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Antisymmetric tensor field and Cheshire Cat smile of the local conformal symmetry

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Abstract The conformal version of the antisymmetric second-order tensor field in four spacetime dimensions does not have gauge invariance extensively discussed in the literature for more than half a century. Our first observation is that, when coupled to fermions, only the conformal version provides renormalizability of the theory at the one-loop level. General considerations are supported by the derivation of one-loop divergences in the fermionic sector, indicating good chances for asymptotic freedom. The arguments concerning one-loop renormalizability remain valid in the presence of self-interactions and the masses for both fermion and antisymmetric tensor fields. In the flat spacetime limit, even regardless the conformal symmetry has gone, there is an expectation to meet renormalizability in all loop orders.

1 Introduction

What we know about fundamental physics converges to the Minimal Standard Model (MSM) with certain modifications related to the neutrino masses. On the other hand, this knowledge comes from the particle physics and the tests on accelerators and in the high-precision experiments, both ways provide the information restricted by available energy scales and possible large masses of the particles (or bound states, or condensates) beyond MSM. The possible particles beyond MSM may belong to the conventional irreducible representations of the Lorentz group, i.e., scalars, spinors and vectors. But there is also a possibility to have other types of fields, such as an antisymmetric second order tensor, which is the subject of the present work.

Starting from the 1966 paper [1] (see also [2]), the study of antisymmetric tensor fields attracted a lot of interest, being an important part of the gauge field theory. Let us start by mentioning a few relevant references. Geometric formulation and relation to gravity have been discussed in [3]. Interaction of antisymmetric field to matter and the formulation of the corresponding renormalizable theory was given in [4]. It is worth noting that the interaction of a Dirac fermion with the electromagnetic field tensor $F_{\mu\nu}$ appears naturally in the effective context of QED (see, e.g., [5–7] and references therein).

The antisymmetric tensor field represents an interesting theoretical model in many respects (see, e.g., [8–11]), including the description of Lorentz violation [12,13]. In the gravitational physics, antisymmetric fields may be a viable (albeit not extensively explored yet) alternative to scalars and vectors in the models of inflation [14] (see also most recent work [15] and further references therein). The last aspect looks especially important owing to the existing expectations to go far beyond the framework of MSM using cosmological and astrophysical data instead of laboratory experiments. In both (cosmology and laboratory) cases, it is important to formulate the new field in a consistent way, that should include interaction with matter (i.e., with fermions, in the first place) and the possibility to incorporate quantum effects, at least in the lowest nontrivial orders of the loop expansion.

In the quantum field theory framework, the effective action of massive antisymmetric fields in curved spacetime was discussed, e.g., in [16–19]. One can say that many different issues related to antisymmetric fields have been discussed and explored in the literature. In the present work, we aim to explore a local conformal symmetry of the second-order antisymmetric tensor field. Previously this issue was addressed in relation to conformal supergravity [20,21] where the conformal actions similar (albeit not equivalent) to those calculated below were obtained. Let us also mention the mathematical works (see, e.g., [22–24] and further references therein) and the preprint [25], where the conformal operators acting on k-forms were constructed and explored. In what follows, we shall obtain the conformal action of this theory with certain generalizations and also show how the requirement of con-

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formal invariance helps to establish the unique general form of the action which is compatible with the quantum consistency. Anticipating the result, let us say that this form does not possess the gauge symmetry [1] and, therefore, requires that the antisymmetric tensor field propagates more than the minimal amount of degrees of freedom required by the irreducible representation of the Lorentz group.

In the following, we show how the conformal symmetry is operational in formulating the consistent action of antisymmetric field coupled to fermions, and confirm our arguments by deriving the one-loop divergences coming from the quantum Dirac field loop. The rest of the article is organized as follows. In Sect. 2 we construct the expressions invariant under the local conformal symmetry and discuss how this symmetry helps to fix the general form of renormalizable action, including in the case when this symmetry is broken by massive terms. Section 3 reports on the derivation of one-loop divergences. This calculation includes a number of reduction relations which are collected in the Appendixes A and B. In Sect. 4 we write down the action of renormalizable (at least at the one-loop level) theory which follows from the considerations of the previous sections and explore renormalization group equations in the new renormalizable model. A short Sect. 5 presents the flat-spacetime limit of the theory and arguments of why this theory may be all-loop renormalizable, even despite the guiding conformal symmetry does not exist in the flat limit. Finally, in Sect. 6 we draw our conclusions and describe the possible extensions of the present work.

The analysis is restricted to the four-dimensional spacetime ($4D$). The conventions include the signature $(+,-,-,-)$, regardless the Wick rotation to the Euclidean space is assumed in the part of the heat-kernel calculations. The definition of the Riemann tensor is $R^{\alpha}_{\ .\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\,\mu} - \Gamma^{\alpha}_{\beta\mu,\,\nu} + ...$, of the Ricci tensor $R_{\alpha\beta} = R^{\mu}_{\ .\alpha\mu\beta}$, and the scalar curvature $R = R^{\alpha}_{\alpha}$. Our notations for derivatives are $\nabla A = A\nabla + (\nabla A)$, except those places where there cannot be misunderstanding.

2 Conformal theory of antisymmetric tensor field

Our first purpose is to construct the theory of antisymmetric tensor field $B_{\mu\nu} = -B_{\nu\mu}$, possessing local conformal symmetry in curved spacetime. The first step is to postulate the transformation rule for this tensor. We define the conformal transformations as

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\, e^{2\sigma}, \qquad B_{\mu\nu} = \bar{B}_{\mu\nu}\, e^{\sigma}, \qquad \sigma = \sigma(x). \tag{1}$$

The indices of the tensors without bars are raised and lowered using the metric, therefore $B^{\mu\nu} = \bar{B}^{\mu\nu}\, e^{-3\sigma}$. We shall see, in what follows, that (1) is a fortunate choice, providing the global conformal symmetry for the bilinear action of the field $B_{\mu\nu}$ with two derivatives. Let us, from the very beginning, agree that the global transformation is when σ is a constant, while local transformation assumes $\sigma = \sigma(x)$. Indeed, local conformal symmetry implies a global one, but not *vice versa*. However, one can separate the terms in the action which have a global symmetry and then look for their combinations possessing local symmetry.

In case the metric $g_{\mu\nu}$ and the antisymmetric tensor field $B_{\mu\nu}$ are coupled to the set of matter fields Φ_i with the conformal weights k_i, the condition of conformal symmetry means that the conformal Noether identity should be satisfied,

$$2g_{\mu\nu}\,\frac{\delta S_c}{\delta g_{\mu\nu}} \,+\, B_{\mu\nu}\,\frac{\delta S_c}{\delta B_{\mu\nu}} + \sum_i k_i\,\Phi_i\,\frac{\delta S_c}{\delta \Phi_i} \,=\, 0, \tag{2}$$

where the transformation rules for the fields Φ_i are

$$\Phi_i \,=\, \bar{\Phi}_i e^{k_i\sigma}. \tag{3}$$

Our purpose is to formulate the conformal action of the metric and the field $B_{\mu\nu}$, and subsequently explore its relevance for the one-loop and higher-loop renormalization. The first step is to write down the list of obvious local conformal invariants, including

$$\begin{aligned} W_1 &= \sqrt{-g}\, B^{\mu\nu} B^{\alpha\beta} C_{\alpha\beta\mu\nu}, \\ W_2 &= \sqrt{-g}\, (B_{\mu\nu} B^{\mu\nu})^2, \\ W_3 &= \sqrt{-g}\, B_{\mu\nu} B^{\nu\alpha} B_{\alpha\beta} B^{\beta\mu}, \end{aligned} \tag{4}$$

and

$$W_{11} = \sqrt{-g}\, B^{\mu\alpha} B^{\nu\beta} C_{\alpha\beta\mu\nu} = \frac{1}{2}\, W_1. \tag{5}$$

where $C_{\alpha\beta\mu\nu}$ is Weyl tensor. The reduction formula (5) can be easily obtained using the cyclic identify for this tensor, $C_{\alpha\beta\mu\nu} + C_{\alpha\nu\beta\mu} + C_{\alpha\mu\nu\beta} = 0$.

The list of global conformal invariants includes

$$\begin{aligned} K_1 &= \sqrt{-g}\, B^{\mu\nu} B^{\alpha\beta} R_{\mu\alpha}\, g_{\nu\beta}, \\ K_2 &= \sqrt{-g}\, B_{\mu\nu} B^{\mu\nu} R, \\ K_3 &= \sqrt{-g}\, (\nabla_\alpha B_{\mu\nu})(\nabla^\alpha B^{\mu\nu}) = \sqrt{-g}\, (\nabla_\alpha B_{\mu\nu})^2, \\ K_4 &= \sqrt{-g}\, (\nabla_\mu B^{\mu\nu})(\nabla^\alpha B_{\alpha\nu}) = \sqrt{-g}\, (\nabla_\mu B^{\mu\nu})^2. \end{aligned} \tag{6}$$

There are also three reducible expressions

$$\begin{aligned} K_{11} &= \sqrt{-g}\, B^{\mu\nu} B^{\alpha\beta} R_{\mu\nu\alpha\beta} = 2K_1 - \frac{1}{3}\, K_2 + W_1. \\ K_{12} &= \sqrt{-g}\, B^{\mu\alpha} B^{\nu\beta} R_{\mu\nu\alpha\beta} = \frac{1}{2}\, K_{11}. \\ K_{31} &= \sqrt{-g}\, (\nabla_\alpha B_{\mu\nu})(\nabla^\mu B^{\alpha\nu}) = K_4 - \frac{1}{6}\, K_2 + \frac{1}{2}\, W_1 \\ &\quad + \nabla_\alpha \big[B_{\mu\nu} (\nabla^\mu B^{\alpha\nu}) - B^{\alpha\nu} (\nabla^\mu B_{\mu\nu}) \big]. \end{aligned} \tag{7}$$

Consider the infinitesimal conformal variations of the irreducible terms. The basic variations are (see, e.g., [26])

$$\delta_c \Gamma^\lambda_{\alpha\beta} = \delta^\lambda_\alpha \sigma_\beta + \delta^\lambda_\beta \sigma_\alpha - \bar{g}_{\alpha\beta}\sigma^\lambda,$$
$$\delta_c R = -2\bar{R}\sigma - 6\bar{\Box}\sigma,$$
$$\delta_c R_{\alpha\beta} = -\bar{g}_{\alpha\beta}\bar{\Box}\sigma - 2\sigma_{\alpha\beta}, \qquad (8)$$

where $\sigma_\alpha = \bar{\nabla}_\alpha \sigma$ and $\sigma_{\alpha\beta} = \bar{\nabla}_\alpha \bar{\nabla}_\beta \sigma$. The covariant derivatives with bars are constructed with the corresponding metric $\bar{g}_{\alpha\beta}$.

After certain algebra, we obtain the variations of the terms (6)

$$\delta_c K_1 = \sqrt{-\bar{g}}\, \bar{B}^{\mu\nu}\big[2\sigma^\lambda(\bar{\nabla}_\lambda \bar{B}_{\mu\nu}) + 2\sigma_\nu(\bar{\nabla}^\lambda \bar{B}_{\mu\lambda}) + 2\sigma^\lambda(\bar{\nabla}_\nu \bar{B}_{\mu\lambda})\big],$$
$$\delta_c K_2 = \sqrt{-\bar{g}}\, \bar{B}^{\mu\nu}\big[12\sigma^\lambda(\bar{\nabla}_\lambda \bar{B}_{\mu\nu})\big],$$
$$\delta_c K_3 = \sqrt{-\bar{g}}\, \bar{B}^{\mu\nu}\big[4\sigma_\nu(\bar{\nabla}^\lambda \bar{B}_{\mu\lambda}) - 4\sigma^\lambda(\bar{\nabla}_\nu \bar{B}_{\mu\lambda}) - 2\sigma^\lambda(\bar{\nabla}_\lambda \bar{B}_{\mu\nu})\big],$$
$$\delta_c K_4 = \sqrt{-\bar{g}}\, \bar{B}^{\mu\nu}\big[2\sigma_\nu(\bar{\nabla}^\lambda \bar{B}_{\mu\lambda})\big]. \qquad (9)$$

It is easy to see that there are only three different variations, which can be called Z_1, Z_2 and Z_3. Then

$$\delta_c K_1 = 2Z_1 + 2Z_2 + 2Z_3,$$
$$\delta_c K_2 = 12Z_1,$$
$$\delta_c K_3 = -2Z_1 + 4Z_2 - 4Z_3,$$
$$\delta_c K_4 = 2Z_2, \qquad (10)$$

from what follows the combination $K_3 - 4K_4 + 2K_1 - \frac{1}{6}K_2$ is invariant. Thus, we arrive at the fourth (and the last) conformal invariant

$$W_4 = \sqrt{-g}\Big\{(\nabla_\alpha B_{\mu\nu})(\nabla^\alpha B^{\mu\nu}) - 4(\nabla_\mu B^{\mu\nu})(\nabla^\alpha B_{\alpha\nu}) + 2B^{\mu\nu}R^\alpha_\nu B_{\mu\alpha} - \frac{1}{6}RB_{\mu\nu}B^{\mu\nu}\Big\}. \qquad (11)$$

The last expression is interesting in various respects, so we can make a list.

(i) A linear combination of the terms (4) and (11) with arbitrary coefficients represents a new conformal theory in $4D$. The expression very similar to (11) was obtained in [20] (see also [21]). Other previously known examples include two- and four-derivative scalars formulated in [20,27–29], one-derivative and three-derivative spinors and gauge vector field. It is also worth mentioning the recent work [30] on the conformal theory of mixed-symmetry tensors.
One of the unusual features of the conformal model (11) is the presence of nonminimal interaction of the CBB-type with an arbitrary coefficient. In all previously known examples in $4D$, the requirement of local conformal symmetry did not leave space for arbitrary nonminimal parameters. Such freedom emerges only in $6D$ for the six-derivative conformal operator acting on a scalar [31,32].

(ii) Out of the known examples of conformal theories mentioned in the first point, only the vector model possesses also gauge symmetry. One could expect the same for the antisymmetric field $B_{\mu\nu}$, but this is not the case. It is known starting from the seminal paper by Ogievetsky and Polubarinov [1], that the "usual" antisymmetric field theory, in flat spacetime, has a gauge symmetry under the transformation

$$\delta b_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \qquad (12)$$

where the vector field ξ_μ satisfies the condition $\partial_\mu \xi^\mu = 0$.
The calculations of the divergences of the vacuum effective action [17–19] had to deal with the degeneracy related to this symmetry, making quantization and the calculations themselves being highly nontrivial and interesting. Our third observation is that the conformal term (11) differs from the gauge invariant theory and, in particular, it is not degenerate.
Let us elaborate on this feature in some detail. One can see that

$$F_{\mu\nu\lambda} = \nabla_\mu b_{\nu\lambda} + \nabla_\lambda b_{\mu\nu} + \nabla_\nu b_{\lambda\mu} = \partial_\mu b_{\nu\lambda} + \partial_\lambda b_{\mu\nu} + \partial_\nu b_{\lambda\mu}. \qquad (13)$$

The same replacement of partial derivative by the covariant one can be performed in (12) and the invariance of $F_{\mu\nu\lambda}$ obviously holds. A small algebra shows that the gauge invariant combination has the form

$$\mathcal{L}_{inv} = \frac{1}{3}\sqrt{-g}\, F_{\mu\nu\lambda}F^{\mu\nu\lambda} = \sqrt{-g}\big(K_3 - 2K_{31}\big)$$
$$= \sqrt{-g}\Big(K_3 - 2K_4 + \frac{1}{3}K_2 - W_1\Big)$$
$$= \sqrt{-g}\Big\{(\nabla_\alpha b_{\mu\nu})^2 - 2(\nabla_\mu b^{\mu\nu})^2 + \frac{1}{3}b^2_{\mu\nu}R - b_{\mu\nu}b_{\alpha\beta}C^{\alpha\beta\mu\nu}\Big\}, \qquad (14)$$

where we used the reduction formula (7). Using the approach utilized in the search for conformal symmetry, one can show that the last Lagrangian is a unique gauge invariant combination with at most two derivatives of the field.
It is easy to note that the linear combinations in (14) and (11) are different. One can also verify that the bilinear form of the gauge invariant Lagrangian (14) is degenerate, while the bilinear form of the conformal invariant Lagrangian (11) is not degenerate. On the other hand, the conformal transformation in the Lagrangian (14)

gives

$$\delta_c\left(\sqrt{-g}F_{\mu\nu\lambda}F^{\mu\nu\lambda}\right) = 6\sqrt{-g}B_{\mu\nu}\sigma^{\lambda}\big(\nabla_\mu B_{\nu\lambda} +\nabla_\lambda B_{\mu\nu} + \nabla_\nu B_{\lambda\mu}\big) \neq 0, \quad (15)$$

that shows its conformal non-invariance, opposite to (11). Let us note that this difference can be seen already in the conformal equations of motion derived in [25].

(iii) It is known that the gauge invariant theory of the antisymmetric field is equivalent to the theory of a real scalar field and the classical [1] and quantum [33,34] levels (the last issue was extensively discussed in different frameworks, e.g., in [17,35,36]. On the other hand, since the conformal model (11) is different from (14), we should not expect the equivalence with the scalar theory. Thus, the conformal model (11) may be an interesting object of study at both classical and quantum levels.

In cosmology, the new conformal theory may be interesting for describing dark radiation or a basis for new models of inflation or of the dark sectors of the matter contents of the late Universe. One of the potentially useful features is that the interaction terms in (4) open the possibility to have broken symmetry at low (IR) or high (UV) energies, such that the same model may have very different conformal properties in the UV and in the IR. Another potentially interesting aspect is that the second order in derivatives nonminimal term W_1 includes the Weyl tensor and therefore is supposed to affect only the cosmic perturbations while it decouples from the conformal factor of the metric.

In quantum theory one can use the new conformal model given by an *arbitrary* linear combination of the terms (4) and (11) to test the important universal features of the trace anomaly, especially the uniformity of signs of the Weyl-squared and Gauss-Bonnet terms (see, e.g., [37–39]). We hope to explore this part in the future works.

iv) Since there are two alternative theories for the antisymmetric tensor field $B_{\mu\nu}$, one can ask which of these theories, i.e. (14) or (11), is “better”. Of course, there is no unique answer to this question, as it depends on the criteria of the choice. However, as we shall see, there are important aspects in which the model (11), based on the conformal symmetry has an advantage.

If considering a conformal theory of another field (e.g., a massless fermion) on the background of an arbitrary metric and antisymmetric field $B_{\mu\nu}$, we should expect the conformal invariance of the one-loop divergences (more precise, the $4D$ limit of the coefficient of the pole in dimensional regularization), as it was proved in [40]. In the present case, this means that the logarithmic one-loop divergences represent a linear combination of the conformal terms. These terms include the square of the Weyl tensor, Gauss-Bonnet topological term, total derivative term $\Box R$, the conformal terms (4) and (11) constructed from $B_{\mu\nu}$ and the metric, plus the total derivatives constructed from the same fields.

One can couple $B_{\mu\nu}$ to the Dirac fermion as follows:

$$S_{1/2} = i\int d^4x\sqrt{-g}\,\bar{\psi}\big\{\gamma^\mu\nabla_\mu - \Sigma^{\mu\nu}B_{\mu\nu} - im\big\}\psi, \quad (16)$$

where gamma-matrices are defined in a usual way as $\gamma^\mu = e^\mu_a\gamma^a$, $\Sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ and m is the mass of the spinor field. The massless version of this theory possesses conformal symmetry under (1) plus the standard transformations for the fermions in $4D$,

$$\psi = \psi_*\,e^{-\frac{3}{2}\sigma}, \qquad \bar{\psi} = \bar{\psi}_*\,e^{-\frac{3}{2}\sigma}. \quad (17)$$

Making the comparison with (3), the last formula implies that the conformal weight of the fermions ψ and $\bar{\psi}$ is $k_f = -3/2$.

According to [40], the conformal symmetry should hold in the one-loop counterterms. Therefore, in the massless case, the one-loop divergences should be of the form (4) and (11), plus surface terms, and not of the form (14). On top of that, the presence of mass means that the violation of the conformal symmetry is soft. As a result, even in the massive case, when $m \neq 0$, the mass-independent one-loop divergences are expected to be exactly as in the massless theory, i.e., a linear combination of (4) and (11).

An additional detail is that fermionic action (16) does not possess symmetry with respect to the gauge transformation (12). Instead, the mass-independent part of this action is invariant under the local conformal transformations (1) and (17). This means the conformal symmetry plays a critical (guiding) role in the construction of renormalizable theory of the antisymmetric field coupled to fermions. Therefore, applications of these fields that do not rule out the interaction of $B_{\mu\nu}$ with leptons and quarks, should be based on the Lagrangian given by a linear combination of (4), (11) and the mass-dependent terms, instead of the gauge invariant expression (14). In the next section, we shall verify this statement by deriving the one-loop divergences in the theory (16).

3 One-loop divergences for fermion fields

In order to check the relation between conformal symmetry and the one-loop renormalizability, let us derive the one-loop divergences for the Dirac fermion coupled to external metric and antisymmetric field $B_{\mu\nu}$. The starting point will be the action (16). Thus, our purpose is to evaluate the divergent part of of the expression

$$\bar{\Gamma}(g, B) = -i\,\mathrm{Tr}\,\log\hat{H}, \quad (18)$$

where

$$\hat{H} = \gamma^\mu \nabla_\mu - \Sigma_{\mu\nu} B_{\mu\nu} + im. \tag{19}$$

To reduce (18) to the standard form, we can introduce the conjugate operator. The simplest choice is to change only the sign of the mass term,

$$\hat{H}^* = \gamma^\mu \nabla_\mu - \Sigma_{\mu\nu} B_{\mu\nu} - im \tag{20}$$

and take into account that $\mathrm{Tr}\,\log \hat{H} = \mathrm{Tr}\,\log \hat{H}^*$ (see, e.g., [39]). After certain algebra (see Appendix A for necessary details concerning gamma-matrices), we get

$$\bar{\Gamma}(g, B) = -\frac{i}{2}\,\mathrm{Tr}\,\log \hat{F}, \tag{21}$$

where

$$\hat{F} = \hat{H}\hat{H}^* = \hat{1}\Box + 2\hat{h}^\alpha \nabla_\alpha + \hat{\Pi}. \tag{22}$$

Starting from this point, we sometimes omit the symbol of the unit matrix $\hat{1}$ and hats over the operators. The remaining elements of the operator can be reduced to the forms

$$\begin{aligned}\hat{h}^\alpha &= 2\gamma^5 \gamma_\beta \tilde{B}^{\alpha\beta},\\ \hat{\Pi} &= m^2 - \frac{1}{4}R + 2B_{\alpha\beta}B^{\alpha\beta} - 2i(\nabla_\alpha B^{\alpha\beta})\gamma_\beta\\ &\quad - 2i\gamma^5 B_{\alpha\beta}\tilde{B}^{\alpha\beta} + 2\gamma^5(\nabla_\alpha \tilde{B}^{\alpha\beta})\gamma_\beta\\ &\quad -4iB_{\alpha\beta}B_{\mu\nu}\Sigma^{\mu\alpha}g^{\nu\beta},\end{aligned} \tag{23}$$

where the last term vanishes and we use the standard notation for the dual tensor,

$$\tilde{B}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}B^{\alpha\beta}. \tag{24}$$

The one-loop divergences are given by the standard heat-kernel expression [41] (see also [39] for introduction and further references), with the sign corresponding to the odd Grassmann parity of the quantum field,

$$\begin{aligned}\bar{\Gamma}^{(1)}_{div} = \frac{\mu^{n-4}}{\varepsilon}\int d^n x\sqrt{-g}\;\mathrm{tr}\Big\{&\frac{\hat{1}}{180}(R^2_{\mu\nu\alpha\beta} - R^2_{\alpha\beta} + \Box R)\\ &+ \frac{1}{2}\hat{P}^2 + \frac{1}{12}\hat{S}^2_{\mu\nu} + \frac{1}{6}(\Box\hat{P})\Big\},\end{aligned} \tag{25}$$

where $\varepsilon = (4\pi)^2(n-4)$ is the parameter of dimensional regularization and the operators $\hat{P}$ and $\hat{S}_{\mu\nu}$ are defined as

$$\begin{aligned}\hat{P} &= \hat{\Pi} + \frac{\hat{1}}{6}R - \nabla_\mu \hat{h}^\mu - \hat{h}_\mu \hat{h}^\mu,\\ \hat{S}_{\mu\nu} &= \hat{\mathcal{R}}_{\mu\nu} + \nabla_\nu \hat{h}_\mu - \nabla_\mu \hat{h}_\nu + \hat{h}_\nu \hat{h}_\mu - \hat{h}_\mu \hat{h}_\nu.\end{aligned} \tag{26}$$

Here $\hat{\mathcal{R}}_{\mu\nu} = [\nabla_\nu, \nabla_\mu]$ in the corresponding space. In the present case, it is the space of Dirac spinors and therefore we get the expressions

$$\begin{aligned}\hat{P} &= m^2 - \frac{1}{12}R - 2B^2_{\mu\nu} - 2i(\nabla_\mu B^{\mu\nu})\gamma_\nu\\ &\quad -2iB_{\mu\nu}\tilde{B}^{\mu\nu}\gamma^5\\ \hat{S}_{\mu\nu} &= -\frac{1}{4}R_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma + 2\gamma^5\gamma^\lambda(\nabla_\nu \tilde{B}_{\mu\lambda} - \nabla_\mu \tilde{B}_{\nu\lambda})\\ &\quad +4(\gamma^\lambda\gamma^\tau - \gamma^\tau\gamma^\lambda)\tilde{B}_{\mu\lambda}\tilde{B}_{\nu\tau}.\end{aligned} \tag{27}$$

Using these building blocks in (25) is simple after deriving the set of reduction formulas presented in Appendix B. The result is the expression for the one-loop divergences

$$\begin{aligned}\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{\varepsilon}\int d^n x\sqrt{-g}\,\Big\{&-\frac{4}{3}(W_1 + W_4) + \frac{40}{3}W_2\\ &-\frac{32}{3}W_3 + 8m^2 B^2_{\mu\nu} - 2m^4 + \frac{1}{3}m^2 R\\ &+\frac{1}{20}C^2_{\mu\nu\alpha\beta} - \frac{11}{360}E_4 + \frac{1}{30}\Box R + \frac{4}{3}\Box B^2_{\mu\nu}\Big\},\end{aligned} \tag{28}$$

where we use the condensed notations (4), (11) and denote the Gauss-Bonnet integrand (Euler density in $4D$) as $E_4 = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$.

Without the $B_{\mu\nu}$-dependent terms, the expression for the divergences has the standard well-known form (see, e.g., [42]). When the field $B_{\mu\nu}$ is present, we note that there are all the terms expected from the dimension and covariance arguments. Remarkably, four of these terms are in the conformal combination (11). It is certainly instructive that the mass-free $B_{\mu\nu}$-dependent terms form conformal invariants $W_{1,2,3,4}$. This property is owing to several cancelations, that can be explained only by the effect of conformal symmetry. The result confirms aforementioned theorem [40] concerning the conformal invariance of the one-loop divergences in the classically conformal invariant theory. Only the mass-dependent terms in the $4D$ limit of the integral in (28) violate Noether identity (2).

4 Renormalization group equations

Taking into account the considerations presented above, we can formulate the curved-space action of the antisymmetric field $B_{\mu\nu}$ which produced a renormalizable theory at the one-loop level. Such an action has the form

$$\begin{aligned}S_B = \int d^4x\sqrt{-g}\,\Big\{&\frac{1}{2}(W_4 + \lambda W_1) - \frac{1}{2}M^2 B_{\mu\nu}\\ &-\frac{1}{4!}(f_2 W_2 + f_3 W_3 + \zeta\Box B_{\mu\nu})\\ &+\text{total derivatives}\Big\} + S_g.\end{aligned} \tag{29}$$

Here λ is an arbitrary nonminimal parameter of interaction with the Weyl tensor and $f_{2,3}$ are arbitrary parameters of the quartic self-coupling of the antisymmetric field. We included one of the possible total derivatives for the sake of generality, but renormalizability in higher loops may require more such terms, especially the ones quoted in Eq. (7). The coefficient in front of W_4 is chosen positive to avoid the ghost states with negative kinetic energy. In this paper, we do not discuss in full details the surface terms related to total derivatives in the action (29). These terms will be considered in the subsequent work devoted to the conformal (trace) anomaly and to the corresponding ambiguities. Finally, the term S_g is the usual metric-dependent vacuum action, which was discussed in many ways (see, e.g., [39,42]), so we avoid repeating well-known things here.

Consider the renormalization and renormalization group running at the one-loop level. For this, let is rewrite the action of fermions (16), introducing new coupling constant g,

$$S_{1/2} = i \int d^4x \sqrt{-g}\, \bar{\psi} \big\{ \gamma^\mu \nabla_\mu - g\, \Sigma^{\mu\nu} B_{\mu\nu} - im \big\} \psi. \tag{30}$$

Then, omitting purely metric-dependent terms, the expression for the $B_{\mu\nu}$-dependent part of the divergences (28) becomes

$$\begin{aligned}\bar{\Gamma}^{(1)}_{div}(B,g) = -\frac{\mu^{n-4}}{\varepsilon} \int d^nx \sqrt{-g}\, \Big\{ &\frac{1}{2}\big(k_1 W_1 + k_4 W_4\big) \\ &- \frac{1}{4!}\big(k_2 W_2 + k_3 W_3\big) \\ &- \frac{1}{2} k_M B^2_{\mu\nu} + k_\zeta \Box B_{\mu\nu} \Big\},\end{aligned} \tag{31}$$

where

$$k_1 = k_4 = -\frac{8g^2}{3}, \quad k_2 = -320 g^4, \quad k_3 = 256 g^4,$$
$$k_M = -16 g^2 m^2, \qquad k_\zeta = \frac{4g^2}{3}. \tag{32}$$

To elucidate the construction presented above, let us consider the renormalization group equations that correspond to the coefficients (32). As usual, we require the equality of the bare and renormalized actions $S_0 = S + \Delta S = S_R$, where the local counterterms are $\Delta S = -\bar{\Gamma}^{(1)}_{div}(B,g)$. All subsequent formulas are restricted to the one-loop approximation, which means, in particular, $\mathcal{O}(1/\varepsilon)$, while the higher orders are neglected.

From the term W_4, we get the renormalization relation between bare $B^{(0)}_{\mu\nu}$ and renormalized $B_{\mu\nu}$ fields,

$$B^{(0)}_{\mu\nu} = \mu^{\frac{n-4}{2}} \Big(1 + \frac{k_4}{2\varepsilon}\Big) B_{\mu\nu}. \tag{33}$$

At this point, we meet an important difference with other theories, such as QED. For the coupling constant g we require

$$g_0 B^0_{\mu\nu} = g B_{\mu\nu}. \tag{34}$$

The mentioned difference is that, in the present case, there is no gauge symmetry that protects the product $gB_{\mu\nu}$ and hence this relation is based only on the possibility of hiding g inside the field and attribute the running to the logarithmic form factor for the W_4 term. Let us note that the corresponding symmetry may be eventually formulated, as it will be discussed in the Conclusions. As a result of (34), in the one-loop approximation we get

$$g^0 = \mu^{\frac{n-4}{2}} \Big(1 - \frac{k_4}{2\varepsilon}\Big) g. \tag{35}$$

Starting from this point, the considerations are quite standard. After small algebra, we arrive at the renormalization relations for the parameters λ, f_2 and f_3 in the form

$$\lambda^0 = \lambda + \frac{k_1 - k_4\lambda}{\varepsilon},$$
$$f^0_{2,3} = \mu^{4-n} \Big(f_{2,3} + \frac{k_{2,3} - 2k_4 f_{2,3}}{\varepsilon} \Big), \tag{36}$$

and for the massive and surface terms

$$M_0^2 = M^2 + \frac{k_M - k_4 M^2}{\varepsilon},$$
$$\zeta_0 = \zeta + \frac{k_\zeta - k_4 \zeta}{\varepsilon}. \tag{37}$$

The beta functions are defined as usual, e.g.,

$$\beta_g = \lim_{n\to 4} \mu \frac{dg}{d\mu}. \tag{38}$$

Direct calculations give

$$\mu \frac{dg}{d\mu} = \frac{n-4}{2} g + g \frac{dk_4}{dg} = \frac{n-4}{2} g - \frac{4}{3(4\pi)^2} g^3. \tag{39}$$

Let us give the list of similar expressions for the parameter λ and coupling constants,

$$\begin{aligned}\mu \frac{d\lambda}{d\mu} &= \frac{1}{2(4\pi)^2} \Big(g \frac{dk_1}{dg} - \lambda g \frac{dk_4}{dg} \Big) \\ &= \frac{8}{3(4\pi)^2} g^2 (\lambda - 1),\end{aligned}$$
$$\mu \frac{d\, f_{2,3}}{d\mu} = (n-4) f_{2,3} + \frac{1}{(4\pi)^2} \Big(k_{2,3} + f_{2,3}\, g \frac{dk_4}{dg} \Big). \tag{40}$$

The derivation of the relations (39) and (40) does not depend on the specific form of the coefficients $k_{1,2,3,4}$ and can be useful in the theory with quantum $B_{\mu\nu}$.

Consider the solutions of the renormalization group equations. For compactness, we denote $t = \log(\mu/\mu_0)$ and attribute index zero to the values of running quantities at the fiducial scale μ_0. For the square of the running coupling $g(t)$ we meet

$$\frac{dg^2}{dt} = -\frac{8}{3(4\pi)^2} g^4 = -a^2 g^4, \qquad a^2 = \frac{1}{6\pi^2}. \tag{41}$$

This is the typical equation for the asymptotic freedom in the UV, with the solution

$$g^2(t) = \frac{g_0^2}{1 + a^2 g_0^2 t}. \tag{42}$$

The following observation is in order. The self-interaction terms of the field $B_{\mu\nu}$ have fourth powers of this field. This means the following consequence. According to the power counting, even if we add the contributions of the quantum antisymmetric tensor field, at the one loop level the equation (41) and the solution (42) do not modify. Of course, this concerns only the effective charge $g(\mu)$. The renormalization group equations for other couplings, namely λ, f_1 and f_2 will get modified.

For the running kinetic nonminimal parameter $\lambda(t)$ we get

$$\lambda(t) = 1 + (\lambda_0 - 1)\big(1 + a^2 g_0^2\, t\big), \tag{43}$$

that means $\lambda = 1$ is UV-unstable fixed point. For an arbitrary initial value $\lambda_0 \neq 1$, this parameter logarithmically runs to infinity with the sign defined by the one of $\lambda_0 - 1$.

For the running couplings $f_{2,3}(t)$ we need to solve the equations

$$\frac{d\,f_{2,3}}{dt} = C_{2,3}g^4 - 2a^2g^2 f_{2,3}, \qquad C_{2,3} = \frac{1}{(4\pi)^2}(-320,\ 256). \tag{44}$$

This equation can be explored using the standard trick [43]. For the ratios between $f_{2,3}(t)$ and $g^2(t)$ we get the solutions

$$\bar{f}_{2,3}(t) = \frac{f_{2,3}(t)}{g^2(t)} = \Big(\bar{f}^0_{2,3} - \frac{C_{2,3}}{a^2}\Big)\big[1 + a^2 g_0^2\, t\big]^{-1}, \tag{45}$$

indicating UV stable fixed points at the values $\bar{f}_{2,3} = \frac{C_{2,3}}{a^2}$. Let us note that the fixed point for $\bar{f}_2$ has a negative sign, which means possible problems with the stability of effective potential in the UV. Indeed, this feature may change after taking the quantum effects of the self-interactions into account, as this is also possible for the scalar potential [43]. In any case, within the given approximation, both running couplings $f_{2,3}(t)$ manifest asymptotic freedom behavior in the UV, independent of the sign of the coefficients $C_{2,3}$.

The beta functions for M^2 and ζ have the form

$$\mu\frac{dM^2}{d\mu} = \frac{1}{2(4\pi)^2}\Big(M^2 g\,\frac{dk_4}{dg} - g\,\frac{dk_M}{dg}\Big) = -\,g^2a^2\big(M^2 - 6m^2\big),$$
$$\mu\frac{d\zeta}{d\mu} = \frac{1}{2(4\pi)^2}\Big(\zeta g\,\frac{dk_4}{dg} - g\,\frac{dk_\zeta}{dg}\Big) = -g^2a^2\left(\zeta + \frac{1}{2}\right). \tag{46}$$

The solutions of the corresponding equations can be easily found and are similar to (45), but we skip these formulas here, as they are not very informative.

5 Flat limit and renormalizability

In the flat limit, the sum of N copies of the fermionic action (30) and (29), gives

$$S_{flat} = \int d^4x \,\Bigg\{ \sum_{k=1}^{N} i\bar{\psi}_k\big(\gamma^\mu\nabla_\mu - g\,\Sigma^{\mu\nu}B_{\mu\nu} - im\big)\psi_k + \frac{1}{2}(\partial_\alpha B_{\mu\nu})^2 - 2(\partial_\mu B^{\mu\nu})^2 - \frac{1}{2}M^2B^2_{\mu\nu} - \frac{f_2}{4!}(B^2_{\mu\nu})^2 - \frac{f_3}{4!}B_{\mu\nu}B^{\nu\alpha}B_{\alpha\beta}B^{\beta\mu}\Bigg\}, \tag{47}$$

where all indices are raised and lowered with the Minkowski metric. The flat-spacetime version of our model gives the possibility to consider several sides of the renormalizability problem, so it is useful to elaborate a list of arguments.

(i) Renormalizability of the theory depends on the types of the counterterms required to cancel divergences in the given order of the loop expansion. The UV divergences in quantum field theory are always removed by adding local counterterms and the model (47) is not supposed to be an exception.

(ii) The possible violation of renormalizability in conformal quantum theory on a curved spacetime is related to the violation of the conformal symmetry because of the trace anomaly. The general structure of anomaly, including of the effective action induced by anomaly are pretty well-known (see, e.g., [39] for a review). The anomaly-induced action [44,45] includes nonlocal and local terms, usually the last are considered irrelevant. On the other hand, it is unlikely that the nonlocal terms produced in the subdiagrams produce local divergent terms in the superficial integration. Thus, the locality of the UV divergences may provide the nonlocal terms in the effective action being non-important for renormalizability. On top of this, we know the structure of the nonlocal terms (see, e.g., [32,38,39,46]) and these terms certainly vanish in the flat background. Therefore, the relevant ones for renormalizability (certainly, in the flat limit) are only the local terms.

(iii) The local terms in the anomaly-induced action usually depend on the scheme of renormalization, as it was recently discussed in [47,48] for scalar fields and in [49] for the axial vector related to torsion.[1] Assuming that the same is true for the theory of antisymmetric field, there may be a renormalization scheme that provides the absence of, at least curvature-independent, nonconformal local terms at higher loops. In this case, the theory (47) may be all-loop renormalizable.

(iv) The amusing feature of the action (47) is that there is no local conformal symmetry in the flat limit. But, on the other hand, the "conformal" restriction on the coefficients of the two kinetic terms in (47) i.e., $(\partial_\alpha B_{\mu\nu})^2$ and

[1] This seems to be the closest analog to the case of $B_{\mu\nu}$, especially in the part concerning local terms.

$(\partial_\mu B^{\mu\nu})^2$, still holds in the one-loop divergences and, according to the previous point, may actually hold even beyond the one-loop approximation, at least in an appropriate renormalization scheme.[2] This situation resembles the smile of Cheshire Cat, which remains seen even when the Cat itself has gone.

(v) According to the general theory of interacting fields in curved spacetime (see, e.g., [38–40]), the renormalization of the curvature-independent "minimal" terms is the same in flat and curved spaces. Therefore, if the relation between $(\partial_\alpha B_{\mu\nu})^2$ and $(\partial_\mu B^{\mu\nu})^2$ really holds beyond one-loop order in flat spacetime, it will be the same for the curved-space analogs $(\nabla_\alpha B_{\mu\nu})^2$ and $(\nabla_\mu B^{\mu\nu})^2$. Then the loss of renormalizability may occur only owing to the curvature-dependent terms, violating the form of the conformal term (11).

6 Conclusions and discussions

We constructed the one-loop renormalizable theory of self-interacting and interacting with fermions, antisymmetric tensor field. Different from the Abelian or non-Abelian vector fields, there is no usual gauge invariance in our model, such that the main symmetry is local conformal invariance. The invariant action includes the terms (11) with fixed coefficients, a qualitatively new nonminimal term W_1 and the two self-interaction terms $W_{2,3}$, which may be used for spontaneous or dynamical symmetry breaking.

The conformal invariance requires the presence of the local metric of spacetime but, even in a flat limit, the effect of the conformal symmetry is sufficient to provide that the relation between two kinetic terms in the action (47) holds at the quantum level. On top of this, there are serious arguments in favor of all-loop renormalizability of the theory, at least in the flat limit and using a specially tuned scheme of renormalization. Our general considerations are confirmed by the one-loop calculation of the fermionic contributions to the divergent part of the effective action of antisymmetric field and metric. These calculations indicate the possibility of the asymptotic freedom in the theory, but this feature should be verified by deriving the divergences in a full theory, including quantum field $B_{\mu\nu}$. We expect to clarify this issue elsewhere.

In this first paper, we leave unexplored several aspects of the new theory, that is supposed to be done in future works. First of all, it would be interesting to write down the trace anomaly and the corresponding effective action. It is especially interesting to consider the ambiguity in the local terms in the anomaly-induced action since this may be relevant for better understanding renormalizability beyond the one-loop order. Another obvious problem to solve is the derivation of full divergences, including the contributions of the proper field $B_{\mu\nu}$, as mentioned above. After this calculation, one can draw a more definite conclusion about the UV limit in this theory, including the asymptotic freedom. At the present stage, we can only claim the asymptotic freedom for a sufficiently large amount of the fermion fields when the fermionic contributions (41) should dominate.

One more interesting aspect is a possible link to the theory of irreducible tensor field $b_{\mu\nu}$ and its action (14). It is well-known that the free version of the gauge invariant antisymmetric field is equivalent to the scalar theory [1]), but such an equivalence does not hold for the conformal version, which is considered in the present work. It might happen that there is a Stückelberg-like procedure linking two kinds of fields. Elaborating on this issue may be helpful, in particular, for a better understanding of the relation (34). Another potentially interesting aspect of the problem is that, different from the theory (14), the model (29) may have problems with the negative-energy states, i.e., ghosts, as discussed in [50]. In this case, we shall meet a new example of the theory which is renormalizable and not unitary, at least at the tree level. Since this theory looks simpler than, e.g., higher derivative quantum gravity, it may serve as a useful model to explore the issues such as instabilities in the classical solutions and resolution of the problem of ghosts at the quantum level. This feature makes the theory of renormalizable antisymmetric field worth investigating. Another possible application is to consider this model in the effective approach, where it has various interesting phenomenological applications to particle physics and astrophysics (see, e.g., [51] and further references therein).

Last, but not least, it would be interesting to explore some applications of the new model (29) to cosmology and maybe even to particle physics. Let us note that the model of an antisymmetric field without gauge invariance has been already applied to inflation in [14,15]. Our analysis may be useful in fixing the coefficients of the terms $(\nabla_\alpha B_{\mu\nu})^2$ and $(\nabla_\mu B^{\mu\nu})^2$ and showing the reason to consider other terms in the action, as they are required to provide consistency at the quantum level.

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[2] The elementary evaluation of power counting shows that the renormalizability holds at the two-loop level.

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Appendix A: Extended algebra of γ-matrices

Our notations include $\gamma^\mu = e^\mu_a \gamma^a$, $\varepsilon^{\mu\nu\alpha\beta} = \epsilon^{abcd} e^\mu_a e^\nu_b e^\alpha_c e^\beta_d$ and

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{24}\,\varepsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta. \tag{48}$$

All Greek indices are raised and lowered with the metric $g^{\mu\nu}$ and its inverse $g_{\mu\nu}$. Furthermore, we denote the antisymmetric combination as

$$\Sigma^{\mu\nu} = i\,\gamma^{[\mu}\gamma^{\nu]} = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right). \tag{49}$$

The Clifford algebra for the curved-space gamma-matrices has the form

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{50}$$

The full basis in the space of spinor matrices is formed by I, γ^μ, γ^5, $\gamma^5\gamma^\mu$, and $\Sigma^{\mu\nu}$. Using covariance and parity arguments, we can write the general relation

$$\gamma^\alpha\gamma^\mu\gamma^\nu = x_1\gamma^\alpha g^{\mu\nu} + x_2\gamma^\mu g^{\alpha\nu} + x_3\gamma^\nu g^{\alpha\mu} + i x_4\gamma^5\varepsilon^{\mu\nu\alpha\beta}\,\gamma_\beta, \tag{51}$$

where $x_{1,2,3,4}$ are unknown coefficients, which can be easily found by contracting (51) with $g_{\mu\alpha}$ and with $\varepsilon_{\mu\nu\alpha\lambda}$. As a result, we arrive at the well-known relation

$$\gamma^\alpha\gamma^\mu\gamma^\nu = \gamma^\alpha g^{\mu\nu} - \gamma^\mu g^{\alpha\nu} + \gamma^\nu g^{\alpha\mu} + i\gamma^5\varepsilon^{\mu\nu\alpha\beta}\,\gamma_\beta, \tag{52}$$

The consequent formulas are

$$\begin{aligned}
&\gamma^\alpha(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = 2(g^{\mu\alpha}\gamma^\nu - g^{\nu\alpha}\gamma^\mu) + 2i\gamma^5\varepsilon^{\mu\nu\alpha\beta}\,\gamma_\beta,\\
&(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^\alpha = 2(g^{\nu\alpha}\gamma^\mu - g^{\mu\alpha}\gamma^\nu) + 2i\gamma^5\varepsilon^{\mu\nu\alpha\beta}\,\gamma_\beta,\\
&\gamma^\alpha(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) + (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^\alpha = 4i\gamma^5\varepsilon^{\mu\nu\alpha\beta}\,\gamma_\beta.
\end{aligned} \tag{53}$$

Using the same approach we used in (51), one can derive the simplified form of the product of four (and more, if necessary) gamma-matrices. For our purposes it is sufficient to restrict the consideration by the antisymmetric version of the product,

$$\gamma^{[\mu}\gamma^{\nu]}\gamma^{[\alpha}\gamma^{\beta]} = i\gamma^5\varepsilon^{\mu\nu\alpha\beta} - \left(g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta}\right) + i\big(\Sigma^{\mu\alpha}g^{\nu\beta} - \Sigma^{\nu\alpha}g^{\mu\beta} + \Sigma^{\nu\beta}g^{\mu\alpha} - \Sigma^{\mu\beta}g^{\nu\alpha}\big). \tag{54}$$

Appendix B: Some algebraic formulas for $\tilde{B}_{\mu\nu}$

To elaborate necessary formulas involving $\tilde{B}_{\mu\nu}$, we shall use the contractions of antisymmetric tensors, e.g., $\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\rho\sigma} = -2\big(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho\big)$ and same method as in Eq. (51), i.e., using symmetries, introducing free coefficients and deriving them using contractions. Skipping the details, let us present just the results, starting from

$$\tilde{B}^{\mu\nu}\tilde{B}^{\alpha\beta} = \frac{1}{3}\,B^{\mu\nu}B^{\alpha\beta} + \frac{2}{3}\left(B^{\mu\alpha}B^{\nu\beta} - B^{\mu\beta}B^{\nu\alpha}\right) - \frac{1}{6}\,B^2_{\rho\sigma}\big(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\big). \tag{55}$$

A partial contraction gives

$$\tilde{B}^{\mu\nu}\tilde{B}^{\alpha\beta}g_{\nu\beta} = B^{\mu\nu}B^{\alpha\beta}g_{\nu\beta} - \frac{1}{2}\,B^{\rho\sigma}B_{\rho\sigma}g^{\mu\alpha}. \tag{56}$$

The consequences include the formulas

$$\begin{aligned}
&\tilde{B}^{\mu\nu}\tilde{B}_{\mu\nu} = -B^{\mu\nu}B_{\mu\nu},\\
&C_{\alpha\beta\mu\nu}\tilde{B}^{\alpha\beta}\tilde{B}^{\mu\nu} = C_{\alpha\beta\mu\nu}B^{\alpha\beta}B^{\mu\nu}
\end{aligned} \tag{57}$$

and, using also the basic reduction relations (7),

$$\begin{aligned}
R_{\alpha\beta\mu\nu}\tilde{B}^{\alpha\beta}\tilde{B}^{\mu\nu} &= 2R_{\alpha\beta\mu\nu}\tilde{B}^{\alpha\mu}\tilde{B}^{\beta\nu}\\
&= C_{\alpha\beta\mu\nu}B^{\mu\nu}B^{\alpha\beta} + 2B^{\mu\nu}B^{\alpha\beta}R_{\mu\alpha}g_{\nu\beta} - \frac{2}{3}\,RB^2_{\mu\nu}\,.
\end{aligned} \tag{58}$$

For the terms with covariant derivatives, after some algebra, we get

$$\begin{aligned}
&(\nabla_\alpha\tilde{B}_{\mu\nu})(\nabla^\alpha\tilde{B}^{\mu\nu}) = -(\nabla_\alpha B_{\mu\nu})^2,\\
&(\nabla_\mu\tilde{B}^{\mu\nu})(\nabla^\alpha\tilde{B}_{\alpha\nu}) = (\nabla_\mu B^{\mu\nu})^2 - \frac{1}{2}(\nabla_\alpha B_{\mu\nu})^2 - \frac{1}{6}\,RB^2_{\mu\nu} + \frac{1}{2}\,C_{\alpha\beta\mu\nu}B^{\mu\nu}B^{\alpha\beta},\\
&(\nabla_\alpha\tilde{B}_{\mu\nu})(\nabla^\mu\tilde{B}^{\alpha\nu}) = -\frac{1}{2}(\nabla_\alpha B_{\mu\nu})^2 + (\nabla_\mu B^{\mu\nu})^2.
\end{aligned} \tag{59}$$

Finally, for the quartic terms, the reduction formulas are $(\tilde{B}_{\mu\nu}\tilde{B}^{\mu\nu})^2 = (B_{\mu\nu}B^{\mu\nu})^2$ and

$$\tilde{B}_{\mu\nu}B^{\mu\nu}\tilde{B}_{\alpha\beta}B^{\alpha\beta} = \frac{4}{3}\,B_{\mu\nu}B^{\nu\alpha}B_{\alpha\beta}B^{\beta\mu},$$

$$\tilde{B}_{\mu\nu}\tilde{B}^{\nu\alpha}\tilde{B}_{\alpha\beta}\tilde{B}^{\beta\mu} = B_{\mu\nu}B^{\nu\alpha}B_{\alpha\beta}B^{\beta\mu}. \tag{60}$$

The last relation can be derived either using (55) or directly, by using the inverse to (24),

$$\frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\,\tilde{B}^{\alpha\beta} = B_{\mu\nu}. \tag{61}$$

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