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Mass spectrum in a six-dimensional SU(n) gauge theory on a magnetized torus

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Abstract: We examine six-dimensional SU(n) gauge theories compactified on a two-dimensional torus with a constant magnetic flux background to obtain a comprehensive low-energy mass spectrum. We introduce general background configurations including the magnetic flux and continuous Wilson line phases, consistent with classical equations of motion. Under the standard gauge fixing procedure, the complete mass spectrum in low-energy effective theory for the SU(n) case is newly presented without imposing restrictions on the gauge fixing parameter. Our analysis confirms the inevitable existence of tachyonic modes, which neither depend on the background configurations of Wilson line phases nor are affected by the gauge fixing parameter. Masses for some low-energy modes exhibit dependence on the gauge fixing parameter, and these modes are identified as would-be Goldstone bosons that are absorbed by massive four-dimensional vector fields. We discuss the phenomenological implications associated with stabilization or condensation of the tachyonic states. Various mass spectra and symmetry-breaking patterns are expected with flux backgrounds in the SU(n) case. They are helpful for constructing phenomenologically viable models beyond the standard model, such as gauge-Higgs unification and grand unified theories.

Keywords: Extra Dimensions, Field Theories in Higher Dimensions, Grand Unification, Flux Compactifications

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Contents

1 Introduction

The idea of a world with extra spatial dimensions compactified into a small volume has been a topic of extensive research in elementary particle physics. In particular, higher-dimensional gauge theories have been widely studied for seeking physics beyond the Standard Model (SM). A remarkable feature of higher-dimensional gauge theories is that extra-dimensional components of gauge fields are four-dimensional (4D) Lorentz scalars. Hence, without contradicting the 4D Lorentz invariance, we can consider non-trivial background configurations of these components, which open up new scenarios for physics beyond the SM.

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A simple possibility is a constant background configuration, namely a vacuum expectation value (VEV) of extra-dimensional gauge fields. The VEV of the Higgs scalar field in the SM is a key ingredient and is forced to have a suitable potential to induce the electroweak symmetry breaking (EWSB). In addition, in grand unified theories (GUTs) [1], large gauge symmetries are often supposed to be broken into the SM gauge symmetry by VEVs of Higgs scalars. Concerning these issues, the Gauge-Higgs Unification (GHU) models, where Higgs scalars are identified to light 4D excitations appearing from the extra-dimensional gauge field [2–5], have been widely studied in the contexts of both the EWSB and unified gauge symmetry breaking [6–32]. In most of these GHU models, VEVs of extra-dimensional gauge fields are related to physical degrees of freedom of Wilson line phases defined with non-contractible cycles on compact spaces [4]. These phases are continuous moduli that parametrize physical vacua along flat directions of tree-level potentials for gauge fields. Thus, in many GHU models, Higgs scalars have flat tree-level potentials and obtain finite effective potentials through quantum corrections, which is thought to be a result of the inherent non-locality of the Wilson line phases, as examined in detail in [33, 34]. Therefore, GHU models are considered to have the advantage of clarifying the origin of the breaking of the electroweak symmetry or unified symmetries.

Another interesting possibility of the background configuration is a constant magnetic flux provided by extra-dimensional gauge fields. For the case with two or more extra dimensions, we can turn on the flux background consistent with classical equations of motion (EOM). The flux backgrounds may play crucial roles in the compactifications in string theory associated with moduli stabilization and breaking of supersymmetry and gauge symmetry [35–38]. In addition, flux background with toroidal or orbifold compactifications yields chiral fermions having a generation structure [37–46], which is one of the fundamental properties of the fermions in the SM, in an effective low-energy theory. The flavor structure of quarks and leptons, such as masses and mixing angles, has been widely examined in this setup [47–54].

Recently, the mass spectrum of low-energy excitations around flux backgrounds has gained much attention and has been examined in detail, including quantum corrections [55–62]. In six-dimensional (6D) models with a T^2 compactification, massless scalar excitations appear from the extra-dimensional gauge field that provides the magnetic flux background. The massless scalars are identified as the Nambu-Goldstone (NG) bosons associated with the translational symmetry broken by the flux background. Recent studies explicitly showed that exact cancellation occurs in one-loop contributions to masses of these scalars [55–58]. Toward applications to phenomenologically viable GHU models, finite masses for the pseudo-NG modes are investigated with the help of explicit breaking of the translational symmetry through interaction terms on T^2 [61] or fixed points on $T^2/\mathbb{Z}_2$ [63].

A non-vanishing flux background taking a direction in the space of a simply-connected gauge group makes some of the 4D parts of gauge fields massive and must induce a spontaneous gauge symmetry breaking at low energy. In addition, such backgrounds have been discussed to accompany tachyonic excitations at a low-energy regime [38, 47, 55], related to the Nielsen-Olesen instabilities [64]. Condensation of these tachyonic modes seems to have an impact on vacuum structure, as examined in superstring models [65, 66], and may give rich phenomena such as further symmetry breaking in a more general field theoretical setup. Despite these phenomenologically interesting features, flux backgrounds associated with an SU(n) gauge group have been less studied, and more comprehensive studies are needed to explore phenomenologically viable models based on the flux background.

This work mainly focuses on clarifying complete tree-level mass spectra appearing in low-energy effective theories with flux backgrounds in a general SU(n) case. We study 6D non-supersymmetric gauge theory compactified on T^2 in detail. We examine the classical EOM to obtain consistent background configurations including both the magnetic flux and Wilson line phases. With the consistent background configurations, boundary conditions for fields associated with the discrete translations on T^2 are studied. The background configurations and the boundary conditions simultaneously change under gauge transformations. With the help of gauge transformations, the Wilson line phases are removed from the background, and the mass spectrum of low-energy 4D excitations is discussed. We explicitly perform a standard R_ξ gauge fixing, keeping the calculations as general as possible, and discuss physically relevant excitations at a low-energy regime. Under a mode expansion of six-dimensional fields, non-trivial mixing among four-dimensional modes appears in their mass terms depending on the gauge fixing terms. We clarify the expressions for the masses depending on the gauge parameter. We confirm that tachyonic scalars inevitably appear in this setup. In addition, some of the scalar excitations are identified as would-be Goldstone modes that are absorbed by 4D vector fields, which become massive due to the gauge symmetry breaking triggered by the flux background. We also discuss the phenomenological implications associated with stabilization or condensation of the tachyonic states. Various mass spectra and symmetry-breaking patterns are expected with the flux background for a simply-connected gauge group and are interesting for constructing phenomenologically viable models beyond the standard model, such as GHU and GUT.

The structure of this paper is as follows. In section 2, we present our definitions and the basic concepts of SU(n) gauge theories. The boundary conditions and consistency conditions for the gauge field are also defined. Taking these conditions into consideration, we obtain a background configuration solution in section 3. In section 4, we discuss a convenient parametrization of this background. In section 5, we examine the Yang-Mills Lagrangian in this setup, fixing the gauge and obtaining the explicit expressions for the quadratic terms. From these terms, we could compute the tree-level mass spectrum for all of the fields, as done in section 6. Finally, phenomenological implications are discussed in section 7, followed by our conclusions. Calculation details can be found in the appendices.

2 SU(n) gauge theories on $\mathcal{M}^4 \times T^2$

We study gauge theories on $\mathcal{M}^4 \times T^2$, where $\mathcal{M}^4$ and T^2 are the Minkowski spacetime and a two-dimensional torus, respectively. We denote coordinates by x^μ ($\mu = 0, 1, 2, 3$) on $\mathcal{M}^4$ and by x^5 and x^6 on T^2. We also use x^M ($M = 0, 1, 2, 3, 5, 6$) and x^m ($m = 5, 6$). A two-dimensional torus T^2 is given by imposing the identification for x^5 and x^6 as

$$(x^5, x^6) \sim \left(x^5 + L(n_5 + n_6\tau_{\mathrm{R}}), x^6 + Ln_6\tau_{\mathrm{I}}\right), \qquad n_5, n_6 \in \mathbb{Z}, \qquad \tau_{\mathrm{I}} > 0, \tag{2.1}$$

where $\tau = \tau_\mathrm{R} + i\tau_\mathrm{I}$ ($\tau_\mathrm{R}, \tau_\mathrm{I} \in \mathbb{R}$) is a complex parameter describing the moduli space of the two-dimensional torus. The size of T^2 is parametrized by L, where the volume of T^2 is given by $\mathcal{V}_{T^2} = L^2\tau_\mathrm{I}$. In the following, we take $L = 1$ without loss of generality.

It is convenient to use complex coordinates defined as

$$z = x^5 + ix^6, \qquad \bar{z} = x^5 - ix^6, \qquad \partial_z = \frac{1}{2}(\partial_5 - i\partial_6), \qquad \bar{\partial}_z = \frac{1}{2}(\partial_5 + i\partial_6), \tag{2.2}$$

where $\partial_M = \partial/\partial x^M$. Then, eq. (2.1) is expressed by $z \sim z + n_5 + n_6\tau$. Let us define the translation operators $\mathcal{T}_p$ ($p = 5, 6$) as

$$\mathcal{T}_5 z = z + 1, \qquad \mathcal{T}_6 z = z + \tau. \tag{2.3}$$

Note that the direction of the translation generated by $\mathcal{T}_6$ is different from the one of x^6 for $\tau_\mathrm{R} \neq 0$, whereas the directions of $\mathcal{T}_5$ and x^5 coincide with each other. Using these operators, eq. (2.1) is rewritten by

$$z \sim \mathcal{T}_6^{n_6}\mathcal{T}_5^{n_5} z. \tag{2.4}$$

We consider gauge theories on $\mathcal{M}^4 \times T^2$ with the gauge group $\mathrm{SU}(n)$, whose Lie algebra is denoted by $su(n)$. The action is given by the 6D volume integral of a Lagrangian density $\mathcal{L}_6$ of an $\mathrm{SU}(n)$ gauge theory. We demand that $\mathcal{L}_6$ is invariant under 6D Lorentz transformations and $\mathrm{SU}(n)$ gauge transformations.

Let $\boldsymbol{A}_M \in su(n)$ be a gauge field, which is a function of x^M. In order to expand the gauge field as $\boldsymbol{A}_M = A_M^a t_a$ ($A_M^a \in \mathbb{R}$, $a = 1, \ldots, n^2 - 1$), we introduce generators $t_a \in su(n)$, which span the vector space $su(n)$. Hereafter, we imply summations over the same upper and lower indices. We refer to A_M^a as a component field. We also introduce the covariant derivative $\boldsymbol{D}_M$ as

$$\boldsymbol{D}_M = \partial_M - ig\boldsymbol{A}_M, \tag{2.5}$$

where g is a gauge coupling constant.

We first discuss the pure Yang-Mills theory; matter fields are discussed in section 7. The Lagrangian is given by

$$\mathcal{L}_\mathrm{YM} = -\frac{1}{2}\eta^{MM'}\eta^{NN'}\mathrm{Tr}(\boldsymbol{F}_{MN}\boldsymbol{F}_{M'N'}), \tag{2.6}$$

where $\boldsymbol{F}_{MN}$ is the field strength, and η^{MN} is the metric of the 6D spacetime. The trace is implied to be taken in a representation space of $su(n)$. The field strength is given by

$$\boldsymbol{F}_{MN} = \frac{i}{g}[\boldsymbol{D}_M, \boldsymbol{D}_N] = \partial_M\boldsymbol{A}_N - \partial_N\boldsymbol{A}_M - ig[\boldsymbol{A}_M, \boldsymbol{A}_N], \tag{2.7}$$

where $[A, B] = AB - BA$, and the metric is defined to be $(\eta^{MN}) = \mathrm{diag}(-1, 1, \ldots, 1)$. Namely, x^M are orthogonal coordinates. We also use $(\eta^{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$ for the Minkowski part. Note that $\boldsymbol{F}_{MN}$ is written by $F_{MN}^a t_a \in su(n)$.

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The Lagrangian in eq. (2.6) is invariant under a gauge transformation, which is defined by

$$\boldsymbol{A}_M \to \boldsymbol{A}_M^{\mathrm{gt}} = \boldsymbol{\Lambda}\boldsymbol{A}_M\boldsymbol{\Lambda}^{-1} + \frac{i}{g}\boldsymbol{\Lambda}\partial_M\boldsymbol{\Lambda}^{-1}, \tag{2.8}$$

where $\boldsymbol{\Lambda} \in \mathrm{SU}(n)$ is a function of x^M and is called a gauge transformation function. From eq. (2.8), one finds $\boldsymbol{D}_M \to \boldsymbol{D}_M^{\mathrm{gt}} = \boldsymbol{\Lambda}\boldsymbol{D}_M\boldsymbol{\Lambda}^{-1}$ and $\boldsymbol{F}_{MN} \to \boldsymbol{F}_{MN}^{\mathrm{gt}} = \boldsymbol{\Lambda}\boldsymbol{F}_{MN}\boldsymbol{\Lambda}^{-1}$, under which the gauge invariance of the Lagrangian in eq. (2.6) is manifested.

For later convenience, we define

$$\boldsymbol{A}_z = \frac{1}{2}(\boldsymbol{A}_5 - i\boldsymbol{A}_6), \qquad \bar{\boldsymbol{A}}_z = \frac{1}{2}(\boldsymbol{A}_5 + i\boldsymbol{A}_6), \tag{2.9}$$

$$\boldsymbol{D}_z = \frac{1}{2}(\boldsymbol{D}_5 - i\boldsymbol{D}_6) = \partial_z - ig\boldsymbol{A}_z, \qquad \bar{\boldsymbol{D}}_z = \frac{1}{2}(\boldsymbol{D}_5 + i\boldsymbol{D}_6) = \bar{\partial}_z - ig\bar{\boldsymbol{A}}_z. \tag{2.10}$$

From the above and eq. (2.2), one naturally defines

$$\boldsymbol{F}_{z\bar{z}} = \partial_z\bar{\boldsymbol{A}}_z - \bar{\partial}_z\boldsymbol{A}_z - ig[\boldsymbol{A}_z, \bar{\boldsymbol{A}}_z] = \frac{i}{2}\boldsymbol{F}_{56}, \qquad \boldsymbol{F}_{\bar{z}z} = -\boldsymbol{F}_{z\bar{z}}, \qquad \boldsymbol{F}_{zz} = \boldsymbol{F}_{\bar{z}\bar{z}} = 0. \tag{2.11}$$

Similar notations are used for component fields, e.g., $F_{z\bar{z}}^a = iF_{56}^a/2$.

Since we consider compact extra dimensions, boundary conditions for fields have to be specified to define gauge theories. In view of the identification in eq. (2.4), we require that $\boldsymbol{A}_M(x^\mu, \mathcal{T}_6^{n_6}\mathcal{T}_5^{n_5}z)$ is equal to $\boldsymbol{A}_M(x^\mu, z)$ up to a gauge transformation as a sufficient condition to make the pure Yang-Mills Lagrangian single-valued on T^2. Hence, the boundary conditions are defined as

$$\boldsymbol{A}_M(\mathcal{T}_p z) = T_p(z)\boldsymbol{A}_M(z)T_p^\dagger(z) + \frac{i}{g}T_p(z)\partial_M T_p^\dagger(z), \tag{2.12}$$

where we have introduced $T_p(z) \in \mathrm{SU}(n)$ $(p = 5, 6)$. As a shorthand notation, x^μ was suppressed, and $\boldsymbol{A}_M(x^\mu, z)$ and $T_p(x^\mu, z)$ are written as $\boldsymbol{A}_M(z)$ and $T_p(z)$, respectively. A similar notation is used for others that depend on x^M. The boundary conditions are specified by T_p, which we refer to as twist matrices hereafter. The twist matrices generally depend on x^M.

Different twist matrices can be physically equivalent since the twist matrices T_p depend on the choice of gauge [5, 67–69]. From eq. (2.8), one sees that

$$\boldsymbol{A}_M^{\mathrm{gt}}(\mathcal{T}_p z) = T_p^{\mathrm{gt}}(z)\boldsymbol{A}_M^{\mathrm{gt}}(z)T_p^{\mathrm{gt}\dagger}(z) + \frac{i}{g}T_p^{\mathrm{gt}}(z)\partial_M T_p^{\mathrm{gt}\dagger}(z), \tag{2.13}$$

where

$$T_p^{\mathrm{gt}}(z) = \boldsymbol{\Lambda}(\mathcal{T}_p(z))T_p(z)\boldsymbol{\Lambda}^\dagger(z). \tag{2.14}$$

The gauge field at $z + 1 + \tau$ is written by

$$\boldsymbol{A}_M(z + 1 + \tau) = \boldsymbol{A}_M(\mathcal{T}_6(z + 1)) = \boldsymbol{A}_M(\mathcal{T}_5(z + \tau)). \tag{2.15}$$

From the above, we obtain

$$[T_\square(z), \boldsymbol{A}_M(z)] = \frac{i}{g}\partial_M T_\square(z), \qquad T_\square(z) = T_6^\dagger(z)T_5^\dagger(z + \tau)T_6(z + 1)T_5(z), \tag{2.16}$$

which is regarded as a consistency condition for the twist matrices.

As seen below, there appear to be additional consistency conditions for the twist matrices if non-trivial background configurations for the gauge field exist. In the next section, we discuss explicit forms of the twist matrices taking background configurations for the gauge field into account.

3 Consistency between background configurations and boundary conditions

Let us introduce non-trivial background configurations for the gauge field $\boldsymbol{A}_M$. For consistency, we demand that the backgrounds satisfy the classical EOM. From the Lagrangian in eq. (2.6), we find that the EOM is given by

$$\boldsymbol{D}^M \boldsymbol{F}_{MN} = \left(\partial^M - ig\mathbf{ad}(\boldsymbol{A}^M)\right) \boldsymbol{F}_{MN} = 0, \tag{3.1}$$

where we have introduced the notation $\mathbf{ad}(X)Y = [X, Y]$. In the following, we replace

$$\boldsymbol{A}_M(z) \to \boldsymbol{B}_M(z) + \boldsymbol{A}_M(z), \tag{3.2}$$

in the Lagrangian in eq. (2.6). On the right-hand side, $\boldsymbol{B}_M$ and $\boldsymbol{A}_M$ are referred to as the background and the fluctuation around the background configuration, respectively. Imposing $\boldsymbol{B}_\mu = 0$ and $\partial_\mu \boldsymbol{B}_M = 0$, we can explicitly keep the 4D Lorentz invariance. Then, non-trivial background configurations can be given by $\boldsymbol{B}_m \neq 0$, which generally depend on the torus coordinates.

It is convenient to introduce the background covariant derivative $\mathcal{D}_m$ and the field strength $\mathcal{F}_{mn}$ as

$$\mathcal{D}_m = \partial_m - ig\mathbf{ad}(\boldsymbol{B}_m), \qquad \mathcal{F}_{mn} = \partial_m \boldsymbol{B}_n - \partial_n \boldsymbol{B}_m - ig[\boldsymbol{B}_m, \boldsymbol{B}_n]. \tag{3.3}$$

According to the above discussion, we require that

$$\mathcal{D}^m \mathcal{F}_{mn} = 0, \tag{3.4}$$

to satisfy the consistency condition for the background field $\boldsymbol{B}_m$.

A solution to eq. (3.4) is written as

$$\boldsymbol{B}_5(z) = \boldsymbol{v}_5 - (1+\gamma)\, \boldsymbol{f} x^6/2, \qquad \boldsymbol{B}_6(z) = \boldsymbol{v}_6 + (1-\gamma)\, \boldsymbol{f} x^5/2, \tag{3.5}$$

$$[\boldsymbol{v}_5, \boldsymbol{v}_6] = [\boldsymbol{v}_m, \boldsymbol{f}] = 0, \tag{3.6}$$

where $\boldsymbol{v}_m, \boldsymbol{f} \in su(n)$ and $\gamma \in \mathbb{R}$ are constants. With this background configuration, one sees that $\mathcal{F}_{56} = \boldsymbol{f}$. We call $\boldsymbol{f}$ a constant magnetic flux, whereas $\boldsymbol{v}_m$ are referred to as continuous Wilson line phases. As discussed in the next section, allowed values of $\boldsymbol{f}$ are quantized. On the other hand, continuous variables $\boldsymbol{v}_m$ are related to the flat directions in the tree-level potential for the gauge field obtained from eq. (2.6). The parameter γ is introduced for clarity of our discussions and is independent of $\mathcal{F}_{56}$. In the literature, $\gamma = \pm 1$ and $\gamma = 0$ are often called the Landau and the symmetric gauge, respectively.

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The background $\boldsymbol{B}_m$ changes by gauge transformations. Using the gauge transformation in eq. (2.8) with a constant $\boldsymbol{\Lambda}$, we can diagonalize $\boldsymbol{f}$ in a representation space of $su(n)$. Then, the last equality in eq. (3.6) implies that we can also diagonalize $\boldsymbol{v}_m$ keeping $\boldsymbol{f}$ diagonal. Thus, without loss of generality, we can expand $\boldsymbol{f}$ and $\boldsymbol{v}_m$ by Cartan generators. We discuss the explicit forms of the $su(n)$ generators in the next section.

Considering the background configuration in eq. (3.5), we examine the boundary conditions for the gauge field in eq. (2.12). The background $\boldsymbol{B}_M$ and the twist matrices T_p must satisfy the relation

$$\boldsymbol{B}_M(\mathcal{T}_p z) = T_p(z)\boldsymbol{B}_M(z)T_p^\dagger(z) + \frac{i}{g}T_p(z)\partial_M T_p^\dagger(z), \tag{3.7}$$

whereas the fluctuations satisfy

$$\boldsymbol{A}_M(\mathcal{T}_p z) = T_p(z)\boldsymbol{A}_M(z)T_p^\dagger(z). \tag{3.8}$$

From eq. (3.7) for $M = \mu$, one sees that $\partial_\mu T_p = 0$. On the other hand, from eq. (3.7) for $M = n$, we obtain

$$\partial_5 T_5 = ig[\boldsymbol{B}_5, T_5], \tag{3.9}$$
$$\partial_6 T_5 = ig[\boldsymbol{B}_6, T_5] + ig\,(1-\gamma)\,\boldsymbol{f}T_5/2, \tag{3.10}$$
$$\partial_5 T_6 = ig[\boldsymbol{B}_5, T_6] - ig\,(1+\gamma)\,\tau_{\mathrm{I}}\boldsymbol{f}T_6/2, \tag{3.11}$$
$$\partial_6 T_6 = ig[\boldsymbol{B}_6, T_6] + ig\,(1-\gamma)\,\tau_{\mathrm{R}}\boldsymbol{f}T_6/2, \tag{3.12}$$

where $\boldsymbol{B}_m$ and T_p are defined at z. The twist matrices must satisfy these conditions in addition to eq. (2.16).

A solution to eqs. (3.9)–(3.12) with the background configuration in eq. (3.5) is given by

$$T_5(z) = e^{ig(1-\gamma)\boldsymbol{f}x^6/2}\tilde{T}_5, \qquad T_6(z) = e^{ig\{-(1+\gamma)\tau_{\mathrm{I}}\boldsymbol{f}x^5/2+(1-\gamma)\tau_{\mathrm{R}}\boldsymbol{f}x^6/2\}}\tilde{T}_6, \tag{3.13}$$

where we have introduced constant matrices $\tilde{T}_p$ that satisfy $[\boldsymbol{f}, \tilde{T}_p] = [\boldsymbol{v}_n, \tilde{T}_p] = 0$. Then, $T_\square(z)$ defined in eq. (2.16) becomes

$$T_\square(z) = e^{-ig\mathcal{V}_{T^2}\boldsymbol{f}}\tilde{T}_6^\dagger\tilde{T}_5^\dagger\tilde{T}_6\tilde{T}_5. \tag{3.14}$$

From the right-hand side of the above equation, one sees that $T_\square(z)$ is a constant matrix. Hence, eq. (2.16) is reduced to $[T_\square, \boldsymbol{A}_M(z)] = 0$. The general solution of this constraint is $T_\square \in \mathbb{Z}_n$, where $\mathbb{Z}_n \subset \mathrm{SU}(n)$ is the center subgroup of $\mathrm{SU}(n)$. Thus, we obtain

$$e^{-ig\mathcal{V}_{T^2}\boldsymbol{f}}\tilde{T}_6^\dagger\tilde{T}_5^\dagger\tilde{T}_6\tilde{T}_5 = e^{2\pi i\tilde{n}/n}\boldsymbol{I}, \qquad \tilde{n} \in \mathbb{Z}, \tag{3.15}$$

where $I \in \mathrm{SU}(n)$ is the identity operator. The integer $\tilde{n}$ modulo n is referred to as the 't Hooft flux [70].

Although there are interesting possibilities of non-trivial choices for $\tilde{T}_p$ and $\tilde{n}$ in eq. (3.15), we restrict our attention to the simplest $\tilde{n} = 0$ and $\tilde{T}_p = \boldsymbol{I}$ in the following discussions. Then, from eq. (3.15), we obtain

$$e^{ig\mathcal{V}_{T^2}\boldsymbol{f}} = \boldsymbol{I}, \tag{3.16}$$

which gives a quantization condition for the flux $\boldsymbol{f}$. We will examine this condition in the next section.

Both the background $\boldsymbol{B}_m$ and the twist matrices T_p have gauge dependence. A notable fact is that the constant terms $\boldsymbol{v}_m$ in eq. (3.5) can be eliminated by a gauge transformation with $\boldsymbol{\Lambda} = e^{-ig(\boldsymbol{v}_5 x^5 + \boldsymbol{v}_6 x^6)}$. From eqs. (2.8) and (2.14), after the gauge transformation, one finds

$$\boldsymbol{B}_5(z) = -(1+\gamma)\boldsymbol{f} x^6/2, \qquad \boldsymbol{B}_6(z) = (1-\gamma)\boldsymbol{f} x^5/2, \tag{3.17}$$

$$T_5(z) = e^{-ig\boldsymbol{v}_5} e^{ig(1-\gamma)\boldsymbol{f} x^6/2}, \qquad T_6(z) = e^{-ig(\tau_{\mathrm{R}}\boldsymbol{v}_5 + \tau_{\mathrm{I}}\boldsymbol{v}_6)} e^{ig\{-(1+\gamma)\tau_{\mathrm{I}}\boldsymbol{f} x^5/2 + (1-\gamma)\tau_{\mathrm{R}}\boldsymbol{f} x^6/2\}}, \tag{3.18}$$

where we have renamed $\boldsymbol{B}_m^{\mathrm{gt}}$ and T_p^{gt} as $\boldsymbol{B}_m$ and T_p to simplify the notation. From the above, it is clear that non-trivial values of the continuous Wilson line phases $\boldsymbol{v}_m$ can be treated as some part of the twist matrices [71]. We choose this gauge in the following discussions.

4 Parametrization of background gauge fields

Considering the background configuration in eq. (3.17), let us discuss the parametrization of $\boldsymbol{B}_m$ by $su(n)$ generators. Let $\{t_a\}$ $(a = 1, \ldots, n^2 - 1)$ be a set of the $su(n)$ generators. It is convenient to use the Cartan-Weyl basis, writing $\{t_a\} = \{H_k\} \cup \{E_{\boldsymbol{\alpha}}\}$, where $\{H_k\}$ $(k = 1, \ldots, n-1)$ are Cartan generators, and $E_{\boldsymbol{\alpha}}$ is a step operator associated to a root vector $\boldsymbol{\alpha}$. The Cartan generators are Hermitian, $H_k^\dagger = H_k$, and step operators $E_{\boldsymbol{\alpha}}$ satisfy $E_{\boldsymbol{\alpha}}^\dagger = E_{-\boldsymbol{\alpha}}$. They obey the commutation relations

$$[H_k, H_\ell] = 0, \qquad [H_k, E_{\boldsymbol{\alpha}}] = \alpha_{[k]} E_{\boldsymbol{\alpha}}, \tag{4.1}$$

where $\alpha_{[k]} \in \mathbb{R}$ is the k-th component of the root vector $\boldsymbol{\alpha}$. As noted before, $\boldsymbol{f}$ and $\boldsymbol{v}_m$ are expanded by $su(n)$ Cartan generators:

$$\boldsymbol{f} = f^k H_k, \qquad \boldsymbol{v}_m = v_m^k H_k, \qquad f^k, v_m^k \in \mathbb{R}. \tag{4.2}$$

To be more concrete, we fix a basis of the generators in a representation space of $su(n)$. Let us denote the fundamental representation of H_k by $\hat{H}_k$ and take

$$\hat{H}_1 = \mathrm{diag}(1, -1, 0, \ldots, 0), \qquad \hat{H}_2 = \mathrm{diag}(0, 1, -1, 0, \ldots, 0), \qquad \ldots, \tag{4.3}$$

$$\hat{H}_{n-2} = \mathrm{diag}(0, \ldots, 0, 1, -1, 0), \qquad \hat{H}_{n-1} = \mathrm{diag}(0, \ldots, 0, 1, -1). \tag{4.4}$$

In the following discussions, we identify any operators of $su(n)$ as their fundamental representation matrices for simplicity.

Let us examine the quantization condition of the constant magnetic flux in eq. (3.16). It is convenient to introduce the unit strength of the magnetic flux $\hat{f} = 2\pi/(g\mathcal{V}_{T^2})$. Then, using the explicit forms of $\hat{H}_k$ in eqs. (4.3) and (4.4), we obtain a general solution to eq. (3.16) as

$$f^k = \frac{2\pi}{g\mathcal{V}_{T^2}} N^k = \hat{f} N^k, \qquad N^k \in \mathbb{Z}. \tag{4.5}$$

The flux $\boldsymbol{f}$ in the fundamental representation is expressed by

$$\boldsymbol{f} = f^k \hat{H}_k = \hat{f}\ \mathrm{diag}(N^1, N^2 - N^1, \ldots, N^{n-1} - N^{n-2}, -N^{n-1}). \tag{4.6}$$

For later convenience, we discuss the step operators in the fundamental representation. We define $n^2 - 1 - (n-1) = n(n-1)$ step operators. To express them, it is convenient to introduce the basis matrices $\hat{e}_{ij}$, whose (i', j') element $(\hat{e}_{ij})_{i'j'}$ is given by $(\hat{e}_{ij})_{i'j'} = \delta_{ii'}\delta_{jj'}$, where $\delta_{ii'}$ is the Kronecker delta. We can denote the $n(n-1)$ step operators in the fundamental representation by

$$E^{(+)}_{ij} = \hat{e}_{ij}, \qquad E^{(-)}_{ij} = \hat{e}_{ji}, \qquad 1 \le i < j \le n. \tag{4.7}$$

The Cartan generators are written by $\hat{H}_k = \hat{e}_{kk} - \hat{e}_{k+1\,k+1}$ with these basis matrices. From the definition of the fundamental representation of the generators, one sees that

$$\mathrm{Tr}[\hat{H}_k \hat{H}_\ell] = (M^{su(n)})_{k\ell}, \qquad \mathrm{Tr}[E^{(+)}_{ij} E^{(-)}_{i'j'}] = \delta_{ii'}\delta_{jj'}, \tag{4.8}$$

$$\mathrm{Tr}[\hat{H}_k E^{(\pm)}_{ij}] = \mathrm{Tr}[E^{(+)}_{ij} E^{(+)}_{i'j'}] = \mathrm{Tr}[E^{(-)}_{ij} E^{(-)}_{i'j'}] = 0, \tag{4.9}$$

where $(M^{su(n)})_{k\ell}$ is the (k, ℓ) element of the $su(n)$ Cartan matrix

$$M^{su(n)} = \begin{pmatrix} 2 & -1 & 0 & \cdots & & & 0 \\ -1 & 2 & -1 & 0 & \cdots & & 0 \\ & & & \ddots & & & \\ 0 & \cdots & & 0 & -1 & 2 & -1 \\ 0 & \cdots & & & 0 & -1 & 2 \end{pmatrix}. \tag{4.10}$$

In the following discussions, commutation relations between the generators often appear, such as

$$[\hat{H}_k, E^{(\pm)}_{ij}] = \pm(\delta_{ki} - \delta_{k+1\,i} - \delta_{kj} + \delta_{k+1\,j}) E^{(\pm)}_{ij}, \tag{4.11}$$

where we note that, on the right-hand side, the indices are not summed. The eigenvalue $\pm(\delta_{ki} - \delta_{k+1\,i} - \delta_{kj} + \delta_{k+1\,j})$ is the k-th component of the root vector corresponding to $E^{(\pm)}_{ij}$. From eqs. (4.6) and (4.11), we obtain

$$[\boldsymbol{f}, E^{(+)}_{ij}] = \hat{f}(N^i - N^{i-1} - N^j + N^{j-1}) E^{(+)}_{ij} \equiv \hat{f} \tilde{N}^{ij} E^{(+)}_{ij}, \tag{4.12}$$

$$[\boldsymbol{v}_m, E^{(+)}_{ij}] = (v^i_m - v^{i-1}_m - v^j_m + v^{j-1}_m) E^{(+)}_{ij} \equiv \tilde{v}^{ij}_m E^{(+)}_{ij}, \tag{4.13}$$

where we have defined

$$\tilde{N}^{ij} = N^i - N^{i-1} - N^j + N^{j-1}, \qquad \tilde{v}^{ij}_m = v^i_m - v^{i-1}_m - v^j_m + v^{j-1}_m. \tag{4.14}$$

We note that we have implied $N^{i-1}, v^{i-1}_m = 0$ for $i = 1$ and $N^i, v^i_m = 0$ for $i = n$. We also have $[\boldsymbol{f}, E^{(-)}_{ij}] = -\hat{f}\tilde{N}^{ij} E^{(-)}_{ij}$, $[\boldsymbol{v}_m, E^{(-)}_{ij}] = -\tilde{v}^{ij}_m E^{(-)}_{ij}$, and $[\boldsymbol{f}, \hat{H}_k] = [\boldsymbol{v}_m, \hat{H}_k] = 0$.

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5 Lagrangian around flux backgrounds

5.1 Lagrangian for the fluctuations

Let us examine the Lagrangian in eq. (2.6) with the background configurations and boundary conditions in eqs. (3.17) and (3.18). We aim to study the tree-level mass spectrum in the effective 4D theory, which describes physics at a sufficiently lower energy scale than the compactification scale $1/L$. Since the quadratic terms of the fields in the Lagrangian are relevant to the mass spectrum, we mainly focus on them here.

As discussed, we take the replacement in eq. (3.2) in the Lagrangian in eq. (2.6). We use the background covariant derivative and the field strength defined in eq. (3.3), which are generalized to

$$\mathcal{D}_M = \begin{cases} \mathcal{D}_\mu = \partial_\mu, \\ \mathcal{D}_m = \partial_m - ig\mathbf{ad}(\boldsymbol{B}_m), \end{cases} \qquad \mathcal{F}_{MN} = \begin{cases} \mathcal{F}_{\mu M} = 0, \\ \mathcal{F}_{mn} = \epsilon_{mn}\boldsymbol{f}, \end{cases} \tag{5.1}$$

where $\epsilon_{56} = -\epsilon_{65} = 1$. Then, the field strength in eq. (2.7) is rewritten as

$$\boldsymbol{F}_{MN} = \mathcal{F}_{MN} + \mathcal{D}_M \boldsymbol{A}_N - \mathcal{D}_N \boldsymbol{A}_M - ig[\boldsymbol{A}_M, \boldsymbol{A}_N], \tag{5.2}$$

where the first term on the right-hand side is a constant.

The Lagrangian in eq. (2.6) is decomposed into

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\sum_{p=0}^{4} \mathrm{Tr}[L^{(p)}], \tag{5.3}$$

where $L^{(p)}$ contains p-th polynomials of the fluctuations. We find

$$L^{(0)} = \mathcal{F}_{MN}\mathcal{F}^{MN}, \tag{5.4}$$

$$L^{(1)} = 4\mathcal{F}^{MN}\mathcal{D}_M\boldsymbol{A}_N, \tag{5.5}$$

$$L^{(2)} = 2\mathcal{D}_M\boldsymbol{A}_N(\mathcal{D}^M\boldsymbol{A}^N - \mathcal{D}^N\boldsymbol{A}^M) - 2ig\mathcal{F}^{MN}[\boldsymbol{A}_M, \boldsymbol{A}_N], \tag{5.6}$$

$$L^{(3)} = -4ig(\mathcal{D}_M\boldsymbol{A}_N)[\boldsymbol{A}^M, \boldsymbol{A}^N], \tag{5.7}$$

$$L^{(4)} = -g^2[\boldsymbol{A}^M, \boldsymbol{A}^N][\boldsymbol{A}_M, \boldsymbol{A}_N]. \tag{5.8}$$

The constant terms $L^{(0)}$ contribute to the cosmological constant, which is irrelevant to our present analysis. The linear terms $L^{(1)}$ vanish in the action.[1] Such terms induce tadpoles and must vanish with background configurations consistent with the EOM. The quadratic terms $L^{(2)}$ describe the tree-level mass spectrum in the 4D effective theory. We study them in detail just below. From the trilinear and quartic terms $L^{(3)}$ and $L^{(4)}$, one finds

$$\begin{aligned} -\frac{1}{2}\mathrm{Tr}[L^{(3)}] &= 2ig\mathrm{Tr}[(\partial_\mu\boldsymbol{A}_\nu)[\boldsymbol{A}^\mu, \boldsymbol{A}^\nu]] + 2ig\mathrm{Tr}[(\partial_\mu\boldsymbol{A}_m)[\boldsymbol{A}^\mu, \boldsymbol{A}^m]] \\ &\quad + 2ig\mathrm{Tr}[(\mathcal{D}_m\boldsymbol{A}_\mu)[\boldsymbol{A}^m, \boldsymbol{A}^\mu]] + 2ig\mathrm{Tr}[(\mathcal{D}_m\boldsymbol{A}_n)[\boldsymbol{A}^m, \boldsymbol{A}^n]], \end{aligned} \tag{5.9}$$

$$-\frac{1}{2}\mathrm{Tr}[L^{(4)}] = \frac{g^2}{2}\mathrm{Tr}[([\boldsymbol{A}_\mu, \boldsymbol{A}_\nu])^2] + g^2\mathrm{Tr}[([\boldsymbol{A}_\mu, \boldsymbol{A}_m])^2] + \frac{g^2}{2}\mathrm{Tr}[([\boldsymbol{A}_m, \boldsymbol{A}_n])^2], \tag{5.10}$$

which give the interactions between the fluctuations.

[1]The linear terms $L^{(1)}$ yield two types of contributions in the Lagrangian. One of them is the term proportional to $\epsilon^{mn}\partial_m\mathrm{Tr}[\boldsymbol{f}\boldsymbol{A}_n]$. Since $\mathrm{Tr}[\boldsymbol{f}\boldsymbol{A}_n]$ is periodic under the shift generated by $\mathcal{T}_p$, this surface term vanishes in the action. Another type is the term proportional to $\mathrm{Tr}[\boldsymbol{f}[\boldsymbol{B}_m, \boldsymbol{A}_n]]$. Using the parametrization of $\boldsymbol{B}_m$ and the relations in eqs. (4.1) and (4.9), we find that such terms vanish due to $\mathrm{Tr}[\hat{H}_k E_{ij}^{(\pm)}] = 0$.

For the quadratic terms, a straightforward calculation shows

$$-\frac{1}{2}\mathrm{Tr}[L^{(2)}]=\mathcal{L}_1^{(2)}+\mathcal{L}_2^{(2)}+\mathcal{L}_3^{(2)}, \tag{5.11}$$

$$\mathcal{L}_1^{(2)}=\mathrm{Tr}[-(\partial_\mu \boldsymbol{A}_\nu)^2-(\mathcal{D}_m \boldsymbol{A}_\mu)^2+(\partial_\mu \boldsymbol{A}_\nu)(\partial^\nu \boldsymbol{A}^\mu)], \tag{5.12}$$

$$\mathcal{L}_2^{(2)}=\mathrm{Tr}\left[-(\partial_\mu \boldsymbol{A}_m)^2-(\mathcal{D}_m \boldsymbol{A}_n)^2+(\mathcal{D}_m \boldsymbol{A}_n)(\mathcal{D}^n \boldsymbol{A}^m)-ig\epsilon^{mn}\boldsymbol{A}_m \mathbf{ad}(\boldsymbol{f})\boldsymbol{A}_n\right], \tag{5.13}$$

$$\mathcal{L}_3^{(2)}=2\mathrm{Tr}[(\mathcal{D}^m \boldsymbol{A}^\mu)(\partial_\mu \boldsymbol{A}_m)], \tag{5.14}$$

where $\mathcal{L}_1^{(2)}$ and $\mathcal{L}_2^{(2)}$ are quadratic terms for $\boldsymbol{A}_\mu$ and $\boldsymbol{A}_m$, respectively. There appears a mixing term between $\boldsymbol{A}_\mu$ and $\boldsymbol{A}_m$ in $\mathcal{L}_3^{(2)}$. As seen below, this mixing term can be canceled by gauge fixing terms.

Using integration by parts, we can rewrite the quadratic terms in eqs. (5.12)–(5.14) as

$$\mathcal{L}_1^{(2)} = \mathrm{Tr}[\boldsymbol{A}^\mu(\eta_{\mu\nu}\Box+\eta_{\mu\nu}\mathcal{D}_m\mathcal{D}^m-\partial_\mu\partial_\nu)\boldsymbol{A}^\nu], \tag{5.15}$$

$$\mathcal{L}_2^{(2)} = \mathrm{Tr}[\boldsymbol{A}^m(\delta_{mn}\Box+\delta_{mn}\mathcal{D}_{m'}\mathcal{D}^{m'}-\mathcal{D}_n\mathcal{D}_m-ig\epsilon_{mn}\mathbf{ad}(\boldsymbol{f}))\boldsymbol{A}^n], \tag{5.16}$$

$$\mathcal{L}_3^{(2)} = 2\mathrm{Tr}[(\partial_\mu \boldsymbol{A}^\mu)(\mathcal{D}_m \boldsymbol{A}^m)], \tag{5.17}$$

where $\Box=\partial_\mu\partial^\mu$ is the 4D d'Alembertian. In the above expressions, we discarded the surface terms. Since we consider the compact extra dimensions, the surface terms have to be carefully treated. In the present case, all of the surface terms actually vanish in the action as discussed in appendix A.

5.2 Gauge fixing

We consider the standard procedure to fix the gauge in quantum theories. The gauge fixing yields the additional contributions to the Lagrangian as

$$\mathcal{L}_{\mathrm{YM}}^{\mathrm{gf}} = \mathcal{L}_{\mathrm{YM}}+\mathcal{L}_{\mathrm{GF}}+\mathcal{L}_c. \tag{5.18}$$

We refer to $\mathcal{L}_{\mathrm{GF}}$ and $\mathcal{L}_c$ as gauge fixing terms and ghost terms, respectively. We choose the gauge fixing terms as

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{\xi}\mathrm{Tr}[(\partial_\mu \boldsymbol{A}^\mu+\xi\mathcal{D}_m\boldsymbol{A}^m)^2] \tag{5.19}$$

$$= \frac{1}{\xi}\mathrm{Tr}[\boldsymbol{A}^\mu\partial_\mu\partial_\nu\boldsymbol{A}^\nu]+\xi\mathrm{Tr}[\boldsymbol{A}^m\mathcal{D}_m\mathcal{D}_n\boldsymbol{A}^n]-2\mathrm{Tr}[(\partial_\mu\boldsymbol{A}^\mu)(\mathcal{D}_m\boldsymbol{A}^m)], \tag{5.20}$$

where $\xi\in\mathbb{R}$ is called the gauge fixing parameter. From the first line to the second line, we have used integration by parts and dropped the surface terms, which vanish in the action as shown in appendix A. One sees that the last term in eq. (5.20) cancels $\mathcal{L}_3^{(2)}$ in $\mathcal{L}_{\mathrm{YM}}^{\mathrm{gf}}$.

From the gauge fixing terms given above, the ghost terms are obtained as

$$\mathcal{L}_c = -2\mathrm{Tr}[\bar{\boldsymbol{c}}\,\{\partial^\mu(\partial_\mu-ig\mathbf{ad}(\boldsymbol{A}_\mu))+\xi\mathcal{D}^m(\mathcal{D}_m-ig\mathbf{ad}(\boldsymbol{A}_m))\}\,\boldsymbol{c}], \tag{5.21}$$

where $\boldsymbol{c}$ and $\bar{\boldsymbol{c}}$ are ghost fields. They can be expanded by the $su(n)$ generators. In addition to the quadratic terms for the ghost fields, there also appear trilinear interactions including the gauge field fluctuations.

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5.3 Quadratic terms

To discuss the tree-level mass spectrum, let us focus on the quadratic terms of the fluctuations $\boldsymbol{A}_M$ and ghost fields. From the discussions in section 5.1 and 5.2, these terms in the Lagrangian in eq. (5.18) are given by

$$\mathcal{L}^{\rm gf}_{\rm YM} \ni \mathcal{L}^{(2)}_{A_\mu} + \mathcal{L}^{(2)}_{A_m} + \mathcal{L}^{(2)}_{c}, \tag{5.22}$$

$$\mathcal{L}^{(2)}_{A_\mu} = {\rm Tr}[\boldsymbol{A}^\mu(\eta_{\mu\nu}\Box + \eta_{\mu\nu}\mathcal{D}_m\mathcal{D}^m - (1-\xi^{-1})\partial_\mu\partial_\nu)\boldsymbol{A}^\nu], \tag{5.23}$$

$$\mathcal{L}^{(2)}_{A_m} = {\rm Tr}[\boldsymbol{A}^m(\delta_{mn}\Box + \delta_{mn}\mathcal{D}_{m'}\mathcal{D}^{m'} - (1-\xi)\mathcal{D}_m\mathcal{D}_n - 2ig\epsilon_{mn}\mathbf{ad}(\boldsymbol{f}))\boldsymbol{A}^n], \tag{5.24}$$

$$\mathcal{L}^{(2)}_{c} = -2{\rm Tr}[\bar{\boldsymbol{c}}(\Box + \xi\mathcal{D}_m\mathcal{D}^m)\boldsymbol{c}]. \tag{5.25}$$

We note that, to obtain the expression for $\mathcal{L}^{(2)}_{A_m}$, we have used

$$\mathcal{D}_n\mathcal{D}_m = [\mathcal{D}_n, \mathcal{D}_m] + \mathcal{D}_m\mathcal{D}_n = ig\epsilon_{mn}\mathbf{ad}(\boldsymbol{f}) + \mathcal{D}_m\mathcal{D}_n. \tag{5.26}$$

The last term in eq. (5.24) being proportional to $\mathbf{ad}(\boldsymbol{f})$ appears since the extra-dimensional components of the gauge fields have non-trivial helicities in the two-dimensional torus [38].

We use the explicit forms of the generators in eqs. (4.3), (4.4), and (4.7) to expand $\boldsymbol{A}_M$ as

$$\boldsymbol{A}_M = A^k_M\hat{H}_k + A^{ij}_M E^{(+)}_{ij} + \bar{A}^{ij}_M E^{(-)}_{ij} = \begin{pmatrix} A^1_M & A^{12}_M & \dots & A^{1n}_M \\ \bar{A}^{12}_M & A^2_M - A^1_M & & \vdots \\ \vdots & & \ddots & \\ \bar{A}^{1n}_M & \dots & & -A^{n-1}_M \end{pmatrix}, \tag{5.27}$$

where we have introduced the component fields A^k_M $(k = 1, \dots, n-1)$, A^{ij}_M, and $\bar{A}^{ij}_M = (A^{ij}_M)^\dagger$ $(1 \le i < j \le n)$. We note that the summations over $1 \le k \le n-1$ or $1 \le i < j \le n$ are implied in each term in the second expression in eq. (5.27). A similar expansion to eq. (5.27) for the ghost fields is also used.

The expansion in eq. (5.27) simplifies the discussion since the component fields are eigenfunctions of the background covariant derivative and the boundary conditions. We first express the Lagrangian by the component fields. From eq. (4.12), one sees that

$$\mathcal{D}_m\boldsymbol{A}_M = \partial_m A^k_M\hat{H}_k + D^{(ij)}_m A^{ij}_M E^{(+)}_{ij} + \bar{D}^{(ij)}_m \bar{A}^{ij}_M E^{(-)}_{ij}, \tag{5.28}$$

where we have defined

$$D^{(ij)}_5 = \partial_5 + ig\hat{f}\tilde{N}^{ij}(1+\gamma)x^6/2, \qquad D^{(ij)}_6 = \partial_6 - ig\hat{f}\tilde{N}^{ij}(1-\gamma)x^5/2, \tag{5.29}$$

$$\bar{D}^{(ij)}_5 = \partial_5 - ig\hat{f}\tilde{N}^{ij}(1+\gamma)x^6/2, \qquad \bar{D}^{(ij)}_6 = \partial_6 + ig\hat{f}\tilde{N}^{ij}(1-\gamma)x^5/2. \tag{5.30}$$

Thus, each component is an eigenfunction of the covariant derivative. Using eqs. (4.8) and (4.9), we obtain

$$\mathcal{L}^{(2)}_{A_\mu} = (M^{su(n)})_{k\ell}A^k_\mu\left[\eta^{\mu\nu}\Box - (1-\xi^{-1})\partial^\mu\partial^\nu + \eta^{\mu\nu}\partial_m\partial^m\right]A^\ell_\nu + \sum_{1\le i<j\le n}\mathcal{L}^{(ij)}_{A_\mu}, \tag{5.31}$$

$$\mathcal{L}^{(2)}_{A_m} = (M^{su(n)})_{k\ell}A^k_m\left[\eta^{mn}\Box + \eta^{mn}\partial_{m'}\partial^{m'} - (1-\xi)\partial^m\partial^n\right]A^\ell_n + \sum_{1\le i<j\le n}\mathcal{L}^{(ij)}_{A_m}, \tag{5.32}$$

where we have introduced

$$\mathcal{L}^{(ij)}_{A_\mu} = 2\bar{A}^{ij}_\mu[\eta^{\mu\nu}\Box - (1-\xi^{-1})\partial^\mu\partial^\nu + \eta^{\mu\nu}(D^{(ij)}_{m'})^2]A^{ij}_\nu, \tag{5.33}$$

$$\mathcal{L}^{(ij)}_{A_m} = 2\bar{A}^{ij}_m[\delta^{mn}\Box + \delta^{mn}(D^{(ij)}_{m'})^2 - (1-\xi)D^{m(ij)}D^{n(ij)} - 2ig\hat{f}\tilde{N}^{ij}\epsilon^{mn}]A^{ij}_n. \tag{5.34}$$

For the ghost fields, a similar discussion holds. To obtain eqs. (5.33) and (5.34), we have used integration by parts and safely discarded surface terms; please refer to appendix A.

In the Lagrangian in eqs. (5.31) and (5.32), the components related to the Cartan generators $\hat{H}_k$ do not couple to flux. Although the Cartan matrix $M^{su(n)}$ is not diagonalized in this basis, we can take linear combinations of A^k_M to diagonalize the Lagrangian. For the components related to the step operators, the Lagrangian is completely separated for each (ij).

Let us discuss the boundary conditions for the component fields. As discussed in section 3, the fluctuations $\boldsymbol{A}_M$ satisfy the boundary conditions in eq. (3.8). Using the expression of the twist matrices in eq. (3.18), we can rewrite the boundary conditions in eq. (3.8) as

$$\boldsymbol{A}_M(\mathcal{T}_5 z) = e^{-ig\,\mathbf{ad}(\boldsymbol{v}_5)}e^{ig(1-\gamma)(x^6/2)\mathbf{ad}(\boldsymbol{f})}\boldsymbol{A}_M(z), \tag{5.35}$$

$$\boldsymbol{A}_M(\mathcal{T}_6 z) = e^{-ig(\tau_{\rm R}\mathbf{ad}(\boldsymbol{v}_5)+\tau_{\rm I}\mathbf{ad}(\boldsymbol{v}_6))}e^{ig\{-(1+\gamma)\tau_{\rm I}x^5/2+(1-\gamma)\tau_{\rm R}x^6/2\}\mathbf{ad}(\boldsymbol{f})}\boldsymbol{A}_M(z), \tag{5.36}$$

where we have used the Campbell-Baker-Hausdorff formula $e^X Y e^{-X} = e^{\mathbf{ad}(X)}Y$. Combining the above and eq. (5.27), one sees that the components obey the following boundary conditions:

$$A^k_M(\mathcal{T}_5 z) = A^k_M(\mathcal{T}_6 z) = A^k_M(z), \tag{5.37}$$

$$A^{ij}_M(\mathcal{T}_5 z) = e^{-ig\tilde{v}^{ij}_5}e^{ig(1-\gamma)(x^6/2)\hat{f}\tilde{N}^{ij}}A^{ij}_M(z), \tag{5.38}$$

$$A^{ij}_M(\mathcal{T}_6 z) = e^{-ig(\tau_{\rm R}\tilde{v}^{ij}_5+\tau_{\rm I}\tilde{v}^{ij}_6)}e^{ig\{-(1+\gamma)\tau_{\rm I}x^5/2+(1-\gamma)\tau_{\rm R}x^6/2\}\hat{f}\tilde{N}^{ij}}A^{ij}_M(z), \tag{5.39}$$

where the ones for $\bar{A}^{ij}_M$ are given by $\bar{A}^{ij}_M = (A^{ij}_M)^\dagger$.

From the Lagrangian and the boundary conditions for the component fields, we can understand that the components A^k_M are decoupled from both the flux and the continuous Wilson lines. On the other hand, the components A^{ij}_M can couple with them if $\tilde{N}^{ij}$ or $\tilde{v}^{ij}_m$ do not vanish. We keep general configurations of the Wilson line phases. Then, for a given magnetic flux configuration, i.e., a value of N^k in eq. (4.5), the masses for A^{ij}_M are basically classified into $\tilde{N}^{ij} = 0$ or $\tilde{N}^{ij} \neq 0$ cases. We will examine the mass spectrum for both cases in the next section.

6 Mass spectrum around flux backgrounds

6.1 Masses for KK modes of A^k_μ

Let us discuss the tree-level mass spectrum obtained from A^k_μ. Their masses are derived from the 6D kinetic terms in eq. (5.31) and the boundary conditions in eq. (5.37). As they

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are periodic under the translations generated by $\mathcal{T}_p$, it is useful to introduce Kaluza-Klein (KK) expansions as

$$A_\mu^k(x^\mu, z) = \sum_{\hat{n},\hat{m}\in\mathbb{Z}} C^{(\hat{n},\hat{m})} e^{2\pi i\hat{n}x^5} e^{2\pi i\tau_{\rm I}^{-1}(\hat{m}-\hat{n}\tau_{\rm R})x^6} A^k_{\mu(\hat{n},\hat{m})}(x^\mu), \tag{6.1}$$

where $C^{(\hat{n},\hat{m})}$ is a normalization constant, and $A^k_{\mu(\hat{n},\hat{m})}(x^\mu)$ $(\hat{n},\hat{m} \in \mathbb{Z})$ is identified as a 4D field.

We denote the KK mass for $A^k_{\mu(\hat{n},\hat{m})}$ as $M^2(A^k_{\mu(\hat{n},\hat{m})})$. The Lagrangian in eq. (5.31) implies that $M^2(A^k_{\mu(\hat{n},\hat{m})})$ is given by the eigenvalue of the operator $-\partial_m\partial^m$ acting on the corresponding mode function. Thus, we obtain

$$M^2(A^k_{\mu(\hat{n},\hat{m})}) = \left(\frac{2\pi}{L}\right)^2 \left[\hat{n}^2 + \tau_{\rm I}^{-2}(\hat{m}-\hat{n}\tau_{\rm R})^2\right], \tag{6.2}$$

where we have temporarily written L for an illustration.

The mass spectrum contains the massless 4D gauge fields $A_\mu^{k(0,0)}$, namely, the zero modes. The massless gauge fields are related to a gauge symmetry manifested in a low-energy effective theory.

There is a clear geometrical interpretation of the KK masses in eq. (6.2). The torus lattice is spanned by the basis vectors $\boldsymbol{e}_1 = (1,0)$ and $\boldsymbol{e}_2 = (\tau_{\rm R}, \tau_{\rm I})$, where the right-hand sides are components in an orthogonal basis. The basis vectors $\boldsymbol{e}^i$ of a dual lattice are defined by $\boldsymbol{e}_i \cdot \boldsymbol{e}^j = 2\pi\delta_i^j$, which gives $\boldsymbol{e}^1 = 2\pi(1, -\tau_{\rm R}/\tau_{\rm I})$ and $\boldsymbol{e}^2 = 2\pi(0, 1/\tau_{\rm I})$. Discretized extra-dimensional momenta correspond to points on the dual lattice, and the squared norm of a dual vector from the origin to a point,

$$|\hat{n}\boldsymbol{e}^1 + \hat{m}\boldsymbol{e}^2|^2 = (2\pi)^2 \left[\hat{n}^2 + \tau_{\rm I}^{-2}(\hat{m}-\hat{n}\tau_{\rm R})^2\right], \tag{6.3}$$

gives the KK mass squared in eq. (6.2).

6.2 Masses for KK modes of A_m^k

Let us study the mass spectrum obtained from A_m^k. Since A_m^k obeys the same boundary conditions as those for A_μ^k, the KK expansion of A_m^k is given by using the same mode functions as in eq. (6.1). We write the KK mode of A_m^k as $A^k_{m(\hat{n},\hat{m})}$.

The KK masses for $A^k_{m(\hat{n},\hat{m})}$, denoted by $M^2(A^k_{m(\hat{n},\hat{m})})$, are determined by the Lagrangian in eq. (5.32). The mass matrices for $(A^k_{5(\hat{n},\hat{m})}, A^k_{6(\hat{n},\hat{m})})$ appear from the operator $-\eta^{mn}\partial_{m'}\partial^{m'} + (1-\xi)\partial^m\partial^n$ in eq. (5.32) and are given by

$$\begin{aligned}(A^k_{5(\hat{n},\hat{m})}\ A^k_{6(\hat{n},\hat{m})}) \left(\frac{2\pi}{L}\right)^2 &\left[\left[\hat{n}^2 + \tau_{\rm I}^{-2}(\hat{m}-\hat{n}\tau_{\rm R})^2\right]\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right.\\ &\left. -(1-\xi)\begin{pmatrix}\hat{n}^2 & \hat{n}\tau_{\rm I}^{-1}(\hat{m}-\hat{n}\tau_{\rm R})\\ \hat{n}\tau_{\rm I}^{-1}(\hat{m}-\hat{n}\tau_{\rm R}) & \tau_{\rm I}^{-2}(\hat{m}-\hat{n}\tau_{\rm R})^2\end{pmatrix}\right]\begin{pmatrix}A^k_{5(\hat{n},\hat{m})}\\ A^k_{6(\hat{n},\hat{m})}\end{pmatrix}.\end{aligned} \tag{6.4}$$

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Diagonalizing the above, we find mass eigenvalues $M^2_{\rm ph}(A^k_{m(\hat{n},\hat{m})})$ and $M^2_\xi(A^k_{m(\hat{n},\hat{m})})$ for each $(\hat{n},\hat{m})$ mode as

$$M^2_{\rm ph}(A^k_{m(\hat{n},\hat{m})}) = \left(\frac{2\pi}{L}\right)^2 \left[\hat{n}^2 + \tau_{\rm I}^{-2}(\hat{m}-\hat{n}\tau_{\rm R})^2\right], \tag{6.5}$$

$$M^2_\xi(A^k_{m(\hat{n},\hat{m})}) = \xi\left(\frac{2\pi}{L}\right)^2 \left[\hat{n}^2 + \tau_{\rm I}^{-2}(\hat{m}-\hat{n}\tau_{\rm R})^2\right]. \tag{6.6}$$

The zero modes $A^k_{m(0,0)}$ are massless scalars whose masses are independent of the gauge parameter ξ. Some of these zero modes may acquire non-zero masses through quantum corrections, except for NG bosons associated with the breaking of the translational invariance on T^2 via non-vanishing flux [55, 57]. We will discuss the quantum corrections in section 6.6. For $(\hat{n},\hat{m}) \neq (0,0)$, there appear massive scalar modes. The massive scalars with masses $M^2_{\rm ph}(A^k_{m(\hat{n},\hat{m})})$ are degenerate with $A^k_{\mu(\hat{n},\hat{m})}$, although the massive modes with masses $M^2_\xi(A^k_{m(\hat{n},\hat{m})})$ are would-be Goldstone modes, which provide physical degrees of freedom to longitudinal modes of massive vector fields $A^k_{\mu(\hat{n},\hat{m})}$.

6.3 Masses for KK modes of A^{ij}_μ and A^{ij}_m with $\tilde{N}^{ij} = 0$

We discuss the mass spectrum that arises from A^{ij}_μ and A^{ij}_m in the $\tilde{N}^{ij}=0$ case. In this case, the Lagrangians in eqs. (5.33) and (5.34) are simplified to

$$\mathcal{L}^{(ij)}_{A_\mu} \to 2\bar{A}^{ij}_\mu[\eta^{\mu\nu}\Box - (1-\xi^{-1})\partial^\mu\partial^\nu + \eta^{\mu\nu}\partial_{m'}\partial^{m'}]A^{ij}_\nu, \tag{6.7}$$

$$\mathcal{L}^{(ij)}_{A_m} \to 2\bar{A}^{ij}_m[\delta^{mn}\Box + \delta^{mn}\partial_{m'}\partial^{m'} - (1-\xi)\partial^m\partial^n]A^{ij}_n. \tag{6.8}$$

The differential operators appearing above are the same ones as in the Lagrangian for A^k_M. On the other hand, the boundary conditions for A^{ij}_M are different from those for A^k_M. From eqs. (5.38) and (5.39), one sees that A^{ij}_M are not periodic under the translations for $\tilde{v}^{ij}_m \neq 0$, while A^k_M are periodic.

Due to the phase factors including $\tilde{v}^{ij}_m$ in eqs. (5.38) and (5.39), the mass spectrum is modified compared to that of A^k_M. These phase factors induce the overall shifts of the momentum lattice spanned by the dual basis vectors $\boldsymbol{e}^i$ discussed in section 6.1. As a result, KK masses for A^{ij}_μ and A^{ij}_m with $\tilde{N}^{ij}=0$ have similar forms to the masses in eqs. (6.2), (6.5), and (6.6), but $\hat{n}$ and $\hat{m}$ are replaced by $\hat{n} - g\tilde{v}^{ij}_5/2\pi$ and $\hat{m} - g\tilde{v}^{ij}_6/2\pi$, respectively.[2] Let us introduce the parametrization $\tilde{a}^{ij}_m = g\tilde{v}^{ij}_m/2\pi$. Then, the KK masses for A^{ij}_μ and A^{ij}_m are given by

$$\begin{aligned} M^2(A^{ij}_{\mu(\hat{n},\hat{m})}) &= M^2_{\rm ph}(A^{ij}_{m(\hat{n},\hat{m})}) & (6.9)\\ &= \left(\frac{2\pi}{L}\right)^2\left[(\hat{n}-\tilde{a}^{ij}_5)^2 + \tau_{\rm I}^{-2}((\hat{m}-\tilde{a}^{ij}_6)-(\hat{n}-\tilde{a}^{ij}_5)\tau_{\rm R})^2\right], & (6.10)\end{aligned}$$

$$M^2_\xi(A^{ij}_{m(\hat{n},\hat{m})}) = \xi\left(\frac{2\pi}{L}\right)^2\left[(\hat{n}-\tilde{a}^{ij}_5)^2 + \tau_{\rm I}^{-2}((\hat{m}-\tilde{a}^{ij}_6)-(\hat{n}-\tilde{a}^{ij}_5)\tau_{\rm R})^2\right]. \tag{6.11}$$

[2] A similar argument in a $T^2/\mathbb{Z}_3$ orbifold model, for instance, is found in [32].

Note that the KK mass spectrum is invariant under integer shifts of $\tilde{a}^{ij}_m$. This property is expected from the boundary conditions in eqs. (5.38) and (5.39), which are also invariant under the integer shifts.

Except for the case with $\tilde{a}^{ij}_m = 0 \bmod 1$, this spectrum has no massless modes. As in the case of A^k_M, massive 4D vector fields from A^{ij}_μ and scalar fields from A^{ij}_m appear. Half of the scalar KK modes have ξ-dependent masses $M^2_\xi(A^{ij}_{m(\hat{n},\hat{m})})$ and are would-be Goldstone modes that provide physical degrees of freedom to the longitudinal modes of massive vector fields.

6.4 Masses for Landau level excitations of A^{ij}_μ with $\tilde{N}^{ij} \neq 0$

We discuss the mass spectrum of the gauge field A^{ij}_μ in the $\tilde{N}^{ij} \neq 0$ case. In this case, A^{ij}_μ couples to the background flux through the covariant derivative in eq. (5.33). The mass spectrum is determined by the eigenvalues of the operator

$$-(D^{(ij)}_m)^2 = [-i\partial_5 + g\hat{f}\tilde{N}^{ij}(1+\gamma)x^6/2]^2 + [-i\partial_6 - g\hat{f}\tilde{N}^{ij}(1-\gamma)x^5/2]^2. \tag{6.12}$$

The mode function is given as the eigenfunction of this operator and should be consistent with the boundary conditions in eqs. (5.38) and (5.39).

This system is an analog to a two-dimensional quantum mechanical system with a constant magnetic flux. The mass spectrum is the well-known Landau levels [72]. Let us denote the quantum mechanical momentum operator $p_m = -i\partial_m$, which satisfies $[x^m, p_n] = i\delta^m_n$. It is convenient to express

$$-iD^{(ij)}_5 = p_5 + g\hat{f}\tilde{N}^{ij}(1+\gamma)x^6/2, \qquad -iD^{(ij)}_6 = p_6 - g\hat{f}\tilde{N}^{ij}(1-\gamma)x^5/2, \tag{6.13}$$

which satisfy $[-iD^{(ij)}_5, -iD^{(ij)}_6] = ig\hat{f}\tilde{N}^{ij}$ independently to γ. Let us define

$$\Pi^{(ij)}_z = -i\sqrt{\frac{2}{g\hat{f}|\tilde{N}^{ij}|}}\frac{D^{(ij)}_5 - iD^{(ij)}_6}{2} = \sqrt{\frac{2}{g\hat{f}|\tilde{N}^{ij}|}}\left[p_z + i\frac{g\hat{f}\tilde{N}^{ij}}{4}(\bar{z} - \gamma z)\right], \tag{6.14}$$

$$\bar{\Pi}^{(ij)}_z = -i\sqrt{\frac{2}{g\hat{f}|\tilde{N}^{ij}|}}\frac{D^{(ij)}_5 + iD^{(ij)}_6}{2} = \sqrt{\frac{2}{g\hat{f}|\tilde{N}^{ij}|}}\left[\bar{p}_z - i\frac{g\hat{f}\tilde{N}^{ij}}{4}(z - \gamma \bar{z})\right], \tag{6.15}$$

where we have used $p_z = -i\partial_z$ and $\bar{p}_z = -i\bar{\partial}_z$.

For the case with $\tilde{N}^{ij} > 0$ ($\tilde{N}^{ij} < 0$), $[\bar{\Pi}^{(ij)}_z, \Pi^{(ij)}_z] = 1$ ($[\Pi^{(ij)}_z, \bar{\Pi}^{(ij)}_z] = 1$) holds. Thus, $\bar{\Pi}^{(ij)}_z$ and $\Pi^{(ij)}_z$ ($\Pi^{(ij)}_z$ and $\bar{\Pi}^{(ij)}_z$) are interpreted as an annihilation and a creation operator, respectively. The operator in eq. (6.12) is expressed by them as

$$-(D^{(ij)}_m)^2 = 2g\hat{f}|\tilde{N}^{ij}|(\hat{a}^\dagger\hat{a} + 1/2), \tag{6.16}$$

where $(\hat{a}, \hat{a}^\dagger) = (\bar{\Pi}^{(ij)}_z, \Pi^{(ij)}_z)$ for $\tilde{N}^{ij} > 0$, and $(\hat{a}, \hat{a}^\dagger) = (\Pi^{(ij)}_z, \bar{\Pi}^{(ij)}_z)$ for $\tilde{N}^{ij} < 0$.

Eigenvalues of the operator in eq. (6.16) are the same as those of a harmonic oscillator. Also, as shown in appendix B, the eigenvalues are derived by introducing the mode functions of the ground state $\zeta^{ij}_{0,d}(z)$ and ℓ-th excited states $\zeta^{ij}_{\hat{\ell},d}(z)$ ($\hat{\ell} \in \mathbb{Z}_{\geq 1}$) as

$$\hat{a}\zeta^{ij}_{0,d}(z) = 0, \qquad \zeta^{ij}_{\hat{\ell},d}(z) = \frac{1}{\sqrt{\hat{\ell}!}}(\hat{a}^\dagger)^{\hat{\ell}}\zeta^{ij}_{0,d}(z). \tag{6.17}$$

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The subscript d takes $d = 0, \ldots, |\tilde{N}^{ij}| - 1$ and labels $|\tilde{N}^{ij}|$ degenerate states in this system. We also have to impose that the mode functions obey the same boundary conditions for A_M^{ij} in eqs. (5.38) and (5.39) as

$$\zeta^{ij}_{\hat{\ell},d}(\mathcal{T}_5 z) = e^{-2\pi i \tilde{a}_5^{ij}} e^{ig(1-\gamma)(x^6/2)\hat{f}\tilde{N}^{ij}} \zeta^{ij}_{\hat{\ell},d}(z), \tag{6.18}$$

$$\zeta^{ij}_{\hat{\ell},d}(\mathcal{T}_6 z) = e^{-2\pi i(\tau_{\mathrm{R}}\tilde{a}_5^{ij}+\tau_{\mathrm{I}}\tilde{a}_6^{ij})} e^{ig\{-(1+\gamma)\tau_{\mathrm{I}}x^5/2+(1-\gamma)\tau_{\mathrm{R}}x^6/2\}\hat{f}\tilde{N}^{ij}} \zeta^{ij}_{\hat{\ell},d}(z). \tag{6.19}$$

The explicit form of the mode function is given by

$$\begin{aligned}\zeta^{ij}_{\hat{\ell},d}(z) = &\left(\frac{2\pi\tilde{N}^{ij}}{\tau_{\mathrm{I}}}\right)^{1/4} \frac{1}{2^{\hat{\ell}}\sqrt{\hat{\ell}!}} e^{\frac{\pi\tilde{N}^{ij}}{2\tau_{\mathrm{I}}}\left[z(z-\bar{z})-\frac{\gamma}{2}(z+\bar{z})(z-\bar{z})\right]} \\ &\times \sum_{n=-\infty}^{+\infty} H_{\hat{\ell}}(w_{n,d}(z)) e^{i\pi\tilde{N}^{ij}\tau(n-(\tilde{a}_5^{ij}-d)/\tilde{N}^{ij})^2} e^{2\pi i(n-(\tilde{a}_5^{ij}-d)/\tilde{N}^{ij})(\tilde{N}^{ij}z+\tau_{\mathrm{R}}\tilde{a}_5^{ij}+\tau_{\mathrm{I}}\tilde{a}_6^{ij}-\gamma\tilde{N}^{ij}\tau_{\mathrm{R}}/2)},\end{aligned} \tag{6.20}$$

for $\tilde{N}^{ij} > 0$, where $H_{\hat{\ell}}(x)$ are the Hermite polynomials, and we have used

$$w_{n,d}(z) = \sqrt{\frac{2\pi N^{ij}}{\tau_{\mathrm{I}}}} \left[\frac{z-\bar{z}}{2i} + \tau_{\mathrm{I}}\left(n - \frac{\tilde{a}_5^{ij}-d}{\tilde{N}^{ij}}\right)\right]. \tag{6.21}$$

The normalization constant has been determined so that the orthogonal relation is satisfied as

$$\int_{\mathcal{V}_{T^2}} dx^5 dx^6 \; \bar{\zeta}^{ij}_{\hat{\ell},d}\zeta^{ij}_{\hat{\ell}',d'} = \delta_{\hat{\ell}\hat{\ell}'}\delta_{dd'}, \tag{6.22}$$

where $\bar{\zeta}^{ij}_{\hat{\ell},d} = (\zeta^{ij}_{\hat{\ell},d})^\dagger$. One can obtain the explicit mode function for $\tilde{N}^{ij} < 0$ by taking the complex conjugate of eq. (6.20). The derivation and details are summarized in appendix B. Through the boundary conditions in eqs. (6.18) and (6.19), the mode function in eq. (6.20) depends on the Wilson line phases $\tilde{a}_m^{ij}$. As explained above, there are $\tilde{N}^{ij}$ independent mode functions for a fixed $\hat{\ell}$. From the right-hand side of eq. (6.20), one sees that the shift of $d \to d + \tilde{N}^{ij}$ leaves the mode function unchanged.

Let us define the mode expansion as

$$A_\mu^{ij}(x^\mu, z) = \sum_{\hat{\ell}=0}^{\infty}\sum_{d=1}^{|\tilde{N}^{ij}|} A^{ij}_{\mu(\hat{\ell},d)}(x^\mu)\zeta^{ij}_{\hat{\ell},d}(z), \tag{6.23}$$

where we refer to $A^{ij}_{\mu(\hat{\ell},d)}(x^\mu)$ as the $\hat{\ell}$-th Landau level. From the operator in eq. (6.12) and the mode expansion in eq. (6.23), we find that the masses for the $\hat{\ell}$-th Landau level, denoted by $M^2(A^{ij}_{\mu(\hat{\ell},d)})$, are given by

$$M^2(A^{ij}_{\mu(\hat{\ell},d)}) = 2g\hat{f}|\tilde{N}^{ij}|(\hat{\ell}+1/2). \tag{6.24}$$

There are no massless modes, and all $A^{ij}_{\mu(\hat{\ell},d)}$ have masses proportional to L^{-1}. Thus, the gauge symmetry is broken by the flux. We note that Wilson line phases do not appear in the mass spectrum, although the mode function in eq. (6.20) depends on the phases.

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6.5 Masses for Landau level excitations of A_m^{ij} with $\tilde{N}^{ij} \neq 0$

In this section, we discuss the mass spectrum for A_m^{ij} when $\tilde{N}^{ij} \neq 0$, that is, the components that couple to the flux. The mass spectrum is determined by the last three terms in the Lagrangian in eq. (5.34), where A_5^{ij} and A_6^{ij} are mixed. After performing the diagonalization of this term, it is convenient to define

$$\begin{pmatrix} A_-^{ij} \\ A_+^{ij} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} A_5^{ij} - iA_6^{ij} \\ A_5^{ij} + iA_6^{ij} \end{pmatrix}, \tag{6.25}$$

where A_-^{ij} and A_+^{ij} correspond to components of $\boldsymbol{A}_z$ and $\bar{\boldsymbol{A}}_z$ in eq. (2.9) related to the generator $E_{ij}^{(+)}$ in eq. (4.7) and $\bar{A}_\pm^{ij} = (A_\pm^{ij})^\dagger$. With this definition, the Lagrangian in eq. (5.34) is rewritten as

$$\begin{aligned}
\mathcal{L}_{A_m}^{(ij)} &= 2\bar{A}_-^{ij} \left[\Box + \frac{1+\xi}{2} (D_m^{(ij)})^2 + \frac{3+\xi}{2} g\hat{f}\tilde{N}^{ij} \right] A_-^{ij} \\
&\quad + 2\bar{A}_+^{ij} \left[\Box + \frac{1+\xi}{2} (D_m^{(ij)})^2 - \frac{3+\xi}{2} g\hat{f}\tilde{N}^{ij} \right] A_+^{ij} \\
&\quad - 2\bar{A}_+^{ij} \left[\frac{1-\xi}{2} (D_5^{(ij)} + iD_6^{(ij)})^2 \right] A_-^{ij} - 2\bar{A}_-^{ij} \left[\frac{1-\xi}{2} (D_5^{(ij)} - iD_6^{(ij)})^2 \right] A_+^{ij},
\end{aligned} \tag{6.26}$$

where the covariant derivatives are defined in eq. (5.29). From the definitions of $\Pi_z^{(ij)}$ and $\bar{\Pi}_z^{(ij)}$ in eqs. (6.14) and (6.15), respectively, the Lagrangian can be written in terms of creation and annihilation operators

$$\mathcal{L}_{A_m}^{(ij)} = 2 \begin{pmatrix} \bar{A}_-^{ij} & \bar{A}_+^{ij} \end{pmatrix} \begin{pmatrix} \Box - g\hat{f}\tilde{N}^{ij}((1+\xi)\hat{a}^\dagger\hat{a} - 1) & g\hat{f}\tilde{N}^{ij}(1-\xi)\hat{a}^\dagger\hat{a}^\dagger \\ g\hat{f}\tilde{N}^{ij}(1-\xi)\hat{a}\hat{a} & \Box - g\hat{f}\tilde{N}^{ij}((1+\xi)\hat{a}^\dagger\hat{a} + \xi + 2) \end{pmatrix} \begin{pmatrix} A_-^{ij} \\ A_+^{ij} \end{pmatrix}, \tag{6.27}$$

where we have set $\tilde{N}^{ij} > 0$ for simplicity. In this case, $(\hat{a}, \hat{a}^\dagger) = (\bar{\Pi}_z^{(ij)}, \Pi_z^{(ij)})$. For the $\tilde{N}^{ij} < 0$ case, we also have a similar expression.

Since A_m^{ij} obeys the same boundary conditions as A_μ^{ij}, they have the same mode functions in the KK expansion in eq. (6.23). Therefore, the mode expansion is defined as

$$A_\pm^{ij}(x^\mu, z) = \sum_{\hat{\ell}=0}^{\infty} \sum_{d=1}^{\tilde{N}^{ij}} A_{\pm(\hat{\ell},d)}^{ij}(x^\mu) \zeta_{\hat{\ell},d}^{ij}(z), \tag{6.28}$$

where $A_{\pm(\hat{\ell},d)}^{ij}$ is the $\hat{\ell}$-th Landau level, and $\zeta_{\hat{\ell},d}^{ij}(z)$ satisfy orthonormal relations as already mentioned in eq. (6.22). After acting the creation and annihilation operators on the mode expansion and integrating out the extra dimensions, the Lagrangian becomes

$$\begin{aligned}
\mathcal{L}_{A_m}^{(ij)} &= 2\bar{A}_{-(0,d)}^{ij} \left[\Box + g\hat{f}\tilde{N}^{ij} \right] A_{-(0,d)}^{ij} + 2\bar{A}_{-(1,d)}^{ij} \left[\Box - \xi g\hat{f}\tilde{N}^{ij} \right] A_{-(1,d)}^{ij} \\
&\quad + 2\sum_{\hat{\ell}=0}^{\infty} \begin{pmatrix} \bar{A}_{+(\hat{\ell},d)}^{ij} & \bar{A}_{-(\hat{\ell}+2,d)}^{ij} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \Box - g\hat{f}\tilde{N}^{ij}[(1+\xi)\hat{\ell} + 2 + \xi] & g\hat{f}\tilde{N}^{ij}(1-\xi)\sqrt{(\hat{\ell}+1)(\hat{\ell}+2)} \\ g\hat{f}\tilde{N}^{ij}(1-\xi)\sqrt{(\hat{\ell}+1)(\hat{\ell}+2)} & \Box - g\hat{f}\tilde{N}^{ij}[(1+\xi)(\hat{\ell}+2) - 1] \end{pmatrix} \begin{pmatrix} A_{+(\hat{\ell},d)}^{ij} \\ A_{-(\hat{\ell}+2,d)}^{ij} \end{pmatrix},
\end{aligned} \tag{6.29}$$

where the sum over d is implied. Taking suitable values of the gauge parameter ξ and $\tilde{N}^{ij}$ for the SU(2) case, the above expression is consistent with similar equations shown in [55]. Diagonalizing the Lagrangian in eq. (6.29) by using the following orthogonal matrix,

$$\frac{1}{\sqrt{2\hat{\ell}+3}}\begin{pmatrix}\sqrt{\hat{\ell}+2} & \sqrt{\hat{\ell}+1}\\ -\sqrt{\hat{\ell}+1} & \sqrt{\hat{\ell}+2}\end{pmatrix}, \tag{6.30}$$

we finally obtain the masses for $A^{ij}_{\pm(\hat{\ell},d)}$:

$$M^2_{\rm ph}(A^{ij}_{-(0,d)}) = 2g\hat{f}\tilde{N}^{ij}(-1/2), \tag{6.31}$$

$$M^2_{\xi}(A^{ij}_{+(0,d)}) = 2g\hat{f}\tilde{N}^{ij}\xi(1/2), \tag{6.32}$$

$$M^2_{\rm ph}(A^{ij}_{\pm(\hat{\ell},d)}) = 2g\hat{f}\tilde{N}^{ij}(\hat{\ell}+1/2), \qquad \hat{\ell}\geq 1, \tag{6.33}$$

$$M^2_{\xi}(A^{ij}_{\pm(\hat{\ell},d)}) = 2g\hat{f}\tilde{N}^{ij}\xi(\hat{\ell}+1/2), \qquad \hat{\ell}\geq 1, \tag{6.34}$$

for $\tilde{N}^{ij} > 0$. We also have the same mass spectrum for the $\tilde{N}^{ij} < 0$ case. The scalars $A^{ij}_{-(0,d)}$ are tachyonic, and all of the other physical scalars are massive. Half of the masses depend on the gauge fixing parameter, corresponding to the masses for would-be Goldstone modes. Their degrees of freedom will be absorbed by the infinite tower of vector fields, leading them to become massive.

6.6 Vanishing one-loop potentials for NG bosons

As discussed in section 6.2, NG bosons appear related to the breaking of the translational symmetry with non-vanishing flux. The translational invariance is realized non-linearly, under which NG bosons shift [55, 57]. In our setup, the NG bosons are identified to some combinations of the zero modes $A^k_{m(0,0)}$. More precisely, for a given set of N^k in eq. (4.5), the Wilson line phase degrees of freedom along the Cartan generator $N^k\hat{H}_k$ are the NG bosons.

From the tree-level mass spectrum, we can understand that one-loop corrections do not induce masses and potentials for the NG bosons, as follows. Remind that we have treated the Wilson line phases $\boldsymbol{v}_m$ in eq. (3.18) as some parts of the boundary condition. On the other hand, as explained in section 3, they can also be treated as the VEVs as in eq. (3.5). Both treatments are gauge equivalent and yield the same mass spectrum in the 4D effective theory. In general, VEVs of the Wilson line phases are not determined at tree level but may be fixed by effective potentials generated by quantum corrections. To obtain the effective potential for the Wilson line phases, which are now considered to be dynamical variables, we can formally evaluate the path integral using the loop expansion, although the tachyonic states exist. The obtained effective potentials for the phases are mainly determined by the dependence of the Wilson line phases on the mass spectrum in the 4D effective theory. The crucial point is that the Wilson line phases corresponding to the NG bosons, along the generator $N^k\hat{H}_k$, completely disappear from the mass spectrum. This follows from the fact that any fields coupled to the Wilson line phases corresponding to the NG bosons are also coupled to the flux and have masses independent of the Wilson line phases. Thus, Wilson line phases along $N^k\hat{H}_k$ have completely flat potential at one-loop level as expected.

The other Wilson line phases have generally non-vanishing one-loop potentials and become massive. However, the result may be disturbed by tachyonic states, which are discussed in the next section.

7 Phenomenological implications

Non-vanishing flux backgrounds in SU(n) gauge theory on $\mathcal{M}^4 \times T^2$ can give a variety of mass spectra in low-energy effective theories, as shown above. In this section, we discuss the phenomenological implications of the magnetized torus. A crucial feature of this setup, as shown in section 6.5, is the existence of tachyonic states [38, 47, 55]. The tachyonic states appear independently of the gauge parameter ξ and imply that the background configuration is unstable; thus, they are expected to evolve non-zero VEVs. The tachyon condensation may induce further breaking or restoration of gauge symmetries. Otherwise, the tachyonic states have to be stabilized by some mechanism.

First, we focus on stabilizing the tachyons by considering additional contributions to the masses of these tachyonic states with the help of Wilson line phases. From a theoretical viewpoint, extending the compactified dimensions by more than two is an interesting possibility. In flux compactifications of superstring theories, where the extra dimensions are usually more than two, stabilization of tachyons by Wilson line phases is often adopted [65]. For example, let us assume a flat 4D torus as the extra dimension. As we have shown, flux backgrounds generated by $\boldsymbol{B}_5$ and $\boldsymbol{B}_6$ break the gauge symmetry G to H. In addition, we can consider non-trivial configurations of Wilson line phases included in the backgrounds of the other extra-dimensional components, denoted by $\boldsymbol{B}_7$ and $\boldsymbol{B}_8$. If the flux background and the Wilson line phases are in the same direction in the representation space of G, all of the tachyonic states appearing in the Landau level excitations receive additional contributions to their masses from the Wilson line phases. Then, the tachyonic states can be eliminated from the tree-level mass spectrum.

Let us now briefly discuss a possible mass spectrum in an eight-dimensional setup. As in the previous sections, the tree-level mass spectrum is determined by the eigenvalues of the differential operator $-(D^{(ij)}_{m'})^2$, with $m' = 5, 6, 7, 8$. The tree-level mass squared for physical excitations of A^{ij}_μ and (A^{ij}_7, A^{ij}_8) are denoted by M^2_+, and equivalently (A^{ij}_5, A^{ij}_6) are denoted by M^2_-. Under a simple setup, if these fields couple to the flux background in $\boldsymbol{B}_5$ and $\boldsymbol{B}_6$, the masses can be given by

$$M^2_\pm = 2g\hat{f}|\tilde{N}|(\hat{\ell} \pm 1/2) + (2\pi\rho)^2 \left[(\hat{n} - \tilde{a}_7)^2 + (\hat{m} - \tilde{a}_8)^2\right], \tag{7.1}$$

where $\hat{f}$ is a flux unit, $\tilde{N}$ is an integer, and ρ parametrizes the relative size of the compact extra dimensions. The Wilson line phases are parametrized by the real degrees of freedom $\tilde{a}_7$ and $\tilde{a}_8$. The non-negative integer $\hat{\ell}$ labels the Landau level excitations, whereas the integers $\hat{n}$ and $\hat{m}$ label KK modes. In the case with vanishing Wilson line phases $\tilde{a}_7 = \tilde{a}_8 = 0$ (mod 1), negative mass squared appears in M^2_- for $\hat{\ell} = \hat{n} = \hat{m} = 0$. Therefore, non-vanishing $\tilde{a}_7$ and $\tilde{a}_8$ can make the mass squared positive, stabilizing the tachyons.

Note that the Wilson line phases are regarded as continuous moduli that parametrize the flat directions of the tree-level potential for the gauge fields, and hence non-vanishing values for the Wilson line phases must be set by hand at tree level. This implies that stabilizing the non-vanishing Wilson line phases arises as an additional issue of this setup. In general, both the flat and non-flat directions of the potential for extra-dimensional gauge fields are disturbed by quantum corrections,[3] and finite potentials for Wilson line phases appear at one-loop level as discussed in the introduction. In a low-energy limit of an intersecting D-brane model and its T-dual, the one-loop effective potentials for Wilson line phases and the vacuum configuration were examined to discuss the stability of tachyons related to flux [65]. The system is shown to be driven to a supersymmetric vacuum, where tachyonic states appear. As a future study, we expect to derive a vacuum configuration that stabilizes the tachyons at one-loop level in a more general field-theoretical setup.

Next, we discuss the possibility of tachyon condensation. If the tachyons are not stabilized, they are expected to evolve non-vanishing VEVs. Justifying the existence of VEVs through tachyonic state condensation is a non-trivial matter. The effective 4D theory contains infinite scalar modes, and the full scalar potential is quite complicated. Since the other massive and massless scalar fields generally receive a backreaction of VEVs of the tachyonic states, the scalar potential in the effective 4D theory is hard to be examined analytically, even at tree level. Still, it could be studied numerically with approximations [75]. From another point of view, their VEVs must also be regarded as a background configuration of a 6D field since the tachyonic states arise from the extra-dimensional gauge fields. Thus, besides the potential analysis in an effective 4D theory, another approach for studying the tachyon condensation is to examine background configurations of the gauge field as a 6D field. This approach has revealed the restoration of supersymmetry, which was previously broken by flux backgrounds in an intersecting D-brane model [66]. In this work, in which we discussed a Yang-Mills theory without supersymmetry, the tree-level EOM seems to prevent non-trivial background configurations that correspond to the tachyon condensation. Some modifications of the EOM through quantum corrections or extensions of the model are expected to provide this condensation. While a more comprehensive analysis of tachyon condensation is necessary, it is still worthwhile to discuss the expected consequences when assuming the condensation.

As an illustrative toy model, we discuss a 6D SU(3) theory with the flux configuration $(N^1, N^2) = (1, 2)$, which gives $\tilde{N}^{12} = 0$ and $\tilde{N}^{13} = \tilde{N}^{23} = 3$. This flux breaks $G = \mathrm{SU}(3)$ to its subgroup $H = \mathrm{SU}(2) \times \mathrm{U}(1)$. Although the non-vanishing continuous Wilson line phases break SU(2) into U(1), let us consider the case with vanishing Wilson line phases, $\tilde{v}^{ij} = 0$, for simplicity. The mass spectrum in this setup is shown in table 1. There appear massless 4D gauge fields that transform under H as the adjoint, $\mathbf{3}_0 \oplus \mathbf{1}_0$, with the representation denoted as $\mathrm{SU}(2)_{\mathrm{U}(1)}$. There are also massless adjoint scalars and tachyonic states that belong to the generator of G/H, i.e., the SU(2) doublets $\mathbf{2}_3 \oplus \mathbf{2}_{-3}$. The remaining fields are either massive or would-be Goldstone modes. If the tachyonic states develop constant VEVs, keeping the other part of the backgrounds invariant, unitary transformations make the VEVs parametrized by a real parameter ϕ as $\langle \mathbf{2}_3 \rangle^T = \langle \mathbf{2}_{-3} \rangle = (0, \phi)$, where T represents

[3]The quantum corrections on non-flat directions in 6D orbifold models are discussed, for instance, in [73]. Also, in supersymmetric five-dimensional models, extra-dimensional gauge fields and scalars belonging to the vector multiplets have tree-level potentials, whose non-flat directions receive quantum corrections [74].

$k=1,2$ and $(i,j)=(1,2)$;	representation of $\mathrm{SU}(2)_{\mathrm{U}(1)}$: $\mathbf{3}_0 \oplus \mathbf{1}_0$	
4D fields	masses	
$A^k_{\mu(0,0)}, A^{ij}_{\mu(0,0)}$	0	massless vectors
$A^k_{m(0,0)}, A^{ij}_{m(0,0)}$	0	massless scalars
$A^k_{\mu(\hat{n},\hat{m})}, A^{ij}_{\mu(\hat{n},\hat{m})}$	$(2\pi)^2\left[\hat{n}^2+\tau_{\mathrm{I}}^{-2}(\hat{m}-\hat{n}\tau_{\mathrm{R}})^2\right]$	massive vectors
$A^k_{m(\hat{n},\hat{m})}, A^{ij}_{m(\hat{n},\hat{m})}$	$(2\pi)^2\left[\hat{n}^2+\tau_{\mathrm{I}}^{-2}(\hat{m}-\hat{n}\tau_{\mathrm{R}})^2\right]$	massive scalars
$A^k_{m(\hat{n},\hat{m})}, A^{ij}_{m(\hat{n},\hat{m})}$	$\xi(2\pi)^2\left[\hat{n}^2+\tau_{\mathrm{I}}^{-2}(\hat{m}-\hat{n}\tau_{\mathrm{R}})^2\right]$	would-be Goldstone modes

$(i,j)=(1,3),(2,3)$;	representation of $\mathrm{SU}(2)_{\mathrm{U}(1)}$: $\mathbf{2}_3 \oplus \mathbf{2}_{-3}$	
4D fields	masses	
$A^{ij}_{-,0,d}$	$-3g\hat{f}$	tachyonic states
$A^{ij}_{\mu,\hat{\ell},d}$	$6g\hat{f}(\hat{\ell}+1/2)$	massive vectors
$A^{ij}_{\pm,\hat{\ell},d}$	$6g\hat{f}(\hat{\ell}+3/2)$	massive scalars
$A^{ij}_{+,0,d}, A^{ij}_{\pm,\hat{\ell},d}$	$6g\hat{f}\xi(\hat{\ell}+1/2)$	would-be Goldstone modes

Table 1. Tree level mass spectrum in the SU(3) model with the flux $(N^1, N^2) = (1, 2)$ and vanishing Wilson line phases. In the table, $\hat{n}, \hat{m}, \hat{\ell} \in \mathbb{Z}_{\geq 0}$ and $(\hat{n}, \hat{m}) \neq 0$ are implied. The subscript d takes 0 to 2. We denote the representations of the 4D fields under H as $\mathrm{SU}(2)_{\mathrm{U}(1)}$.

the transpose. For a non-zero value of ϕ, some of the massless adjoint fields acquire masses, resulting in the symmetry breaking $\mathrm{SU}(2) \times \mathrm{U}(1) \to \mathrm{U}(1)'$. This presents the possibility of the EWSB triggered by the Nielsen-Olsen type instability [64], where the SM Higgs scalar originates from the lowest Landau level of the extra-dimensional gauge fields. However, various aspects such as vacuum stability, the matter sector, and other details of the model must be examined in detail.

With enlarged gauge symmetries, flux backgrounds combined with tachyon condensation may provide several possibilities for symmetry-breaking patterns. For example, in the SU(5) case with $(N^1, N^2, N^3, N^4) = (2, 4, 6, 3)$, the flux breaks $G = \mathrm{SU}(5)$ to its subgroup $H = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. For vanishing Wilson line phases, there appear tachyonic states belonging to the generator of G/H, namely $(\mathbf{3}, \mathbf{2})_5$, where the representation is written as $(\mathrm{SU}(3), \mathrm{SU}(2))_{\mathrm{U}(1)}$. Their VEVs can induce the symmetry breaking $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \to \mathrm{SU}(2)_{\mathrm{D}} \times \mathrm{U}(1)'$, where $\mathrm{SU}(2)_{\mathrm{D}}$ is the diagonal part of $\mathrm{SU}(2)' \times \mathrm{SU}(2) \subset \mathrm{SU}(3) \times \mathrm{SU}(2)$, and $\mathrm{U}(1)'$ is the linear combination of a U(1) generator and a diagonal generator in $\mathrm{SU}(3)/\mathrm{SU}(2)'$. The resultant symmetry may be identified as the electroweak symmetry. Flux backgrounds and tachyon condensation can also be utilized to break unified gauge symmetries in GUT models. If we consider GUT models with $G = \mathrm{SU}(7)$, SU(8), SO(10), or E_6, flux backgrounds can break G to $H = \mathrm{SU}(5) \times G'$, and the condensation of tachyons belonging to G/H seems to provide similar symmetry-breaking patterns in known five-dimensional models [26]. In this case, the Wilson line phases belonging to G/H induce the symmetry breaking into the SM gauge symmetry, in contrast to the tachyon condensation.

Finally, we briefly mention matter fields under flux backgrounds. It is well known that when fermion fields are present, the lowest Landau level excitations of the fermions that couple to flux backgrounds become massless states and give rise to chiral mass spectra. The strength of the coupling is expressed by the integer $\tilde{N}^{ij}$ in eq. (4.14) and determines the generation number of the chiral fermions at a low-energy regime. While we have mainly focused on $\mathrm{SU}(n)$ so far, examining the example of E_6 GUT models can provide insight into the generation structure. If there is a flux background along the $\mathrm{U}(1)_X$ generator, which is a part of the maximal subgroup $\mathrm{SO}(10) \times \mathrm{U}(1)_X$ of E_6, then the flux can break E_6 into $\mathrm{SO}(10) \times \mathrm{U}(1)_X$. Furthermore, this symmetry can be broken down to $\mathrm{SU}(5) \times \mathrm{U}(1)'$ or the SM gauge group with the help of tachyons or VEVs of Wilson line phases. In terms of matter fields, we can incorporate a 27-plet of E_6, which is decomposed into the representations of $\mathrm{SO}(10)_{\mathrm{U}(1)_X}$ as $\mathbf{27} \to \mathbf{16}_{-1} + \mathbf{10}_2 + \mathbf{1}_{-4}$. When the generation of light modes from $\mathbf{16}_{-1}$ is three, which have fermion content suitable to the SM matter fields and right-handed neutrinos, the generations from $\mathbf{10}_2$ and $\mathbf{1}_{-4}$ are six and twelve, respectively. We note that fermions in $\mathbf{10}_2$ and $\mathbf{1}_{-4}$ are vector-like under both $\mathrm{SU}(5) \times \mathrm{U}(1)'$ and the SM gauge group, and thus are expected to acquire masses during the symmetry breaking to the SM. Light chiral fermions could be mixed states from three generations of light fields from $\mathbf{16}_{-1}$ and six generations of light fields from $\mathbf{10}_2$. Additionally, the matter contents may be regarded as an extension of the twisted flavor structure discussed in E_6 GUT models [76–78]. Comprehensive studies of GUT models, including the prediction of the flavor structure, are interesting issues left for future exploration.

8 Conclusions

In this study, we have investigated an $\mathrm{SU}(n)$ gauge theory in a 6D spacetime with a constant magnetic flux in the two-dimensional torus. By analyzing the classical equations of motion, a general form of consistent background configurations incorporating both the magnetic flux and Wilson line phases has been obtained. Then, we have derived the appropriate boundary conditions for fields associated with the discrete translations on the torus. There are many possible setups since the boundary conditions adjust to changes in the background configuration induced by gauge transformations. We have chosen a gauge that incorporates the Wilson line phases into the twist matrices. In addition, we have performed a standard R_ξ gauge fixing, demonstrating the dependence of the masses for the KK modes on the gauge fixing parameter.

Keeping the gauge parameter and the background configurations arbitrary, the complete expressions of the tree-level mass spectrum in the effective low-energy theory is newly obtained for a general $\mathrm{SU}(n)$ case. Our analysis confirmed the existence of tachyonic modes, which appear independently of the gauge fixing parameter ξ or the background configuration of Wilson line phases with flux. Some of the remaining scalar fields coupled to the flux were found to be massive, while others that showed dependence on ξ were identified as would-be

Goldstone modes. Consequently, the degrees of freedom of the latter are absorbed by the infinite tower of 4D vector fields, rendering them massive. As expected, only the masses of flux-blind scalar fields in the non-Cartan directions receive contributions from the Wilson line phases. We have also discussed vanishing one-loop potentials for NG bosons related to the violation of the translational symmetry. The one-loop potential for the NG bosons is not generated since the VEVs of Wilson line phases corresponding to the NG bosons do not appear in the mass parameters for any 4D modes.

Based on our findings, we have discussed the phenomenological implications associated with stabilization or condensation of the tachyonic states. The elimination of tachyonic modes through Wilson line phases is still viable by increasing the number of extra dimensions, where the stabilization of the vacuum at a quantum level provides an interesting topic for future research. Also, we have discussed implications related to the tachyon condensation. In this setup, it is expected to be possible to generate various mass spectra and explore different patterns of symmetry breaking. Therefore, extending the gauge group of the magnetic torus setup to simply-connected ones enables our results to be applied in the construction of diverse models within GUT and GHU. The exploration of phenomenologically viable models, considering the prediction of the flavor structure through the inclusion of matter fields, is another intriguing topic left for future studies.

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A Surface terms

We discuss surface terms related to integration by parts in our setup. We examine the background covariant derivative in eq. (3.3), which depends on the torus coordinates and is denoted by $\mathcal{D}_m(z)$. With the background configurations and the twist matrices in eqs. (3.17) and (3.18), one obtains the boundary conditions for $\mathcal{D}_m(z)$ as

$$\mathcal{D}_m(\mathcal{T}_p z) = T_p(z)\mathcal{D}_m(z)T_p^\dagger(z). \tag{A.1}$$

Let $\boldsymbol{\phi}(z), \boldsymbol{\phi}'(z) \in su(n)$ be general fields in the adjoint representation that obey the following relations:

$$\mathcal{D}_m\boldsymbol{\phi} = (\partial_m - ig\,\mathbf{ad}(\boldsymbol{B}_m))\boldsymbol{\phi}, \qquad \boldsymbol{\phi}(\mathcal{T}_p z) = T_p(z)\boldsymbol{\phi}(z)T_p^\dagger(z). \tag{A.2}$$

For $\boldsymbol{\phi}'(z)$, the same relations are applied. In this case, one sees that

$$\mathrm{Tr}[(\mathcal{D}_m\boldsymbol{\phi})(\mathcal{D}_m\boldsymbol{\phi}')] = -\mathrm{Tr}[\boldsymbol{\phi}(\mathcal{D}_m\mathcal{D}_n\boldsymbol{\phi}')] + \partial_m\mathrm{Tr}[\boldsymbol{\phi}(\mathcal{D}_n\boldsymbol{\phi}')], \tag{A.3}$$

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where the last term gives the vanishing contribution in the action as

$$\int_{\mathcal{M}^4} d^4x \int_{T^2} d^2x\ \partial_m \mathrm{Tr}[\phi(\mathcal{D}_n\phi')] \propto \left[\mathrm{Tr}[\phi(\mathcal{D}_n\phi')]\right]_z^{\mathcal{T}_p z} = 0. \tag{A.4}$$

Here, we have used the boundary conditions in eqs. (A.1) and (A.2) and the cyclic property of traces.

Using similar discussions as the above, we obtain eqs. (5.15)–(5.17), (5.20), (5.33), and (5.34) without contributions from surface terms.

B Mode functions

In section 6.4, we showed the explicit form of the mode functions without derivation. In this appendix, we show the derivation.

B.1 Zero mode functions

First, let us derive the zero-mode function $\zeta_{0,d}$ appearing in eq. (6.17). For the case with $N^{ij} > 0$, we have to solve the differential equation

$$\left(\bar{\partial}_z + \frac{g\hat{f}\tilde{N}}{4}(z - \gamma\bar{z})\right)\zeta_{0,d}(z) = 0, \tag{B.1}$$

with the boundary conditions in eqs. (6.18) and (6.19). Here, we omit the upper script ij for simplicity of notation. A solution of eq. (B.1) is given by

$$\zeta_{0,d}(z) = Ce^{-\frac{g\hat{f}\tilde{N}}{4}(z\bar{z}-\frac{1}{2}\gamma\bar{z}^2)}\tilde{\zeta}_{0,d}(z), \tag{B.2}$$

where C is a constant, and the function $\tilde{\zeta}_{0,d}(z)$ depends only on z and does not depend on $\bar{z}$. To determine the function $\tilde{\zeta}_{0,d}(z)$ completely, we have to impose the boundary conditions. It is convenient to introduce a new function $\chi_{0,d}$ by

$$\tilde{\zeta}_{0,d}(z) = e^{\frac{g\hat{f}\tilde{N}}{4}(1-\frac{\gamma}{2})z^2}\chi_{0,d}(z), \tag{B.3}$$

to express the solution as

$$\zeta_{0,d}(z) = Ce^{-\frac{g\hat{f}\tilde{N}}{4}(z\bar{z}-z^2+\frac{\gamma}{2}(z^2-\bar{z}^2))}\chi_{0,d}(z). \tag{B.4}$$

From the conditions in eqs. (6.18) and (6.19), we obtain the boundary conditions for $\chi_{0,d}(z)$ as

$$\chi_{0,d}(z+1) = e^{-2\pi i\tilde{a}_5}\chi_{0,d}(z), \tag{B.5}$$

$$\chi_{0,d}(z+\tau) = e^{-2\pi i(\tau_{\mathrm{R}}\tilde{a}_5+\tau_{\mathrm{I}}\tilde{a}_6)}e^{-\pi i\tilde{N}\tau}e^{\gamma\pi i\tilde{N}\tau_{\mathrm{R}}}e^{-2\pi i\tilde{N}z}\chi_{0,d}(z), \tag{B.6}$$

where we have used $\hat{f} = 2\pi/(g\mathcal{V}_{T^2}) = 2\pi/(g\tau_{\mathrm{I}})$ to obtain the second relation. The function $\chi_{0,d}$ satisfying these boundary conditions can be expressed by the Jacobi theta function [47]. The Jacobi theta function $\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z'|\tau')$ is defined by

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z'|\tau') = \sum_{n\in\mathbb{Z}} e^{i\pi\tau'(n+a)^2}e^{2\pi i(n+a)(z'+b)}, \qquad a,b\in\mathbb{R}, \qquad z',\tau'\in\mathbb{C}, \tag{B.7}$$

which satisfies

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z' + k|\tau') = e^{2\pi i k a} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z'|\tau') , \qquad k \in \mathbb{Z}, \tag{B.8}$$

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z' + \tau'|\tau') = e^{-i\pi\tau'} e^{-2\pi i(z'+b)} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z'|\tau') . \tag{B.9}$$

Using it, we can express $\chi_{0,d}$ as

$$\chi_{0,d}(z) = \vartheta \begin{bmatrix} -(\tilde{a}_5 + d)/\tilde{N} \\ \tau_{\mathrm{R}}\tilde{a}_5 + \tau_{\mathrm{I}}\tilde{a}_6 - \gamma\tau_{\mathrm{R}}\tilde{N}/2 \end{bmatrix} (\tilde{N}z|\tilde{N}\tau). \tag{B.10}$$

From eqs. (B.8) and (B.9), we can show that eq. (B.10) satisfies the boundary conditions in eqs. (B.5) and (B.6).

Finally, the zero mode function is expressed as

$$\zeta_{0,d}(z) = C e^{-\frac{\pi\tilde{N}}{2\tau_{\mathrm{I}}}\left(z\bar{z} - zz + \frac{\gamma}{2}(zz - \bar{z}\bar{z})\right)} \vartheta \begin{bmatrix} -(\tilde{a}_5 + d)/\tilde{N} \\ \tau_{\mathrm{R}}\tilde{a}_5 + \tau_{\mathrm{I}}\tilde{a}_6 - \gamma\tau_{\mathrm{R}}\tilde{N}/2 \end{bmatrix} (\tilde{N}z|\tilde{N}\tau). \tag{B.11}$$

The constant C is determined by a normalization condition. Here, we impose

$$\int_{\mathcal{V}_{T^2}} dx^5 dx^6 \; \bar{\zeta}_{0,d}(z)\zeta_{0,d'}(z) = \delta_{d,d'}, \tag{B.12}$$

which yields

$$C = \left(\frac{2\pi\tilde{N}}{\tau_{\mathrm{I}}}\right)^{1/4} . \tag{B.13}$$

B.2 Excited mode functions

The excited mode functions are obtained by

$$\zeta_{\hat{\ell},d}(z) = \frac{1}{\sqrt{\hat{\ell}!}} (\hat{a}^\dagger)^{\hat{\ell}} \zeta_{0,d}(z), \tag{B.14}$$

where $\hat{a}^\dagger$ is given by

$$\hat{a}^\dagger = \Pi_z = -i\sqrt{\frac{\tau_{\mathrm{I}}}{2\pi\tilde{N}}} \left[\partial_z - \frac{\pi\tilde{N}}{2\tau_{\mathrm{I}}}(\bar{z} - \gamma z)\right], \tag{B.15}$$

for $\tilde{N}^{ij} > 0$. Let us show that the excited mode functions are given by

$$\begin{aligned} \zeta_{\hat{\ell},d}(z) = \frac{C}{2^{\hat{\ell}}\sqrt{\hat{\ell}!}} \sum_{n=-\infty}^{+\infty} & Z_{\hat{\ell}}(w_{n,d}(z)) e^{-\frac{\pi\tilde{N}}{2\tau_{\mathrm{I}}}\left(z\bar{z} - zz + \frac{\gamma}{2}(zz - \bar{z}\bar{z})\right)} \\ & \times e^{i\pi\tilde{N}\tau(n-(\tilde{a}_5+d)/\tilde{N})^2} e^{2\pi i\tilde{N}(n-(\tilde{a}_5+d)/\tilde{N})(z+(\tau_{\mathrm{R}}\tilde{a}_5+\tau_{\mathrm{I}}\tilde{a}_6)/\tilde{N}-\gamma\tau_{\mathrm{R}}/2)}. \end{aligned} \tag{B.16}$$

Here, the function $w_{n,d}(z)$ is defined by

$$w_{n,d}(z) = \sqrt{\frac{2\pi\tilde{N}}{\tau_{\mathrm{I}}}} \left[\frac{z - \bar{z}}{2i} + \tau_{\mathrm{I}}\left(n - \frac{\tilde{a}_5 + d}{\tilde{N}}\right)\right], \tag{B.17}$$

and $Z_{\hat{\ell}}$ is defined by

$$Z_0(w_{n,d}) = 1, \qquad Z_{\hat{\ell}+1}(w_{n,d}) = 2w_n Z_{\hat{\ell}}(w_{n,d}) - \frac{dZ_{\hat{\ell}}(w_{n,d})}{dw_{n,d}}. \tag{B.18}$$

To derive the excited mode functions, let us express the zero-mode function in eq. (B.11) as

$$\zeta_{0,d}(z) = C \sum_{n=-\infty}^{+\infty} Z_0(w_{n,d}) F_{n,d}(z), \tag{B.19}$$

$$F_{n,d}(z) \equiv e^{-\frac{\pi\tilde{N}}{2\tau_{\rm I}}\left(z\bar{z}-zz+\frac{\gamma}{2}(zz-\bar{z}\bar{z})\right)} e^{i\pi\tilde{N}\tau(n-(\tilde{a}_5+d)/\tilde{N})^2} e^{2\pi i(n-(\tilde{a}_5+d)/\tilde{N})(\tilde{N}z+\tau_{\rm R}\tilde{a}_5+\tau_{\rm I}\tilde{a}_6-\gamma\tau_{\rm R}\tilde{N}/2)}. \tag{B.20}$$

By using the expression above and

$$\zeta_{1,d}(z) = \frac{1}{\sqrt{1!}}\hat{a}^\dagger \zeta_{0,d}(z) = -i\sqrt{\frac{\tau_{\rm I}}{2\pi\tilde{N}}}\left[\partial_z - \frac{\pi\tilde{N}}{2\tau_{\rm I}}(\bar{z}-\gamma z)\right]\zeta_{0,d}(z), \tag{B.21}$$

the first-excited mode is written as

$$\zeta_{1,d}(z) = -iC\sqrt{\frac{\tau_{\rm I}}{2\pi\tilde{N}}}\sum_n \left[(\partial_z Z_0)F_{n,d} + Z_0(\partial_z F_{n,d}) - \frac{\pi\tilde{N}}{2\tau_{\rm I}}(\bar{z}-\gamma z)Z_0 F_{n,d}\right]. \tag{B.22}$$

The last two terms are rearranged as

$$\begin{aligned} &Z_0(\partial_z F_{n,d}) - \frac{\pi\tilde{N}}{2\tau_{\rm I}}(\bar{z}-\gamma z)Z_0 F_{n,d} \\ &= \frac{2\pi\tilde{N}}{\tau_{\rm I}} i\left(\frac{z-\bar{z}}{2i} + \tau_{\rm I}\left(n - \frac{g\tilde{v}_5}{2\pi\tilde{N}} - \frac{d}{\tilde{N}}\right)\right) Z_0 F_{n,d} = i\sqrt{\frac{2\pi\tilde{N}}{\tau_{\rm I}}} w_{n,d} Z_0 F_{n,d}, \end{aligned} \tag{B.23}$$

and the first term can be calculated as

$$(\partial_z Z_0)F_{n,d} = \frac{\partial w_{n,d}}{\partial z}\frac{dZ_0}{dw_{n,d}}F_{n,d} = \frac{1}{2i}\sqrt{\frac{2\pi\tilde{N}}{\tau_{\rm I}}}\frac{dZ_0}{dw_{n,d}}F_{n,d}. \tag{B.24}$$

By using these results, eq. (B.22) becomes

$$\zeta_{1,d}(z) = \frac{C}{2}\sum_{n=-\infty}^{+\infty}\left(2w_{n,d}(z)Z_0(w_{n,d}) - \frac{dZ_0(w_{n,d})}{dw_{n,d}}\right)F_{n,d}(z). \tag{B.25}$$

Identifying the square bracket with Z_1, i.e.,

$$Z_1 = 2w_{n,d}Z_0 - \frac{dZ_0}{dw_{n,d}}, \tag{B.26}$$

the first excited mode can be written as

$$\zeta_{1,d}(z) = \frac{C}{2}\sum_{n=-\infty}^{+\infty} Z_1(w_{n,d}(z))F_{n,d}(z). \tag{B.27}$$

Therefore, eq. (B.16) is correct for $\hat{\ell} = 1$.

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Let us assume eq. (B.16) for $\hat{\ell} = k$ and show it for $\hat{\ell} = k+1$. Using

$$\zeta_{k+1,d}(z) = \frac{1}{\sqrt{k+1}} a^\dagger \zeta_{k,d}(z), \tag{B.28}$$

and repeating the same calculation we did for the $\hat{\ell} = 0$ case, we get

$$\zeta_{k+1,d}(z) = \frac{C}{2^{k+1}\sqrt{(k+1)!}} \sum_{n=-\infty}^{+\infty} \left(2w_{n,d} Z_k - \frac{dZ_k}{dw_{n,d}} \right) F_{n,d}(z). \tag{B.29}$$

Hence, eq. (B.16) is satisfied for $\hat{\ell} = k+1$. Therefore, the assumption holds for any $\hat{\ell}$.

The function $Z_{\hat{\ell}}(w_n)$, which satisfies the differential equation in eq. (B.18), can be expressed by Hermite polynomials $H_{\hat{\ell}}(x)$. Therefore, we obtain the expression in eq. (B.16) for $\tilde{N}^{ij} > 0$. In the same way, we can obtain the explicit form of the mode function for $\tilde{N}^{ij} < 0$ as

$$\begin{aligned} \zeta_{\hat{\ell},d}(z) = \frac{C}{2^{\hat{\ell}}\sqrt{\hat{\ell}!}} \sum_{n=-\infty}^{+\infty} & H_{\hat{\ell}}(w_{n,d}(z)) e^{-\frac{\pi \tilde{N}}{2\tau_{\mathrm{I}}}\left(\bar{z}z - \bar{z}\bar{z} + \frac{\gamma}{2}(\bar{z}\bar{z} - zz)\right)} \\ & \times e^{-i\pi\tilde{N}\bar{\tau}(n-(\tilde{a}_5+d)/\tilde{N})^2} e^{-i2\pi\tilde{N}(n-(\tilde{a}_5+d)/\tilde{N})(\bar{z}+(\tau_{\mathrm{R}}\tilde{a}_5+\tau_{\mathrm{I}}\tilde{a}_6)/\tilde{N}-\gamma\tau_{\mathrm{R}}/2)}. \end{aligned} \tag{B.30}$$

These functions correctly satisfy the boundary conditions in eqs. (6.18) and (6.19) and the orthogonal relation in eq. (6.22).

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