Published for SISSA by Springer

Received: November 15, 2023
Revised: January 31, 2024
Accepted: February 13, 2024
Published: March 4, 2024

JHEP03(2024)019

Domain-wall Skyrmion phase in a rapidly rotating QCD matter

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Abstract: Based on the chiral perturbation theory at the leading order, we show a signal of the presence of a new phase in rapidly rotating QCD matter with two flavors, that is a domain-wall Skyrmion phase. Based on the chiral Lagrangian with a Wess-Zumino-Witten (WZW) term responsible for the chiral anomaly and chiral vortical effect, it was shown that the ground state is a chiral soliton lattice(CSL) consisting of a stack of η-solitons in a high density region under rapid rotation. In a large parameter region, a single η-soliton decays into a pair of non-Abelian solitons, each of which carries $\mathrm{SU}(2)_\mathrm{V}/\mathrm{U}(1) \simeq \mathbb{C}P^1 \simeq S^2$ moduli as a consequence of the spontaneously broken vector symmetry $\mathrm{SU}(2)_\mathrm{V}$. In such a non-Abelian CSL, we construct the effective world-volume theory of a single non-Abelian soliton to obtain a $d = 2+1$ dimensional $\mathbb{C}P^1$ model with a topological term originated from the WZW term. We show that when the chemical potential is larger than a critical value, a topological lump supported by the second homotopy group $\pi_2(S^2) \simeq \mathbb{Z}$ has negative energy and is spontaneously created, implying the domain-wall Skyrmion phase. This lump corresponds in the bulk to a Skyrmion supported by the third homotopy group $\pi_3[\mathrm{SU}(2)] \simeq \mathbb{Z}$ carrying a

https://doi.org/10.1007/JHEP03(2024)019

baryon number. This composite state is called a domain-wall Skyrmion, and is stable even in the absence of the Skyrme term. An analytic formula for the effective nucleon mass in this medium can be written only in terms of the meson's constants as $4\sqrt{2}\pi f_\pi f_\eta / m_\pi \sim 1.21\,\mathrm{GeV}$ with the decay constants f_π and f_η of the pions and η meson, respectively, and the pion mass m_π. This is reasonably heavier than the nucleon mass in the QCD vacuum.

Keywords: Solitons Monopoles and Instantons, Chiral Lagrangian, Effective Field Theories of QCD

ArXiv ePrint: 2310.17511

Contents

1 Introduction

Quantum Chromodynamics (QCD) is the fundamental theory of the strong interaction described by quarks and gluons. QCD at extreme conditions such as high baryon density, strong magnetic field, and rapid rotation has been paid much attention since it is relevant for neutron star interiors and heavy-ion collisions [1]. Lattice QCD cannot be extended to finite baryon density because of the notorious sign problem. Instead, at least at low energy, the chiral Lagrangian or the chiral perturbation theory (ChPT) offers a powerful tool since the theory is thoroughly determined by symmetry up to some constants, the pion's decay constant, quark masses, and so on [2, 3]. When the chiral symmetry mixing different species of quarks (up-quarks, down-quarks and so on) is spontaneously broken, there appear Nambu-Goldstone(NG) bosons or pions. Thus, the low-energy dynamics can be described by the aforementioned ChPT.

One of the most important extreme conditions for QCD is strong magnetic fields because of the interior of neutron stars and heavy-ion collisions. In the presence of an external magnetic field, the chiral Lagrangian is accompanied by the Wess-Zumino-Witten (WZW) term containing an anomalous coupling of the neutral pion π_0 to the magnetic field via the chiral anomaly [4, 5] in terms of the Goldstone-Wilczek (GW) current [6, 7]. It was determined to reproduce the so-called chiral separation effect (CSE) [1, 4, 8–10] in terms of the neutral pion π_0. Then, at a finite baryon chemical potential μ_{B} under a sufficiently strong magnetic field, the ground state of QCD with two flavors (up and down quarks) was found to be a chiral soliton lattice (CSL) consisting of a stack of domain walls or solitons carrying a baryon number [5, 11, 12]. However, such a CSL state was found to be unstable against a charged pion condensation in a region of higher density and/or stronger magnetic field [12]. In such a region, there appears a new phase, the domain-wall Skyrmion phase in which Skyrmions are created on top of the solitons in the ground state [13, 14].[1] To show this, the effective world-volume theory on a single soliton was constructed as an O(3) sigma model or the $\mathbb{C}P^1$ model with topological terms induced from the WZW term. Then,

[1] A possibility of an Abrikosov's vortex lattice was also proposed in the unstable region [15]. See also a recent paper [16].

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topological lumps (or baby Skyrmions) supported by $\pi_2(\mathbb{C}P^1) \simeq \mathbb{Z}$ on the world volume, corresponding to 3D Skyrmions supported by $\pi_3[\mathrm{SU}(2)] \simeq \mathbb{Z}$ in the bulk point of view, appear in the ground state for a sufficiently large chemical potential. The composite states of a domain wall and Skyrmions are called domain-wall Skyrmions. Such domain-wall Skyrmions were previously proposed and studied in field theory [17–22].[2] Domain-wall Skyrmions of a 2+1 dimensional version were also proposed in field theory [24–26] and have been recently observed experimentally [27, 28] in chiral magnets [29–33] (see also refs. [34–36]).

Another important extreme condition for QCD that we focus on in this paper is a rapid rotation. Quark-gluon plasmas produced in non-central heavy-ion collision experiments at the Relativistic Heavy Ion Collider (RHIC) reach the largest vorticity observed thus far, of the order of 10^{22}/s [37, 38]. This has triggered significant attention to rotating QCD matter in recent years [39–53]. In particular, similar but different type of CSL appears in QCD at finite density under rapid rotation instead of a strong magnetic field.[3] The anomalous term for the η' meson was obtained [50, 51] by matching with the chiral vortical effect (CVE) [10, 64–68] in terms of mesons. Although the full WZW term is not known unlike the case of the magnetic field, this CVE term is sufficient to yield a CSL phase made of the η' meson as the ground state in a certain parameter region [50–52], instead of that of the neutral pion π_0 in the case of the magnetic field.[4] In the two-flavor case, there appears a CSL phase made of the η meson. However, it was shown in ref. [53] that in a large parameter region, a single η-soliton energetically decays into a pair of non-Abelian solitons, around which the neutral pion condensation occurs. A single non-Abelian soliton spontaneously breaks the vector symmetry $\mathrm{SU}(2)_\mathrm{V}$ into its U(1) subgroup, resulting in NG modes $\mathrm{SU}(2)_\mathrm{V}/\mathrm{U}(1) \simeq \mathbb{C}P^1 \simeq S^2$ localized in the vicinity of the soliton. Thus, as the case of a π_0 soliton in the magnetic field, each non-Abelian soliton carries $\mathbb{C}P^1$ moduli and are called non-Abelian sine-Gordon solitons [21, 69] (see also refs. [22, 70, 71]). In a lattice of non-Abelian solitons, the $\mathbb{C}P^1$ modes at two neighboring solitons repel each other, and thus they are antialigned. The lattice behaves as a Heisenberg anti-ferromagnet, in which we call one soliton an up-soliton and its neighbors down-solitons, and then up and down solitons appear alternately. Such a non-Abelian CSL can be classified into the two cases, the deconfined and dimer phases. In the deconfined phase, an up-soliton and down-soliton repel each other, and they are separated with the equal distance. In the dimmer phase, they attract each other at large distances and repel at short distances, and thus constitute a molecule. The lattice can be regarded as a lattice of molecules. On the other hand, in the confined phase, the up and down-solitons attract and are completely overlapped to become η solitons. This is the previously known η-CSL [51], in which the vector symmetry $\mathrm{SU}(2)_\mathrm{V}$ is unbroken, and no soliton carries $\mathbb{C}P^1$ modes.

In this paper, we establish the presence of a new phase in rapidly rotating QCD matter, namely a domain-wall Skyrmion phase inside the non-Abelian CSL, similar to the case of a

[2] The term "domain-wall Skyrmions" was first used in ref. [23] for Yang-Mills instantons absorbed into a domain wall, which can be described as Skyrmions in the domain-wall effective theory. This term is different from ours.

[3] Now CSLs appear in various situations in QCD: CSLs under thermal fluctuation [54–57], quantum nucleation of CSLs [58, 59] and quasicrystals [60]. Possible relations between Skyrmion crystals at zero magnetic field and the CSL phase was discussed in refs. [61–63].

[4] A different type of inhomogeneity of rotating matter was also discussed in refs. [48, 49].

strong magnetic field. Within non-Abelian CSLs, either in the deconfined or dimer phase, the vector symmetry $\mathrm{SU}(2)_\mathrm{V}$ is spontaneously broken into its U(1) subgroup, thus being accompanied by NG modes $\mathrm{SU}(2)_\mathrm{V}/\mathrm{U}(1) \simeq \mathbb{C}P^1 \simeq S^2$ as mentioned above. For our purpose we concentrate on a single non-Abelian soliton in the deconfined phase. We construct the effective theory on a single non-Abelian soliton and obtain a $\mathbb{C}P^1$ model with a topological term originated from the WZW term. It admits topological lumps (baby Skyrmions) ensured by the second homotopy group $\pi_2(S^2) \simeq \mathbb{Z}$ [72]. We find that when the chemical potential is larger than a critical value, the topological lumps have negative energy due to the WZW term. The lumps on the non-Abelian soliton are Skyrmions supported by the third homotopy group $\pi_3(S^3) \simeq \mathbb{Z}$ in the bulk point of view [21], and they carry baryon numbers. This implies the domain-wall Skyrmion phase in which lumps are spontaneously created in the ground state, where the both chiral solitons and Skyrmions carry baryon numbers. The lump energy is obtained as $4\sqrt{2}\pi f_\pi f_\eta / m_\pi$ which can be interpreted as the effective nucleon mass in this medium (inside a soliton at finite density under rapid rotation) and is evaluated as $\sim 1.21\,\mathrm{GeV}$. This value is reasonably heavier than the nucleon mass 938 MeV in the QCD vacuum.

This paper is organized as follows. In section 2, we review non-Abelian CSLs. In section 3, we construct the effective worldvolume theory of a single non-Abelian soliton in the deconfined phase of non-Abelian CSL. In section 4, we constuct domain-wall Skyrmions and find the presence of the domain-wall Skyrmion phase. Section 5 is devoted to a summary and discussion.

2 Non-Abelian chiral soliton lattices under rotation

We focus on the phase in which the $\mathrm{U}(2)_\mathrm{L} \times \mathrm{U}(2)_\mathrm{R}$ chiral symmetry is spontaneously broken down. The low-energy dynamics can thus be described by an effective field theory of the pions — ChPT. A 2×2 unitary matrix represents the pion fields

$$U = \Sigma \mathrm{e}^{\mathrm{i}\chi_0}\,, \quad \Sigma = \mathrm{e}^{\mathrm{i}\tau_a \chi_a}\,, \tag{2.1}$$

where τ_a ($a = 1, 2, 3$) are the Pauli matrices with the normalization $\mathrm{tr}(\tau_a \tau_b) = 2\delta_{ab}$. This field Σ transforms under the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ chiral symmetry as $\Sigma \to L\Sigma R^\dagger$, where L and R are 2×2 unitary matrices, while χ_0 transforms under the axial $\mathrm{U}(1)_\mathrm{A}$ symmetry as $\chi_0 \to \chi_0 + 2\theta_0$.

Then, the effective Lagrangian at the leading order is ($\mu = 0, \cdots, 3$)

$$\begin{aligned} \mathcal{L}_\mathrm{ChPT} = {}& \frac{f_\pi^2}{4} g^{\mu\nu}\,\mathrm{tr}(\partial_\mu U \partial_\nu U^\dagger) - \frac{f_\eta^2 - f_\pi^2}{8} g^{\mu\nu}\,\mathrm{tr}(U^\dagger \partial_\mu U)\,\mathrm{tr}(U^\dagger \partial_\nu U) \\ & + \frac{Bm}{2}\,\mathrm{tr}(U + U^\dagger - 2\mathbf{1}_2) + \frac{A}{2}(\det U + \det U^\dagger - 2)\,, \end{aligned} \tag{2.2}$$

where f_π and f_η are the decay constants of the pions and the $\mathrm{U}(1)_\mathrm{A}$ singlet (η) meson, respectively, m is the quark mass, and A and B are parameters that cannot be determined by symmetry alone. The first and second terms are the kinetic terms of the χ_a and χ_0, respectively. The third term is the mass term of the mesons, stemming from the explicit chiral symmetry breaking due to the finite quark masses. Then, the pion mass m_π is related to the quark mass by the Gell-Mann-Oakes-Renner relation $Bm = f_\pi^2 m_\pi^2/4$. The fourth term represents the QCD anomaly: $\mathrm{U}(1)_\mathrm{A} \to \mathbb{Z}_4$. The parameter A gives an additional mass term for χ_0 meson given by $\delta m_{\chi_0}^2 = A/f_\eta^2$. Here, $g_{\mu\nu}$ is a spacetime metric representing the rotating coordinates, and $g^{\mu\nu}$ is its inverse:

$$g_{\mu\nu} = \begin{pmatrix} 1-\Omega^2(x^2+y^2) & \Omega y & -\Omega x & 0 \\ \Omega y & -1 & 0 & 0 \\ -\Omega x & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.3}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & \Omega y & -\Omega x & 0 \\ \Omega y & \Omega^2 y^2 - 1 & -\Omega^2 xy & 0 \\ -\Omega x & -\Omega^2 xy & \Omega^2 x^2 - 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.4}$$

where Ω stands for the rotation velocity. Here we have denoted $(x, y, z) = (x^1, x^2, x^3)$.

The external $\mathrm{U(1)_B}$ gauge field A_μ^{B} can couple to Σ through the GW current [6, 7]

$$j_{\mathrm{GW}}^\mu = -\frac{\epsilon^{\mu\nu\alpha\beta}}{24\pi^2}\,\mathrm{tr}\, L_\nu L_\alpha L_\beta = \frac{\epsilon^{\mu\nu\alpha\beta}}{24\pi^2}\,\mathrm{tr}\, R_\nu R_\alpha R_\beta\,, \tag{2.5}$$

where we have introduced the standard notation $L_\mu \equiv \Sigma\partial_\mu\Sigma^\dagger$ and $R_\mu \equiv \partial_\mu\Sigma^\dagger\Sigma$. Then, the effective Lagrangian coupling to A_μ^{B} can be written as

$$\mathcal{L}_{\mathrm{GW}} = A_\mu^{\mathrm{B}} j_{\mathrm{GW}}^\mu \subseteq \mathcal{L}_{\mathrm{WZW}}\,. \tag{2.6}$$

In order to introduce finite baryon chemical potential μ_{B}, we choose the $\mathrm{U(1)_B}$ gauge field as $A_\mu^{\mathrm{B}} = (\mu_{\mathrm{B}}, \mathbf{0})$. This is a WZW term but the full expression for rotation $\mathcal{L}_{\mathrm{WZW}}$ is not known thus far, in contrast to the case of the magnetic field in which case the full expression is known [4, 5]. In the external electromagnetic field, the gauge-invariant and conserved baryon current can be derived by the "trial and error" $\mathrm{U(1)_{em}}$ gauging [7]. The coupling of $\mathrm{U(1)_{em}}$ gauged GW current to A_μ^{B} is calculated as [5, 6]

$$\tilde{\mathcal{L}}_{\mathrm{WZW}} = A_\mu^{\mathrm{B}} \tilde{j}_{\mathrm{GW}}^\mu\,, \tag{2.7}$$

$$\tilde{j}_{\mathrm{GW}}^\mu = -\frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\{\mathrm{tr}(L_\nu L_\alpha L_\beta) - 3\mathrm{i}e\partial_\nu[A_\alpha\,\mathrm{tr}(QL_\beta + QR_\beta)]\}\,, \tag{2.8}$$

The second term with $\Sigma = \mathrm{e}^{\mathrm{i}\chi_3\tau_3}$ becomes $e\boldsymbol{B}\cdot\boldsymbol{\nabla}\chi_3/(4\pi^2)$. In fact, the above equation can be derived by reproducing the chiral separation effect (CSE) [1, 4, 8–10] in terms of the χ_3 meson. This procedure is justified by the fact that the chiral anomaly coefficient determines the transport coefficient of the CSE [66, 73]. Therefore, the anomaly matching of the CSE can derive the part of $\tilde{\mathcal{L}}_{\mathrm{WZW}}$. Unfortunately, the GW current is already invariant under general coordinate transformations, so the method applied to the electromagnetic field cannot be used. Hence, the method to derive the full expression is not known. However, when applying this method to a rotating system, it is evident that at least the following terms exist. QCD at finite baryon chemical potential under global rotation is known to exhibit the anomalous current in the direction of the rotation, which is the so-called CVE [8, 10, 64, 67, 68]:

$$\boldsymbol{j}_5 = \langle\bar{q}\boldsymbol{\gamma}\gamma_5 q\rangle = \frac{\mu_{\mathrm{B}}^2}{\pi^2 N_{\mathrm{c}}}\boldsymbol{\Omega}\,, \tag{2.9}$$

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where q is a quark field. We note that the chiral anomaly determines the transport coefficient of CVE. Therefore, due to the exactness of the chiral anomaly coefficient, it must be reproduced in terms of the χ_0 meson in the ChPT. The anomaly matching for the CVE gives us the anomalous coupling of the χ_0 meson to the rotation [50, 51]:

$$\mathcal{L}_{\rm CVE} = \frac{\mu_{\rm B}^2}{2\pi^2 N_{\rm c}} \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \chi_0 \,. \tag{2.10}$$

Hereafter, we interpret that $\mathcal{L}_{\rm CVE}$ is a part of $\mathcal{L}_{\rm WZW}$.

To derive an effective Lagrangian, we adopt a modification to the conventional power counting scheme in ChPT [56]:

$$\partial_\mu \,, m_\pi \sim \sqrt{Bm} \,, A_\mu \,, \Omega \,, \in \mathcal{O}(p^1), \tag{2.11}$$

$$A_\mu^{\rm B} \sim \mu_{\rm B} \in \mathcal{O}(p^{-1}). \tag{2.12}$$

In this power counting, eq. (2.6) is of order $\mathcal{O}(p^2)$, consistent with eq. (2.2). The sole appearance of $\mu_{\rm B}$ in the WZW term in eq. (2.6) permits the assignment of a negative power counting to $\mu_{\rm B}$. The effective field theory up to $\mathcal{O}(p^2)$ encompasses the terms in eq. (2.5); however, eq. (2.5) has been overlooked in prior studies of the CSLs under rotation. For discussions related to the magnetic field, we refer to our previous work [13, 14]. In the QCD vacuum, the effects of the QCD anomaly are generally not suppressed. Hence, we note that it is not feasible to incorporate the QCD anomaly's effects into ChPT (Of course, in the large-$N_{\rm c}$ expansion, the effects of the QCD anomaly are of the order of $1/N_{\rm c}$, allowing them to be treated perturbatively [74, 75]). We underscore that an $\mathcal{O}(p^4)$ term, such as a Skyrme term, is unnecessary for our findings.

Our effective theory when ignoring the charged pions $\chi_{1,2}$ and assuming one-dimensional dependence in the x^3 coordinate reduces to

$$\frac{\mathcal{H}}{C} = \frac{1-\epsilon}{2}(\chi_3')^2 + \frac{1}{2}(\chi_0')^2 + \sin\beta(1-\cos 2\chi_0) + \cos\beta(1-\cos\chi_0\cos\chi_3) - S\chi_0' \,, \tag{2.13}$$

where we have introduced the following quantities,

$$\sin\beta \equiv \frac{A}{C} \,, \qquad \cos\beta \equiv \frac{2Bm}{C} \,, \qquad C \equiv \sqrt{A^2 + (2Bm)^2} \,. \tag{2.14}$$

and dimensionless variables as follows:

$$\zeta \equiv \frac{\sqrt{C} x^3}{f_\eta} \,, \qquad \epsilon \equiv 1 - \left(\frac{f_\pi}{f_\eta}\right)^2 \,, \qquad S \equiv \frac{\Omega \mu_{\rm B}^2}{2\pi^2 N_{\rm c} \sqrt{C}} \,. \tag{2.15}$$

In eq. (2.13), the prime denotes a differentiation with respect to ζ. The third and fourth terms in eq. (2.13) are the potential terms of the χ_0 and χ_3:

$$\frac{V_{\rm pot}}{C} = \cos\beta(1-\cos\chi_0\,\cos\chi_3) + \sin\beta(1-\cos 2\chi_0) \,. \tag{2.16}$$

For later convenience, we introduce new fields defined as

$$\chi_\pm \equiv \chi_0 \pm \chi_3 \,. \tag{2.17}$$

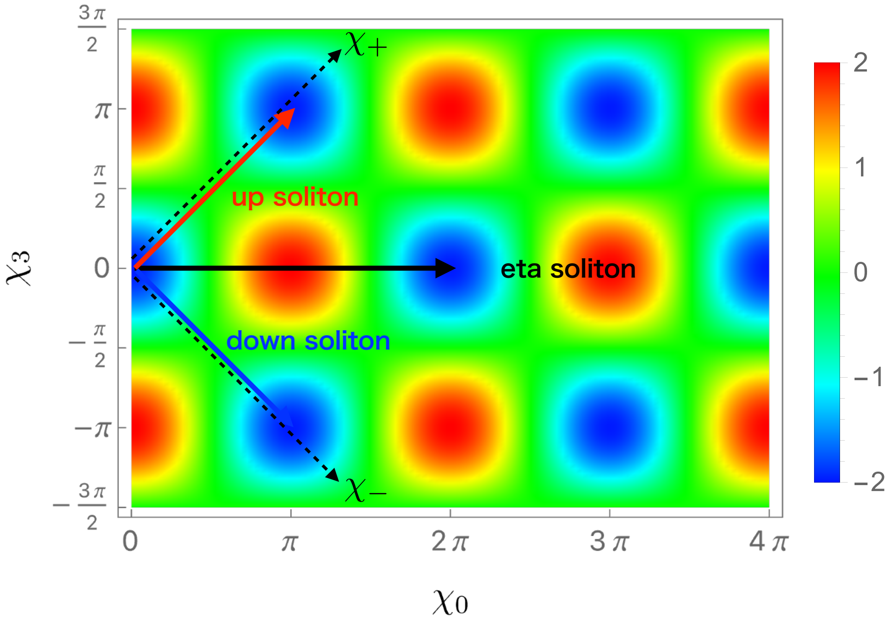

Figure 1. Single η soliton and up and down non-Abelian solitons in the field space. The color denotes the height of the potential and blues are vacua. Fixing the state $(\phi_0, \phi_3) = (0, 0)$ at $x^3 = -\infty$, the η soliton denoted by the black arrow connects it to $(\phi_0, \phi_3) = (2\pi, 0)$ the up soliton denoted by the red arrow connects it to $(\phi_0, \phi_3) = (\pi, \pi)$ and the down soliton denoted by the blue arrow connects it to $(\phi_0, \phi_3) = (\pi, -\pi)$ at $x^3 = +\infty$.

In terms of χ_+ and χ_-, the Hamiltonian is reduced as

$$\begin{aligned}\frac{\mathcal{H}}{C} = &\frac{1}{2}\left[\frac{1}{2}\left(\frac{\mathrm{d}\chi_+}{\mathrm{d}\zeta}\right)^2 + (1-\cos\chi_+) - S\frac{\mathrm{d}\chi_+}{\mathrm{d}\zeta}\right] + \frac{1}{2}\left[\frac{1}{2}\left(\frac{\mathrm{d}\chi_-}{\mathrm{d}\zeta}\right)^2 + (1-\cos\chi_-) - S\frac{\mathrm{d}\chi_-}{\mathrm{d}\zeta}\right] \\ &- \frac{\epsilon}{8}(\chi_+')^2 - \frac{\epsilon}{8}(\chi_-')^2 + \frac{\epsilon}{4}\chi_+'\chi_-' - \sin\beta(1-\cos(\chi_+ + \chi_-)) - S\chi_0' \,. \end{aligned} \tag{2.18}$$

Eq. (2.13) gives the equation of motions as follows:

$$\chi_+'' - \frac{2-\epsilon}{2(1-\epsilon)}\cos\beta\,\sin\chi_+ + \frac{\epsilon}{2(1-\epsilon)}\cos\beta\,\sin\chi_- - 2\sin\beta\,\sin(\chi_+ + \chi_-) = 0\,, \tag{2.19}$$

$$\chi_-'' - \frac{2-\epsilon}{2(1-\epsilon)}\cos\beta\,\sin\chi_- + \frac{\epsilon}{2(1-\epsilon)}\cos\beta\,\sin\chi_+ - 2\sin\beta\,\sin(\chi_+ + \chi_-) = 0\,. \tag{2.20}$$

Let us first consider the case of $\epsilon = 0$ and $\beta = 0$. The potential term at $\beta = 0$ is sketched as figure 1. The configuration connecting $(0, 0)$ and $(2\pi, 0)$ is well-known as a single sine-Gordon soliton,

$$\chi_0 = 4\tan^{-1} \mathrm{e}^{\zeta - \zeta_0}\,, \tag{2.21}$$

which has the transnational moduli, ζ_0. On the other hand, the configuration connecting $(0, 0)$ and (π, π) is given by

$$U_+ = \mathrm{diag}(u^{\mathrm{i}\theta}, 1)\,, \tag{2.22}$$

$$\theta = 4\tan^{-1}\mathrm{e}^{\zeta-\zeta_0}\,. \tag{2.23}$$

which also has the translational moduli. We note that this soliton spontaneously breaks the $\mathrm{SU}(2)_\mathrm{V}$ symmetry to a U(1) subgroup:

$$U_+ \to gU_+g^\dagger = U_+\,, \tag{2.24}$$

$$g = \mathrm{e}^{\mathrm{i}\alpha\tau_3}\,. \tag{2.25}$$

Therefore, this soliton has not only translational moduli $\mathbb{R}$ but also $\mathrm{SU}(2)/\mathrm{U}(1) \cong \mathbb{C}P^1 \cong S^2$ moduli. We call this soliton an up soliton. The configuration connecting $(0,0)$ and $(\pi,-\pi)$ is given by the $\mathrm{SU}(2)_\mathrm{V}$ transformation:

$$U_- = \mathrm{e}^{\mathrm{i}\pi\tau_1/2}U_+\mathrm{e}^{-\mathrm{i}\pi\tau_1/2} = \mathrm{diag}(1, u^{\mathrm{i}\theta})\,, \tag{2.26}$$

which is referred to as a down soliton. The up soliton and down soliton are connected by the $\mathbb{C}P^1$ moduli.

Next, let us consider the effects of non-zero ϵ and β. The term depending on ϵ is

$$-\frac{\epsilon}{2}(\chi_3')^2 = -\frac{\epsilon}{8}(\chi_+')^2 - \frac{\epsilon}{8}(\chi_-')^2 + \frac{\epsilon}{4}\chi_+'\chi_-'\,. \tag{2.27}$$

Since $\chi_+(\chi_-)$ has a peak at the center of the up (down) soliton, the energy density of eq. (2.27) becomes lower when the up soliton and down soliton are separated. Therefore, there is a repulsive (attractive) force between the up and down solitons due to the finite positive (negative) ϵ. As β increases from 0, the χ_3 dependence of the potential decreases. Therefore, the up and down soliton overlap to become the ordinary sine-Gordon soliton. The effect of finite β is the attractive interaction between the up and down solitons.

From the preceding discussion, we identify three distinct cases concerning the arrangement of the up and down solitons: the deconfined phase, the dimer phase, and the confined phase.

1. *Deconfined Phase*: if the repulsive force significantly exceeds the attractive force, the up and down solitons fully separate. This state is termed the *deconfined phase*. For this condition, the relationship between the distance d between the up and down solitons and the distance ℓ between the same type of soliton is given by $d = \frac{\ell}{2}$.

2. *Confined Phase*: conversely, when the attractive force strongly prevails over the repulsive force, the up and down solitons overlap entirely, denoted as $d = 0$. This case is referred to as the *confined phase*.

3. *Dimer Phase*: when the attractive and repulsive forces counteract each other equally, a molecular state of the up and down solitons forms, satisfying the condition $0 < d < \frac{\ell}{2}$.

When $\epsilon < 0$ the inter-soliton force is attractive, so that the CSL is in the confined phase. Namely, the CSL is of the Abelian type with $\chi_3 = 0$ and χ_0 is the same as that given in eq. (2.21). Then the Hamiltonian reads

$$\frac{\mathcal{H}}{C} = \frac{1}{2}(\chi_0')^2 + (1-\cos\chi_0) - S\chi_0' = 4\,\mathrm{sech}^2\zeta - 2S\,\mathrm{sech}\,\zeta\,. \tag{2.28}$$

The tension (the mass per unit area) is given by integrating $\mathcal{H}$ over z as

$$\sigma\big|_{\text{1-soliton}} = \frac{f_\eta}{\sqrt{C}}\int_{-\infty}^{\infty} d\zeta\,\mathcal{H} = f_\eta\sqrt{C}\,(8-2\pi S)\,. \tag{2.29}$$

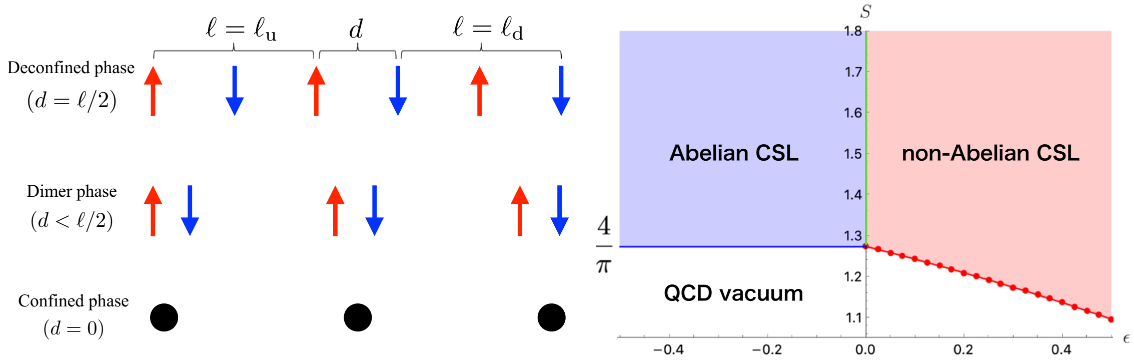

Figure 2. The schemetic picture of the three phases (left) and the phase diagram for $\beta = 0$ (right). (Left panel) The deconfined, dimer and confined phases in which up and down solitons repel, form a molecule, and are completely overlapped, respectively. (Right panel) In this parameter choice, the dimer phase does not appear, and the whole non-Abelian CSL implies the deconfined phase. The green line corresponds to the noninteractive case in which the up and down solitons do not interact each other and form lattices independently.

This becomes zero at $S = 4/\pi$, and therefore the Abelian CSL becomes the ground state for $S \geq 4/\pi$. This is shown in figure 2. When $\epsilon > 0$, the inter-soliton force is repulsive and therefore the ground state is the non-Abelian CSL. There are no analytic solutions to eqs. (2.19) and (2.20), so that we numerically solve them and find the phase diagram. The numerical determination of the phase diagram is done by finding out the minimum energy state in the following way. When the ground state is a CSL, then we read the distance d from numerical solutions for given parameter sets of ϵ and S. The numerical solutions are independent of S since the EOMs do not includes S. We thus first solve the EOMs by fixing ϵ and the lattice constant ℓ. Then, plugging the numerical solutions into the Hamiltonian given in eq. (2.13) including S, we calculate the energy per one period ℓ. Repeating this for various ℓ and ϵ, we obtain the energy

$$\mathcal{M}(\epsilon, \ell, S) = \int_0^\ell \mathcal{H}\, d\zeta \tag{2.30}$$

as a function of ϵ, ℓ and S. Next, we look for ℓ minimizing $\mathcal{M}(\epsilon, \ell, S)$ for a fixed S and ϵ, and we specify the period $\ell(\epsilon, S)$ as a function of ϵ and S. At this stage, we can judge whether the ground state is CSL or not. If the minimized energy is negative, the ground state is CSL. Otherwise, the ground state is homogeneous QCD vacuum. Finally, we read the distance d to specify the type of CSLs: it is the Abelian confined phase ($d = 0$), the non-Abelian deconfined phase ($d = \ell/2$), or the non-Abelian dimer phase ($0 < d < \ell/2$). As an example, the phase diagram in the case of $\beta = 0$ is shown in figure 2 (the right panel). In the case of $\epsilon < 0$ denoted by the blue shaded region in figure 2, the CSL is the Abelian confined CSL. The critical angular velocity in the Abelian CSL is known to be $S = 4/\pi$ [12, 53]. In the case of $\epsilon = 0$ denoted by the green line in figure 2, the up and down solitons do not interact, and they form lattices independently. The total configuration is a superposition of the up CSL and down CSL, and consequently, the confined, dimer, and deconfined CSLs are all energetically degenerated. In the case of $\epsilon > 0$ denoted by the red shaded region in figure 2, the ground

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state is the non-Abelian deconfined CSL. It is important that the critical angular velocity is less than $4/\pi$ for $\epsilon > 0$. The phases for more generic ϵ and β can be found in ref. [53].

3 Non-Abelian sine-Gordon soliton and its effective world-volume theory

In this section, we construct the effective field theory of the U(2) non-Abelian sine-Gordon soliton under rotation by using the moduli (Manton) approximation [76–78]. On the phase transition line between the vacuum and the deconfined phases, the single soliton enters the system alternately, and their distance is infinite. Then, we focus only on a single up soliton:

$$U_0 \simeq \mathrm{diag}(\mathrm{e}^{\mathrm{i}\chi_+}, 1) = \mathrm{e}^{\mathrm{i}(\chi_+/2+\chi_+\tau_3/2)} \,. \tag{3.1}$$

Of course, one can choose the down soliton, but since they are infinitely apart, it is sufficient to choose one. Considering a sufficiently small ϵ and small β, we can approximate $\chi_+ \simeq 4\tan^{-1} e^{\zeta}$.

So far, we have neglected the charged pions. Its general solution containing the charged pions can be obtained from U_0 by an $\mathrm{SU}(2)_\mathrm{V}$ transformation,

$$U = gU_0 g^\dagger \,, \tag{3.2}$$

where g is an SU(2) matrix. Since g in eq. (3.2) is redundant with respect to a U(1) subgroup generated by τ_3, it takes a value in a coset space, $\mathrm{SU}(2)_\mathrm{V}/\mathrm{U}(1) \simeq \mathbb{C}P^1 \simeq S^2$. Together with the translational modulus Z, the single sine-Gordon soliton has the moduli

$$\mathcal{M} \simeq \mathbb{R} \times \mathbb{C}P^1 \,. \tag{3.3}$$

Such a soliton with non-Abelian moduli is called a non-Abelian sine-Gordon soliton [21, 69].

Let us parameterize the $\mathbb{C}P^1$ moduli by the homogeneous coordinates $\phi \in \mathbb{C}^2$ of $\mathbb{C}P^1$, satisfying [21]

$$\phi^\dagger \phi = 1 \,, \qquad g\tau_3 g^\dagger = 2\phi\phi^\dagger - \mathbf{1}_2 \,. \tag{3.4}$$

In terms of ϕ, eq. (3.2) is represented as

$$U = \exp(\mathrm{i}\chi_+ \phi\phi^\dagger) = \mathbf{1}_2 + (u-1)\phi\phi^\dagger \,, \tag{3.5}$$

where we define $u \equiv \mathrm{e}^{\mathrm{i}\chi_+}$. Since $\mathbb{C}P^1 \simeq S^2$, the moduli space is also parameterized by the three-component real vector $\boldsymbol{n}$ with the unit length condition, $|\boldsymbol{n}| = 1$. ϕ and n_a are related by the following formula:

$$n_a = \phi^\dagger \tau_a \phi \,. \tag{3.6}$$

The condition $\phi^\dagger\phi = 1$ is solved by using the inhomogeneous coordinate $f \in \mathbb{C}$ as follow:

$$\phi = \frac{1}{\sqrt{1+|f|^2}} \begin{pmatrix} 1 \\ f \end{pmatrix} \,. \tag{3.7}$$

Then, the up soliton corresponds to $n_3 = 1$ $(f = 0)$ and the down solitons to $n_3 = -1$ $(f = \infty)$.

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Now, we are prepared to formulate the low-energy effective theory for a single soliton using the moduli approximation [76–78]. Consider a single sine-Gordon soliton perpendicular to the x^3-coordinate. In what follows, the moduli parameter ϕ will be treated as fields on the soliton's 2 + 1-dimensional worldvolume. However, we will not do the same for the translational modulus Z as its transverse motion is not pertinent to our investigation. By substituting eq. (3.5) into $\mathcal{L}$, we get

$$\mathcal{L}_{\text{kin}} = \frac{f_\pi^2 + f_\eta^2}{8} g^{\mu\nu}\partial_\mu \chi_+ \partial_\nu \chi_+ + \frac{f_\pi^2}{2}|1-u|^2 g^{\mu\nu}(\phi^\dagger \partial_\mu \phi \phi^\dagger \partial_\nu \phi + \partial_\mu \phi^\dagger \partial_\nu \phi)\,, \tag{3.8}$$

$$\mathcal{L}_{\text{mass}} = (Bm + A)(-1 + \cos\chi_+)\,, \tag{3.9}$$

$$\mathcal{L}_{\text{GW}} = \frac{\mu_{\text{B}} q}{4\pi}|1-u|^2 \partial_3 \chi_+\,. \tag{3.10}$$

Here, q is the baby Skyrmion (lump) charge density defined by

$$q \equiv -\frac{\mathrm{i}}{2\pi}\epsilon^{ij}\partial_i\phi^\dagger\partial_j\phi = \frac{1}{8\pi}\epsilon^{ij}\boldsymbol{n}\cdot(\partial_i\boldsymbol{n}\times\partial_j\boldsymbol{n}) = \frac{i}{2\pi}\operatorname{tr}\left([\partial_{\bar z}\mathcal{P},\partial_z\mathcal{P}]\,\mathcal{P}\right), \tag{3.11}$$

the integration of which over the two-dimensional space defines the topological lump number

$$k = \int \mathrm{d}^2x\, q \in \pi_2(S^2) \simeq \mathbb{Z}\,. \tag{3.12}$$

In eq. (3.11), $\mathcal{P}$ is the projection operator defined by

$$\mathcal{P} \equiv \phi\phi^\dagger \tag{3.13}$$

and satisfies $\mathcal{P}^2 = \mathcal{P}$ [79, 80]. The integrations of the above Lagrangian (3.8), (3.9) and (3.10) over the codimension x^3 give us the total effective world-volume theory of the non-Abelian sine-Gordon soliton:

$$\begin{aligned}\mathcal{L}_{\text{DW}} &= \int \mathrm{d}x^3(\mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{mass}} + \mathcal{L}_{\text{GW}} + \mathcal{L}_{\text{CVE}})\\ &= -\frac{8\sqrt{C}(f_\pi^2+f_\eta^2)}{8f_\eta} - \frac{4f_\eta(Bm+A)}{\sqrt{C}} + \frac{\mu_{\text{B}}^2\Omega}{2\pi N_{\text{c}}}\\ &\quad + \frac{4f_\pi^2 f_\eta}{\sqrt{C}} g^{\alpha\beta}(\phi^\dagger\partial_\alpha\phi\phi^\dagger\partial_\beta\phi + \partial_\alpha\phi^\dagger\partial_\beta\phi)\\ &\quad + \mu_{\text{B}} q\,,\end{aligned} \tag{3.14}$$

where $\alpha, \beta = 0, 1, 2$ and we have used the integration formulas

$$\int_{-\infty}^{\infty}\mathrm{d}x\left[\frac{f_\eta^2+f_\pi^2}{8}(\partial_3\chi_+)^2 + (Bm+A)(1-\cos\chi_+)\right] = \frac{8\sqrt{C}(f_\pi^2+f_\eta^2)}{8f_\eta} + \frac{4f_\eta(Bm+A)}{\sqrt{C}}\,, \tag{3.15}$$

$$\int_{-\infty}^{\infty}\mathrm{d}x\,|1-u|^2 = \frac{8f_\eta}{\sqrt{C}}\,, \tag{3.16}$$

$$\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{1}{4\pi}|1-u|^2\partial_3\chi_+ = 1\,. \tag{3.17}$$

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The terms in the first line in eq. (3.14) represent the tension of the domain wall. These are constant terms irrelevant in our study; thus we ignore them hereafter. The second and third lines in eq. (3.14) denote the Lagrangian density for the $\mathbb{C}P^1$ theory in the rotating coordinates and the lump charge density, respectively. The Lagrangian can be rewritten in terms of the inhomogeneous coordinate f as

$$\mathcal{L}_{\rm DW} = \frac{f_\pi^2 f_\eta}{\sqrt{C}} g^{\alpha\beta} \frac{\partial_\alpha f \partial_\beta f^*}{(1+|f|^2)^2} + \mu_{\rm B} q \,. \tag{3.18}$$

In order to determine the ground state, let us calculate the momentum conjugate for $\partial_0 f$ and $\partial_0 f^*$:

$$\pi_f \equiv \frac{\partial \mathcal{L}_{\rm DW}}{\partial(\partial_0 f)} = \partial_0 f^* + \Omega(y\partial_1 f^* - x\partial_2 f^*)\,, \tag{3.19}$$

$$\pi_{f^*} \equiv \frac{\partial \mathcal{L}_{\rm DW}}{\partial(\partial_0 f^*)} = \partial_0 f + \Omega(y\partial_1 f - x\partial_2 f)\,. \tag{3.20}$$

Then, the Hamiltonian can be calculated as follows:

$$\begin{aligned}\mathcal{H}_{\rm DW} &= \pi_f \partial_0 f + \pi_{f^*}\partial_0 f^* - \mathcal{L}_{\rm DW} \\ &= \frac{f_\pi^2 f_\eta}{\sqrt{C}} \frac{\partial_0 f \partial_0 f^* + \partial_i f \partial_i f^*}{(1+|f|^2)^2} - \mu_{\rm B} q \\ &\quad - \frac{f_\pi^2 f_\eta \Omega^2}{\sqrt{C}} \frac{(y^2\partial_1 f\partial_1 f^* + x^2\partial_2 f\partial_2 f^*) + xy(\partial_1 f\partial_2 f^* + \partial_2 f \partial_1 f^*)}{(1+|f|^2)^2}\,.\end{aligned} \tag{3.21}$$

The last term proportional to Ω^2 is at higher order $\mathcal{O}(p^4)$ which we will omit in the following.

4 Domain-wall Skyrmion phase

In this section, we construct topological lumps in the domain-wall world-volume theory, and show that they correspond to Skyrmions in the bulk, implying the domain-wall Skyrmion phase. For this purpose, let us introduce the complex coordinate:

$$w \equiv x + {\rm i}y\,, \qquad \bar{w} \equiv x - {\rm i}y\,. \tag{4.1}$$

Using these coordinates, we have

$$\mathcal{H}_{\rm DW} = \frac{f_\pi^2 f_\eta}{\sqrt{C}} \frac{|\partial_w f|^2 + |\partial_{\bar{w}} f|^2}{(1+|f|^2)^2} - \mu_{\rm B} q\,. \tag{4.2}$$

When the Bogomol'nyi-Prasad-Semmerfield (BPS) equation for $k > 0$ [72]

$$\partial_{\bar{w}} f = 0 \tag{4.3}$$

or the anti-BPS equation for $k < 0$

$$\partial_w f = 0 \tag{4.4}$$

holds, the energy (4.2) saturates the minimum (the Bogomol'nyi bound) of the following inequality:

$$E_{\rm DW} = \int {\rm d}^2x\, \mathcal{H}_{\rm DW} \geq \frac{4\pi f_\pi^2 f_\eta}{\sqrt{C}} \left| \int {\rm d}^2x\, q \right| - \mu_{\rm B} \int {\rm d}^2x\, q\,. \tag{4.5}$$

Due to the second term, anti-BPS lumps have more energy than BPS lumps. Let us consider BPS k-lump solution [72]

$$f = \frac{b_{k-1}w^{k-1} + \cdots + b_0}{w^k + a_{k-1}w^{k-1} + \cdots + a_0}\,, \tag{4.6}$$

with moduli $a_i, b_i \in \mathbb{C}$ $(i = 0, \cdots k-1)$.

Now, let us show the relation between the topological lumps in the domain-wall worldvolume theory and Skyrmions in the bulk. To this end, the baryon (Skyrmion) number B in the bulk taking a value in $\pi_3[{\rm SU}(2)] \simeq \mathbb{Z}$ can be calculated as $(i = 1, 2, 3)$ [21]

$$\begin{aligned} B &= \frac{1}{24\pi^2} \int d^3x\, \epsilon_{ijk}\, {\rm tr}(R_i R_j R_k) \\ &= -\frac{1}{8\pi^2} \int d^3x\ \ {\rm tr} \left[\left(\partial_1 \Sigma^\dagger \partial_2 \Sigma - \partial_2 \Sigma^\dagger \partial_1 \Sigma \right) \Sigma^\dagger \partial_3 \Sigma \right] \\ &= -\frac{1}{8\pi^2} \int dx^1 dx^2\ \ {\rm tr} \left(\left[\partial_1 \mathcal{P}, \partial_2 \mathcal{P} \right] \mathcal{P} \right) \int dx^3\ |u-1|^2 u^* \partial_3 u \\ &= \frac{i}{2\pi} \int dw d\bar{w}\ \ {\rm tr} \left(\left[\partial_{\bar{z}} \mathcal{P}, \partial_z \mathcal{P} \right] \mathcal{P} \right) \times \frac{1}{2\pi} \int dx^3\ (1 - \cos\theta) \partial_3 \theta \\ &= k\,, \end{aligned} \tag{4.7}$$

with the lump number k defined in eq. (3.12) and the projection operator $\mathcal{P}$ defined in eq. (3.13). Therefore, we have found that k topological lumps on the non-Abelian sine-Gordon soliton carry a baryon number k and represent k Skyrmions in the bulk. This one-to-one correspondence between lumps on the soliton and Skyrmions in the bulk has a sharp contrast to the domain-wall Skyrmion in the strong magnetic field, in which case one lump in the domain wall corresponds to two Skyrmions in the bulk.

Figure 3 shows a three-dimensional configuration of a domain-wall Skyrmion. This can be compared with the case of that in strong magnetic field, in which a single lump has two peaks corresponding to two Skyrmions in the bulk.

Finally, let us evaluate the energy of the lump. Substituting eq. (4.6) into eq. (4.2), we obtain the energy of BPS k-lump configurations as[5]

$$E_{\rm DW} = \left(\frac{4\pi f_\pi^2 f_\eta}{\sqrt{C}} - \mu_{\rm B} \right) k. \tag{4.8}$$

[5]In the case of domain-wall Skyrmions in the magnetic field, there is an additional term proportional to $|b_{k-1}|^2$ in the k-lump energy due to the full WZW term in the domain-wall theory. By contrast, there is not such a term in eq. (4.8). While the full WZW for the rotation is not available yet, we can understand this by recalling the origin of such a term in the case of magnetic field. It comes from the topological term $\partial_3\pi_0$ in the WZW term, counting chiral solitons. It flips its sign, $-\partial_3\pi_0$, when the orientation of the $\mathbb{C}P^1$ moduli is at the south pole $n_3 = -1$, while the vacuum of the domain-wall worldvolume theory is at the north pole $n_3 = +1$. Therefore, the center $(n_3 = -1)$ of a lump costs energy. Contrary to this, in the case of rotation, the topological term supporting the η-solitons is $\partial_3\eta$ as in eq. (2.10). Since all solitons with different $\mathbb{C}P^1$ moduli have the same boundary condition from $\eta = 0$ to $\eta = \pi$, the topological term $\int dx^3 \partial_3 \eta = \pi$ does not depend on the $\mathbb{C}P^1$ moduli, in contrast to the case of the magnetic field.

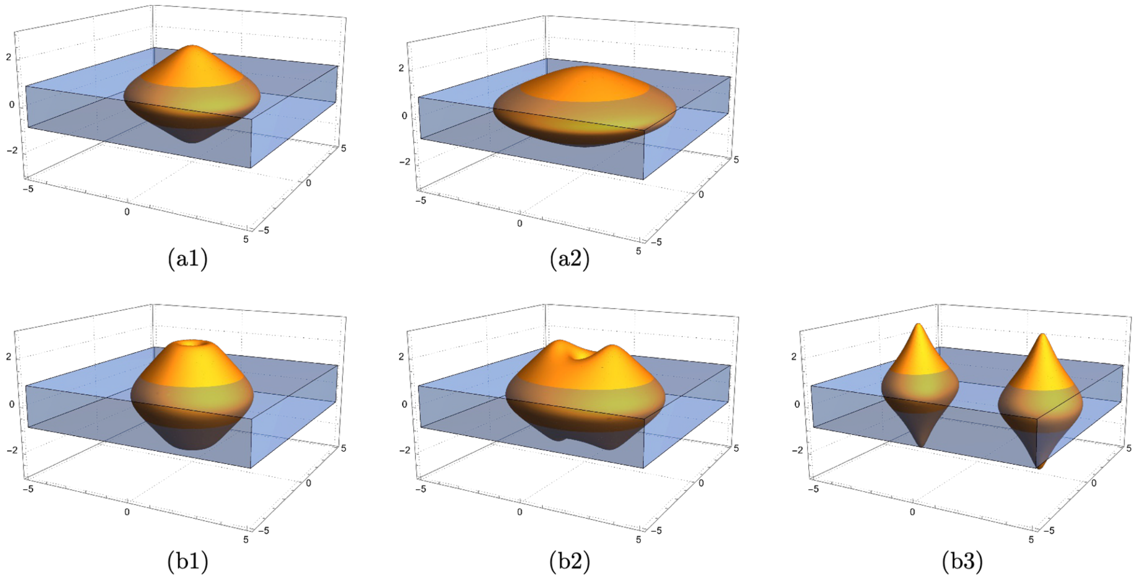

Figure 3. Three-dimensional configuration of $k = 1$ and $k = 2$ domain-wall Skyrmions for QCD at finite density under rapid rotation. An isosurface of the baryon density $(1/4\pi^2)\,\mathrm{tr}\,[R_1 R_2 R_3]$ is plotted by orange surface and the blue region denotes a non-Abelian soliton. The vertical coordinate is dimensionless as $\zeta \equiv \sqrt{C}x^3/f_\eta$. The horizontal coordinate is also dimensionless with $\tilde{w} = \sqrt{C}w/f_\eta$. The top row shows $k = 1$ with $|b_0| = 1$ for (a1) and $|b_0| = 2$ for (a2) both of which are energetically degenerated. The bottom row corresponds to $f = b_0/(\tilde{w} - d)(\tilde{w} + d)$. (b_0, d) is taken as $(1, 0)$ for (b1), $(2, 1)$ for (b2), and $(3, 2)$ for (b3). They are also energetically degenerated.

Thus, when the chemical potential is larger than the critical value,

$$\mu_{\mathrm{B}} \geq \mu_c \equiv \frac{4\pi f_\pi^2 f_\eta}{\sqrt{C}} = \frac{4\sqrt{2}\pi f_\pi f_\eta}{m_\pi} \sim 1.21[\mathrm{GeV}], \tag{4.9}$$

the energy of lumps are negative, $E_{\mathrm{DW}} \leq 0$ so that lumps are spontaneously created. This implies the presence of the domain-wall Skyrmion phase for rotation. In the evaluation in eq. (4.9), we have used the vacuum values of the physical quantities $f_\pi \approx 93\,\mathrm{MeV}$ and $m_\pi \approx 140\,\mathrm{MeV}$, and the relation $f_{\eta'}/f_\pi = 1.1$ valid for the three flavors [81], assuming that the same $f_\eta/f_\pi = 1.1$ holds for the two flavors. The value of μ_c implies the effective nucleon mass in this environment inside the soliton at finite density under rapid rotation. It is interesting to note that this value is reasonably heavier than the nucleon mass $\sim 938\,\mathrm{MeV}$ in the QCD vacuum.

5 Summary and discussion

We have shown a signal of the presence of a new phase of rapidly rotating QCD matter in high density region, that is a domain-wall Skyrmion phase. It was previously known based on the chiral Lagrangian with the CVE term [50] that the ground state is a CSL consisting of a stack of η-solitons for two flavors (η'-solitons for three flavors) in a high density region under rapid rotation [51]. In a large parameter region, a single η-soliton decays into a pair of non-Abelian sine-Gordon solitons [21, 69], each of which carries $\mathrm{SU}(2)_{\mathrm{V}}/\mathrm{U}(1) \simeq \mathbb{C}P^1 \simeq S^2$

moduli as a consequence of the spontaneous breaking of the vector symmetry $\mathrm{SU}(2)_\mathrm{V}$ in the vicinity of each soliton [53]. In such a non-Abelian CSL, we have shown that the effective world-volume theory of a single non-Abelian soliton is a $d = 2+1$ dimensional $\mathbb{C}P^1$ model [O(3) model] with a topological term originated from the WZW term, eq. (3.14). We have shown that when the chemical potential is larger than a critical value in eq. (4.9), a lump has negative energy to be spontaneously created on the soliton world-volume, implying the domain-wall Skyrmion phase. This lump on the soliton world-volume corresponds to a Skyrmion carrying a baryon number in the bulk point of view, in contrast to the domain-wall Skyrmions in the magnetic field [13, 14], in which case one lump on the soliton worldvolume corresponds to two Skyrmions in the bulk. The effective nucleon mass has been found to be $\sim 1.21\,\mathrm{GeV}$, which is reasonably heavier than the nucleon mass $\sim 938\,\mathrm{MeV}$ in the QCD vacuum. We have worked out at the leading order $\mathcal{O}(p^2)$ of the ChPT and do not need higher derivative terms such as the Skyrme term.

We concentrated on the deconfined phase in which the up and down solitons appear altnately with equal distance. Then, we focused on a single up(or down) soliton. There is another a non-Abelian CSL which is dimer phase as in the left panel of figure 2. In such a case, up and down solitons form a pair and such pairs constitute a lattice. Therefore, we can discuss effective theory on a pair of up and down sotlions. Then, we can construct domain-wall Skyrmions on a pair. This remains a future problem.

In our recent paper [14], we found a similar domain-wall Skyrmion phase in the case of strong magnetic field. In this case, we studied a chain of Skyrmions in *multiple* solitons without taking a signle-soliton approximation. In such a case, the domain-wall Skyrmion phase extends to the lower density (with stronger magnetic field) region. The same discussion can be repeated for the case of rapid rotation in this paper. We thus expect the domain-wall Skymion phase is extended to a region of lower density with more rapid rotation.

We have studied the domain-wall Skyrmion phase at the leading order $\mathcal{O}(p^2)$ of the ChPT. At this order, the topological lumps are BPS, and thus there are no forces between them. For instance, the configurations of two lumps at arbitrary separations in figure 3 are all energetically degenerated. This would not be the case if we go to the next leading order $\mathcal{O}(p^4)$. At this order, one has to include four derivative terms such as the Skyrme term in ChPT. The more important is the inclusion of the terms proportional to the angular velocity Ω in the domain-wall effective action in eq. (3.21) that will exert centrifugal force on lumps. It is an important problem to investigate the next leading order $\mathcal{O}(p^4)$, because at that order we should be able to find Skyrmion lattice configurations as the ground state of the domain-wall Skyrmion phase.

Next, let us discuss the limit of zero rotation. The Skyrmion energy in eq. (4.8) itself does not depend on the rotation as a consequence of the fact that the GW current in eq. (2.5) does not. Therefore, in the limit of zero rotation, the solitons disappear and there should remain conventional Skyrmions in the bulk (at least in the large N_c). However, this limit can be discussed with the Skyrme term in the next leading order $\mathcal{O}(p^4)$. Otherwise Skyrmions are unstable in the bulk. We leave this limit as one of the future problems.

Let us discuss the possibility that the non-Abelian CSL and domain-wall Skyrmion phase may be reached in the near future low-energy non-central heavy-ion collision experiments.

The magnitude of the largest vorticity of the current experiment is the order of 10^{22}/s [37, 38]. Let us roughly estimate the critical S for a rotating nuclear matter made from $^{197}_{79}$Au with saturation density $n \approx 0.16/\,\text{fm}^3$. The corresponding μ_B is $\mu_\text{B} \approx 1\,\text{GeV}$. Then, we estimate the critical S as

$$S \approx 0.04\,, \tag{5.1}$$

where we have used the vacuum values of $f_\pi \approx 93\,\text{MeV}$, $f_\eta/f_\pi \approx 1.1$, $m_\pi \approx 140\,\text{MeV}$, $m_\eta \approx 770\,\text{MeV}$ [81, 82], and experimental angular velocity $\Omega_\text{exp} \approx 10\,\text{MeV}$ [37, 38], together with the Gell-Mann-Oakes-Renner relation for the two-flavor case,

$$f_\pi^2 m_\pi^2 = 2Bm\,, \qquad f_\eta^2 m_\eta^2 = 2Bm + 4A\,. \tag{5.2}$$

Although the critical rotation velocity of the η'-CSL for three flavors is larger by one order of magnitude [51], that of the non-Abelian CSL is smaller than that of the η'-CSL [53]. Our results can be extended to more realistic case of the three-flavors. Also, the $\Lambda(\bar{\Lambda})$ hyperon polarization increases as the collision energy $\sqrt{s}$ decreases, implying the larger angular velocity [37, 38]. Another important issue is temperature effects. The collider experiments are done at high temperature while our analysis is based on zero temperature. As this regards, it was shown in the case of strong magnetic fields that finite temperatures rather increase the stability of the CSL [56, 57]. We thus can expect that the same holds for the case of the CSL under rapid rotation. Thus, the non-Abelian CSL and domain-wall Skyrmion phase may be reached in the near future low-energy heavy-ion collision experiments.

In this paper, we have not considered the electromagnetism. There are two electromagnetic couplings: the minimal and anomalous couplings. First, the non-Abelian CSL is made of the eta meson and neutral pion and are neutral in the electromagnetism. By contrast, the charged pions have nontrivial profiles around the lumps studied in this paper, and thus the lumps are charged. The electromagnetic U(1) symmetry is spontaneously broken in the vicinity of the lumps, and consequently the lumps are superconducting. More precisely, the lumps are superconducting rings, so that their sizes are quantized if there is an external magnetic field. Second, there is an anomalous coupling to the electromagnetism [4, 5]. Consequently, magnetizations appear on the non-Abelian CSLs through the topological term under an external magnetic field $\boldsymbol{B}$ [4, 5, 83]: $\mathcal{L}_\text{top} = \frac{q_\text{u}\mu_\text{B}}{4\pi^2}\nabla\phi_+ \cdot \boldsymbol{B} + \frac{q_\text{d}\mu_\text{B}}{4\pi^2}\nabla\phi_- \cdot \boldsymbol{B}$. The rotation induced ferro(ferri) magnetism was discussed in our previous paper [53]. It is an open question how Skyrmions considered in this paper affect on the magnetization.

If we introduce the isospin chemical potential μ_I, there is also an anomalous coupling to the neutral pion π_0 given by $\frac{\mu_\text{B}\mu_\text{I}}{2\pi^2 f_\pi}\Omega \cdot \nabla\pi_0$ [50], giving an another topological term in addition to $\Omega \cdot \nabla\eta$ that we have considered in this paper. In this case, both the neutral pion π_0 and η meson try to constitute a lattice with different periodicities, but it is impossible because these solitons interact. Consequently, the ground state is not periodic anymore and is rather a quasicrystal, as discussed in the case of strong magnetic fields [60]. In such a case, we still can discuss domain-wall Skyrmions forcusing on an η or π_0 soliton as a constituent of the quasicrystal at least when solitons are well separated for very rapid rotation. There is an additional term on the worldvolume theory on non-Abelian η soliton due to the above topological term (see footnote 5). Then, the lumps would have constraint as the case of those for the strong magnetic field [13, 14].

While we have considered two flavors in this paper, more realistic case is three flavors including the strange quark. In this case, the chiral symmetry $\mathrm{SU}(3)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{L}}$ is spontaneously broken to $\mathrm{SU}(3)_{\mathrm{V}}$ as well as the axial $\mathrm{U}(1)_{\mathrm{A}}$ symmetry, and the order parameter manifold is $[\mathrm{SU}(3)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{R}} \times \mathrm{U}(1)_{\mathrm{A}}]/[\mathrm{SU}(3)_{\mathrm{V}} \times \mathbb{Z}_3] \simeq \mathrm{U}(3)$. Then, a single U(3) non-Abelian soliton spontaneously breaks $\mathrm{SU}(3)_{\mathrm{V}}$ to its subgroup $\mathrm{SU}(2) \times \mathrm{U}(1)$, and thus there appear $\mathrm{SU}(3)/[\mathrm{SU}(2) \times \mathrm{U}(1)] \simeq \mathbb{C}P^2$ NG modes in the vicinity of the soliton. Then, $\mathbb{C}P^2$ lumps on the soliton corresponds to SU(3) Skyrmions in the bulk [21].

Now, we make comments on the domain-wall Skyrmions in quark matter at large μ_{B}. The ground state in the two-flavors case without rotation is the two-flavor superconducting (2SC) phase [84] in which the chiral symmetry is unbroken. Instead, for the three flavors, the ground state without rotation is the color-flavor locked (CFL) phase [85, 86] (see ref. [84] as a review), in which the chiral symmetry $\mathrm{SU}(3)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{R}}$ is spontaneously broken as well as $\mathrm{U}(1)_{\mathrm{A}}$. As the case of three-flavor nuclear matter, an η'-CSL phase appears under the rapid rotation [51], in which a continuity to the three-flavor CSL in nuclear matter was also studied. The instanton effect is suppressed due to the Debye screening [87, 88] in the large μ_{B} region, implying small β so that non-Abelian CSL is favored. Then, domain-wall Skyrmions can be constructed as $\mathbb{C}P^2$ lumps on a U(3) non-Abelian soliton. It was discussed in ref. [89] that Skyrmions in the CFL phase can be regarded as quarks instead of baryons and are called qualitons. Thus, quarks condensed outside the non-Abelian solitons which is in the CFL phase may not be condensed inside the non-Abelian solitons, similar to Andreev bound states in superconductors.

Apart from application to QCD, there are also interesting points for physics of topological solitons. For instance, more general non-BPS solutions of the $\mathbb{C}P^{N-1}$ model can be constructed by the Din and Zakrzewski's projection method [79, 80, 90]. One of questions is what these correspond to in the bulk. The other is fractional $\mathbb{C}P^{N-1}$ lumps in a twisted boundary condition [91, 92]. What is the meaning of fractional baryons in the bulk perspective?

Acknowledgments

This work is supported in part by JSPS KAKENHI [Grants No. JP19K03839 (ME) No. JP21H01084 (KN), and No. JP22H01221 (ME and MN)] and the WPI program "Sustainability with Knotted Chiral Meta Matter (SKCM^2)" at Hiroshima University.

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