, then

$$b_1 = \frac{y_1 - y_2}{x_1 - x_2} + x_1 + x_2, \qquad b_2 = \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2} - x_1 x_2, \quad \text{for } x_1 \neq x_2, \tag{3.17}$$

$$b_1 = \frac{P_4'(x_1)}{2y_1} + 2x_1, \qquad b_2 = y_1 + x_1^2 - b_1 x_1, \qquad \text{for } x_1 = x_2, \tag{3.18}$$

---

[11]This example can e.g. be found in [109].

and

$$x_3 = \frac{b_1^2 - 2b_2 - a_2}{2b_1 + a_3} - x_1 - x_2 \,, \qquad y_3 = -\sqrt{P_4(x_3)} \,. \tag{3.19}$$

Since $z_c^+ = \int_{-\infty}^c \mathrm{d}x/y$, Abel's addition theorem tells us

$$z_{x_1}^+ + z_{x_2}^+ \equiv z_{x_3}^+ \bmod \Lambda \,, \qquad\qquad \text{for } b_1 \neq -a_3/2 \,, \tag{3.20a}$$

$$z_{x_1}^+ + z_{x_2}^+ \equiv 0 \bmod \Lambda \,, \qquad\qquad \text{for } b_1 = -a_3/2 \,, \tag{3.20b}$$

which is the well-known group law on the elliptic curve. Furthermore, if we take $b_2 = (a_3^2 - 4a_2)/8$ aside $b_1 = -a_3/2$, then $\mathcal{C}$ and $\mathcal{C}'$ only intersect at one point,

$$\chi = \frac{a_3^4 - 8a_2 a_3^2 + 16a_2^2 - 64a_0}{8\left(a_3^3 - 4a_2 a_3 + 8a_1\right)} \,. \tag{3.21}$$

Together with a little divisor theory, this gives[12]

$$2z_\infty^- \equiv \omega_1 + z_\chi^+ \bmod \omega_2 \,. \tag{3.22}$$

Similarly, for the integral $\int \psi_{-1}(c,x)\mathrm{d}x$, the same procedure gives

$$\int_*^{x_1} \frac{y_c\,\mathrm{d}x}{y(x-c)} + \int_*^{x_2} \frac{y_c\,\mathrm{d}x}{y(x-c)} - \int_*^{x_3} \frac{y_c\,\mathrm{d}x}{y(x-c)} = \log\frac{c^2 - b_1 c - b_2 + y_c}{c^2 - b_1 c - b_2 - y_c} + \text{const.} \tag{3.23}$$

If $z_{x_1}^\pm + z_{x_2}^\pm = z_{x_3}^\pm$, applying (2.16b) to (3.23) gives

$$\sum_{i=1}^2 \Omega^{(1)}\left(w_c^+ - w_{x_i}^+\right) - \Omega^{(1)}\left(w_c^+ - w_{x_i}^-\right) \tag{3.24}$$

$$= \Omega^{(1)}\left(w_c^+ - w_{x_3}^+\right) - \Omega^{(1)}\left(w_c^+ - w_{x_3}^-\right) + \Omega^{(1)}\left(w_c^+\right) - \Omega^{(1)}\left(w_c^-\right) + \log\frac{c^2 - b_1 c - b_2 + y_c}{c^2 - b_1 c - b_2 - y_c} \,.$$

If $z_{x_1}^\pm + z_{x_2}^\pm \equiv z_{x_3}^\pm \bmod \Lambda$, a corresponding identity can be found from (3.24) using the quasi double periodicity of $\Omega^{(1)}$ (3.2).

Three boundary cases of (3.24) require special care:

(i) taking $c \to \infty$ gives

$$\sum_{i=1}^2 \Omega^{(1)}(w_{x_i}^+) - \Omega^{(1)}(w_{x_i}^-) = \Omega^{(1)}(w_{x_3}^+) - \Omega^{(1)}(w_{x_3}^-)$$

$$- \Omega^{(1)}(\omega_\infty^- - \omega_\infty^+) - \log\frac{2b_1 + a_3}{4} + \frac{1}{12}\log\Delta \,, \tag{3.25}$$

---

[12]For any meromorphic function F on a torus, by using $\oint \mathrm{d}\log F(z) = 0$ and $\oint z\,\mathrm{d}\log F(z) = 0$, one can conclude that the number and the sum of its poles are the same as of its zeros, where poles and zeros of order n are counted n times. Now consider the function

$$F = -\kappa'(z) - \kappa(z)^2 - a_3\kappa(z)/2 + (a_3^2 - 4a_2)/8 \,,$$

which has poles at lattice points but vanishes only at z_χ^- and z_∞^-, two intersection points of the curve $y = -x^2 - a_3 x/2 + (a_3^2 - 4a_2)/8$ and the elliptic curve. We then obtain (3.22) by using (2.10).

(ii) taking $x_3 \to \infty$ gives

$$\sum_{i=1}^{2} \Omega^{(1)} \left(w_c^+ - w_{x_i}^+ \right) - \Omega^{(1)} \left(w_c^+ - w_{x_i}^- \right) \tag{3.26}$$

$$= \Omega^{(1)} \left(w_c^+ - w_\infty^+ \right) - \Omega^{(1)} \left(w_c^+ - w_\infty^- \right) + \Omega^{(1)} \left(w_c^+ \right) - \Omega^{(1)} \left(w_c^- \right) + \log \frac{c^2 + a_3 c/2 - b_2 + y_c}{c^2 + a_3 c/2 - b_2 - y_c} \,.$$

(iii) taking $c \to \infty$ and $x_3 \to \infty$ gives

$$\sum_{i=1}^{2} \left[\Omega^{(1)} \left(w_\infty^+ - w_{x_i}^+ \right) - \Omega^{(1)} \left(w_\infty^+ - w_{x_i}^- \right) \right] = -2\Omega^{(1)} \left(w_\infty^+ - w_\infty^- \right) + \Omega^{(0)} \left(w_\infty^- - w_\infty^+ \right)$$

$$+ \frac{1}{6} \log \Delta - \log \frac{4a_2 - a_3^2 + 8b_2}{16} \,. \tag{3.27}$$

Eqs. (3.20), (3.3)–(3.5) as well as (3.24)–(3.27) explain the subset of the identities numerically found in [33] which only involve $\Omega^{(0)}$'s and $\Omega^{(1)}$'s.

Note that the identities we presented in this subsection can be equivalently formulated in terms of divisor theory, see e.g. [104].

3.2 Elliptic Bloch relation and the symbol prime

In [33], also five identities involving $\Omega^{(2)}$'s were observed which are much lengthier than the other identities; each of these five identities contains at least 100 terms in the form that they were found. It turns out all these identities are consequences of the so-called elliptic Bloch relation [102, 103], an elliptic generalization of the five-term identity for dilogarithms,

$$D(x) + D(y) + D\left(\frac{1-x}{1-xy} \right) + D(1-xy) + D\left(\frac{1-y}{1-xy} \right) = 0 \,, \tag{3.28}$$

where $D(z) = \Im(\mathrm{Li}_2(z)) + \arg(1-z) \log |z|$ is the Bloch-Wigner function.[13]

In practice, it is difficult to simplify even expressions containing dilogarithms by using the above five-term identity directly. Instead, we introduce the symbol map [63] for polylogarithms as an assistance; we associate to each polylogarithm a tensor product whose entries satisfy simpler identities. We then exploit that the symbol of a combination of polylogarithms vanishes if that combination of polylogarithms vanishes.

A similar strategy can be used for the elliptic letters $\Omega^{(2)}(w) = (2\pi i)^{-1} \widetilde{\Gamma} \left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix} ; w \right)$, although they already serve as entries of the symbol for elliptic multiple polylogarithms. Inspired by the proof of the elliptic Bloch relation for $\widetilde{\Gamma} \left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix} ; w \right)$ in [104], we associate to $\Omega^{(2)}(w)$ a rank-two tensor through the *symbol prime* map,

$$\mathcal{S}'\big(\Omega^{(2)}(w)\big) = \Omega^{(0)}(w) \otimes' \Omega^{(1)}(w) \,, \tag{3.29}$$

---

[13]To show this concretely, one would need to do the divisor-theory analog of finding a curve that intersects the elliptic curve at precisely the points given by the more than 100 terms in the identities. An algorithm for doing this is given in [105].

where we have added a prime on "$\otimes$" to distinguish it from the tensor product in the symbol. This map has a property similar to that of the symbol map:

$$\sum_j c_j \Omega^{(2)}(w_j) = 0 \quad \text{"}\Rightarrow\text{"} \quad \sum_j c_j \mathcal{S}'\left(\Omega^{(2)}(w_j)\right) \equiv \sum_j c_j \Omega^{(0)}(w_j) \otimes' \Omega^{(1)}(w_j) = 0 \tag{3.30}$$

for some rational coefficients c_j. To show this, consider the sum $\sum_j c_j \widetilde{\Gamma}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}; w_j\right)$. According to (2.19),

$$\mathcal{S}\left(2\pi i \widetilde{\Gamma}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}; w\right)\right) = \Omega^{(0)}(w) \otimes \Omega^{(1)}(w) - \Omega^{(2)}(w) \otimes (2\pi i \tau), \tag{3.31}$$

where we used that $\widetilde{\Gamma}\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}; w\right) = w$, $\widetilde{\Gamma}\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}; w\right) = 2\pi i \Omega^{(2)}(w)$ and $\Omega^{(-1)} = -2\pi i \tau$. If the arguments w_j and coefficients c_j are such that $\sum_j c_j \Omega^{(2)}(w_j) = 0$ due to an elliptic Bloch relation, $\sum_j c_j \widetilde{\Gamma}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}; w_j\right) = 0$ according to an analogous elliptic Bloch relation [104], which in turn implies that the second term on the right-hand side of (3.31), $\sum_j c_j \mathcal{S}'(\Omega^{(2)}(w_j))$, drops out in the sum. In this sense, the symbol prime makes the elliptic Bloch relations manifest.

Note that we have assumed that $\sum_j c_j \Omega^{(2)}(w_j) = 0$ vanishes *due to an elliptic Bloch relation* here in order to show that $\sum_j c_j \mathcal{S}'(\Omega^{(2)}(w_j)) = 0$. We have indicated this in (3.30) as "$\Rightarrow$". However, we currently have no way of proving that all identities $\sum_j c_j \Omega^{(2)}(w_j) = 0$ are due to an elliptic Bloch relation [102, 103]. This is similar to the case of dilogarithms, where (3.28) is only conjectured but not proven to generate all functional identities among dilogarithms.

The symbol map itself has a kernel, and the same is true for the symbol prime. If $\sum_j \Omega^{(2)}(w_j)$ is in the kernel of the symbol prime, i.e. $\mathcal{S}'(\sum_j \Omega^{(2)}(w_j)) = 0$, the first term on the right-hand side of (3.31) drops out in the sum. This implies that $\sum_j \widetilde{\Gamma}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}; w_j\right)$ and thus $\sum_j \Omega^{(2)}(w_j)$ is a function of τ *only*. However, not all functions only of τ are in the kernel of the symbol prime; for example, $\Omega^{(2)}(\tau/n)$ with some positive integer n only depends on τ but has a non-vanishing symbol prime.[14]

One can find the action of the symbol prime map on the letters $\Omega^{(n<2)}$ by expressing them in terms of $\Omega^{(2)}$ using the quasi periodicity (3.2) of $\Omega^{(n)}$:

$$\Omega^{(1)}(w) = \tfrac{1}{6}\Omega^{(2)}(w+2\tau) - \Omega^{(2)}(w+\tau) + \tfrac{1}{2}\Omega^{(2)}(w) + \tfrac{1}{3}\Omega^{(2)}(w-\tau), \tag{3.32}$$

$$\Omega^{(0)}(w) = \Omega^{(2)}(w+\tau) + \Omega^{(2)}(w-\tau) - 2\Omega^{(2)}(w), \tag{3.33}$$

$$\Omega^{(-1)} = -\Omega^{(2)}(w+2\tau) + 3\Omega^{(2)}(w+\tau) - 3\Omega^{(2)}(w) + \Omega^{(2)}(w-\tau). \tag{3.34}$$

This yields

$$\mathcal{S}'\left(\Omega^{(1)}(w)\right) = \Omega^{(0)}(w) \otimes' \Omega^{(0)}(w) + \Omega^{(-1)} \otimes' \Omega^{(1)}(w), \tag{3.35a}$$

$$\mathcal{S}'\left(\Omega^{(0)}(w)\right) = \Omega^{(0)}(w) \otimes' \Omega^{(-1)} + 2\Omega^{(-1)} \otimes' \Omega^{(0)}(w), \tag{3.35b}$$

$$\mathcal{S}'\left(\Omega^{(-1)}\right) = 3\Omega^{(-1)} \otimes' \Omega^{(-1)}, \tag{3.35c}$$

---

[14]However, the occurrence of such a letter means that the point on the elliptic curve that corresponds to τ/n on the torus, namely $\kappa(\omega_1 \tau/n)$, should occur in the calculation of the E_4 functions and hence is algebraic in kinematics. This is not the case for the examples of the unequal-mass sunrise integral and the ten-point double-box integral studied in sections 4–5, but it is the case for the *equal*-mass sunrise integral.

where we have moreover used quasi periodicity to simplify the entries of the symbol prime. In particular, by expressing a logarithm in terms of $\Omega^{(1)}$'s and $\Omega^{(0)}$'s either through (3.3) or (3.24), one finds

$$\mathcal{S}'\left(\log c\right) = \Omega^{(-1)} \otimes' \log c. \tag{3.36}$$

Thus, for a combination of $\Omega^{(n \leq 2)}$'s and logarithms, one can compute its symbol prime. It involves only $\Omega^{(n \leq 1)}$ and can thus be simplified using the techniques discussed in subsection 3.1. If the symbol prime is not zero, one may search for a simpler combination of $\Omega^{(n \leq 2)}$'s and logarithms with the same symbol prime according to (3.29), (3.35) and (3.36).[15] The difference of these two combinations has to be a function of τ only, and a simple expression for this function can be obtained by sending the independent w-variables in the difference to any values, say 0. In this way, we have proven the five identities involving $\Omega^{(2)}$'s found in [33].

The current definitions for $\Omega^{(n \leq 2)}$ are sufficient for the two examples treated in this paper. For $n > 2$, one might similarly define the symbol prime for $\Omega^{(n)}$'s as

$$\mathcal{S}^{(n-1)}\left(\Omega^{(n)}(w)\right) = \frac{1}{n-1}\Omega^{(0)}(w) \otimes^{(n-1)} \Omega^{(n-1)}(w), \tag{3.37}$$

due to the fact that,

$$\mathcal{S}\left((2\pi i)^{2-n} \widetilde{\Gamma}\left(\begin{smallmatrix} n-1 & 0 \\ 0 & 0 \end{smallmatrix}; w\right)\right) = \Omega^{(0)}\left(w\right) \otimes \Omega^{(n-1)}\left(w\right) - (n-1)\left(\Omega^{(n)}(w) - \Omega^{(n)}(0)\right) \otimes (2\pi i\tau), \tag{3.38}$$

where $\Omega^{(n)}(0)$ is either zero or a function only depending on τ for even or odd n, respectively. With the knowledge of the identities among $\Omega^{(n-1)}$, we can then find identities among $\Omega^{(n)}$ recursively. We leave the exploration of the symbol prime for $\Omega^{(n>2)}$ to future work.

4 Example I: unequal-mass sunrise integral

Two particularly interesting cases of elliptic Feynman integrals are the unequal-mass sunrise integral in two dimensions and the double-box integral in four dimensions. We will investigate these two integrals through the tools developed so far. The main focus will be on the sunrise integral treated in this section, since this integral is simple enough such that the main results can be written within a couple of lines. After applying the symbol prime map, we will see that several properties, such as double periodic invariance, modular invariance (covariance) and part of integrability are manifest.

We calculate the unequal-mass sunrise integral in terms of elliptic multiple polylogarithms $\mathcal{E}_4$ in appendix A. This integral was originally calculated in terms of iterated integrals on the moduli space $\overline{\mathcal{M}}_{1,3}$ in [28]. We closely follow the Feynman-parameter approach of [20] for the equal-mass case.

The resulting expression when rescaling the torus by the period ω_1 is

$$I_{\ominus} = \frac{\omega_1}{2\pi i m_1^2}\left(2\pi i T_{\ominus}^{(1)}\right), \tag{4.1}$$

---

[15]In particular, if the first entry of the symbol prime is only τ, then the function is the sum of logarithms and a function of τ.

where the periods were defined in figure 2 and $T_{\ominus}^{(1)}$ is a pure combination of elliptic multiple polylogarithms of weight one and length two,

$$T_{\ominus}^{(1)} = \mathcal{E}_4 \left(\begin{smallmatrix} 0 & -1 \\ 0 & -1 \end{smallmatrix} ; \infty | \tau \right) - \mathcal{E}_4 \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix} ; \infty | \tau \right) + \mathcal{E}_4 \left(\begin{smallmatrix} 0 & -1 \\ 0 & r \end{smallmatrix} ; \infty | \tau \right) - \mathcal{E}_4 \left(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix} ; \infty | \tau \right)$$

$$+ 4\pi i \mathcal{E}_4 \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} ; \infty | \tau \right) - \mathcal{E}_4 \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} ; \infty | \tau \right) \log \frac{t_2^2}{t_3^2} , \tag{4.2}$$

where we introduced $t_i^2 = m_i^2/p^2$ and $r = -t_3^2/t_1^2$. Note that we have included seemingly redundant factors of $(2\pi i)$ in the numerator and denominator of (4.1) that ensure that the prefactor degenerates to an algebraic function in the limit where the elliptic curve degenerates, and the term in parentheses degenerates to a pure logarithm of transcendental weight two; see subsection 4.1.

However, we can also rescale the torus by the period $-\omega_2$, finding

$$I_{\ominus} = \frac{-\omega_2}{2\pi i m_1^2} \left(2\pi i T_{\ominus}^{(2)} \right) , \tag{4.3}$$

with

$$T_{\ominus}^{(2)} = \mathcal{E}_4 \left(\begin{smallmatrix} 0 & -1 \\ 0 & -1 \end{smallmatrix} ; \infty | \tau' \right) - \mathcal{E}_4 \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix} ; \infty | \tau' \right) + \mathcal{E}_4 \left(\begin{smallmatrix} 0 & -1 \\ 0 & r \end{smallmatrix} ; \infty | \tau' \right) - \mathcal{E}_4 \left(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix} ; \infty | \tau' \right)$$

$$- \mathcal{E}_4 \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} ; \infty | \tau' \right) \log \frac{t_2^2}{t_3^2} , \tag{4.4}$$

(Recall from subsection 2.1 that $\tau' = -\omega_1/\omega_2$.)

According to (4.1) and (4.3), the values of $T_{\ominus}^{(1)}$ and $T_{\ominus}^{(2)}$ are related by $T_{\ominus}^{(1)} = -\tau T_{\ominus}^{(2)}$, but this relation is not obvious from their expressions in terms of eMPLs. In general, eMPLs transform non-trivially under the modular S-transformation $\tau \to \tau' = -1/\tau$; for example,

$$\mathcal{E}_4 \left(\begin{smallmatrix} -1 \\ c \end{smallmatrix} ; x | \tau \right) = \mathcal{E}_4 \left(\begin{smallmatrix} -1 \\ c \end{smallmatrix} ; x | \tau' \right) - \frac{2\pi i \left(\xi_c^+ - \xi_c^- \right)}{\tau'} \mathcal{E}_4 \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} ; x | \tau' \right) . \tag{4.5}$$

See [30] for the cases of iterated integrals of modular forms. The same is true for the symbol, as we will see soon. However, we will see that the application of the symbol prime map makes the behavior under the modular S-transformation manifest.

4.1 Symbol of the sunrise integral

The symbol of $T_{\ominus}^{(1,2)}$ can be calculated by first rewriting $\mathcal{E}_4$'s in terms of $\widetilde{\Gamma}$'s via (2.18) and then applying (2.19)–(2.26). For example,

$$\mathcal{E}_4 \left(\begin{smallmatrix} 0 & -1 \\ 0 & c \end{smallmatrix} ; x \right) = \widetilde{\Gamma} \left(\begin{smallmatrix} 0 & 1 \\ 0 & w_c^+ - w_0^+ \end{smallmatrix} ; w_x^+ - w_0^+ \right) - \widetilde{\Gamma} \left(\begin{smallmatrix} 0 & 1 \\ 0 & w_c^- - w_0^+ \end{smallmatrix} ; w_x^+ - w_0^+ \right) \tag{4.6}$$

and

$$\mathcal{S} \left(2\pi i \widetilde{\Gamma} \left(\begin{smallmatrix} 0 & 1 \\ 0 & w_1 \end{smallmatrix} ; w_2 \right) \right) = \left(\Omega^{(2)}(-w_1) - \Omega^{(2)}(w_2 - w_1) \right) \otimes \Omega^{(-1)} - \Omega^{(0)}(w_2) \otimes \Omega^{(1)}(w_1)$$

$$+ \left(\Omega^{(1)}(w_2 - w_1) - \Omega^{(1)}(-w_1) \right) \otimes \Omega^{(0)}(w_2 - w_1) . \tag{4.7}$$

The simplification of the symbols in this case is slightly non-trivial: it involves some non-trivial relations of $\Omega^{(1)}$'s, $\Omega^{(0)}$'s and logarithms as described in section 3.1; for instance,

$$\log \frac{t_1}{t_3} = \Omega^{(1)} \left(w_{-1}^+ - w_0^+ \right) - \Omega^{(1)} \left(w_{-1}^+ - w_\infty^+ \right) + \Omega^{(1)} \left(w_{-1}^+ - w_0^- \right) - \Omega^{(1)} \left(w_{-1}^+ - w_\infty^- \right),$$
(4.8)

$$\log \frac{t_2}{t_3} = \Omega^{(1)} \left(w_0^+ - w_\infty^+ \right) - \Omega^{(1)} \left(w_{-1}^+ - w_\infty^+ \right) + \Omega^{(1)} \left(w_{-1}^+ - w_0^- \right) - \Omega^{(1)} \left(w_\infty^- - w_\infty^+ \right).$$
(4.9)

(Recall from subsection 2.1 that $w_c^+ = \omega_1^{-1} \int_{-\infty}^c \mathrm{d}x/y$). All the relations involving $\Omega^{(2)}$ in this case are comparably trivial; they are the consequences of (3.2). We present the full simplification in the attached file `sunrise_symbol.nb`.

The final result is

$$\mathcal{S} \left(2\pi i T_{\ominus}^{(1)} \right) = \log \frac{t_2^2}{t_1^2} \otimes \Omega^{(0)}(w_0^+) + \log \frac{t_1^2}{t_3^2} \otimes \Omega^{(0)}(w_{-1}^+)$$
$$+ \left[\frac{1}{2\pi i} \left(2\mathcal{E}_4 \left(\begin{smallmatrix} -2 \\ -1 \end{smallmatrix} ; \infty \right) - \mathcal{E}_4 \left(\begin{smallmatrix} -2 \\ 0 \end{smallmatrix} ; \infty \right) - \mathcal{E}_4 \left(\begin{smallmatrix} -2 \\ \infty \end{smallmatrix} ; \infty \right) \right) + \log \frac{t_3^2}{t_2^2} \right] \otimes (2\pi i \tau),$$
(4.10)

where we have moreover used

$$\frac{\mathcal{E}_4 \left(\begin{smallmatrix} -n \\ c \end{smallmatrix} ; \infty \right)}{(2\pi i)^{n-1}} = \Omega^{(n)} \left(w_\infty^+ - w_c^+ \right) - \Omega^{(n)} \left(w_0^+ - w_c^+ \right) - \Omega^{(n)} \left(w_\infty^+ - w_c^- \right) + \Omega^{(n)} \left(w_0^+ - w_c^- \right).$$
(4.11)

Similarly,

$$\mathcal{S} \left(2\pi i T_{\ominus}^{(2)} \right) = \log \frac{t_2^2}{t_1^2} \otimes \Omega^{(0)} \left(\xi_0^+ \right) + \log \frac{t_1^2}{t_3^2} \otimes \Omega^{(0)} \left(\xi_{-1}^+ \right)$$
$$+ \left[\frac{1}{2\pi i} \left(2\mathcal{E}_4 \left(\begin{smallmatrix} -2 \\ -1 \end{smallmatrix} ; \infty \right) - \mathcal{E}_4 \left(\begin{smallmatrix} -2 \\ 0 \end{smallmatrix} ; \infty \right) - \mathcal{E}_4 \left(\begin{smallmatrix} -2 \\ \infty \end{smallmatrix} ; \infty \right) \right) \right.$$
$$\left. + \log \frac{t_1}{t_3} + \Omega^{(1)} \left(\xi_\infty^+ - \xi_{-1}^+ \right) - \Omega^{(1)} \left(\xi_{-1}^+ - \xi_0^+ \right) \right] \otimes (2\pi i \tau'),$$
(4.12)

where we used (4.11) in terms of ξ-coordinates.

At this point, the symbols of the sunrise integral partially show some desired properties; for example, the first entries of the first two terms in (4.10) and (4.12) indicate the physical first-entry conditions known from the massless case, and their last entries are related by simple S-transformations $w \to \xi$. However, the first two terms on their own are neither double periodic nor integrable.

The first entries of the last terms in (4.10) and (4.12), i.e., $\partial_\tau T_{\ominus}^{(1)}$ and $\partial_{\tau'} T_{\ominus}^{(2)}$, are rather complicated and the main obstacles to understanding the entire symbols, since it is hard to see how they render the whole symbol double periodic and integrable. In this respect, it is instructive to consider the symbol primes of these entries:

$$\mathcal{S}' \left(\partial_\tau T_{\ominus}^{(1)} \right) = \Omega^{(0)} \left(w_0^+ \right) \otimes' \log \frac{t_2^2}{t_1^2} + \Omega^{(0)} \left(w_{-1}^+ \right) \otimes' \log \frac{t_1^2}{t_3^2},$$
(4.13)

$$\mathcal{S}' \left(\partial_{\tau'} T_{\ominus}^{(2)} \right) = \Omega^{(0)} \left(\xi_0^+ \right) \otimes' \log \frac{t_2^2}{t_1^2} + \Omega^{(0)} \left(\xi_{-1}^+ \right) \otimes' \log \frac{t_1^2}{t_3^2}.$$
(4.14)

They have the following advantageous properties:

1. It is obvious that $\mathcal{S}'(\partial_\tau T_{\ominus}^{(1)})$ differs from $\mathcal{S}'(\partial_{\tau'} T_{\ominus}^{(2)})$ only by a modular S-transformation $w \to \xi$, i.e. the symbol prime makes modular covariance manifest.

2. If we shift w_{-1}^+ or w_0^+ by τ, then $\partial_\tau T_{\ominus}^{(1)}$ changes by $\log \frac{t_3^2}{t_1^2}$ or $\log \frac{t_1^2}{t_2^2}$, respectively, (and similarly for $\partial_{\tau'} T_{\ominus}^{(2)}$) since

$$\mathcal{S}'\left(\partial_\tau T_{\ominus}^{(1)}\big|_{w_{-1}^+ \to w_{-1}^+ + \tau} - \partial_\tau T_{\ominus}^{(1)}\right) = (2\pi i \tau) \otimes' \log \frac{t_1^2}{t_3^2} = \mathcal{S}'\left(-\log \frac{t_1^2}{t_3^2}\right), \qquad (4.15)$$

$$\mathcal{S}'\left(\partial_\tau T_{\ominus}^{(1)}\big|_{w_0^+ \to w_0^+ + \tau} - \partial_\tau T_{\ominus}^{(1)}\right) = (2\pi i \tau) \otimes' \log \frac{t_2^2}{t_1^2} = \mathcal{S}'\left(-\log \frac{t_2^2}{t_1^2}\right). \qquad (4.16)$$

The first two terms in the symbol change by corresponding terms with opposite sign that cancel these. Thus, $\mathcal{S}(2\pi i T_{\ominus}^{(1,2)})$ are *double periodic*.

3. Moreover, the symbol prime also makes integrability with respect to τ manifest. This is slightly trivial in the case of the length-two sunrise integral, and will thus be discussed in full generality for the case of the double-box integral in section 5.1.

Finally, note that the equal-mass case can be obtained smoothly by taking $t_1 = t_2 = t_3$; we will briefly comment on this case in section 6.

4.2 Degeneration at $p^2 = 0$ and pseudo-thresholds

Next, let us see how the symbol of the unequal-mass sunrise integral behaves in kinematic limits where the elliptic curve degenerates.

The kinematic configurations where the elliptic curve degenerates can be easily read off from the discriminant

$$\Delta_{\ominus} = \frac{t_2^4 t_3^4}{t_1^{20}}\left((t_1+t_2+t_3)^2 - 1\right)\left((t_1+t_2-t_3)^2 - 1\right)\left((t_1-t_2+t_3)^2 - 1\right)\left((-t_1+t_2+t_3)^2 - 1\right),$$
$$(4.17)$$

where $t_i^2 = m_i^2/p^2$ as before. In particular, the sunrise integral remains finite at $p^2 = 0$, at the pseudo-thresholds $p^2 = \{(m_1+m_2-m_3)^2, (m_1+m_3-m_2)^2, (m_2+m_3-m_1)^2\}$ and at the threshold $p^2 = (m_1+m_2+m_3)^2$, while it diverges for $m_i = 0$. The values at $p^2 = 0$ and the pseudo-thresholds were given in terms of MPLs in [10]. In what follows, we will show how the symbols $\mathcal{S}(2\pi i T_{\ominus}^{(1,2)})$ reproduce the corresponding symbols in these two limits.[16]

Null-momentum limit. As $p^2 \to 0$, t_1, t_2 and t_3 approach infinity while their ratios remain finite. The elliptic curve degenerates in a way that $r_1 \to r_3$ and $r_2 \to r_4$; cf. figure 3a. In this case, $\omega_1 \to \infty$ since the roots pinch the corresponding integration contour γ_1, while

$$\omega_2 \to \int_{-\infty}^{\infty} \frac{dx}{x^2 + (1 + (t_3/t_1)^2 - (t_2/t_1)^2)x + (t_3/t_1)^2} = \frac{2\pi i}{\sqrt{\left(1 - \frac{t_2^2}{t_1^2} - \frac{t_3^2}{t_1^2}\right)^2 - 4\frac{t_2^2 t_3^2}{t_1^4}}}. \qquad (4.18)$$

---

[16]The threshold can be treated in a similar way.
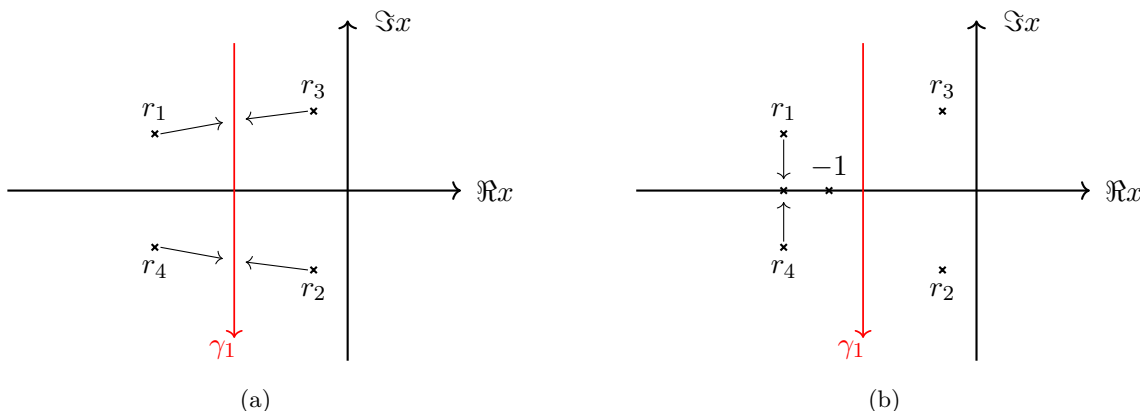
Figure 3. The roots of $y^2(x)$ for the sunrise integral coincide in the null-momentum limit (a) and pseudo-thresholds (b). In the latter case, the position of the coinciding roots relative to -1 is shown for the case $t_3 > t_1$.

Then $-(2\pi i)^{-1} m_1^{-2} \omega_2$ reproduces the same normalization factor as in [10] (up to a sign). Thus, we should expect that $\mathcal{S}(2\pi i T_{\ominus}^{(2)})$ reduces to the corresponding symbol. In this limit, $q = \exp(2\pi i \tau')$ vanishes, and hence all $\mathcal{E}_4\left(\begin{smallmatrix} -2 \\ c \end{smallmatrix}; x\right)$ in (4.10) vanish; cf. (4.11) and (2.22). Furthermore,

$$\Omega^{(0)}\left(\xi_c^+\right) \to \log \frac{2c + 1 - u + v + \sqrt{(1-u-v)^2 - 4uv}}{2c + 1 - u + v - \sqrt{(1-u-v)^2 - 4uv}}, \tag{4.19}$$

$$\log \frac{t_1}{t_3} + \Omega^{(1)}\left(\xi_\infty^+ - \xi_{-1}^+\right) - \Omega^{(1)}\left(\xi_{-1}^+ - \xi_0^+\right) \to 0, \tag{4.20}$$

where we have introduced $u = (t_2/t_1)^2 = z\bar{z}$ and $v = (t_3/t_1)^2 = (1-z)(1-\bar{z})$. Then,

$$\mathcal{S}\left(2\pi i T_{\ominus}^{(2)}\right) \to \log u \otimes \log \frac{1 - \bar{z}}{1 - z} - \log v \otimes \log \frac{\bar{z}}{z}, \tag{4.21}$$

which is the symbol of $-4i$ times the Bloch-Wigner dilogarithm $D(z)$, in perfect agreement with [10].

Pseudo-thresholds. Without loss of generality, we consider the pseudo-threshold $p^2 = (m_1 + m_2 - m_3)^2$. In terms of t_i, this pseudo-threshold is equal to the condition $(t_1 + t_2 - t_3)^2 = 1$. We only consider the solution $t_3 = t_1 + t_2 - 1$ since the treatment for the other solution is similar. At $t_3 = t_1 + t_2 - 1$, the roots r_1 and r_4 pinch the real axis; cf. figure 3b. Thus ω_2 diverges and we should consider the rescaling of the torus by ω_1, $T_{\ominus}^{(1)}$. We can then close the contour γ_1 with a large semi-circle in the left half-plane and evaluate the integral via residues:

$$\int_{\gamma_1} \frac{t_1 \mathrm{d}x}{\left(x + \frac{t_3}{t_1}\right)\sqrt{\left(t_1^2 x^2 + (t_1^2 + t_3^2 - (t_2 + 1)^2)x + t_3^2\right)}} = -2\pi i \sqrt{\frac{t_1^3}{4 t_2 t_3}}. \tag{4.22}$$

Again $(2\pi i)^{-1}m_1^{-2}\omega_1$ reproduces the same normalization factor as in [10] at $p^2 = (m_1 + m_2 - m_3)^2$. Thus, we should expect that $\mathcal{S}(2\pi i T_{\ominus}^{(1)})$ reduces to the corresponding symbol. At the pseudo-threshold, $q = \exp(2\pi i \tau)$ vanishes. Furthermore, we assume $t_3 > t_1$;[17] then all $\mathcal{E}_4 \left(\begin{smallmatrix} -2 \\ c \end{smallmatrix} ; x \right)$ appearing in (4.10) also vanish in this limit. Each of the three last entries in (4.10) is divergent in this limit since the roots pinch the integration contour, but all these divergences cancel in the following combinations

$$\Omega^{(0)} \left(w_{\infty}^{+} - w_{-1}^{+} \right) = 2\pi i \int_{-1}^{\infty} \frac{\mathrm{d}x}{y} \rightarrow \log \frac{1 + \sqrt{\frac{t_3}{t_1 t_2}}}{1 - \sqrt{\frac{t_3}{t_1 t_2}}} \,, \tag{4.23}$$

$$\Omega^{(0)} \left(w_{\infty}^{+} - w_{0}^{+} \right) = 2\pi i \int_{0}^{\infty} \frac{\mathrm{d}x}{y} \rightarrow \log \frac{1 + \sqrt{\frac{t_2}{t_1 t_3}}}{1 - \sqrt{\frac{t_2}{t_1 t_3}}} \,, \tag{4.24}$$

$$\Omega^{(0)} \left(w_{0}^{+} - w_{-1}^{+} \right) = 2\pi i \int_{-1}^{0} \frac{\mathrm{d}x}{y} \rightarrow \log \frac{1 + \sqrt{\frac{t_1}{t_2 t_3}}}{1 - \sqrt{\frac{t_1}{t_2 t_3}}} \,; \tag{4.25}$$

cf. figure 3b. Correspondingly, the symbol is reduced to

$$\mathcal{S} \left(2\pi i T_{\ominus}^{(1)} \right) \rightarrow \log \frac{t_1}{t_2 t_3} \otimes \log \frac{1 - \sqrt{\frac{t_1}{t_2 t_3}}}{1 + \sqrt{\frac{t_1}{t_2 t_3}}} + (t_1 \leftrightarrow t_2) - (t_1 \leftrightarrow t_3) \,, \tag{4.26}$$

where each term is the symbol of $\mathrm{Li}_2(z) - \mathrm{Li}_2(-z) + \log(z) \log((1-z)/(1+z))$ — in perfect agreement with [10].

5 Example II: double-box integral

As a second example of Feynman integrals that evaluate to elliptic multiple polylogarithms, we consider the two-loop ten-point double-box integral:

$$I_{\text{⊞⊞}} = \quad = \int \mathrm{d}^4 x_l \mathrm{d}^4 x_k \frac{(x_1 - x_5)^2}{(x_1 - x_l)^2 (x_3 - x_l)^2 (x_5 - x_l)^2}$$
$$\times \frac{(x_3 - x_8)^2 (x_6 - x_{10})^2}{(x_l - x_k)^2 (x_6 - x_k)^2 (x_8 - x_k)^2 (x_{10} - x_k)^2} \,, \tag{5.1}$$

where we have introduced the dual coordinates x_i defined as $x_i - x_{i+1} = p_i$ and the notation $x_{i,j} = x_i - x_j$. This integral is a particular component of the two-loop ten-point N^3MHV amplitude in planar maximally supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ sYM theory) [110], and was recently integrated in terms of elliptic multiple polylogarithms in [33]. It depends on seven dual conformal cross-ratios

$$u_1 = \frac{x_{1,3}^2 x_{5,8}^2}{x_{1,5}^2 x_{3,8}^2} \,, \qquad u_2 = \frac{x_{3,6}^2 x_{8,10}^2}{x_{3,8}^2 x_{6,10}^2} \,, \qquad v_1 = \frac{x_{1,8}^2 x_{3,5}^2}{x_{1,5}^2 x_{3,8}^2} \,, \qquad v_2 = \frac{x_{3,10}^2 x_{6,8}^2}{x_{3,8}^2 x_{6,10}^2} \,,$$

$$u_3 = \frac{x_{1,3}^2 x_{5,10}^2}{x_{1,5}^2 x_{3,10}^2} \,, \qquad u_4 = \frac{x_{1,6}^2 x_{3,5}^2}{x_{1,5}^2 x_{3,6}^2} \,, \qquad u_5 = \frac{x_{1,5}^2 x_{6,10}^2}{x_{1,6}^2 x_{5,10}^2} \tag{5.2}$$

---

[17]The case $t_1 < t_3$ can be obtained by an analytic continuation.

and contains the elliptic curve defined by

$$y^2 = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$
$$= \left(\frac{v_1}{u_4} \big((1-u_4)(x+1-v_2) - u_1 + u_3 v_2 \big) + h_1 + h_2 \right)^2 - 4 h_1 h_2, \tag{5.3}$$

where

$$h_1 = \frac{u_2 u_4}{v_1} \big(x^2 + (1-u_1+v_1)x + v_1 \big),$$
$$h_2 = \left(x + \frac{v_1}{u_4} \right) \left((1+x-u_1) \Big(\frac{u_2 u_4}{v_1} - 1 \Big) + (1-u_3) v_2 \right). \tag{5.4}$$

As shown in [33], the elliptic double-box integral can be evaluated in terms of $\mathcal{E}_4$ functions whose arguments make up the set

$$
\{c_i\} = \bigg\{ 0, -1, \infty, -u_2, -v_1, -\frac{v_1}{u_4}, -1 + \frac{u_1}{u_3}, -u_2 u_4 u_5, -u_2(1-u_4) - v_1,
$$
$$
\frac{u_2(u_3+u_4-1)-v_1}{1-u_3}, \frac{u_2 u_3 u_4 u_5 - v_1}{1-u_3}, \frac{u_2 u_3 u_4 u_5 - v_1}{u_4(1-u_3 u_5)}, \frac{u_2(u_3 u_4 u_5 v_2 - u_1)}{u_3 v_2 - u_1},
$$
$$
\frac{v_1(u_3 u_4 u_5 v_2 - u_1)}{u_4(u_1 - u_3 u_5 v_2)}, \frac{u_4 u_5(u_2(u_4-1)-v_1)+v_1}{u_4(u_5-1)}, \frac{u_1 u_2(u_4-1)-v_1(u_1-u_3 v_2)}{u_1 - u_3 v_2},
$$
$$
z_1 - 1, \bar{z}_1 - 1, z_{1,3,6,8} - 1, \bar{z}_{1,3,6,8} - 1, -z_{3,5,8,10}, -\bar{z}_{3,5,8,10}, -z_2, -\bar{z}_2,
$$
$$
\frac{u_2 u_3 u_4 u_5 - v_1 + r_+}{1-u_3}, \frac{u_2 u_3 u_4 u_5 - v_1 + r_-}{1-u_3} \bigg\}, \tag{5.5}
$$

where $z_{a,b,c,d} \bar{z}_{a,b,c,d} = x_{a,b;c,d}$, $(1 - z_{a,b,c,d})(1 - \bar{z}_{a,b,c,d}) = x_{d,a;b,c}$, with the abbreviation

$$x_{a,b;c,d} = \frac{x_{a,b}^2 x_{c,d}^2}{x_{a,c}^2 x_{b,d}^2}. $$

Note that $u_1 = x_{1,3;5,8}, v_1 = x_{3,5;8,1}$ and $u_2 = x_{3,6;8,10}, v_1 = x_{6,8;10,3}$, and we thus have abbreviated $z_1 \equiv z_{1,3,5,8}$ and $z_2 \equiv z_{3,6,8,10}$. The expressions for $r_\pm$ are slightly lengthy,

$$r_\pm = \frac{\mathbf{G}_{45}^{-1} \det \mathbf{G} \pm \sqrt{\det \mathbf{G}^{(45)}} \sqrt{-\det \mathbf{G}}}{2(1-u_5) x_{1,5}^2 x_{3,10}^2 x_{1,6}^2 x_{3,8}^2 x_{5,10}^2}, \tag{5.6}$$

where $\mathbf{G}$ denotes the Gram matrix $(x_{a,b}^2)$ with a and b running over the index set of dual coordinates $\{1, 3, 5, 6, 8, 10\}$, $\mathbf{G}_{ij}^{-1}$ denotes the elements of the inverse of the Gram matrix $(\mathbf{G}^{-1})_{ij}$, and $\mathbf{G}^{(ij)}$ denotes the matrix obtained from $\mathbf{G}$ by deleting the i'th and j'th rows and columns. By using (2.16), the double-box integral was also expressed in terms of $\tilde{\Gamma}$ functions on the torus $[1 : \tau = \omega_2/\omega_1]$ [33].

The last two arguments in (5.5) can be written in a slightly more compact form if we introduce the *momentum twistor* variables [111],

$$Z_i^a = (\lambda_i^\alpha, x_i^{\alpha \dot{\alpha}} \lambda_{i\alpha}), \qquad \alpha, \dot{\alpha} = 1, 2, \tag{5.7}$$

as well as the SL(4)-invariant $\langle ijkl \rangle = \epsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d$, where λ_i is the usual spinor-helicity variable $p_i^\mu \sigma_\mu^{\alpha \dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}$. In terms of these variables, the last two arguments in (5.5) can be written as

$$\frac{\langle 9, 10, 1, (7, 8) \cap (2, 3, 5) \rangle}{\langle 1, 5, 9, 10 \rangle \langle 2, 3, 7, 8 \rangle}, \frac{\langle 4, 5, 6, (2, 3) \cap (7, 8, 10) \rangle}{\langle 2, 3, 7, 8 \rangle \langle 4, 5, 6, 10 \rangle}, \tag{5.8}$$

where $(ab) \cap (ijk) = Z_a \langle bijk \rangle + Z_b \langle ijka \rangle$ is the intersection point of the line (ab) and the plane (ijk) [112].

In the present paper, we compute $I_{\boxplus}$ in terms of $\tilde{\Gamma}$ functions on the torus rescaled as $[1 : \tau' = -\omega_1/\omega_2]$, as well as the corresponding symbol. The computation is straightforward; however, both expressions are too lengthy to be recorded in the main text and are given in the file `doublebox_omega2` in the supplementary material attached to this paper. In what follows, we will give a schematic expression of the symbol as well as the symbol prime and comment on several interesting aspects, such as the integrability condition and the soft limit $p_{10} \to 0$.

5.1 Symbol of the double-box integral

Let us first review some results and notations from [33]. After normalization by ω_1, $T_{\boxplus}^{(1)} = I_{\boxplus}/\omega_1$ is a pure combination of $\tilde{\Gamma}$ functions of length four and weight three. The corresponding symbol $\mathcal{S}(2\pi i T_{\boxplus}^{(1)})$ satisfies the same physical first-entry conditions as in the MPL cases [67]; namely, the first entries are given by $\log u_i$ and $\log v_i$. The first two entries are $\mathrm{Li}_2(1 - x_{a,b;c,d})$, $\log x_{a,b;c,d} \log x_{a',b';c',d'}$ or four-mass-box functions,

$$
- \mathrm{Li}_2(z_{a,b,c,d}) + \mathrm{Li}_2(\bar{z}_{a,b,c,d}) - \frac{1}{2} \log(x_{a,b;c,d}) \log\left(\frac{1 - z_{a,b,c,d}}{1 - \bar{z}_{a,b,c,d}}\right). \tag{5.9}
$$

The last entries of this symbol consist of seven letters of elliptic type $\Omega^{(0)}$. According to these last entries, the symbol can be organized as[18]

$$
\begin{aligned}
\mathcal{S}\left(2\pi i T_{\boxplus}^{(1)}\right) = {} & \mathcal{S}\left(I_{\mathrm{hex}}\right) \otimes \Omega^{(0)}\left(w_{c_{25}}^+\right) + \mathcal{S}\left(F_-\right) \otimes \Omega^{(0)}\left(w_\infty^-\right) + \mathcal{S}\left(F_\tau\right) \otimes (2\pi i \tau) \\
& + \mathcal{S}\left(F_{z_1-1}\right) \otimes \Omega^{(0)}\left(w_{z_1-1}^+\right) + \mathcal{S}\left(F_{\bar{z}_1-1}\right) \otimes \Omega^{(0)}\left(w_{\bar{z}_1-1}^+\right) \\
& + \mathcal{S}\left(F_{-z_2}\right) \otimes \Omega^{(0)}\left(w_{-z_2}^+\right) + \mathcal{S}\left(F_{-\bar{z}_2}\right) \otimes \Omega^{(0)}\left(w_{-\bar{z}_2}^+\right),
\end{aligned} \tag{5.10}
$$

where I_{hex} is the 6D hexagon integral (normalized to be pure), $c_{25} = \frac{\langle 9,10,1,(7,8)\cap(2,3,5)\rangle}{\langle 1,5,9,10\rangle\langle 2,3,7,8\rangle}$ is the 25'th element of the set (5.5) and we used $\Omega^{(0)}(w_\infty^+) = 2\pi i \tau$. In particular, the symbols of all weight-three functions except F_τ, that are I_{hex}, F_-, etc., are *polylogarithmic*; their symbol entries are logarithms of kinematics. Furthermore, I_{hex} and F_- are invariant up to a sign under the two reflections

$$
R_1: \quad p_i \to p_{15-i}, \tag{5.11}
$$

$$
R_2: \quad p_i \to p_{10-i}, \tag{5.12}
$$

where $p_i \equiv p_{i+10}$, while F_{z_1-1}, $F_{\bar{z}_1-1}$, F_{-z_2} and $F_{-\bar{z}_2}$ form an orbit (up to a sign) under both reflections. Besides the seven elliptic last entries, elliptic letters only appear at the third entry of $\mathcal{S}(F_\tau)$ and come in only 13 linear independent combinations of $\Omega^{(2,1,0)}$'s. For the full symbol alphabet and a form that is manifestly invariant under the two reflections, see [33].

---

[18] Here we use a slightly different notation for $\mathcal{S}(2\pi i T_{\boxplus}^{(1)})$ than the one in [33] for bookkeeping.

In the other normalization, $T_{\boxplus}^{(2)} = -I_{\boxplus}/\omega_2$, the structure of the symbol is almost the same as (5.10),

$$\mathcal{S}\left(2\pi i T_{\boxplus}^{(2)}\right) = \mathcal{S}\left(I_{\text{hex}}\right) \otimes \Omega^{(0)}\left(\xi_{c_{25}}^+\right) + \mathcal{S}\left(F_-\right) \otimes \Omega^{(0)}\left(\xi_\infty^-\right) + \mathcal{S}\left(F_{\tau'}\right) \otimes \left(2\pi i \tau'\right)$$
$$+ \mathcal{S}\left(F_{z_1-1}\right) \otimes \Omega^{(0)}\left(\xi_{z_1-1}^+\right) + \mathcal{S}\left(F_{\bar{z}_1-1}\right) \otimes \Omega^{(0)}\left(\xi_{\bar{z}_1-1}^+\right) \tag{5.13}$$
$$+ \mathcal{S}\left(F_{-z_2}\right) \otimes \Omega^{(0)}\left(\xi_{-z_2}^+\right) + \mathcal{S}\left(F_{-\bar{z}_2}\right) \otimes \Omega^{(0)}\left(\xi_{-\bar{z}_2}^+\right).$$

Besides the last entries, which are simple modular S-transformations of the last entries of $\mathcal{S}(2\pi i T_{\boxplus}^{(1)})$, the only difference arises from the third entry of $F_{\tau'}$ which consist of *only* $\Omega^{(2)}$'s. Meanwhile, the $\Omega^{(2)}$'s appearing in the third entry of $\mathcal{S}(F'_\tau)$ also come in 13 linear independent combinations.

It is not surprising that all weight-three functions appearing in $\mathcal{S}(2\pi i T_{\boxplus}^{(1)})$ except F_τ are the same as in $\mathcal{S}(2\pi i T_{\boxplus}^{(2)})$. Actually, for any elliptic multiple polylogarithm I given by an integral of the form (2.7), its two normalizations, $T^{(1)} = I/\omega_1$ and $T^{(2)} = -I/\omega_2$, are related by $T^{(2)} = \tau' T^{(1)}$. With the last entry of $T^{(1)}$ being of elliptic type $\Omega^{(0)}$, i.e.,

$$dT^{(1)} = \sum_i F_i dw_i + F_\tau d\tau, \tag{5.14}$$

where F_i and F_τ are elliptic multiple polylogarithms of lower length, a straightforward computation gives

$$dT^{(2)} = \sum_i F_i d\xi_i + \underbrace{\left(T^{(1)} - \sum_i F_i w_i - \tau F_\tau\right)}_{F_{\tau'}} d\tau'. \tag{5.15}$$

While it is not manifest from (5.15), this particular expression for $F_{\tau'}$ has length three.

Once we evaluate the symbol prime for the third entry of F_τ and F'_τ, a simple connection emerge. Both $\mathcal{S}(2\pi i T_{\boxplus}^{(1)})$ and $\mathcal{S}(2\pi i T_{\boxplus}^{(2)})$ have similar structures as in the case of the sunrise integral:

$$\mathcal{S}(2\pi i T_{\boxplus}^{(1)}) = \sum_{ij} \mathcal{S}(f_i) \otimes \left(\log a_{ij} \otimes \Omega^{(0)}(w_j) + \mathbf{\Omega}_i \otimes (2\pi i \tau)\right), \tag{5.16}$$

with

$$\mathcal{S}'(\mathbf{\Omega}_i) = \Omega^{(0)}(w_j) \otimes' \log a_{ij}, \tag{5.17}$$

and a similar expression for $\mathcal{S}(2\pi i T_{\boxplus}^{(2)})$ with ξ_j and τ' in place of w_j and τ. Here w_j are the six last entries except τ, a_{ij} are some algebraic functions of kinematics, f_i are of the form $\text{Li}_2(1 - x_{ab;cd})$, $\log x_{ab;cd} \log x_{a'b';c'd'}$ or four-mass-box functions, and $\mathbf{\Omega}_i$ are combinations of $\Omega^{(2,1,0)}$ that occur in the last entry of F_τ associated with $\mathcal{S}(f_i)$.

Let us close this subsection by remarking some advantages of the form (5.16). Firstly, the symbol prime is manifestly double-periodic due to the same argument used in section 4.1: under the translation $w_j \to w_j + \tau$,

$$\underbrace{[\Omega^{(-1)} \otimes' \log a_{ij}]}_{\mathcal{S}'(\log a_{ij})} \otimes \Omega^{(0)}(w_j) + \underbrace{[\Omega^{(0)}(w_j) \otimes' \log a_{ij}]}_{\mathcal{S}'(\mathbf{\Omega}_i)} \otimes (2\pi i \tau)$$
$$\to [\Omega^{(-1)} \otimes' \log a_{ij}] \otimes \Omega^{(0)}(w_j + \tau) + [\Omega^{(0)}(w_j + \tau) \otimes' \log a_{ij}] \otimes (2\pi i \tau) \tag{5.18}$$
$$= [\Omega^{(-1)} \otimes' \log a_{ij}] \otimes \Omega^{(0)}(w_j) + [\Omega^{(0)}(w_j) \otimes' \log a_{ij}] \otimes (2\pi i \tau),$$

where we have used (3.36) and $\Omega^{(-1)} = -\Omega^{(0)}(\tau) = -2\pi i\tau$. Secondly, this form makes a part of the integrability conditions manifest and sheds light on the (13 linearly independent) combinations of $\Omega^{(2,1,0)}$ that occur in the last entry of F_τ. The integrability condition requires

$$\partial_{w_j}\mathbf{\Omega}_i + \partial_\tau \log a_{ij} = 0\,, \tag{5.19}$$

which is a consequence of

$$\mathcal{S}'(\mathbf{\Omega}_i) = \sum_j \Omega^{(0)}(w_j) \otimes' \log a_{ij}\,, \tag{5.20}$$

since

$$\mathbf{\Omega}_i \otimes \tau - \sum_j \Omega^{(0)}(w_j) \otimes \log a_{ij} = \mathbf{\Omega}_i \otimes \tau + \sum_j \log a_{ij} \otimes \Omega^{(0)}(w_j) - \mathcal{S}\left(\log a_{ij}\Omega^{(0)}(w_j)\right) \tag{5.21}$$

is an integrable symbol by the definition of the symbol prime. We will discuss the remaining integrability conditions in upcoming work [113].

5.2 Soft limit

In this subsection, we will consider the soft limit $p_{10} \to 0$, in which the elliptic double-box integral remains finite and becomes polylogarithmic:

In terms of momentum twistors, this limit amounts to first setting

$$Z_{10} \to Z_9 + \alpha Z_1 + \epsilon(Z_8 + \beta Z_2) \tag{5.23}$$

with finite α and β, and then taking the limit $\epsilon \to 0$ [114]. In this limit, the seven cross-ratios become

$$u_1 = \frac{x_{1,3}^2 x_{5,8}^2}{x_{1,5}^2 x_{3,8}^2}\,, \qquad u_2 = \frac{x_{3,6}^2 x_{8,1}^2}{x_{3,8}^2 x_{6,1}^2}\,, \qquad v_1 = \frac{x_{1,8}^2 x_{3,5}^2}{x_{1,5}^2 x_{3,8}^2}\,, \qquad v_2 = \frac{x_{3,1}^2 x_{6,8}^2}{x_{3,8}^2 x_{6,1}^2}\,,$$

$$u_3 = 1\,, \qquad u_4 = \frac{x_{1,6}^2 x_{3,5}^2}{x_{1,5}^2 x_{3,6}^2} = \frac{v_1}{u_2}\,, \qquad u_5 = 1\,. \tag{5.24}$$

Correspondingly, the elliptic curve (5.3) degenerates to

$$y^2 = \left(x^2 + (1 - u_1 + u_2)x + u_2 - u_1 u_2 + v_1 v_2\right)^2 = \left((x - r)(x - \bar{r})\right)^2\,, \tag{5.25}$$

where

$$r = -\frac{1}{2}\left(1 - u_1 + u_2 + \sqrt{(1 - u_1 - u_2)^2 - 4v_1 v_2}\right)\,, \tag{5.26}$$

$$\bar{r} = -\frac{1}{2}\left(1 - u_1 + u_2 - \sqrt{(1 - u_1 - u_2)^2 - 4v_1 v_2}\right)\,. \tag{5.27}$$

The double-box integral in this soft limit can be easily integrated to multiple polylogarithms starting from the one-fold integral representation in [33] by using, for instance, HYPERINT [115] or POLYLOGTOOLS [116]. We record both the function and the symbol result in the file `doublebox_soft` in the supplementary material attached to this paper. In the following, we demonstrate how the same result for the symbol is obtained as a limit of the elliptic symbol of the ten-point double-box integral (5.10).

We work in the region given by positive momentum-twistor kinematics [117], where the four roots of $y^2(x)$ in (5.3) come in complex conjugate pairs as shown in figure 2, but both r and $\bar{r}$ are negative real numbers, i.e., $r_1, r_4 \to r$, while $r_2, r_3 \to \bar{r}$ in the soft limit. Therefore, $\omega_2 \to \infty$ in this limit, and

$$\omega_1 \to \int_{\gamma_1} -\frac{\mathrm{d}x}{(x-r)(x-\bar{r})} = \frac{2\pi i}{r - \bar{r}} \tag{5.28}$$

where we made the following choice for the degenerated curve (5.25) in the different regions:

$$y = \begin{cases} -(x-r)(x-\bar{r}), & \text{for} \quad r < \operatorname{Re} x < \bar{r} \\ (x-r)(x-\bar{r}), & \text{otherwise}. \end{cases} \tag{5.29}$$

The reason for this choice is that we want to keep y non-negative on the real x-axis for the degenerated curve, following the convention we chose for the original curve in (5.3). Again, $(2\pi i)^{-1}\omega_1$ gives the correct normalization factor for this nine-point double-box integral, and we should expect that $\mathcal{S}(2\pi i T_{\boxplus}^{(1)})$ will reproduce the correct symbol in this soft limit.

The immediate problem we encounter in taking the soft limit for $\mathcal{S}(2\pi i T_{\boxplus}^{(1)})$ is that not all of the seven last entries in (5.10) have smooth, definite and finite limits. While the soft limit of c_{25} depends on the arbitrary parameters α and β in (5.23), the term $\mathcal{S}(I_{\text{hex}}) \otimes \Omega^{(0)}(w_{c_{25}}^+)$ does not introduce a problem since $\mathcal{S}(I_{\text{hex}})$ vanishes in the soft limit. The integration contours for w_∞^-, $w_{z_1-1}^+$ as well as $w_{-\bar{z}_2}^+$ go through the pole at $x = r$, and the integration contours for $w_{z_1-1}^+$ as well as $w_\infty^+ = \tau$ go through both poles at $x = r$ and $x = \bar{r}$; see figure 4. To cancel the resulting singularities, we reorganize $\mathcal{S}(2\pi i T_{\boxplus}^{(1)})$ as

$$\begin{aligned} \mathcal{S}\left(2\pi i T_{\boxplus}^{(1)}\right) = {} & \mathcal{S}\left(F_{-z_2}\right) \otimes \Omega^{(0)}\left(w_{-z_2}^+\right) + \mathcal{S}\left(F_{z_1-1}\right) \otimes \Omega^{(0)}\left(w_{z_1-1}^+ - w_\infty^+\right) \\ & + \mathcal{S}\left(F_{\bar{z}_1-1}\right) \otimes \Omega^{(0)}\left(w_{\bar{z}_1-1}^+ - w_\infty^-\right) + \mathcal{S}\left(F_{-\bar{z}_2}\right) \otimes \Omega^{(0)}\left(w_{-\bar{z}_2}^+ - w_\infty^-\right) \\ & + \mathcal{S}\left(F_\tau + F_{z_1-1}\right) \otimes (2\pi i \tau) + \mathcal{S}\left(F_- + F_{\bar{z}_1-1} + F_{-\bar{z}_2}\right) \otimes \Omega^{(0)}\left(w_\infty^-\right) \\ & + \mathcal{S}\left(I_{\text{hex}}\right) \otimes \Omega^{(0)}\left(w_{c_{25}}^+\right), \end{aligned} \tag{5.30}$$

cf. figure 4. One can easily check that not only the last term but the last three terms do not contribute in the soft limit since the three weight-three symbols making up their first three entries vanish in the soft limit. The first four terms yield the correct polylogarithmic symbol in the soft limit with last entries

$$\Omega^{(0)}\left(w_{-z_2}^+\right) \to (r - \bar{r}) \int_{-\infty}^{-z_2} \frac{\mathrm{d}x}{(x-r)(x-\bar{r})} = \log\frac{r + z_2}{\bar{r} + z_2},$$

$$\Omega^{(0)}\left(w_{z_1-1}^+ - w_\infty^+\right) \to (r - \bar{r}) \int_{+\infty}^{z_1-1} \frac{\mathrm{d}x}{(x-r)(x-\bar{r})} = \log\frac{1 + r - z_1}{1 + \bar{r} - z_1},$$
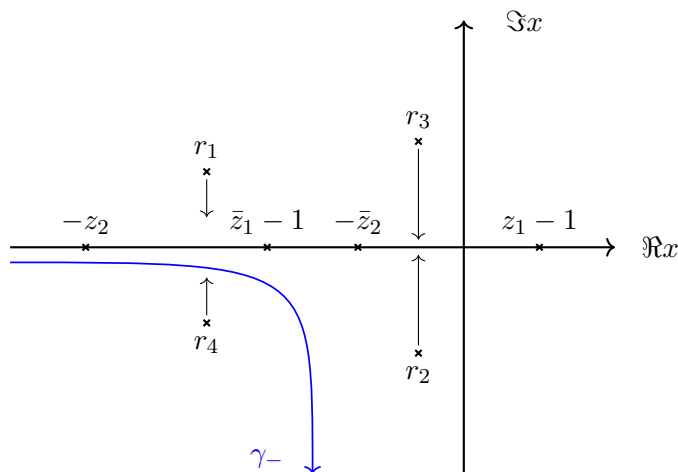
Figure 4. In the soft limit $p_{10} \to 0$, the roots of $y^2(x)$ pairwise pinch the integration contours for $w^+_{\bar{z}_1-1}, w^+_{-\bar{z}_2}$ and $w^+_{z_1-1}$, which run along the real axis. By subtracting w^-_∞ and w^+_∞, respectively, we obtain integration contours that can be deformed such they are not pinched, thus resulting in finite integrals in the soft limit.

$$\Omega^{(0)} \left(w^+_{\bar{z}_1-1} - w^-_\infty \right) \to (r - \bar{r}) \int_{-i\infty}^{\bar{z}_1-1} \frac{-\mathrm{d}x}{(x-r)(x-\bar{r})} = \log \frac{1 + \bar{r} - \bar{z}_1}{1 + r - \bar{z}_1} \,,$$

$$\Omega^{(0)} \left(w^+_{-\bar{z}_2} - w^-_\infty \right) \to (r - \bar{r}) \int_{-i\infty}^{-\bar{z}_2} \frac{-\mathrm{d}x}{(x-r)(x-\bar{r})} = \log \frac{\bar{r} + \bar{z}_2}{r + \bar{z}_2} \,. \tag{5.31}$$

Note that in the soft limit $z_1 \equiv z_{1,3,5,8}$ and $z_2 \equiv z_{3,6,8,1}$ and the reflection symmetry R_1 is broken while R_2 survives; thus, the symbol for this nine-point double-box integrals can be expressed as

$$\mathcal{S}\left((r - \bar{r}) I_{\boxplus}\right) = \mathcal{S}\left(F_{-z_2}|_{p_{10}\to 0}\right) \otimes \log \frac{r + z_2}{\bar{r} + z_2} + \mathcal{S}\left(F_{-\bar{z}_2}|_{p_{10}\to 0}\right) \otimes \log \frac{\bar{r} + \bar{z}_2}{r + \bar{z}_2}$$

$$+ \text{(images under } R_2) \,, \tag{5.32}$$

where R_2 acts on the last entries via $r \leftrightarrow -(1+\bar{r})$, $z_1 \leftrightarrow z_2$ and $\bar{z}_1 \leftrightarrow \bar{z}_2$. Furthermore, $F_{-z_2}|_{p_{10}\to 0}$ and $F_{-\bar{z}_2}|_{p_{10}\to 0}$ are related by exchanging z_2 and $\bar{z}_2$, same as the corresponding last entries. The reason is that z and $\bar{z}$ occur symmetrically in their definition $\{z\bar{z} = u, (1-z)(1-\bar{z}) = v\}$, and thus have to occur symmetrically in the symbol as well.

The symbol alphabet of the nine-point double-box integral consists of 10 rational letters and 11 algebraic letters:

1. Rational letters:

$$u_1, \ u_2, \ v_1, \ v_2, \ u_1 - v_2, \ v_1 - u_2, \ u_1 u_2 - v_1 v_2, \ \Delta_1, \ \Delta_2,$$
$$\frac{\langle 5(91)(23)(78)\rangle \langle \bar{5}(91)(23)(78)\rangle \langle 1239\rangle \langle 1789\rangle}{\langle 1459\rangle^2 \langle 1569\rangle^2 \langle 2378\rangle^3} \,, \tag{5.33}$$

where we introduced the following notations:[19]

$$\langle a(bc)(de)(fg) \rangle = \langle abde \rangle \langle acfg \rangle - \langle acde \rangle \langle abfg \rangle, \qquad (5.34)$$

$$\langle \bar{a}(i\,i{+}1)(j\,j{+}1)(k\,k{+}1) \rangle = \langle (i\,i{+}1) \cap (\bar{a})\, j\,j{+}1\,(k,k{+}1) \cap (\bar{a}) \rangle \qquad (5.35)$$

and

$$\Delta_i = (1 - u_i - v_i)^2 - 4u_iv_i = (z_i - \bar{z}_i)^2, \qquad i = 1, 2. \qquad (5.36)$$

2. Algebraic letters:

- $\dfrac{z_1}{\bar{z}_1}, \dfrac{1-z_1}{1-\bar{z}_1}, \dfrac{1+\bar{r}-z_1}{1+r-z_1}, \dfrac{1+\bar{r}-\bar{z}_1}{1+r-\bar{z}_1}, \dfrac{\frac{\langle 5(23)(46)(78) \rangle \langle 1239 \rangle}{\langle 5(19)(23)(78) \rangle \langle 2378 \rangle} - z_1}{\frac{\langle 5(23)(46)(78) \rangle \langle 1239 \rangle}{\langle 5(19)(23)(78) \rangle \langle 2378 \rangle} - \bar{z}_1},$

  and five others generated by the reflection R_2.

- $\dfrac{(z_1 - 1 + \bar{z}_2)(\bar{z}_1 - 1 + z_2)}{(z_1 - 1 + z_2)(\bar{z}_1 - 1 + \bar{z}_2)}.$

We find that there are three different square roots in this alphabet; two of them are of four-mass-box type and the other, that is the square root in r and $\bar{r}$, arises from the leading singularity of the whole Feynman diagram. Furthermore, the new type of square root *only* appears in the last entries. The symbol alphabet is organized such that the symbol is manifestly invariant (up to a sign) under the reflection R_2, as well as under each of the three transformations $z_1 \leftrightarrow \bar{z}_1$, $z_2 \leftrightarrow \bar{z}_2$, and $r \leftrightarrow \bar{r}$. For an analysis of these letters through Schubert problems, see [118].

6 Discussion and outlook

In this paper, we have investigated various techniques for manipulating and simplifying the symbol of Feynman integrals that evaluate to elliptic multiple polylogarithms. In particular, we studied identities between the elliptic symbol letters $\Omega^{(i)}$.

In contrast to ordinary multiple polylogarithms, the length of an elliptic multiple polylogarithm is not necessarily equal to its weight. A symbol letter $\Omega^{(i)}$, whose length is by definition one, can have weight $i \neq 1$. Identities for $\Omega^{(0)}$ follow from the well-known group law on the elliptic curve. Moreover, we found that various identities for $\Omega^{(1)}$ can be derived from Abel's theorem, which generalize the identity $\log(a) + \log(b) = \log(ab)$ in the polylogarithmic case. The higher-weight letters $\Omega^{(2)}$ satisfy significantly more intricate identities, closer to those of $\mathrm{Li}_2(a)$ than those of $\log(a)$, which are harder to exploit in a direct fashion. We have thus introduced the *symbol prime* $\mathcal{S}'$ for elliptic symbol letters $\Omega^{(2)}$, which plays the same role the symbol $\mathcal{S}$ plays for $\mathrm{Li}_2(a)$. We also introduced a symbol prime for $\Omega^{(i>2)}$ but leave its exploration for future work.

We studied two concrete examples at two-loop order, namely the sunrise integral in two dimensions and the ten-point double-box integral in four dimensions. In particular, we provided proofs for the identities between symbol letters numerically found in [33].

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[19]Here we use $(\bar{a}) \equiv Z_{a-1} \wedge Z_a \wedge Z_{a+1}$ to denote the dual plane of Z_a. Then a vanishing $\langle \bar{a}(i\,i{+}1)(j\,j{+}1)(k\,k{+}1) \rangle$ means that the three intersection points $(i\,i{+}1) \cap \bar{a}$, $(j\,j{+}1) \cap \bar{a}$ and $(k\,k{+}1) \cap \bar{a}$ are on the same line, which is the dual picture of the vanishing of $\langle a(i\,i{+}1)(j\,j{+}1)(k\,k{+}1) \rangle$. We are grateful to Cristian Vergu for pointing this out.

In addition to identities between symbol letters, we also studied how the symbol behaves under kinematic limits in which the elliptic curve degenerates. We recovered the known symbols of the sunrise integral in the null-momentum limit $p^2 \to 0$ and the pseudo threshold $p^2 \to (m_1 + m_2 - m_3)^2$ [10], as well as the nine-point limit of the double box, which has not previously appeared in the literature.

The numeric values of elliptic Feynman integrals are of course independent of whether we rescale the torus by the period ω_1 or $-\omega_2$; the corresponding modular parameters $\tau = \omega_2/\omega_1$ and $\tau' = -1/\tau$ are related by a modular S-transformation. In particular, for an elliptic integral of the form $\int \mathcal{G}(x,y)\,\mathrm{d}x/y$ as in (2.7), its two normalizations $T^{(1)}$ and $T^{(2)}$, which are obtained by dividing by ω_1 and $-\omega_2$ respectively, are simply related by $T^{(2)} = \tau' T^{(1)}$. However, this property is not manifest when expressed in terms of elliptic multiple polylogarithms or their symbols. Instead, we found that the application of the symbol prime to the two examples in this paper yields symbols of the form

$$\sum_{ij} \mathcal{S}(f_i) \otimes \Big(\log a_{ij} \otimes \Omega^{(0)}(w_j) + \mathbf{\Omega}_i \otimes (2\pi i \tau) \Big), \qquad (6.1)$$

with

$$\mathcal{S}'(\mathbf{\Omega}_i) = \Omega^{(0)}(w_j) \otimes' \log a_{ij}. \qquad (6.2)$$

Not only the modular covariance is manifest in this form, but also the double-periodic invariance and the integrability conditions involving τ. However, one could *not* expect that the application of the symbol prime to the one-fold integral of general polylogarithms, of the form $\int \mathcal{G}(x,y)\,\mathrm{d}x/y$, yields such a structure. As a simple counter-example, consider the symbol of the integral $\int_0^c \log(x+a)\,\mathrm{d}x/y$, with arbitrary values of a and c as well as the elliptic curve given by (5.3), which does *not* follow the above structure after applying the symbol prime map. It would be very interesting to investigate why the two elliptic Feynman integrals we considered in this paper turn out to exhibit such a structure after applying the symbol prime map, and to study whether this property extends to further Feynman integrals.

Already the polylogarithmic symbol has a kernel, which is given by $i\pi$, multiple zeta values (MZVs) and their products with MPLs. Similarly, also the symbol prime has a kernel. As discussed in section 3.2, all functions in the kernel necessarily depend only on the modular parameter τ, but not all functions that depend only on τ are in the kernel. To see that the kernel can be non-trivial, consider the symbol of the *equal-mass* sunrise integral, which is an iterated integral of modular forms and *only* depends on τ:

$$\mathcal{S}\left(2\pi i T_{\ominus}^{(1)}\right) = \left[\frac{1}{2\pi i} \left(2\mathcal{E}_4 \left(\begin{smallmatrix} -2 \\ -1 \end{smallmatrix} ; \infty \right) - \mathcal{E}_4 \left(\begin{smallmatrix} -2 \\ 0 \end{smallmatrix} ; \infty \right) - \mathcal{E}_4 \left(\begin{smallmatrix} -2 \\ \infty \end{smallmatrix} ; \infty \right) \right) \right] \otimes (2\pi i \tau), \qquad (6.3)$$

where the $\mathcal{E}_4$ are specific combinations of $\Omega^{(2)}$'s given in (4.11). The application of the symbol prime map to the first entry in (6.3) yields 0. We leave a comprehensive treatment of the kernel of the symbol prime map to a future study.

Another interesting problem is to lift simplified symbols to simplified functions for elliptic multiple polylogarithms. As a primary example, let us consider how to lift the

simplified symbol (4.10) to a simplified function for the sunrise integral. By rewriting the logarithms in (4.10) in terms of $\Omega^{(1)}$'s, we find a (slightly) simpler expression for $T_{\oplus}^{(1)}$,

$$
\begin{aligned}
T_{\oplus}^{(1)} = {} & 2\mathcal{E}_4 \left(\begin{smallmatrix} 0 & -1 \\ 0 & -1 \end{smallmatrix} ; \infty | \tau \right) - \mathcal{E}_4 \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix} ; \infty | \tau \right) - \mathcal{E}_4 \left(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix} ; \infty | \tau \right) \\
& - \left(2 \log \frac{t_2}{t_3} + \mathcal{E}_4 \left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix} ; \infty \right) \right) \mathcal{E}_4 \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} ; \infty | \tau \right) ,
\end{aligned}
\tag{6.4}
$$

(and a similar expression for $T_{\oplus}^{(2)}$), in which the $\mathcal{E}_4$ functions only contain $c = 0, -1, \infty$ but not the fourth argument r that occurs in (4.2). However, the general prescription of uplifting more complicated elliptic symbols to functions is still underexplored.[20]

Although the simplified symbols of the elliptic Feynman integrals manifest some desired properties, such as double periodicity and modular covariance, after applying the symbol prime map, the singularity structures are not completely manifest. For example, the sunrise integral becomes singular at $m_i = 0$ as well as at the threshold $p^2 = (m_1 + m_2 + m_3)^2$, as can be seen through a Landau analysis [120]. One can see the branch cut at $m_i = 0$ explicitly from eq. (4.10) or (4.12); however, the branch cut at the threshold is not manifest from the symbol. In general, the logarithmic letter $\log(a)$ has a logarithmic singularity if $a = 0$ or $a = \infty$. In contrast, the elliptic letter $\Omega^{(1)}(w)$ has a logarithmic singularity at all lattice points, while $\Omega^{(j \geq 2)}(w)$ has a logarithmic singularity at all lattice points except for the origin. However, w is a function of the kinematics; typically, $w = w_c^+ = 1/\omega_1 \int_{-\infty}^{c} \mathrm{d}x/y$, where c is an algebraic function of the kinematics. If the configuration of roots in y does not change as we vary c, $w_c^+ = 0$ if $c = -\infty$ and $w_c^+ = \tau$ if $c = +\infty$. However, the configuration of roots may also vary as we vary c; we leave a comprehensive analysis to a future study.

It would be interesting to apply the techniques used in this paper to bootstrap the symbol of scattering amplitudes or Feynman integrals that can be expressed in terms of elliptic multiple polylogarithms, such as the twelve-point elliptic double box. On top of the integrability condition for the final entry τ, which is made manifest by the symbol prime, this requires understanding the integrability condition for the other last entries [113]. Moreover, it requires an educated guess for the alphabet of symbol letters that occur in them. For six- and seven-point amplitudes in $\mathcal{N} = 4$ sYM theory, the symbol alphabet was shown to be given by cluster algebras [80, 91, 121–123], and similar techniques have recently been extended to Feynman integrals and amplitudes [124–127] containing symbol letters that are given by logarithms of algebraic functions of the kinematics [87, 95, 128–134]. It would be interesting to extend these techniques also to the elliptic case.

Acknowledgments

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[20]In particular, there are symbols that can be written in terms of only logarithms as letters that can nevertheless not be lifted to polylogarithmic functions, only to elliptic ones [119].

A   Calculation of the unequal-mass sunrise integral

In this appendix, we calculate the unequal-mass sunrise integral in two dimensions in terms of eMPLs. This integral was originally calculated in terms of iterated integrals on the moduli space $\overline{\mathcal{M}}_{1,3}$ in [28]. We use a Feynman-parameter approach, closely following the calculation of [20] in the equal-mass case.

The Feynman parameter representation for the unequal-mass sunrise in two dimensions is (see e.g. [20])

$$I_{\ominus} = \int_0^\infty \frac{\mathrm{d}x_1 \, \mathrm{d}x_2 \quad \delta(x_3 - 1)}{-p^2 \, x_1 x_2 x_3 + \left(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3\right)\left(x_1 x_2 + x_1 x_3 + x_2 x_3\right)} \,. \tag{A.1}$$

Furthermore, we introduce $t_i^2 = m_i^2 / p^2$ and integrate out x_2. This gives

$$I_{\ominus} = \frac{1}{m_1^2} \int \frac{\mathrm{d}x}{y} \log(R_{\ominus}(x,y)), \tag{A.2}$$

where we have denoted x_1 by x and $R_{\ominus}(x,y)$ is a rational function of x and y,

$$R_{\ominus}(x,y) = \frac{t_3^2 + x\left(t_1^2 + t_2^2 + t_3^2 - 1\right) + t_1^2 x^2 + t_1^2 y}{t_3^2 + x\left(t_1^2 + t_2^2 + t_3^2 - 1\right) + t_1^2 x^2 - t_1^2 y} \tag{A.3}$$

with the elliptic curve defined by

$$\begin{aligned}
\left(t_1^2 y\right)^2 &= \left(t_1^2 x^2 + \left(t_1^2 + t_2^2 + t_3^2 - 1\right)x + t_3^2\right)^2 - 4t_2^2\left(1 + x\right)\left(t_1^2 x + t_3^2\right)x \tag{A.4} \\
&= \left(t_1^2 x^2 + \left(t_1^2 + t_3^2 - (t_2 - 1)^2\right)x + t_3^2\right)\left(t_1^2 x^2 + \left(t_1^2 + t_3^2 - (t_2 + 1)^2\right)x + t_3^2\right).
\end{aligned}$$

It is straightforward to rewrite the logarithm in (A.2) in terms of E_4 functions by expanding $\partial_x \log R_{\ominus}(x,y)$ on the ψ-basis (2.5) and integrating up again:[21]

$$\begin{aligned}
\log R_{\ominus}(x,y) = {}&-\mathrm{E}_4\left(\begin{smallmatrix}-1\\0\end{smallmatrix};x\right) + \mathrm{E}_4\left(\begin{smallmatrix}-1\\\infty\end{smallmatrix};x\right) + \mathrm{E}_4\left(\begin{smallmatrix}-1\\-1\end{smallmatrix};x\right) + \mathrm{E}_4\left(\begin{smallmatrix}-1\\r\end{smallmatrix};x\right) \\
&+ \frac{1 - t_2^2}{t_1^2}\mathrm{E}_4\left(\begin{smallmatrix}0\\0\end{smallmatrix};x\right) - \log\frac{t_2^2}{t_3^2}, \tag{A.5}
\end{aligned}$$

where $r = -t_3^2 / t_1^2$.

In the conventions introduced in section 2, we have two rescaled tori. By using (2.16) and (2.18), we find

$$\begin{aligned}
\log R_{\ominus}(x,y) = {}&\mathcal{E}_4\left(\begin{smallmatrix}-1\\-1\end{smallmatrix};x|\tau\right) - \mathcal{E}_4\left(\begin{smallmatrix}-1\\0\end{smallmatrix};x|\tau\right) + \mathcal{E}_4\left(\begin{smallmatrix}-1\\r\end{smallmatrix};x|\tau\right) \\
&- \mathcal{E}_4\left(\begin{smallmatrix}-1\\\infty\end{smallmatrix};x|\tau\right) + 4\pi i \mathcal{E}_4\left(\begin{smallmatrix}0\\0\end{smallmatrix};x|\tau\right) - \log\frac{t_2^2}{t_3^2} \tag{A.6}
\end{aligned}$$

---

[21]To fix the integration constant, we can consider the difference of the left and the right hand side in the limit $x \to 0$.

on the torus $[1 : \tau = \omega_2/\omega_1]$ with coordinates w, while, by using the analog of (2.16) and (2.18),

$$
\begin{aligned}
\log R_{\ominus}(x,y) = \mathcal{E}_4 \left({-1 \atop -1}; x|\tau' \right) - \mathcal{E}_4 \left({-1 \atop 0}; x|\tau' \right) + \mathcal{E}_4 \left({-1 \atop r}; x|\tau' \right) \\
- \mathcal{E}_4 \left({-1 \atop \infty}; x|\tau' \right) - \log \frac{t_2^2}{t_3^2}
\end{aligned}
\tag{A.7}
$$

on the torus $[1 : \tau' = -\omega_1/\omega_2]$ with coordinates ξ. The equality of (A.6) and (A.7) is given by the S-transformation of $g^{(1)}$,

$$
g^{(1)}(w|\tau) = \tau' g^{(1)}(\xi|\tau') + 2\pi i \xi \,,
\tag{A.8}
$$

and the following identities on the (rescaled) tori,

$$
z_{-1}^+ - z_0^+ + z_r^+ = 0 \,, \quad w_\infty^+ = \tau \,, \quad \xi_\infty^+ = -1 \,.
\tag{A.9}
$$

Furthermore, there are three independent variables $\{w_{-1}^+, w_0^+, \tau\}$ and $\{\xi_{-1}^+, \xi_0^+, \tau'\}$ on each torus, respectively, since (3.22) here gives

$$
z_\infty^- = z_0^+ - \frac{1}{2}\omega_1 \,.
\tag{A.10}
$$

Now the integration in (A.2) can be easily performed and gives the result (4.1)–(4.4) in the main text.

References

[1] K.-T. Chen, *Iterated path integrals*, *Bull. Am. Math. Soc.* **83** (1977) 831 [INSPIRE].

[2] A. Goncharov, *Geometry of configurations, polylogarithms, and motivic cohomology*, *Advances in Mathematics* **114** (1995) 197.

[3] A.B. Goncharov, *Multiple polylogarithms, cyclotomy and modular complexes*, *Math. Res. Lett.* **5** (1998) 497 [arXiv:1105.2076] [INSPIRE].

[4] E. Remiddi and J.A.M. Vermaseren, *Harmonic polylogarithms*, *Int. J. Mod. Phys. A* **15** (2000) 725 [hep-ph/9905237] [INSPIRE].

[5] J.M. Borwein, D.M. Bradley, D.J. Broadhurst and P. Lisonek, *Special values of multiple polylogarithms*, *Trans. Am. Math. Soc.* **353** (2001) 907 [math/9910045] [INSPIRE].

[6] S. Moch, P. Uwer and S. Weinzierl, *Nested sums, expansion of transcendental functions and multiscale multiloop integrals*, *J. Math. Phys.* **43** (2002) 3363 [hep-ph/0110083] [INSPIRE].

[7] S. Laporta and E. Remiddi, *Analytic treatment of the two loop equal mass sunrise graph*, *Nucl. Phys. B* **704** (2005) 349 [hep-ph/0406160] [INSPIRE].

[8] S. Müller-Stach, S. Weinzierl and R. Zayadeh, *From motives to differential equations for loop integrals*, *PoS* **LL2012** (2012) 005 [arXiv:1209.3714] [INSPIRE].

[9] F.C.S. Brown and A. Levin, *Multiple elliptic polylogarithms*, `arXiv:1110.6917`.

[10] S. Bloch and P. Vanhove, *The elliptic dilogarithm for the sunset graph*, *J. Number Theor.* **148** (2015) 328 [`arXiv:1309.5865`] [ɪɴSPIRE].

[11] L. Adams, C. Bogner and S. Weinzierl, *The two-loop sunrise graph with arbitrary masses*, *J. Math. Phys.* **54** (2013) 052303 [`arXiv:1302.7004`] [ɪɴSPIRE].

[12] L. Adams, C. Bogner and S. Weinzierl, *The two-loop sunrise graph in two space-time dimensions with arbitrary masses in terms of elliptic dilogarithms*, *J. Math. Phys.* **55** (2014) 102301 [`arXiv:1405.5640`] [ɪɴSPIRE].

[13] L. Adams, C. Bogner and S. Weinzierl, *The two-loop sunrise integral around four space-time dimensions and generalisations of the Clausen and Glaisher functions towards the elliptic case*, *J. Math. Phys.* **56** (2015) 072303 [`arXiv:1504.03255`] [ɪɴSPIRE].

[14] L. Adams, C. Bogner and S. Weinzierl, *The iterated structure of the all-order result for the two-loop sunrise integral*, *J. Math. Phys.* **57** (2016) 032304 [`arXiv:1512.05630`] [ɪɴSPIRE].

[15] L. Adams, C. Bogner, A. Schweitzer and S. Weinzierl, *The kite integral to all orders in terms of elliptic polylogarithms*, *J. Math. Phys.* **57** (2016) 122302 [`arXiv:1607.01571`] [ɪɴSPIRE].

[16] L. Adams and S. Weinzierl, *Feynman integrals and iterated integrals of modular forms*, *Commun. Num. Theor. Phys.* **12** (2018) 193 [`arXiv:1704.08895`] [ɪɴSPIRE].

[17] L. Adams, E. Chaubey and S. Weinzierl, *Simplifying Differential Equations for Multiscale Feynman Integrals beyond Multiple Polylogarithms*, *Phys. Rev. Lett.* **118** (2017) 141602 [`arXiv:1702.04279`] [ɪɴSPIRE].

[18] C. Bogner, A. Schweitzer and S. Weinzierl, *Analytic continuation and numerical evaluation of the kite integral and the equal mass sunrise integral*, *Nucl. Phys. B* **922** (2017) 528 [`arXiv:1705.08952`] [ɪɴSPIRE].

[19] J. Broedel, C. Duhr, F. Dulat and L. Tancredi, *Elliptic polylogarithms and iterated integrals on elliptic curves. Part I: general formalism*, *JHEP* **05** (2018) 093 [`arXiv:1712.07089`] [ɪɴSPIRE].

[20] J. Broedel, C. Duhr, F. Dulat and L. Tancredi, *Elliptic polylogarithms and iterated integrals on elliptic curves. II. An application to the sunrise integral*, *Phys. Rev. D* **97** (2018) 116009 [`arXiv:1712.07095`] [ɪɴSPIRE].

[21] E. Remiddi and L. Tancredi, *An Elliptic Generalization of Multiple Polylogarithms*, *Nucl. Phys. B* **925** (2017) 212 [`arXiv:1709.03622`] [ɪɴSPIRE].

[22] L.-B. Chen, J. Jiang and C.-F. Qiao, *Two-Loop integrals for CP-even heavy quarkonium production and decays: Elliptic Sectors*, *JHEP* **04** (2018) 080 [`arXiv:1712.03516`] [ɪɴSPIRE].

[23] J.L. Bourjaily, A.J. McLeod, M. Spradlin, M. von Hippel and M. Wilhelm, *Elliptic Double-Box Integrals: Massless Scattering Amplitudes beyond Polylogarithms*, *Phys. Rev. Lett.* **120** (2018) 121603 [`arXiv:1712.02785`] [ɪɴSPIRE].

[24] L. Adams and S. Weinzierl, *The ε-form of the differential equations for Feynman integrals in the elliptic case*, *Phys. Lett. B* **781** (2018) 270 [`arXiv:1802.05020`] [ɪɴSPIRE].

[25] J. Broedel, C. Duhr, F. Dulat, B. Penante and L. Tancredi, *Elliptic symbol calculus: from elliptic polylogarithms to iterated integrals of Eisenstein series*, *JHEP* **08** (2018) 014 [`arXiv:1803.10256`] [ɪɴSPIRE].

[26] J. Broedel, C. Duhr, F. Dulat, B. Penante and L. Tancredi, *Elliptic Feynman integrals and pure functions*, *JHEP* **01** (2019) 023 [arXiv:1809.10698] [INSPIRE].

[27] I. Hönemann, K. Tempest and S. Weinzierl, *Electron self-energy in QED at two loops revisited*, *Phys. Rev. D* **98** (2018) 113008 [arXiv:1811.09308] [INSPIRE].

[28] C. Bogner, S. Müller-Stach and S. Weinzierl, *The unequal mass sunrise integral expressed through iterated integrals on* $\overline{\mathcal{M}}_{1,3}$, *Nucl. Phys. B* **954** (2020) 114991 [arXiv:1907.01251] [INSPIRE].

[29] J. Broedel, C. Duhr, F. Dulat, B. Penante and L. Tancredi, *Elliptic polylogarithms and Feynman parameter integrals*, *JHEP* **05** (2019) 120 [arXiv:1902.09971] [INSPIRE].

[30] C. Duhr and L. Tancredi, *Algorithms and tools for iterated Eisenstein integrals*, *JHEP* **02** (2020) 105 [arXiv:1912.00077] [INSPIRE].

[31] M. Walden and S. Weinzierl, *Numerical evaluation of iterated integrals related to elliptic Feynman integrals*, *Comput. Phys. Commun.* **265** (2021) 108020 [arXiv:2010.05271] [INSPIRE].

[32] S. Weinzierl, *Modular transformations of elliptic Feynman integrals*, *Nucl. Phys. B* **964** (2021) 115309 [arXiv:2011.07311] [INSPIRE].

[33] A. Kristensson, M. Wilhelm and C. Zhang, *Elliptic Double Box and Symbology Beyond Polylogarithms*, *Phys. Rev. Lett.* **127** (2021) 251603 [arXiv:2106.14902] [INSPIRE].

[34] H. Frellesvig, *On epsilon factorized differential equations for elliptic Feynman integrals*, *JHEP* **03** (2022) 079 [arXiv:2110.07968] [INSPIRE].

[35] A. Sabry, *Fourth order spectral functions for the electron propagator*, *Nuclear Physics* **33** (1962) 401.

[36] D.J. Broadhurst, J. Fleischer and O.V. Tarasov, *Two loop two point functions with masses: Asymptotic expansions and Taylor series, in any dimension*, *Z. Phys. C* **60** (1993) 287 [hep-ph/9304303] [INSPIRE].

[37] S. Müller-Stach, S. Weinzierl and R. Zayadeh, *A Second-Order Differential Equation for the Two-Loop Sunrise Graph with Arbitrary Masses*, *Commun. Num. Theor. Phys.* **6** (2012) 203 [arXiv:1112.4360] [INSPIRE].

[38] E. Remiddi and L. Tancredi, *Schouten identities for Feynman graph amplitudes; The Master Integrals for the two-loop massive sunrise graph*, *Nucl. Phys. B* **880** (2014) 343 [arXiv:1311.3342] [INSPIRE].

[39] S. Bloch, M. Kerr and P. Vanhove, *A Feynman integral via higher normal functions*, *Compos. Math.* **151** (2015) 2329 [arXiv:1406.2664] [INSPIRE].

[40] S. Bloch, M. Kerr and P. Vanhove, *Local mirror symmetry and the sunset Feynman integral*, *Adv. Theor. Math. Phys.* **21** (2017) 1373 [arXiv:1601.08181] [INSPIRE].

[41] E. Remiddi and L. Tancredi, *Differential equations and dispersion relations for Feynman amplitudes. The two-loop massive sunrise and the kite integral*, *Nucl. Phys. B* **907** (2016) 400 [arXiv:1602.01481] [INSPIRE].

[42] F.C.S. Brown, *On the periods of some Feynman integrals*, arXiv:0910.0114 [INSPIRE].

[43] F. Brown and O. Schnetz, *A K3 in* ϕ^4, *Duke Math. J.* **161** (2012) 1817 [arXiv:1006.4064] [INSPIRE].

[44] J.L. Bourjaily, A.J. McLeod, M. von Hippel and M. Wilhelm, *Bounded Collection of Feynman Integral Calabi-Yau Geometries*, *Phys. Rev. Lett.* **122** (2019) 031601 [arXiv:1810.07689] [InSPIRE].

[45] J.L. Bourjaily, Y.-H. He, A.J. Mcleod, M. Von Hippel and M. Wilhelm, *Traintracks through Calabi-Yau Manifolds: Scattering Amplitudes beyond Elliptic Polylogarithms*, *Phys. Rev. Lett.* **121** (2018) 071603 [arXiv:1805.09326] [InSPIRE].

[46] D. Festi and D. van Straten, *Bhabha Scattering and a special pencil of K3 surfaces*, *Commun. Num. Theor. Phys.* **13** (2019) 463 [arXiv:1809.04970] [InSPIRE].

[47] J. Broedel, C. Duhr, F. Dulat, R. Marzucca, B. Penante and L. Tancredi, *An analytic solution for the equal-mass banana graph*, *JHEP* **09** (2019) 112 [arXiv:1907.03787] [InSPIRE].

[48] M. Besier, D. Festi, M. Harrison and B. Naskręcki, *Arithmetic and geometry of a K3 surface emerging from virtual corrections to Drell-Yan scattering*, *Commun. Num. Theor. Phys.* **14** (2020) 863 [arXiv:1908.01079] [InSPIRE].

[49] J.L. Bourjaily, A.J. McLeod, C. Vergu, M. Volk, M. Von Hippel and M. Wilhelm, *Embedding Feynman Integral (Calabi-Yau) Geometries in Weighted Projective Space*, *JHEP* **01** (2020) 078 [arXiv:1910.01534] [InSPIRE].

[50] C. Vergu and M. Volk, *Traintrack Calabi-Yaus from Twistor Geometry*, *JHEP* **07** (2020) 160 [arXiv:2005.08771] [InSPIRE].

[51] C.F. Doran, A.Y. Novoseltsev and P. Vanhove, *Mirroring Towers: Calabi-Yau Geometry of the Multiloop Feynman Sunset Integrals*, to appear.

[52] D.J. Broadhurst, *The Master Two Loop Diagram With Masses*, *Z. Phys. C* **47** (1990) 115 [InSPIRE].

[53] L. Adams, E. Chaubey and S. Weinzierl, *Analytic results for the planar double box integral relevant to top-pair production with a closed top loop*, *JHEP* **10** (2018) 206 [arXiv:1806.04981] [InSPIRE].

[54] L. Adams, E. Chaubey and S. Weinzierl, *Planar Double Box Integral for Top Pair Production with a Closed Top Loop to all orders in the Dimensional Regularization Parameter*, *Phys. Rev. Lett.* **121** (2018) 142001 [arXiv:1804.11144] [InSPIRE].

[55] R. Huang and Y. Zhang, *On Genera of Curves from High-loop Generalized Unitarity Cuts*, *JHEP* **04** (2013) 080 [arXiv:1302.1023] [InSPIRE].

[56] A. Klemm, C. Nega and R. Safari, *The l-loop Banana Amplitude from GKZ Systems and relative Calabi-Yau Periods*, *JHEP* **04** (2020) 088 [arXiv:1912.06201] [InSPIRE].

[57] K. Bönisch, F. Fischbach, A. Klemm, C. Nega and R. Safari, *Analytic structure of all loop banana integrals*, *JHEP* **05** (2021) 066 [arXiv:2008.10574] [InSPIRE].

[58] K. Bönisch, C. Duhr, F. Fischbach, A. Klemm and C. Nega, *Feynman integrals in dimensional regularization and extensions of Calabi-Yau motives*, *JHEP* **09** (2022) 156 [arXiv:2108.05310] [InSPIRE].

[59] H. Müller and S. Weinzierl, *A Feynman integral depending on two elliptic curves*, *JHEP* **07** (2022) 101 [arXiv:2205.04818] [InSPIRE].

[60] E. Chaubey, M. Kaur and A. Shivaji, *Master integrals for $\mathcal{O}(\alpha\alpha_s)$ corrections to $H \to ZZ^*$*, *JHEP* **10** (2022) 056 [arXiv:2205.06339] [InSPIRE].

[61] J.L. Bourjaily et al., *Functions Beyond Multiple Polylogarithms for Precision Collider Physics*, in *2022 Snowmass Summer Study* Seattle U.S.A., July 17–26 2022 [arXiv:2203.07088] [InSPIRE].

[62] A.B. Goncharov, *Galois symmetries of fundamental groupoids and noncommutative geometry*, *Duke Math. J.* **128** (2005) 209 [math/0208144] [InSPIRE].

[63] A.B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Classical Polylogarithms for Amplitudes and Wilson Loops*, *Phys. Rev. Lett.* **105** (2010) 151605 [arXiv:1006.5703] [InSPIRE].

[64] F. Brown, *On the decomposition of motivic multiple zeta values*, arXiv:1102.1310 [InSPIRE].

[65] C. Duhr, H. Gangl and J.R. Rhodes, *From polygons and symbols to polylogarithmic functions*, *JHEP* **10** (2012) 075 [arXiv:1110.0458] [InSPIRE].

[66] C. Duhr, *Hopf algebras, coproducts and symbols: an application to Higgs boson amplitudes*, *JHEP* **08** (2012) 043 [arXiv:1203.0454] [InSPIRE].

[67] D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, *Pulling the straps of polygons*, *JHEP* **12** (2011) 011 [arXiv:1102.0062] [InSPIRE].

[68] O. Steinmann, *Über den Zusammenhang zwischen den Wightmanfunktionen und der retardierten Kommutatoren*, *Helv. Phys. Acta* **33** (1960) 257.

[69] O. Steinmann, *Wightman-Funktionen und retardierten Kommutatoren. II*, Helv. Phys. Acta **33** (1960) 347.

[70] L.J. Dixon, J.M. Drummond and J.M. Henn, *Bootstrapping the three-loop hexagon*, *JHEP* **11** (2011) 023 [arXiv:1108.4461] [InSPIRE].

[71] L.J. Dixon, J.M. Drummond and J.M. Henn, *Analytic result for the two-loop six-point NMHV amplitude in N = 4 super Yang-Mills theory*, *JHEP* **01** (2012) 024 [arXiv:1111.1704] [InSPIRE].

[72] L.J. Dixon, J.M. Drummond, M. von Hippel and J. Pennington, *Hexagon functions and the three-loop remainder function*, *JHEP* **12** (2013) 049 [arXiv:1308.2276] [InSPIRE].

[73] L.J. Dixon, J.M. Drummond, C. Duhr and J. Pennington, *The four-loop remainder function and multi-Regge behavior at NNLLA in planar N = 4 super-Yang-Mills theory*, *JHEP* **06** (2014) 116 [arXiv:1402.3300] [InSPIRE].

[74] L.J. Dixon and M. von Hippel, *Bootstrapping an NMHV amplitude through three loops*, *JHEP* **10** (2014) 065 [arXiv:1408.1505] [InSPIRE].

[75] J.M. Drummond, G. Papathanasiou and M. Spradlin, *A Symbol of Uniqueness: The Cluster Bootstrap for the 3-Loop MHV Heptagon*, *JHEP* **03** (2015) 072 [arXiv:1412.3763] [InSPIRE].

[76] L.J. Dixon, M. von Hippel and A.J. McLeod, *The four-loop six-gluon NMHV ratio function*, *JHEP* **01** (2016) 053 [arXiv:1509.08127] [InSPIRE].

[77] S. Caron-Huot, L.J. Dixon, A. McLeod and M. von Hippel, *Bootstrapping a Five-Loop Amplitude Using Steinmann Relations*, *Phys. Rev. Lett.* **117** (2016) 241601 [arXiv:1609.00669] [InSPIRE].

[78] L.J. Dixon, M. von Hippel, A.J. McLeod and J. Trnka, *Multi-loop positivity of the planar $\mathcal{N} = 4$ SYM six-point amplitude*, *JHEP* **02** (2017) 112 [arXiv:1611.08325] [InSPIRE].

[79] L.J. Dixon, J. Drummond, T. Harrington, A.J. McLeod, G. Papathanasiou and M. Spradlin, *Heptagons from the Steinmann Cluster Bootstrap*, *JHEP* **02** (2017) 137 [arXiv:1612.08976] [InSPIRE].

[80] J. Drummond, J. Foster, O. Gürdoğan and G. Papathanasiou, *Cluster adjacency and the four-loop NMHV heptagon*, *JHEP* **03** (2019) 087 [arXiv:1812.04640] [InSPIRE].

[81] S. Caron-Huot, L.J. Dixon, F. Dulat, M. von Hippel, A.J. McLeod and G. Papathanasiou, *Six-Gluon amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills theory at six and seven loops*, *JHEP* **08** (2019) 016 [arXiv:1903.10890] [InSPIRE].

[82] Y. Li and H.X. Zhu, *Bootstrapping Rapidity Anomalous Dimensions for Transverse-Momentum Resummation*, *Phys. Rev. Lett.* **118** (2017) 022004 [arXiv:1604.01404] [InSPIRE].

[83] O. Almelid, C. Duhr, E. Gardi, A. McLeod and C.D. White, *Bootstrapping the QCD soft anomalous dimension*, *JHEP* **09** (2017) 073 [arXiv:1706.10162] [InSPIRE].

[84] L.J. Dixon, A.J. McLeod and M. Wilhelm, *A Three-Point Form Factor Through Five Loops*, *JHEP* **04** (2021) 147 [arXiv:2012.12286] [InSPIRE].

[85] L.J. Dixon, O. Gürdoğan, A.J. McLeod and M. Wilhelm, *Bootstrapping a stress-tensor form factor through eight loops*, *JHEP* **07** (2022) 153 [arXiv:2204.11901] [InSPIRE].

[86] S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, *Two-Loop Integrals for Planar Five-Point One-Mass Processes*, *JHEP* **11** (2020) 117 [arXiv:2005.04195] [InSPIRE].

[87] D. Chicherin, J.M. Henn and G. Papathanasiou, *Cluster algebras for Feynman integrals*, *Phys. Rev. Lett.* **126** (2021) 091603 [arXiv:2012.12285] [InSPIRE].

[88] L.J. Dixon, C. Duhr and J. Pennington, *Single-valued harmonic polylogarithms and the multi-Regge limit*, *JHEP* **10** (2012) 074 [arXiv:1207.0186] [InSPIRE].

[89] V. Chestnov and G. Papathanasiou, *Hexagon bootstrap in the double scaling limit*, *JHEP* **09** (2021) 007 [arXiv:2012.15855] [InSPIRE].

[90] S. He, Z. Li, Y. Tang and Q. Yang, *Bootstrapping octagons in reduced kinematics from A_2 cluster algebras*, *JHEP* **10** (2021) 084 [arXiv:2106.03709] [InSPIRE].

[91] J. Drummond, J. Foster and O. Gürdoğan, *Cluster Adjacency Properties of Scattering Amplitudes in $N = 4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **120** (2018) 161601 [arXiv:1710.10953] [InSPIRE].

[92] D. Chicherin, J. Henn and V. Mitev, *Bootstrapping pentagon functions*, *JHEP* **05** (2018) 164 [arXiv:1712.09610] [InSPIRE].

[93] S. Caron-Huot, L.J. Dixon, M. von Hippel, A.J. McLeod and G. Papathanasiou, *The Double Pentaladder Integral to All Orders*, *JHEP* **07** (2018) 170 [arXiv:1806.01361] [InSPIRE].

[94] J. Henn, E. Herrmann and J. Parra-Martinez, *Bootstrapping two-loop Feynman integrals for planar $\mathcal{N} = 4$ sYM*, *JHEP* **10** (2018) 059 [arXiv:1806.06072] [InSPIRE].

[95] S. He, Z. Li and Q. Yang, *Truncated cluster algebras and Feynman integrals with algebraic letters*, *JHEP* **12** (2021) 110 [arXiv:2106.09314] [InSPIRE].

[96] S. He, Z. Li and Q. Yang, *Kinematics, cluster algebras and Feynman integrals*, arXiv:2112.11842 [InSPIRE].

[97] M. Heller, A. von Manteuffel and R.M. Schabinger, *Multiple polylogarithms with algebraic arguments and the two-loop EW-QCD Drell-Yan master integrals*, *Phys. Rev. D* **102** (2020) 016025 [arXiv:1907.00491] [INSPIRE].

[98] M. Heller, *Planar two-loop integrals for μe scattering in QED with finite lepton masses*, arXiv:2105.08046 [INSPIRE].

[99] C. Duhr, V.A. Smirnov and L. Tancredi, *Analytic results for two-loop planar master integrals for Bhabha scattering*, *JHEP* **09** (2021) 120 [arXiv:2108.03828] [INSPIRE].

[100] F. Brown, *Notes on motivic periods*, arXiv:1512.06410.

[101] *Mémoire sur une propriété générale d'une classe très étendue de fonctions transcendantes*, in *Oeuvres complètes de Niels Henrik Abel*, Cambridge University Press (2012), pp. 145–211 [DOI].

[102] D. Zagier and H. Gangl, *Classical and elliptic polylogarithms and special values of l-series*, in *The Arithmetic and Geometry of Algebraic Cycles*, Springer Netherlands (2000), pp. 561–615 [DOI].

[103] S. Bloch, *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves*, American Mathematical Society (2011) [DOI].

[104] J. Broedel and A. Kaderli, *Functional relations for elliptic polylogarithms*, *J. Phys. A* **53** (2020) 245201 [arXiv:1906.11857] [INSPIRE].

[105] V. Bolbachan, *On functional equations for the elliptic dilogarithm*, arXiv:1906.05068 [INSPIRE].

[106] J. Broedel, C.R. Mafra, N. Matthes and O. Schlotterer, *Elliptic multiple zeta values and one-loop superstring amplitudes*, *JHEP* **07** (2015) 112 [arXiv:1412.5535] [INSPIRE].

[107] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons Inc., New York (1994) [DOI].

[108] O.V. Tarasov, *Methods for deriving functional equations for Feynman integrals*, *J. Phys. Conf. Ser.* **920** (2017) 012004 [arXiv:1709.07058] [INSPIRE].

[109] O.V. Tarasov, *Functional equations for feynman integrals and abel's addition theorem*, http://theor.jinr.ru/sqs17/Talks/Tarasov.pdf.

[110] S. Caron-Huot and K.J. Larsen, *Uniqueness of two-loop master contours*, *JHEP* **10** (2012) 026 [arXiv:1205.0801] [INSPIRE].

[111] A. Hodges, *Eliminating spurious poles from gauge-theoretic amplitudes*, *JHEP* **05** (2013) 135 [arXiv:0905.1473] [INSPIRE].

[112] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, S. Caron-Huot and J. Trnka, *The All-Loop Integrand For Scattering Amplitudes in Planar N = 4 SYM*, *JHEP* **01** (2011) 041 [arXiv:1008.2958] [INSPIRE].

[113] R. Morales, A. Spiering, M. Wilhelm, Q. Yang, and C. Zhang, *Bootstrapping elliptic Feynman integrals using Schubert analysis*, in progress.

[114] J.M. Drummond and J.M. Henn, *Simple loop integrals and amplitudes in N = 4 SYM*, *JHEP* **05** (2011) 105 [arXiv:1008.2965] [INSPIRE].

[115] E. Panzer, *Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals*, *Comput. Phys. Commun.* **188** (2015) 148 [arXiv:1403.3385] [INSPIRE].

[116] C. Duhr and F. Dulat, *PolyLogTools — polylogs for the masses*, *JHEP* **08** (2019) 135 [arXiv:1904.07279] [INSPIRE].

[117] N. Arkani-Hamed and J. Trnka, *The Amplituhedron*, *JHEP* **10** (2014) 030 [arXiv:1312.2007] [INSPIRE].

[118] Q. Yang, *Schubert problems, positivity and symbol letters*, *JHEP* **08** (2022) 168 [arXiv:2203.16112] [INSPIRE].

[119] C. Duhr and F. Brown, *A double integral of dlog forms which is not polylogarithmic*, *PoS* **MA2019** (2022) 005 [arXiv:2006.09413] [INSPIRE].

[120] L. Okun and A. Rudik, *On a method of finding singularities of feynman graphs*, *Nucl. Phys.* **15** (1960) 261.

[121] J. Golden, A.B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Motivic Amplitudes and Cluster Coordinates*, *JHEP* **01** (2014) 091 [arXiv:1305.1617] [INSPIRE].

[122] J. Golden and M. Spradlin, *A Cluster Bootstrap for Two-Loop MHV Amplitudes*, *JHEP* **02** (2015) 002 [arXiv:1411.3289] [INSPIRE].

[123] J. Drummond, J. Foster and O. Gürdoğan, *Cluster adjacency beyond MHV*, *JHEP* **03** (2019) 086 [arXiv:1810.08149] [INSPIRE].

[124] S. He, Z. Li and C. Zhang, *Two-loop octagons, algebraic letters and $\bar{Q}$ equations*, *Phys. Rev. D* **101** (2020) 061701 [arXiv:1911.01290] [INSPIRE].

[125] S. He, Z. Li and C. Zhang, *The symbol and alphabet of two-loop NMHV amplitudes from $\bar{Q}$ equations*, *JHEP* **03** (2021) 278 [arXiv:2009.11471] [INSPIRE].

[126] S. He, Z. Li, Q. Yang and C. Zhang, *Feynman Integrals and Scattering Amplitudes from Wilson Loops*, *Phys. Rev. Lett.* **126** (2021) 231601 [arXiv:2012.15042] [INSPIRE].

[127] Z. Li and C. Zhang, *The three-loop MHV octagon from $\overline{Q}$ equations*, *JHEP* **12** (2021) 113 [arXiv:2110.00350] [INSPIRE].

[128] S. He, Z. Li and Q. Yang, *Notes on cluster algebras and some all-loop Feynman integrals*, *JHEP* **06** (2021) 119 [arXiv:2103.02796] [INSPIRE].

[129] J. Drummond, J. Foster, O. Gürdoğan and C. Kalousios, *Algebraic singularities of scattering amplitudes from tropical geometry*, *JHEP* **04** (2021) 002 [arXiv:1912.08217] [INSPIRE].

[130] N. Arkani-Hamed, T. Lam and M. Spradlin, *Non-perturbative geometries for planar $\mathcal{N} = 4$ SYM amplitudes*, *JHEP* **03** (2021) 065 [arXiv:1912.08222] [INSPIRE].

[131] N. Henke and G. Papathanasiou, *How tropical are seven- and eight-particle amplitudes?*, *JHEP* **08** (2020) 005 [arXiv:1912.08254] [INSPIRE].

[132] A. Herderschee, *Algebraic branch points at all loop orders from positive kinematics and wall crossing*, *JHEP* **07** (2021) 049 [arXiv:2102.03611] [INSPIRE].

[133] N. Henke and G. Papathanasiou, *Singularities of eight- and nine-particle amplitudes from cluster algebras and tropical geometry*, *JHEP* **10** (2021) 007 [arXiv:2106.01392] [INSPIRE].

[134] L. Ren, M. Spradlin and A. Volovich, *Symbol alphabets from tensor diagrams*, *JHEP* **12** (2021) 079 [arXiv:2106.01405] [INSPIRE].