Exploring holography with boundary fractal-like structures

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In this paper, we study the holographic quantum error correcting code properties in different boundary fractal-like structures. We construct and explore different examples of the uberholographic bulk reconstruction corresponding to these structures in higher dimensions for Cantor-like sets, thermal states, and $T\bar{T}$-deformed conformal field theories. We show how the growth of the dimensionality of the system emphasizes the role of the Cantor set, due to the special bound naturally arising in this context.

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I. INTRODUCTION

The study of the entanglement entropy in the context of holographic duality has led to fruitful investigations of how various aspects of quantum information including the quantum error correction (QEC) theory fit into the gravitational picture of quantum information [1–3]. Extraordinary connection between such different, at first sight, theories like gravity and QEC has recently attracted considerable interest. This connection is essentially based on the notion of the entanglement wedge and hypotheses related to it [4–6]. The QEC theory was first applied to resolve the apparent inconsistency in the relation between subregion-subregion duality and some properties of operator algebra in quantum field theory, paving the way for progress in our understanding of holographic duality through QEC [7–11]. Assuming the connection of quantum error correction and AdS/CFT correspondence, different problems arising from particular logical subalgebras can be formulated as the AdS bulk geometry problem. In [11], the so-called uberholographic proposal has been made. The (sub)linear scaling of the price and distance of the logical subalgebra corresponding to a particular holographic code leads to the fact that the logical subalgebra can have support on the fractal set with the dimensionality less than the boundary dimension. This fractal dimension is expected to be a universal feature of code and shows how the bulk emerges from the system with lower dimension, as it typically occurs in holography (see also a recent proposal about the so-called "wedge holography," where a system with $d-2$ dimensions gives rise to a d-dimensional bulk).

In [11], the uberholographic property and how it works have been demonstrated on the Cantor set defined on a constant time slice of the Poincaré AdS patch, and an explicit estimate of the code distance has been obtained. This construction has been extended to the case of AdS_4/CFT_3 duality and reconstruction of the Sierpínski triangle defined on the boundary [12]. In [13], uberholography has been discussed in the context of black hole evaporation. Paper [14] studies uberholography for special Hamiltonians motivated by the HAPPY code. In this paper, we study many explicit examples, in which variations of the construction based on Cantor-like sets can be implemented. One of the main results of [11] is the estimate of the distance $d(\mathcal{A}_X)$ of a logical subalgebra associated with the fractal region X (which is chosen to be a Cantor set) in the form

$$d(\mathcal{A}_X) \leq \left(\frac{|R|}{a}\right)^{\alpha}, \qquad \alpha = \frac{\log 2}{\log(2/r)}, \qquad r = 2(\sqrt{2}-1), \qquad (1)$$

where R is the size of the set,[1] and a is the cutoff (elementary lattice spacing). We generalize this bound to different setups of excited states and different dimensions, noting some universal features corresponding to each case. Let us briefly list our results:

(i) We slice a $(d-1)$-dimensional spatial boundary of the Poincaré AdS_{d+1} patch by infinite hyperplanes and obtain that $r = 3-\sqrt{5}$ for $d = 3$ and $r = \sqrt{3}-1$ for $d = 4$. We reaffirm that this estimate is universal for $d = 3$ by considering other slices in a region of finite size—thin concentric annuli. For larger d, we

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[1]We mean the set from which we constructed the fractal, using some iterative procedures.

find that $r = 2/3$ serves as the bound for r, i.e., exactly the value of r corresponding to the Cantor set.

(ii) We consider the holographic dual of finite temperature 2D CFT and the Cantor set boundary region. As a result, we find an explicit dependence of r on the combination $\gamma = T \cdot R$ (where R is assumed to be relatively small, while temperature T is large, such that γ is finite). The generalization to a rotating Banados-Teitelboim-Zanelli (BTZ) black hole is also presented.

(iii) We find that asymmetry in the fractal noise (i.e., asymmetry added on each step of the iterative fractal construction) tends to increase r. However, a finite temperature reduces the effects of asymmetry making r more stable against spatial perturbations.

(iv) Finally, we consider $T\bar{T}$-deformed 2D CFT—the integrable nonlocal deformation, which can be seen as an increase in width of the elementary "particles" (or a coupling to dynamical gravity), obtain the dependence for r, and comment on it.

The paper is organized as follows. In Sec. II, we recall how uberholography works using the example of AdS_3/CFT_2. In Sec. III, we discuss higher-dimensional generalization of Cantor-like structures. In Sec. IV, we consider different excited states in the uberholographic context.

II. CANONICAL UBERHOLOGRAPHY

Let us review the uberholographic proposal discovered in [11], the ultimate recoverability property inherent to holographic duality. The ultimate recoverability means that operators in the bulk can be recovered for zero measure sets on the boundary. In [11], this was demonstrated for the Cantor set example and extended on the Sierpínski fractal in [12]. In the holographic correspondence, the cornerstone of any discussion is the Ryu-Takayanagi (RT) formula, relating the minimal surface area spanned by a subregion A on the AdS boundary and hanging into the bulk, with the entanglement entropy of the subregion A in the boundary CFT

$$S(A) = \frac{\text{Area}(A)}{4G}, \tag{2}$$

where G is Newton's constant. The holographic dual of 2D CFT defined on the line is the metric of the AdS_3 Poincaré patch, which has the form

$$ds^2 = \frac{L_{\text{AdS}}^2}{z^2}(-dt^2 + dx^2 + dz^2). \tag{3}$$

The main example of the boundary subregion A to study here is a Cantor set—a recursively constructed fractal object of zero measure.

Ryu-Takayanagi surfaces in AdS_3 are geodesics in the Poincaré patch, namely, the semicircles spanned by A. If A is a single interval of length ℓ, then one can obtain the entanglement entropy of A via the RT formula, namely,

$$S(\ell) = \frac{c}{3}\log\frac{\ell}{\varepsilon}, \tag{4}$$

where the central charge c is identified with L_{AdS}/G as $c = 3/2 \cdot L_{\text{AdS}}/G$, and ε is the cutoff. A Cantor set is a special set, which has a number of properties that go beyond the standard intuition, namely, it has "dimensionality" less than 1—its Hausdorff measure is equal to $\log 2/\log 3$, and Lebesgue measure is zero. A Cantor set is constructed recursively by poking "holes" of a certain size in the interval of interest. The main "building bricks" necessary to understand the entanglement features of Cantor sets are simply two equivalent disjoint intervals of lengths ℓ_1 and ℓ_2. For simplicity, we begin with $\ell_1 = \ell_2 = rR/2$ separated by the distance $h = (1-r)R$, such that a total length of the system "intervals + hole" is R. It is well known that in a system of two disjoint intervals of lengths $\ell_{1,2}$ there are two competing configurations of RT surfaces—the disjoint one with the entropy given by

$$S_{\text{disj}} = \frac{c}{3}\log\frac{\ell_1}{\varepsilon} + \frac{c}{3}\log\frac{\ell_2}{\varepsilon}, \tag{5}$$

and the one with "connected" topology,

$$S_{\text{conn}} = \frac{c}{3}\log(\ell_1 + \ell_2 + h) + \frac{c}{3}\log h - \frac{2c}{3}\log\varepsilon. \tag{6}$$

We are interested in the transition point of RT surfaces, which is determined by the minimality condition, i.e., when the connected RT topology is equal to the disconnected one, $S_{\text{conn}} = S_{\text{dis}}$. The phase transition of this type for various models has been considered in [15,16]. The solution to this equation for the iterative building bricks described above with $\ell_1 = \ell_2$ has the form

$$r = 2(\sqrt{2} - 1). \tag{7}$$

Now let us consider the total system and make symmetric holes on each step of the iteration. The iterations terminate at some length a, which can be associated with the natural cutoff or lattice spacing. After m iterations, we have that

$$a = \left(\frac{r}{2}\right)^m |R|,$$

and the region we get (let us call it R_{min}) has 2^m components each of the length a. Following the proposal of [11], the estimate of the distance of the logical operator (sub)algebra $\mathcal{A}_X$ associated with the bulk region X is derived as follows. The distance $d(\mathcal{A}_X)$ is defined as the minimal size of boundary region R, not correctable with respect to X, with the following estimate proposed in [11]:

$$d(\mathcal{A}_X) \leq \frac{|R_{\min}|}{a} = 2^m = \left(\frac{|R|}{a}\right)^{\alpha}, \tag{8}$$

where

$$\alpha = \frac{\log 2}{\log(2/r)} = \frac{1}{\log_2(\sqrt{2}+1)} \approx 0.786. \tag{9}$$

This implies that the distance for any logical subalgebra is bounded by $d(\mathcal{A}_X) \leq n^{\alpha}$, where n could be considered as the number of boundary "sites" [11].

III. HIGHER-DIMENSIONAL GENERALIZATION

A. Straight slicing

Now let us consider $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ duality, which relates $(d+1)$-dimensional gravity and d-dimensional boundary conformal field theory. The metric of the AdS_{d+1} Poincaré patch has the form

$$ds^2 = \frac{L_{\mathrm{AdS}}}{z^2}(-dt^2 + dz^2 + d\bar{x}^2), \tag{10}$$

thus the dual theory is defined on a d-dimensional plane. The simplest generalization of the construction described in the previous section to an arbitrary dimension case is to slice $(d-1)$-dimensional constant time sections of the boundary into "Cantor lasagna"—infinitely "long" spatial slices with the Cantor set spatial organization in the preferred direction. The holographic entanglement entropy of this single infinitely long slice of the size ℓ in some spatial direction (let us call it x_1 for definiteness) is well known and has the form

$$S(\ell) = \frac{1}{4G^{d+1}}\left[\frac{2L_{\mathrm{AdS}}^{d-1}}{d-2}\left(\frac{\ell_\perp}{\varepsilon}\right)^{d-2} - \frac{2^{d-1}\pi^{\frac{d-1}{2}}L_{\mathrm{AdS}}^{d-1}}{d-2}\left(\frac{\Gamma(\frac{d}{2d-2})}{\Gamma(\frac{1}{2d-2})}\right)^{d-1}\left(\frac{\ell_\perp}{\ell}\right)^{d-2}\right], \tag{11}$$

where, again, ε is the cutoff, and $\ell_\perp$ is the diverging size of the stripe in the spatial directions complementary to x_1. Following the logic as in the case of one spatial direction in the boundary system and solving the equation $S_{\mathrm{dis}} = S_{\mathrm{conn}}$ for the entanglement entropy given by (11), we obtain that now the scrambling length is given by the solution of the equation

$$2^d r^2\left(\frac{L_{\mathrm{AdS}}}{rR}\right)^d - 2(r-1)^2\left(\frac{L_{\mathrm{AdS}}}{R-rR}\right)^d - 2\left(\frac{L_{\mathrm{AdS}}}{R}\right)^d = 0. \tag{12}$$

For a particular d, this equation has analytic solutions r, namely,

$$d = 3:\ r = 3 - \sqrt{5} \approx 0.763932, \tag{13}$$

$$d = 4:\ r = \sqrt{3} - 1 \approx 0.732051. \tag{14}$$

The dependence of r on d is monotonic, and for large $d \to \infty$ one can show (at least numerically) that r exactly converges to $r = 2/3$. Also, $r = 2/3$ delimits the generalized Cantor sets with zero Lebesgue measure and the one with a positive measure,

$$d \to \infty:\ r = 2/3. \tag{15}$$

This means that the "Cantor slicing" corresponds to an infinite number of dimensions d and limits the transition point for the entanglement wedge.

B. Annular slicing

Now let us consider another Cantor-like slicing of the plane (to be more precise, the bulk is given by AdS_4 with the boundary at a constant time being a two-dimensional plane). Instead of straight higher-dimensional belts, now we organize the set under consideration by poking holes in the form of annular regions. The entanglement entropy of the annular region with two radii R_1 and R_2 was studied in [17] and is given by

$$S(R_1,R_2) = \frac{c}{6}\left(\frac{2\pi R_1}{\delta} + \frac{2\pi R_2}{\delta} - \frac{4\pi}{\sqrt{2\kappa^2-1}}(\mathbb{E}(\kappa^2) - (1-\kappa^2)\mathbb{K}(\kappa^2)) + \cdots\right), \tag{16}$$

where κ is a constant determined by the ratio R_2/R_1,

$$\log\frac{R_1}{R_2} = 2\kappa\sqrt{\frac{1-2\kappa^2}{\kappa^2-1}}(\mathbb{K}(\kappa^2) - \Pi(1-\kappa^2|\kappa^2)),$$

where $\mathbb{E}$, $\mathbb{K}$ and Π are incomplete elliptic integrals. This entanglement is given by the RT surface of the hemitorus form stretched over the annulus. There is a second phase described by a disconnected solution, which dominates for large R_2/R_1. Since we are interested in small size elementary annular regions, we discard this solution. Another expression for annular regions has been obtained in [18] via the entanglement contour proposal,

$$S(R_1,R_2) = \frac{c}{6}\left(\frac{2\pi R_2}{\varepsilon_2} + \frac{2\pi R_1}{\varepsilon_1} - 4\pi\frac{(R_2^2+R_1^2)}{(R_2^2-R_1^2)} + \cdots\right), \tag{17}$$

where $\varepsilon_{1,2}$ are cutoffs, which also indicate the difference in the entanglement contour. Since this expression is simpler, let us use it first to estimate r. We are interested in thin annular regions, i.e., small $R_2 - R_1$. Taking the sizes in terms of R and r as in the previous calculations, it is straightforward to obtain an equation on r for small R,

$$\frac{4\pi d(r^2 - 6r + 4)}{(r-1)rR} + \frac{2\pi(r^2 - 6r + 4)}{(r-1)r} = 0. \tag{18}$$

It has the same solution as for the "straight" slicing considered above,

$$r = 3 - \sqrt{5}, \tag{19}$$

which confirms the universality of this quantity.

As we can see, in general, the distance $d(\mathcal{A}_X)$ is defined by r, and since we are interested in the reconstruction of the regions with a fractal dimension, only the UV behavior corresponding to small basic fractal building blocks matters. For the higher-dimensional "straight-belt" slicing, the result for r is fixed solely by the dimension, leading to the universal estimate for $d(\mathcal{A}_X)$ on all scales, while the radial slicing in $d = 3$ gives us the same r only in the UV limit.

IV. EXCITED STATES: FINITE TEMPERATURE $T\bar{T}$

A. Finite temperature

It is interesting to consider some tractable examples of excited states in the similar context. One of the simplest generalizations of the Poincaré metric is the BTZ black hole,

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2\right),$$
$$f(z) = 1 - z^2/z_h^2, \tag{20}$$

which is dual to 2D CFT at finite temperature $T = 1/(2\pi z_h)$ fixed by the location of the horizon. The entanglement entropy for a single interval is given by the corresponding RT surface, leading to the expression

$$S(\ell) = \frac{c}{3}\log\left(\frac{\sinh(\pi T\ell)}{\pi T\varepsilon}\right). \tag{21}$$

For the spatial organization of the boundary region considered previously, the parameter r now has the form explicitly depending on scales through R and T, namely,

$$r = \frac{2\coth^{-1}(\coth(\pi\gamma) + \sqrt{2}\mathrm{csch}(\pi\gamma))}{\pi\gamma}, \tag{22}$$

where $\gamma = R \cdot T$. Now the distance $d(\mathcal{A}_X)$ is controlled not only by the size R but also by the temperature. In principle, according to this estimate, one can obtain the corresponding corrections to the distance $d(\mathcal{A}_X)$ by taking a large enough temperature. The dependence of r on γ is presented in Fig. 1, indicating that for large T (and small R) the distance converges to the estimate $d(\mathcal{A}_X) < n$.

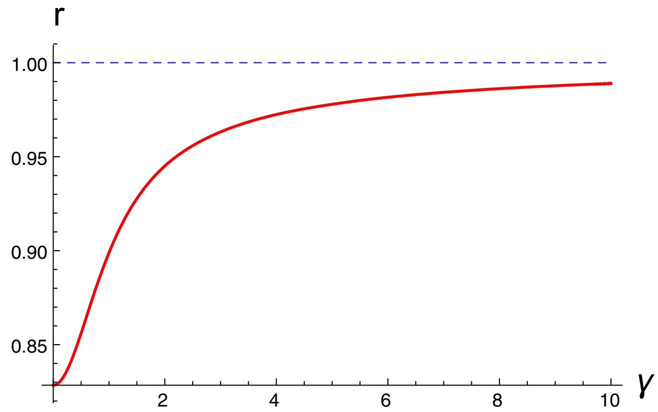

FIG. 1. The dependence of r on γ given by (22).

Now consider how rotating and extremal black holes fit into this construction. Assuming that $z = 1/r$, the canonical form of the rotating BTZ black hole metric has the form

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}dt^2 + \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)}dr^2 + r^2\left(d\phi - \frac{r_+ r_-}{2r^2}dt\right)^2. \tag{23}$$

Black hole rotation leads to two different temperatures $T_\pm$ for the left and right moving modes of the dual CFT (apart from the ordinary black hole temperature),

$$T_+ = \frac{r_+ + r_-}{2\pi}, \quad T_- = \frac{r_+ - r_-}{2\pi}, \quad T = \frac{r_+^2 - r_-^2}{2\pi r_+}. \tag{24}$$

The entanglement entropy for a single interval of the dual theory now depends on the temperatures $T_\pm$ and is given by

$$S(\ell) = \frac{c}{6}\log\left(\frac{\sinh(T_+\pi\ell)}{T_+\pi\epsilon}\right) + \frac{c}{6}\log\left(\frac{\sinh(T_-\pi\ell)}{T_-\pi\epsilon}\right). \tag{25}$$

This leads to the equation determining r,

$$\frac{\sinh(\pi\gamma_-)\sinh(\pi\gamma_+)\sinh(\pi(1-r)\gamma_-)\sinh(\pi(1-r)\gamma_+)}{\sinh^2(\frac{1}{2}\pi r\gamma_-)\mathrm{csch}^2(\frac{1}{2}\pi r\gamma_+)} = 1.$$

In Fig. 2, we plot the dependence of r on γ_+ for different fixed γ_- and one can see that decreasing γ_- (i.e., closer to extremality) decreases the growth of r.

B. Asymmetric regions: Thermal stabilization and smoothness property?

Now let us turn to the question about what happens if we slightly deviate from the Cantor set and how the temperature

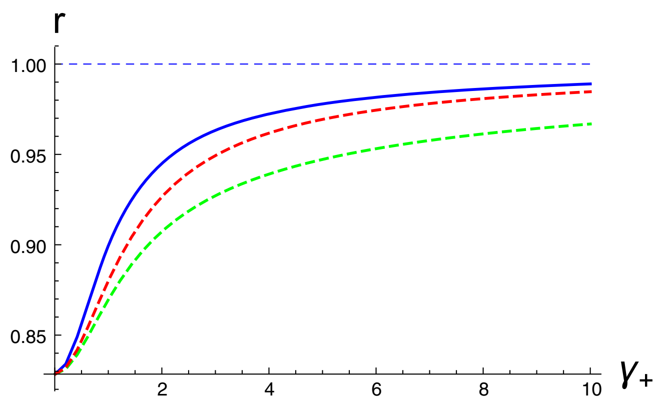

FIG. 2. The dependence of r on γ_+ for different γ_-. The green dashed curve corresponds to $\gamma_- \to 0$ and the red one to $\gamma_- = \gamma_+/2$. The blue solid curve corresponds to a nonrotating black hole.

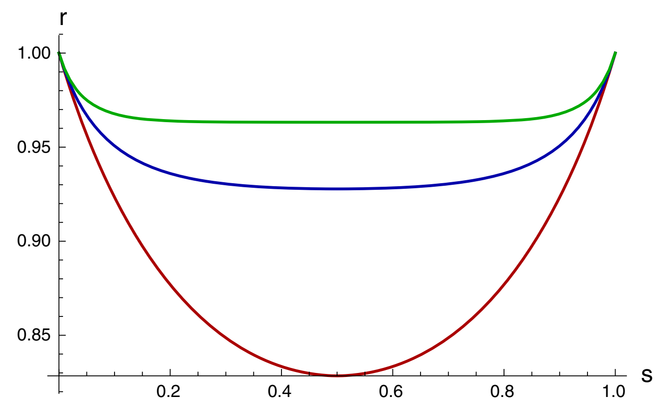

FIG. 3. The dependence of r on the asymmetry parameter s for different values of γ (red curve corresponds to zero temperature, blue one to $\gamma = 1.5$, and green one to $\gamma = 3$).

affects such construction in the context of uberholography. The Cantor set is constructed by iterative withdraw exactly of the middle third of each subinterval. In fact, even small variations in the parameters of the withdraw procedure can lead us to different topological characteristics of the final set. First, let us consider the following asymmetric partition of the interval with the length R,

$$\ell_1 = (1-s)Rr, \qquad \ell_2 = sRr, \qquad h = (1-r)R, \tag{26}$$

i.e., on the each step of iteration, we obtain the partition of the system into unequal parts separated by the distance $(1-r)R$. For the ground state, i.e., when the entropy is given by (4), we obtain the solution for r corresponding to the entanglement wedge phase transition,

$$r = \frac{2}{\sqrt{1-4(s-1)s}+1}. \tag{27}$$

It is straightforward to see that r has the minimum at $s = 1/2$, i.e., precisely for the Cantor set. The generalization of Eq. (27) to the case of finite temperature is straightforward, although, it does not lead to an analytical formula that can be written in any readable way. In Fig. 3, we plot the dependence of r on s for different temperatures. Although this does not lead to the generalization of $d(\mathcal{A}_X)$ and α (for arbitrary s) in a straightforward manner, one can try to estimate qualitative features solely by the behavior of $r(s)$. From this plot, one can see that the dependence of r on s becomes more stable, i.e., large enough temperature deviation from $s = 1/2$ does not lead to a significant change in r.

C. $T\bar{T}$ deformation

Finally, let us consider the special kind of nonlocal exactly solvable quantum field theories—$T\bar{T}$ deformed 2D CFT [19–22]. The bulk dual of $T\bar{T}$ deformation of 2D CFT is given by locating the boundary theory at a radial cutoff of the bulk at some finite z_c [23]. This deformation admits different interpretations, for example, it could be considered as a coupling of a field theory to Jackiw-Teitelboim (JT) gravity [24,25] or a dynamical coordinate transformation applied to the field. Also, one can consider the deformation as the broadening of the fundamental particle's "width" (extension/reduction of the phase space or wave function support in the nonrelativistic case) [26,27]. The computation of the entanglement entropy[2] in different ways has been shown to be consistent with the prescription given by the cutoff proposal.

The entanglement entropy of the interval with the length ℓ is now given by

$$S(\ell) = \log\left(\sqrt{\frac{\ell_c^2}{4}+1}+\frac{\ell_c}{2}\right), \qquad \ell_c = \frac{\ell}{z_c}, \tag{28}$$

which vanishes when $\ell \to 0$ in contrast to the logarithmic divergence in the undeformed theory. At the leading nonvanishing order, the solution to the equation defining r for $R_c = R/z_c \to 0$ is given by

$$T\bar{T}\text{-deformed 2d CFT}: \ r = 1, \qquad R_c \to 0, \tag{29}$$

due to linearly decreasing entanglement entropy $S(\ell) \approx \ell/(2z_c)$. From Fig. 4, one can observe that the coefficient r being equal to 1 for $R \to 0$ is size dependent, and for large ℓ_c it decreases to its ordinary value, $r = 2(\sqrt{2}-1)$, corresponding to the 2D CFT vacuum. The interpretation in terms of the phase space deformation (or broadening the width of fundamental particles) seems to be consistent with this change of r. For small ℓ_c, the fine structure of the fractal cannot be taken into account due to the finite size of particles. It would be interesting to clarify the effect of $T\bar{T}$ deformations in explicit quantum error correcting codes

[2]The holographic entanglement entropy in $T\bar{T}$ deformed theories has been considered in [28–30].

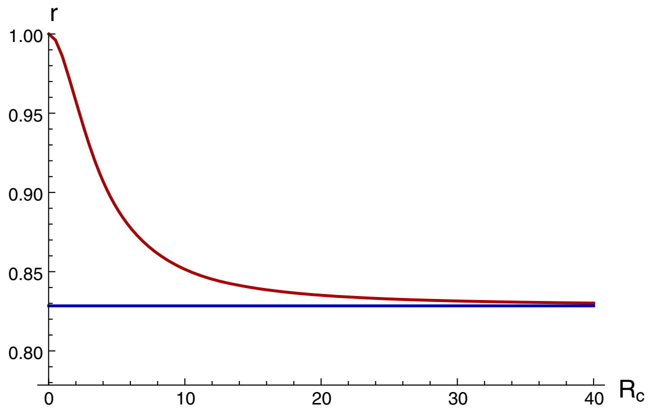

FIG. 4. The dependence of r on the $R_c = R/z_c$ for $T\bar{T}$-deformed theory.

and systems using objects of “zero size” (like in the Gotesman-Preskill-Kitaev protocol, which uses an infinite sequence of Dirac delta functions to encode a qubit).

V. CONCLUSION

In this paper, we studied some of the quantum error-correcting properties of the holographic correspondence, namely, the estimate of logical subalgebra distance $d(\mathcal{A}_X)$ corresponding to fractal regions X. We obtain a generalization of estimates proposed in [11] of the form

$$d(\mathcal{A}_X) \leq \left(\frac{|R|}{a}\right)^{\alpha}, \qquad \alpha = \frac{\log 2}{\log(2/r)},$$

on different fractal regions and for different dimensions and excited states, like black holes and $T\bar{T}$ deformed theories by calculating the “index” r for each case. We find that the property of holographic reconstruction of fractal regions, called in [11] uberholography, also takes place in higher dimensions. We calculate r corresponding to higher dimensions explicitly for AdS_{d+1} duals with $d = 3$ ($r = 3 - \sqrt{5}$) and $d = 4$ ($r = \sqrt{3} - 1$). For $d = 3$, we show that r is independent of some features of the X structure—namely, the set obtained by slicing a plane into strips organized in some direction as a Cantor set, and a similar annular slicing leads to the same r. This is consistent with the earlier observations in [11–13] that many properties of the uberholographic reconstruction are defined by the vacuum structure of a certain theory (and, as we have shown, by its dimension). For a finite temperature 2D CFT (i.e., the dual of the BTZ black hole), we find that similar properties also take place, for a certain scaling limit of R and temperature T. Naively, one can think of this limit as a selection of some special part of the Hilbert space corresponding to a Cantor set (or, probably, some special state) because the straightforward consideration of the black hole background (see, for example, [13]) is also defined by the vacuum structure. It would be interesting to understand this aspect of our work better in future research. After submitting our paper, an interesting proposal was made in [31], where the bulk entropy is taken into account, resulting in a state-dependent holographic reconstruction, which allows us to deal with the black hole in a slightly different way.

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