Topological index associated with transverse axial vector and vector anomalies in QED *

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Abstract: It is shown that a novel anomaly associated with transverse Ward-Takahashi identity exists for a pseudo-tensor current in QED, and the anomaly gives rise to a topological index of a Dirac operator in terms of an Atiyah-Singer index theorem.

Key words: anomaly, Ward-Takahashi, topological index, Atiyah-Singer index theorem, pseudo-tensor current

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1 Introduction

Some time ago Takahashi made the argument for the plausible existence of a transverse Ward-Takahashi (WT) relation in canonical field theory, which has the potential to restrict the transverse vertex function from gauge symmetry alone [1]. Subsequently these transverse WT relations for the fermion-boson vertex in coordinate space (or in momentum space) are cast in four-dimensional Abelian gauge theory by computing the curl of the time ordered products of three-point Green functions [2, 3]. In addition, the proposed transverse WT relation holds at one-loop order level in four dimensions gauge theory [4]. Up to the effect of quantum anomaly, the possible anomaly for the transverse Ward-Takahashi relations in four dimensional gauge theories is studied by He using the point-splitting method [5]. Recently, the anomaly issue was reexamined by means of the perturbative method. The conclusion is that there are no transverse anomalies for both the axial vector and vector current [6]. Also the path-integral derivation of the transverse WT relation for the vector vertex is presented due to a set of infinitesimal transformations of field variables in QED in Refs. [7, 8], wherein topological property of the axial-vector current has been illustrated in terms of the Atiyah-Singer index theorem. Based on the validity of Fujikawa's analysis, we reevaluate in detail the transverse anomaly of the transverse WT identity for the axial-vector and vector vertex in the QED, and we find that a careful application of Fujikawa's approach leads to a transverse quantum anomaly for the axial vector current [9]. In this paper, we point out that the mathematical explanation of such an anomaly is directly related to the Atiyah-Singer index theorem. The topological index for the Dirac operator corresponding to the anomaly will be expressed.

2 Calculation of the anomaly factor in Ward-Takahashi identity

From the point of view of the path-integral formulation, we proposed an infinitesimal transverse transformation of field variables to derive the WT identities [7, 9]. Let us consider a set of infinitesimal local transformations in the QED

$$\begin{aligned}\psi'(x) &= \mathrm{e}^{-\frac{1}{4}\theta(x)\omega_{\mu\nu}\sigma^{\mu\nu}}\psi(x)\\ \overline{\psi}'(x) &= \overline{\psi}(x)\gamma^0\mathrm{e}^{-\frac{1}{4}\theta(x)\omega_{\mu\nu}\sigma^{\mu\nu\dagger}}\gamma^0\\ B'_\mu(x) &= B_\mu(x)+\omega_{\mu\nu}\partial^\nu\theta(x),\end{aligned} \tag{1}$$

where $\omega_{\mu\nu}$ stands for the antisymmetry tensor, $\psi(x)$ and $B_\mu(x)$ are the fermion and gauge fields, respectively. In principle, the variation of the generating functional itself under the transformation of field variables Eq. (1) can lead to Ward-Takahashi-type identities. The change of

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the function integral due to the transformation (choosing $\delta B_\mu(x)=0$ for simplicity) gives the relation in momentum space in QED

$$\begin{aligned}&-\mathrm{i}q^\mu\Gamma_V^\nu(p_1,p_2)+\mathrm{i}q^\nu\Gamma_V^\mu(p_1,p_2)+(p_{1\lambda}+p_{2\lambda})\\&\times\varepsilon^{\lambda\mu\nu\rho}q_\lambda\Gamma_{A\rho}(p_1,p_2)+S^{-1}(p_2)\sigma^{\mu\nu}+S^{-1}(p_1)\sigma^{\mu\nu}\\&-\int\frac{\mathrm{d}^4k}{(2\pi)^4}2k_\lambda\epsilon^{\lambda\mu\nu\rho}\Gamma_{A\rho}(p_1,p_2;k)=0,\end{aligned}\tag{2}$$

This WT relation for the vector current has been listed in Ref. [9]. The integral term in Eq. (2) may be called the integral-term involving the vertex function $\Gamma_{A\rho}(p_1,p_2;k)$ with the internal momentum k of the gauge boson appearing in the Wilson line [10]. The Fourier transformation for the vertex function $\Gamma_{A\rho}(p_1,p_2;k)$ is defined as

$$\begin{aligned}&\int\mathrm{d}^4x\mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{e}^{\mathrm{i}(p_1x-p_2x_2-qx)}\\&\quad\times\langle0|T(\overline{\psi}(x)\varepsilon^{\lambda\mu\nu\rho}\gamma_\rho\gamma^5\psi(x)\overline{\psi}(x_1)\psi(x_2)U_P(x',x))|0\rangle\\&=(2\pi)^4\delta^4(p_1-p_2-q)\mathrm{i}S_F(p_1)\Gamma_{A\rho}(p_1,p_2,k)\mathrm{i}S_F(p_2),\end{aligned}\tag{3}$$

where $q=(p_1-k)-(p_2-k)$. Obviously, the full vector function and the full axial -vector function are coupled with each other. As shown in Ref. [3], the apparent feature of this transverse identity Eq. (2) is that the vertex function Γ_V^μ (fermion's three point function) has the transverse component of itself.

Completely analogous to the calculations above, let us consider the other transverse transformation

$$\begin{aligned}\psi'(x)&=\mathrm{e}^{-\frac{1}{4}\theta(x)\omega_{\mu\nu}\sigma^{\mu\nu}\gamma^5}\psi(x)\\\overline{\psi}'(x)&=\overline{\psi}(x)\gamma^0\mathrm{e}^{-\frac{1}{4}\theta(x)\omega_{\mu\nu}\gamma^{5\dagger}}\sigma^{\mu\nu\dagger}\gamma^0.\end{aligned}\tag{4}$$

We obtain the identity for the axial -vector current in momentum space as

$$\begin{aligned}&-\mathrm{i}q^\mu\Gamma_A^\nu(p_1,p_2)+\mathrm{i}q^\nu\Gamma_A^\mu(p_1,p_2)\\&+(p_{1\lambda},p_{2\lambda})\varepsilon^{\lambda\mu\nu\rho}q_\lambda\Gamma_{V\rho}(p_1+p_2)+\\&+S^{-1}(p_2)\sigma^{\mu\nu}\gamma^5-S^{-1}(p_1)\sigma^{\mu\nu}\gamma^5\\&-\int\frac{\mathrm{d}^4k}{(2\pi)^4}2k_\lambda\epsilon^{\lambda\mu\nu\rho}\Gamma_{V\rho}(p_1,p_2;k)=0.\end{aligned}\tag{5}$$

According to Fujikawa's interpretation, it is argued that the appearance of the quantum anomaly in WT identity is a symptom of the impossibility of defining a suitably invariant functional integral measure due to the relevant transformations on fermionic field variables.The regularization procedure for the variations of the integral measure can provide access to a wider class of such anomaly objects [11–14].To see how the change of the measure corresponding to the transverse transformation Eq. (1) and Eq. (4) gives rise to a possible anomaly factor, let us consider an Abelian gauge field to show our argument. The Lagrangian density for massive QED, which is of the form

$$\begin{aligned}L_{\text{eff}}&=\overline{\psi}(x)\gamma^\mu(\partial_\mu-\mathrm{i}eB_\mu(x))\psi(x)-\overline{\psi}(x)m\psi(x)\\&\quad-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2\xi}(\partial^\mu B_\mu)^2,\end{aligned}\tag{6}$$

where e and m denote, respectively, the charge and mass of the electron. In this case, the gauge field is just the photon field $B_\mu(x)$. Thus Jacobian $J^{[\alpha\beta]}(x)$ of the integral measure due to the transformations Eq. (1) is evaluated below

$$J^{[\alpha\beta]}(x)=\mathrm{e}^{\int\mathrm{d}^4xA^{[\alpha\beta]}(x)\omega_{[\alpha\beta]}\theta(x)}\mathrm{e}^{\int\mathrm{d}^4(x)\bar{A}^{[\alpha\beta]}(x)\omega_{[\alpha\beta]}\theta(x)}.\tag{7}$$

This is what we set out to calculate. Due to the transverse transformation (2.1), the anomaly functions can be written as the limit of a manifestly convergent integral

$$\begin{aligned}A^{[\alpha\beta]}(x)&=\lim_{M\to\infty}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}kx}\left(\frac{-\sigma^{\alpha\beta}}{4}\right)f\left(\left(\frac{\mathrm{i}\gamma^\mu D_\mu}{M}\right)^2\right)\mathrm{e}^{\mathrm{i}kx}\\\overline{A}^{[\alpha\beta]}(x)&=\lim_{M\to\infty}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}kx}\gamma^0\left(\frac{-\sigma^{\alpha\beta}}{4}\right)^\dagger\gamma^0f\left(\left(\frac{\mathrm{i}\gamma^\mu D_\mu}{M}\right)^2\right)\mathrm{e}^{\mathrm{i}kx},\end{aligned}\tag{8}$$

where $D_\mu=\partial_\mu-\mathrm{i}eB_\mu(x)$ is the covariant derivative. In addition, the transformation of the field $B_\mu(x)$ is a translation, so that its Jacobian is trivial. The anomaly function $A^{[\alpha\beta]}(x)$ requires regulation, which is achieved by inserting a regulator

$$f\left(\frac{-(\gamma^\mu D_\mu)^2}{M^2}\right)=\left(\mathrm{e}^{-\frac{(\gamma^\mu D_\mu)^2}{M^2}}\right)^2.\tag{9}$$

The expression of the transverse vector anomaly function $A^{[\alpha\beta]}(x)$ can be put in the regulating form

$$\begin{aligned}A^{[\alpha\beta]}(x)&=\lim_{M\to\infty}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}kx}\left(\frac{-1}{4}\sigma^{\alpha\beta}\right)f\left(\left(\frac{\mathrm{i}\gamma^\mu D_\mu}{M}\right)^2\right)\mathrm{e}^{\mathrm{i}kx}\\&=\lim_{M\to\infty}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}kx}\left[\left(\frac{-1}{4}\sigma^{\alpha\beta}\right)\left(\sum_n\frac{1}{n!}f^{(n)}\left(-\frac{D^2}{M^2}\right)\left(\frac{\mathrm{i}}{4M^2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)^n\right)\right.\end{aligned}$$

$$\times\left[\sum_m \frac{1}{m!} f^{(m)}\left(-\frac{D^2}{M^2}\right)\left(\frac{\mathrm{i}}{4M^2}[\gamma^\rho,\gamma^\sigma]F_{\rho\sigma}\right)^m\right]\mathrm{e}^{\mathrm{i}kx}$$
$$=\lim_{M\to\infty}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\left(\frac{-1}{4}\sigma^{\alpha\beta}\right)\left(f^{(')}\left(-\frac{D^2}{M^2}\right)\left(\frac{\mathrm{i}}{4M^2}[\gamma_\mu,\gamma_\nu]F_{\mu\nu}\right)^1\right)\left(f^{(')}\left(-\frac{D^2}{M^2}\right)\left(\frac{\mathrm{i}}{4M^2}[\gamma_\rho,\gamma_\sigma]F_{\rho\sigma}\right)^1\right)\right.$$
$$=\frac{1}{128\pi^2}\left(g^{\alpha\mu}g^{\beta\rho}g^{\nu\sigma}-g^{\alpha\mu}g^{\beta\sigma}g^{\nu\rho}+g^{\alpha\nu}g^{\beta\sigma}g^{\mu\rho}-g^{\alpha\nu}g^{\beta\rho}g^{\mu\sigma}\right.$$
$$\left.+g^{\beta\mu}g^{\alpha\sigma}g^{\nu\rho}-g^{\beta\mu}g^{\alpha\rho}g^{\nu\sigma}+g^{\beta\nu}g^{\alpha\rho}g^{\nu\sigma}-g^{\beta\nu}g^{\alpha\sigma}g^{\mu\rho}\right)\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}). \tag{10}$$

In terms of the symmetry of the metric and antisymmetry of the 4-dimensional field strength tensor, we expand the anomaly function and find that it equals zero. Thus the Jacobian Eq. (7) becomes

$$J^{[\alpha\beta]}(x)=\mathrm{e}^{2\int\mathrm{d}^4xA^{[\alpha\beta]}(x)\omega_{[\alpha\beta]}\theta(x)}=1. \tag{11}$$

By the parallel procedure, for the case of the transformation Eq. (4), the transverse axial vector anomaly function is given by

$$A^{[\alpha\beta5]}(x)=\lim_{M\to\infty}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}kx}\left(\frac{-1}{4}\sigma^{\alpha\beta}\gamma^5\right)f\left(\left(\frac{\mathrm{i}\gamma^\mu D_\mu}{M}\right)^2\right)\mathrm{e}^{\mathrm{i}kx}$$
$$=\lim_{M\to\infty}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}kx}\left[\left(\frac{-1}{4}\sigma^{\alpha\beta}\gamma^5\right)\left(\sum_n\frac{1}{n!}f^{(n)}\left(-\frac{D^2}{M^2}\right)\left(\frac{\mathrm{i}}{4M^2}[\gamma_\mu,\gamma_\nu]F_{\mu\nu}\right)^n\right)\right.$$
$$\left.\times\left[\sum_m\frac{1}{m!}f^{(m)}\left(-\frac{D^2}{M^2}\right)\left(\frac{\mathrm{i}}{4M^2}[\gamma^\rho,\gamma^\sigma]F_{\rho\sigma}\right)^m\right]\mathrm{e}^{\mathrm{i}kx}\right.$$
$$=\lim_{M\to\infty}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\left(\frac{-1}{4}\sigma^{\alpha\beta}\gamma^5\right)\left(f^{(')}\left(-\frac{D^2}{M^2}\right)\left(\frac{\mathrm{i}}{4M^2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)^1\right)\left(f^{(')}\left(-\frac{D^2}{M^2}\right)\left(\frac{\mathrm{i}}{4M^2}[\gamma^\rho,\gamma^\sigma]F_{\rho\sigma}\right)^1\right)\right.$$
$$=\frac{\mathrm{i}}{128\pi^2}\left(-g^{\alpha\mu}\epsilon^{\beta\nu\rho\sigma}+g^{\alpha\nu}\epsilon^{\beta\mu\rho\sigma}+g^{\beta\mu}\epsilon^{\alpha\nu\rho\sigma}-g^{\beta\nu}\epsilon^{\alpha\mu\rho\sigma}\right.$$
$$\left.+g^{\nu\rho}\epsilon^{\alpha\beta\mu\sigma}-g^{\mu\rho}\epsilon^{\alpha\beta\nu\sigma}+g^{\mu\sigma}\epsilon^{\alpha\beta\nu\rho}-g^{\nu\sigma}\epsilon^{\alpha\beta\mu\rho}\right)\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma})=-\bar{A}^{[\alpha\beta5]}(x). \tag{12}$$

The corresponding Jacobian is non-trivial in the form

$$J^{[\alpha\beta5]}(x)=\mathrm{e}^{2\int\mathrm{d}^4xA^{[\alpha\beta]}(x)\omega_{[\alpha\beta]}\theta(x)}. \tag{13}$$

Obviously Eq. (11) and Eq. (13) are perfectly consistent with the result of the derivation of the transverse vector and axial vector anomalies in four-dimensional $U(1)$ gauge theory using perturbative methods [5].

3 Topological index for the anomaly

From the topological viewpoint, the topological property of the quantum anomaly Eq. (13) is addressed by the Atiyah-Singer index theorem for the Dirac operator in Eq. (6) in a gauge background [15]. Since D is an elliptic operator, the Atiyah-Singer local index formula for the topological index $\mathrm{Ind}_{\mathrm{t}}D$ of D in terms of the chern character of $\mathrm{Ch}E$ and the $\widehat{A}$-genus of E^4 gives

$$\mathrm{Ind}_{\mathrm{a}}D=\mathrm{Ind}_{\mathrm{t}}D=\int\widehat{A}(M)\cdot\mathrm{Ch}(E). \tag{14}$$

Here the more natural definition of the $\widehat{A}$-genus form with the Riemannian curvature R is

$$\widehat{A}(M)=\det{}^{\frac{1}{2}}\frac{\frac{R}{2}}{\sinh\left(\frac{R}{2}\right)}. \tag{15}$$

The chern character form is

$$\mathrm{Ch}(E)=\mathrm{Str}\left(\mathrm{e}^{-\nabla^2}\right). \tag{16}$$

Substituting this into Eq. (14) we have

$$\mathrm{Ind}_{\mathrm{t}}D=-\frac{4}{32\pi^2}\int\mathrm{d}^4x\mathrm{Tr}\partial_\mu(\varepsilon^{\mu\nu\rho\sigma}B_\nu\partial_\rho B_\rho). \tag{17}$$

Obviously the quantity on the right of Eq. (17) is known as the chen-pontrjagin term. Eq. (17) can be expressed by a complete set of eigenstates ϕ_n of the Dirac operater D

$$\mathrm{Ind}_{\mathrm{t}}(D)=\lim_{M\to\infty}\int\mathrm{d}^4x\langle x|\Gamma^{[5]}f\left(-\frac{D^2}{M^2}\right)|x\rangle$$
$$=\int\mathrm{d}^4xA^{[5]}(x). \tag{18}$$

Here, the result in Eq. (18) is no other than the topological index for the corresponding anomaly function $A^{[5]}(x)$ of the axial-vector current [8]. Furthermore we generalize the above approach to the pseudo-tensor current. According to the local Atiyah-Singer index theorem, the existence of an asymptotic expansion for $\langle x|\Gamma^{[\alpha\beta5]}f(-D^2/M^2)|x\rangle$ at $M\to\infty$ is presented by the index $\mathrm{Ind}(\Gamma^{[\alpha\beta5]},D)$ of the Dirac operator D on the manifold M, which is calculated below

$$\begin{aligned}\mathrm{Ind}(\Gamma^{[\alpha\beta5]},D) &= \lim_{M\to\infty}\int\mathrm{d}^4x\langle x|\Gamma^{[\alpha\beta5]}f\left(-\frac{D^2}{M^2}\right)|x\rangle\\ &= \int\mathrm{d}^4xA^{[\alpha\beta5]}(x). \qquad (19)\end{aligned}$$

Here the kernel function $A^{[\alpha\beta5]}(x)$ in Eq. (20) is none other than the anomaly function Eq. (12) for the Jacobian of the functional measure Eq. (8). This is what we do. It is shown that the index theorem for the operator $\Gamma^{[\alpha\beta5]}\mathrm{e}^{(-D^2/M^2)}$ associated with the square of Dirac operator D is a statement about the relationship between the kernel $A^{[\alpha\beta5]}(x)$ and the topological character in gauge theory. Fortunately, the asymptotic expansion of the kernel $A^{[\alpha\beta]}(x)$ of the operator $\Gamma^{[\alpha\beta5]}\mathrm{e}^{(-D^2/M^2)}$ on the right of Eq. (5) has been evaluated in Eq. (12) by Fujikawa's method for these fermion pseudo-currents.

4 Concluding remarks

As already described, the topological index of the Dirac operator, reflecting the property of the fermion current coupling gauge field, is indeed understood on the Atiyah-Singer index theorem in quantum field theory as a consequence of the fact that the Jacobian of the integration measure possesses anomaly terms, which is associated with Ward-Takahashi identity for the pseudo-tensor current in QED.

References

1 Takahashi Y, Mancini F. In Quantum Field Theory. Amsterdam: Elsevier Scince Publisher, 1986. 19
2 KondoK-I, Maris. Phys. Rev. D, 1995, **52**: 1212–1233
3 HE H X, Khanna F C, Takahashi Y. Phys. Lett. B, 2000, **480**: 222–228
4 Pennnington M R, Williams R. J. Phys. G: Nucl. Part. Phys., 2006, **32**: 2219–2233
5 HE H X. Phys. Lett. B, 2001, **507**: 351–355
6 SUN W S, ZONG H S, CHEN X S et al. Phys. Lett. B, 2003, **569**(2): 211–238
7 BAO A D, WU S S. Int. J. Theor. Phys., 2007, **46**: 12—27
8 BAO A D, YAO H B, WU S S. Chin. Phys. C, 2009, **33**(3): 177–182
9 BAO A D, SUN Y Q, WAN D. Chin. Phys. Lett., 2012, **29**: 121101–121104
10 HE H X. An Introduction to Nuclear Chromodynamics. Hefei: China University of Sci and Tech. Press, 2009. 12
11 HE H X. Int. J. Mod. Phys. A, 2007, **22**: 2119–2132
12 Fujikawa K. Phys. ReV. Lett., 1997, **42**: 1195–1198
13 EinhornM B, T Jones D R. Phys. Rev. D, 1984, **29**: 331–333
14 Umezswa M. Phys. Rev. D, 1989, **39**: 3672–3683
15 Verline N, Getzle E, Michele, Vergne. Heat Kernels and Dirac Operators. Veilag Berlin Heidelberg: Springer, 2004. 18–169