# $1/L^2$ corrected soft photon theorem from a CFT$_3$ Ward identity

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ABSTRACT: Classical soft theorems applied to probe scattering processes on AdS$_4$ spacetimes predict the existence of perturbative $1/L^2$ corrections to the soft photon and soft graviton factors of asymptotically flat spacetimes. In this paper, we establish that the $1/L^2$ corrected soft photon theorem can be derived from a large $N$ CFT$_3$ Ward identity. We derive a perturbed soft photon mode operator on a flat spacetime patch in global AdS$_4$ in terms of an integrated expression of the boundary CFT current. Using the same in the CFT$_3$ Ward identity, we recover the $1/L^2$ corrected soft photon factor derived from classical soft theorems.

# Contents

## 1 Introduction

Soft theorems relate amplitudes with soft emission to the amplitude without the soft particles through a soft factor [1, 2]. In the case of the single soft photon theorem, a scattering amplitude $\mathcal{A}_{n+1}$ involving $m$ incoming hard particles, $n-m$ outgoing hard particles and a single soft external photon can be expressed as

$$
\begin{aligned}
\mathcal{A}_{n+1} &= \langle p_{m+1} \cdots p_n | a_\alpha(q) \mathcal{S} | p_1 \cdots p_m \rangle \\
&= S^{\text{photon}} \langle p_{m+1}, \cdots, p_n | \mathcal{S} | p_1, \cdots, p_m \rangle = S^{\text{photon}} \mathcal{A}_n \,.
\end{aligned}
\tag{1.1}
$$

In eq. (1.1), $\mathcal{S}$ is the $S$-matrix that relates incoming and outgoing states, $p_1, \cdots, p_n$ are the momenta of the hard particles with charges $Q_1, \cdots Q_n$, $a_\alpha(q)$ is the soft photon creation operator in the outgoing state with momentum $q$ and polarization $\epsilon_\alpha$ and $S^{\text{photon}}$ is the soft photon factor which is a function of all the charges, momenta and the soft photon polarization. The soft factor admits an expansion in soft momentum $q$, with the leading pole contribution being the Weinberg soft photon factor [2]

$$
S^{\text{photon}}_{(0)} = \sum_{i=m+1}^{n} Q_i \frac{p_i . \epsilon}{p . q} - \sum_{i=1}^{m} Q_i \frac{p_i . \epsilon}{p . q} \,.
\tag{1.2}
$$

Remarkably, soft theorems have also been recently realized as a symmetry of the $S$-matrix along future and past null infinity on asymptotically flat spacetimes [3–8]. More specifically, there exist soft charges that generate large gauge transformations of asymptotic massless fields, with the $S$-matrix satisfying the corresponding large gauge Ward identity [9–18]. Thus soft theorems are equivalent to large gauge Ward identities, with the intepretation of

soft particles as Goldstone modes. This equivalence is part of a larger web of relations known as the 'infrared triangle' for interacting theories with massless fields on asymptotically flat spacetimes [8].

It is currently unknown if similar infrared structures of scattering processes are present on non-asymptotically flat spacetimes, particularly those with a cosmological constant. While certain generalizations of BMS symmetries on spacetimes with a cosmological constant are known to result from modified boundary conditions [19–21], their relevance in scattering processes remain obscure. In large part, this is due to the absence of well defined scattering amplitudes on these spacetimes. However there exists an approach to derive classical contributions in soft factors without recourse to an $S$-matrix which comes from classical soft theorems [22–27]. To be specific one can write the classical soft photon theorem in four spacetime dimensions as,

$$\lim_{\omega \to 0} \epsilon^\mu \tilde{a}_\mu \left( \omega \, , \vec{x} \right) = -\frac{i}{4\pi R} e^{i\omega R} S_{\text{em}} \tag{1.3}$$

where $\tilde{a}_\mu$ is the radiative component of the electromagnetic field in frequency space, $\epsilon^\mu$ is the polarization vector, $R$ is the distance of the soft photon from the scatterer and $S_{\text{em}}$ denotes the soft photon factor. These theorems provide universal soft factor contributions from classical scattering processes whenever the soft radiation has a wavelength much larger than the impact parameter and total energy far less than that of the scatterer. Using a probe scattering process in the small cosmological constant limit, this approach was generalized to derive universal soft graviton [28] and soft photon [29] factor contributions on asymptotically $\text{AdS}_4$ spacetimes up to the first subleading order in frequency.

The AdS radius ($L$) dependent corrections are a consequence of a double scaling limit required on spacetimes with a cosmological constant. The spectrum of massless fields on AdS spacetimes is actually discrete and a typical $\omega \to 0$ limit does not exist. As further elaborated in [28, 29], one have to consider a double scaling limit wherein $\omega \to 0$ as $L \to \infty$ while leaving $\omega L = \gamma$ constant and large. We have found out in [28, 29] that retaining all $1/L^2$ corrections, the soft radiation involve corrections at this order while the correction in massive probe particle asymptotic trajectory becomes relevant at the next subleading order ($1/L^4$). This is a consequence of the $\text{AdS}_4$ potential considered perturbatively about flat spacetime up to $1/L^2$ order, which affects the trajectories of massless soft particles while preserving the flat spacetime geodesics of massive particles. This also ensures that the relation in eq. (1.3) still holds at this order. At both leading ($\omega^{-1}$) and subleading ($\ln \omega^{-1}$) orders in frequency, there exist AdS radius specific corrections to the known universal asymptotically flat spacetime soft factor results.[1] The corrected soft factors provide evidence for asymptotic interactions due to AdS potential, that distinguish between asymptotically flat spacetimes and a flat region in a larger spacetime.

It is well understood that the flat spacetime $S$-matrix involving the hard particles can be recovered from CFT correlation functions in the large AdS radius limit [30–36].

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[1]It should also be noted that such corrections can't be replicated by higher curvature contributions to General Relativity on asymptotically flat spacetimes, as the corresponding soft factors in this case appear at subleading order in soft frequency.

The scattering in this context takes place in a small locally asymptotically flat region, centrally located within a larger AdS spacetime. In addition, some infrared properties of asymptotically flat spacetime $S$-matrices have been recovered in this large AdS radius limit, including the derivation of BMS symmetries [37] and the soft photon theorem from CFT Ward identities [38]. Let us summarize how the correspondence works:

- Within the Lorentzian approach Applying the HKLL (Hamilton, Kabat, Lifschytz, and Lowe) bulk reconstruction method [39], one first constructs bulk AdS operators from the boundary CFT operators.

- Subsequently, the large AdS radius limit of these suitably constructed operators provide the corresponding flat spacetime creation and annihilation operators.

In particular for boundary dual U(1) current operators smeared around a small window of global time, one gets the photon creation and annihilation modes defined on the null infinity of the flat patch at the center of the AdS spacetime. Hence aspects of the asymptotically flat spacetime infrared triangle can in principle be derived from CFT correlation functions on asymptotically AdS spacetimes.

In this paper, we derive the leading $1/\gamma^2$ corrected soft photon factor from the U(1) symmetry Ward identities of the CFT sitting at the boundary of AdS$_4$ spacetimes. The leading soft photon factor at $1/L^2$ order was already shown to be equivalent to a perturbed large gauge Ward identity on the flat spacetime patch in AdS spacetimes [29]. We will follow the approach in [38] and extend the reconstruction process of [38] to $1/L^2$ contributions, which provide $1/\gamma^2$ corrections to flat spacetime soft photon factor in double scaling limit. The general expression for the corrected soft photon modes involve an integration over angles on the flat spacetime patch and CFT boundary. We show that a leading contribution from this integral precisely agrees with the $1/\gamma^2$ corrected soft photon factor obtained from classical soft theorems. It is important to note that our analysis do not provide any AdS radius corrections to the $S$-matrix involving hard particles.

The organization of our paper is as follows. In the next section, we review essential features and derivation of the Weinberg soft photon theorem from a large $N$ CFT$_3$ Ward identity on AdS$_4$ spacetimes following [38]. In section 3, we then consider $1/L^2$ corrections to the flat spacetime limit of the CFT Ward identity. We first review the result for $1/\gamma^2$ corrections to the flat spacetime soft factors coming from classical soft theorems. We then proceed to derive the corrected soft photon theorem from the CFT$_3$ Ward identity. We conclude the paper with some interesting open questions.

## 2 Soft photon theorem from CFT Ward identities

In this section, we will review the derivation of Weinberg's soft photon theorem from a CFT$_3$ Ward identity at the boundary of AdS$_4$ spacetimes closely following [38]. Experts familiar with notations and other relevant details may skip this part. We first address certain preliminaries needed for this derivation before turning to the result from the Ward identity. The AdS$_4$ spacetime metric in global coordinates is

$$ds^2 = \frac{L^2}{\cos^2(\rho)} \left[ -d\tau^2 + d\rho^2 + \sin^2(\rho)d\Omega_2^2 \right], \tag{2.1}$$

where the 2-sphere metric $d\Omega_2^2$ will be described using complex stereographic coordinates $\{z, \bar{z}\}$. The metric for the asymptotically flat spacetime patch follows from eq. (2.1) by defining

$$\frac{r}{L} = \tan(\rho) \qquad ; \qquad \frac{t}{L} = \tau \,, \tag{2.2}$$

and taking $L \to \infty$. This patch is centrally located at global time $\tau = 0$.

The Lorentzian analysis in [38] is based on relating Fock states on the asymptotic boundary of the flat spacetime patch with CFT states on the boundary of the AdS spacetime. This is achieved through global Cauchy slices that foliate the spacetime, whose $L \to \infty$ limit recovers the asymptotic slices of the flat spacetime patch. In particular, the scattering process is confined to the flat patch in a region of $\epsilon \sim \mathcal{O}(L)^{-1}$ around global time $\tau = 0$. Beyond this region, the bulk fields are asymptotically free and their reconstruction from boundary operators can be carried out using the HKLL reconstruction. The outgoing (ingoing) states can be defined on Cauchy slices for $\tau \in \{\epsilon, \pi - \epsilon\}$ ($\tau \in \{-\pi + \epsilon, -\epsilon\}$) denoted by $\Sigma^+$ ($\Sigma^-$).

We consider bulk gauge field operators $\hat{\mathcal{A}}_\mu(\rho, \tau, z, \bar{z})$ defined on early and late time slices, with the general fall-off conditions

$$\hat{\mathcal{A}}_\mu(\rho, x') \xrightarrow[\rho \to \frac{\pi}{2}]{} (\cos\rho)^1 \hat{\alpha}_\mu(x') + (\cos\rho)^0 \hat{\beta}_\mu(x') \,, \tag{2.3}$$

where $x' = \{\tau', z', \bar{z}'\}$ denotes global boundary coordinates, while $\hat{\alpha}_\mu(x')$ and $\hat{\beta}_\mu(x')$ are primary operators of the CFT that are sources. In the following, we choose 'magnetic boundary conditions' with $\hat{\beta}_\mu(x') = 0$ and $\hat{\alpha}_\mu(x') = j_\mu(x')$ being a U(1) conserved current of conformal dimension $\Delta = 2$. With these assumptions, the boundary limit of the bulk gauge field is simply

$$\hat{\mathcal{A}}_\mu(\rho, x') \xrightarrow[\rho \to \frac{\pi}{2}]{} (\cos\rho) j_\mu(x') \,. \tag{2.4}$$

The choice in eq. (2.4) will provide us with Weinberg's soft photon theorem in the absence of magnetic charges. We further assume the absence of Coulombic fields, with the conserved current dual to radiative modes. Hence $\mu = z, \bar{z}$ provide the only non-vanishing current components.

In the $L \to \infty$ limit, bulk radiative fields must also satisfy the mode expansion on constant time slices of asymptotically flat spacetimes. To this end, we adopt the conventions of [8] and write the mode expansion as,

$$\hat{\mathcal{A}}_\mu(y) = \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{2\omega_q} \sum_{\lambda=\pm} \left[ \varepsilon_\mu^{(\lambda)*} \hat{a}_{\vec{q}}^{(\lambda)} e^{iqy} + \varepsilon_\mu^{(\lambda)} \hat{a}_{\vec{q}}^{(\lambda)\dagger} e^{-iqy} \right] \,, \tag{2.5}$$

where $y = \{t, \vec{y}\} = \{t, r, z, \bar{z}\}$ are flat spacetime coordinates, $q = \{q^0, \vec{q}\}$ is the 4-momentum of the radiative fields satisfying $q^2 = 0$ with frequency $\omega_q$, and $\varepsilon_\mu^{(\lambda)}$ are the polarization vectors normalized according to $\varepsilon_\mu^{(+)} \varepsilon^{(-)\mu} = 1$. The expression in eq. (2.5) can be used to derive the creation and annihilation operators

$$\hat{a}_{\vec{q}}^{(\lambda)} = \lim_{t \to \pm\infty} i \int d^3\vec{y}\, \varepsilon^{(\lambda)\mu} e^{-iq\cdot y} \overleftrightarrow{\partial_0} \hat{\mathcal{A}}_\mu(y) \,, \tag{2.6}$$

$$\hat{a}_{\vec{q}}^{(\lambda)\dagger} = \lim_{t \to \pm\infty} -i \int d^3\vec{y}\, \varepsilon^{(\lambda)*\mu} e^{iq\cdot y} \overleftrightarrow{\partial_0} \hat{\mathcal{A}}_\mu(y) \,. \tag{2.7}$$

In eq. (2.6) and eq. (2.7) the outgoing modes follow from $t \to \infty$ while the ingoing modes are those from $t \to -\infty$. The creation and annihilation modes of the ingoing and outgoing states satisfy

$$\left[ \hat{a}_{\vec{q}}^{(\lambda)}, \hat{a}_{\vec{q}'}^{(\lambda')\dagger} \right] = \delta^{\lambda\lambda'}(2\pi)^3 2\omega_q \delta^{(3)}(\vec{q} - \vec{q}') \tag{2.8}$$

We can recover soft photon modes from eq. (2.6) and eq. (2.7) in the $\omega_q \to 0$ limit. The outgoing positive helicity soft photon mode will result from eq. (2.6), for which we have the following polarization vector and plane wave expressions

$$\varepsilon^{(+)z} = \frac{1 + z\bar{z}}{\sqrt{2}r}, \quad e^{-iq \cdot y} = e^{i\omega_{\vec{q}} t} e^{-i\vec{p} \cdot \vec{r}} = e^{i\omega_{\vec{q}} t} 4\pi \sum_{l',m'} (-i)^{l'} j_{l'}(r\omega_{\vec{q}}) Y_{l'm'}(\Omega) Y_{l'm'}^*(\Omega_q). \tag{2.9}$$

In order to derive flat spacetime soft modes from a CFT, we must also relate the gauge field appearing in eq. (2.6) and eq. (2.7) with the CFT current at the boundary. This can be facilitated by using the HKLL prescription [39], with the reconstruction of bulk gauge fields in global coordinates that satisfy eq. (2.4) taking the form

$$\hat{\mathcal{A}}_{\mu}(X) = \int d^3x' \left[ K_{\mu}^V(X;x')\epsilon_{\tau'}^{a'b'}\nabla_{a'}j_{b'}^+ + K_{\mu}^S(X;x')\gamma^{a'b'}\nabla_{a'}j_{b'}^+ \right.$$
$$\left. + \left(K_{\mu}^V\right)^*(X;x')\epsilon_{\tau'}^{a'b'}\nabla_{a'}j_{b'}^- + \left(K_{\mu}^S\right)^*(X;x')\gamma^{a'b'}\nabla_{a'}j_{b'}^- \right], \tag{2.10}$$

where $x' = \{\tau', z', \bar{z}'\}$ are boundary coordinates while $X = \{\tau, \rho, z, \bar{z}\}$ is a bulk point in global AdS$_4$ coordinates, $\epsilon^{a'b'c'}$ and $\nabla_{a'}$ are respectively the Levi-Civita tensor and covariant derivatives on the boundary, and $j_{b'}^{\pm}$ represent current components at the boundary with the $\pm$ signs indicating positive and negative frequency solutions. The explicit form of the boundary integral is

$$\int d^3x' = \int_{\mathcal{T}} d\tau' \int d\Omega' \tag{2.11}$$

with the domain of integration $\mathcal{T}$ in the $\tau'$ integral being $\{-\pi, 0\}$ for ingoing states and $\{0, \pi\}$ for outgoing states. Lastly, $K_{\mu}^V$ and $K_{\mu}^S$ appearing in eq. (2.10) are respectively the HKLL kernels for 'vector' and 'scalar' type components of the Maxwell field. For the purely radiative modes we have the components

$$K_z^V(X;x') = \frac{1}{\pi} \sum_{\kappa,l,m} \mathcal{N}^V Y_{lm}^*(\Omega') \partial_z Y_{lm}(\Omega) \Xi_{\kappa l}(\rho,\tau,\tau')\Big|_{\Delta=2} \tag{2.12}$$

$$K_{\bar{z}}^V(X;x') = -\frac{1}{\pi} \sum_{\kappa,l,m} \mathcal{N}^V Y_{lm}^*(\Omega') \partial_{\bar{z}} Y_{lm}(\Omega) \Xi_{\kappa l}(\rho,\tau,\tau')\Big|_{\Delta=2} \tag{2.13}$$

$$K_z^S(X;x') = \frac{1}{\pi} \sum_{\kappa,l,m} \mathcal{N}^S Y_{lm}^*(\Omega') \partial_z Y_{lm}(\Omega) \Xi_{\kappa l}(\rho,\tau,\tau')\Big|_{\Delta=1} \tag{2.14}$$

$$K_{\bar{z}}^S(X;x') = \frac{1}{\pi} \sum_{\kappa,l,m} \mathcal{N}^S Y_{lm}^*(\Omega') \partial_{\bar{z}} Y_{lm}(\Omega) \Xi_{\kappa l}(\rho,\tau,\tau')\Big|_{\Delta=1}, \tag{2.15}$$

with

$$\Xi_{\kappa l}(\rho,\tau,\tau') = e^{i\omega_{\kappa}(\tau-\tau')} \sin^{l+1}\rho \cos^{\Delta-1}\rho \,_2F_1\left(-\kappa, \kappa+\Delta+l, \Delta-\frac{1}{2}\Big|\cos^2\rho\right), \tag{2.16}$$

and $\kappa, l, m$ are positive integers. Our expressions follow from the free Maxwell field solutions [40], which we review in appendix A. The scaling dimensions $\Delta = 2$ and $\Delta = 1$ are those of the vector and scalar type solutions. The frequency modes of fields in AdS$_4$ are discrete and related to the scaling dimension in the above solutions by

$$\omega_\kappa = 2\kappa + \Delta + l \tag{2.17}$$

The normalizations $\mathcal{N}^V$ and $\mathcal{N}^S$ appearing in eqs. (2.12)–(2.15) are

$$\mathcal{N}^V = -\frac{1}{4l(l+1)} \qquad ; \qquad \mathcal{N}^S = -\frac{1}{4l(l+1)} \frac{i}{\omega_\kappa|_{\Delta=1}} \,. \tag{2.18}$$

This choice is consistent with the normalization of the CFT current in [38] and provides canonically normalized creation and annihilation operators in the flat spacetime patch. We can now substitute the $L \to \infty$ limit of eq. (2.10) in eq. (2.6) and eq. (2.7) to find expressions for the flat spacetime annihilation and creation operators in terms of derivatives of the boundary current. The evaluation of the $L \to \infty$ limit involves substituting for $\rho$ and $\tau$ using eq. (2.2). Consistency with the flat spacetime mode solutions also require that the discrete frequency modes $\omega_\kappa$ in eq. (2.17) scale with $L$ in the flat spacetime limit. This can be achieved by requiring that modes in $L \to \infty$ limit are dominated by large values of $\kappa$, with $\omega_\kappa \approx \omega L$ and where $\omega$ is the continuous frequency of modes in flat spacetime. In this way, the sum over $\kappa$ gets traded for an integral over $\omega$ in the expressions eqs. (2.12)–(2.15). Explicitly we have

$$2\kappa = \omega_\kappa - \Delta - l \,, \qquad \sum_\kappa \to \frac{1}{2} \int d\omega L \tag{2.19}$$

This procedure leads to a solution $\hat{\mathcal{A}}_\mu(y)$, with $y$ the Minkowski coordinates, from eq. (2.10). The flat spacetime modes $\hat{a}_{\vec{q}}^{(\pm)\text{out}}$ create photons with positive (+) and negative (−) helicity in the outgoing state and respectively result from $\hat{\mathcal{A}}_z^{\text{out}}(y)$ and $\hat{\mathcal{A}}_{\bar{z}}^{\text{out}}(y)$ in our conventions for flat spacetime modes. Denoting the corresponding annihilation modes as $\hat{a}_{\vec{q}}^{\text{out}(\pm)}$, we find that eq. (2.6) gives the result

$$\hat{a}_{\vec{q}}^{\text{out}(-)} = \frac{1}{4\omega_q} \frac{1 + z_q \bar{z}_q}{\sqrt{2}} \int d^3x' \, e^{i\omega_{\vec{q}} L\left(\frac{\pi}{2} - \tau'\right)} \frac{1}{\bar{z}_q - \bar{z}'} D^{z'} j_{z'}^-(x') \,,$$
$$\hat{a}_{\vec{q}}^{\text{out}(+)} = \frac{1}{4\omega_q} \frac{1 + z_q \bar{z}_q}{\sqrt{2}} \int d^3x' \, e^{i\omega_{\vec{q}} L\left(\frac{\pi}{2} - \tau'\right)} \frac{1}{z_q - z'} D^{\bar{z}'} j_{\bar{z}'}^-(x') \tag{2.20}$$

The $\omega_q$ frequency is defined below eq. (2.5). We get the result in eq. (2.20) after integrating over the general flat spacetime frequency $\omega$.[2]

The expressions for creation modes in the outgoing states and all modes in the ingoing states can be similarly derived from the bulk gauge field solution. The association of flat spacetime modes with current operators at the boundary has also been identified for massless and massive scalar fields in [38]. For the outgoing modes in eq. (2.20) we see that the dominant contribution of the phase in the large $L$ limit comes around $\tau = \frac{\pi}{2}$. More generally

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[2]There is a delta function for the frequency from integrating over the spherical Bessel functions that picks the frequency $\omega_q$.
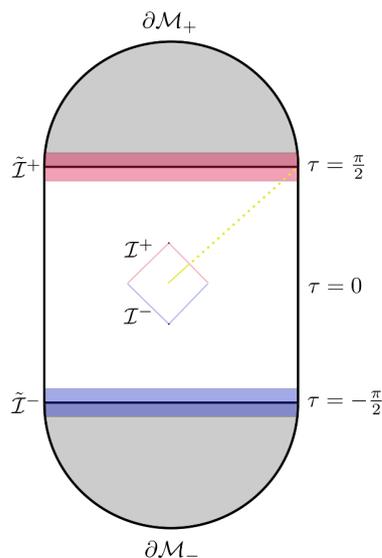
**Figure 1**. The future (pink) and past (blue) null infinities of the central flat spacetime patch, $\mathcal{I}^+$ and $\mathcal{I}^-$ respectively, can be identified as the $L \to \infty$ limits of boundary regions $\tilde{\mathcal{I}}^\pm$ around $\tau' = \pm\frac{\pi}{2}$. Massless particles approaching the soft limit on the flat spacetime patch more closely approximate the $\tau' = \pm\frac{\pi}{2}$ surfaces. As an example, we have drawn a soft particle trajectory (yellow). The dashed line up to the boundary indicates the global trajectory outside the patch, but has no role in mapping the boundaries of AdS to those of the flat spacetime patch. Timelike infinities $i^\pm$ on the flat spacetime patch are identified with the Euclidean caps $\partial\mathcal{M}_\pm$, while spatial infinity $i_0$ is identified with $\tau \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$.

for ingoing and outgoing massless fields in the large $L$ limit, the dominant contribution comes from a $\mathcal{O}(L)^{-1}$ region around $\tau = \pm\frac{\pi}{2}$. This provides a correspondence between a small window around $\tau = \pm\frac{\pi}{2}$, denoted as $\tilde{\mathcal{I}}^\pm$, and null infinity on the flat spacetime patch $\mathcal{I}^\pm$. In the case of massive fields in the large $L$ limit, the phase has complex saddles around $\tau = \pm\frac{\pi}{2} \pm if(\omega_p, m)$ with $f$ a function of the massive particle energy $\omega_p$ and mass $m$. This indicates that $i^\pm$ of the flat spacetime patch can be associated at the AdS$_4$ boundary with Euclidean caps that are analytic continuations in the global time from $\pm\frac{\pi}{2}$. The mapping between asymptotic regions of the flat spacetime patch and that of the AdS$_4$ boundary is indicated in figure 1.

A feature of the Maxwell field modes which distinguish them from massless scalar field modes are the appearance of specific functions of the boundary angular coordinates $\{z', \bar{z}'\}$. Denoting the parameter $\epsilon(\hat{x}')$ (with $\hat{x}'$ indicating dependence on angles) for the two helicity choices as

$$\epsilon(\hat{x}') = \frac{1}{z_q - z'} \text{ (+ve helicity)}, \qquad \epsilon(\hat{x}') = \frac{1}{\bar{z}_q - \bar{z}'} \text{ (−ve helicity)}, \qquad (2.21)$$

we can express the $\omega_q \to 0$ limit of eq. (2.20) as

$$\lim_{\omega_q \to 0} \omega_q \frac{\sqrt{2}}{1 + z_q \bar{z}_q} \hat{a}_{\vec{q}}^{\text{out}(-)} = \frac{1}{4} \int d^3x' \epsilon(\hat{x}') D^{z'} j_{z'}^-(x'),$$

$$\lim_{\omega_q \to 0} \omega_q \frac{\sqrt{2}}{1 + z_q \bar{z}_q} \hat{a}_{\vec{q}}^{\text{out}(+)} = \frac{1}{4} \int d^3x' \epsilon(\hat{x}') D^{\bar{z}'} j_{\bar{z}'}^-(x') \qquad (2.22)$$

The parameters in eq. (2.21) are precisely those that are chosen in the large gauge Ward identity on asymptotically flat spacetimes to recover the Weinberg soft photon theorem [8]. We note that in taking the soft limit in eq. (2.22), the $\tau'$ dependent phase drops out. The soft limit hence has a boundary description on the $\tau = \pm\frac{\pi}{2}$ slices, providing a 2 dimensional realization on the 3 dimensional boundary. We also note that the outgoing positive (negative) helicity flat spacetime soft photon modes are mapped to $D^{\bar{z}}j_{\bar{z}}^-$ $(D^z j_z^-)$ current derivatives on the AdS$_4$ boundary.

With the above results, we can now derive Weinberg's soft photon theorem from the Ward identity of a large $N$ CFT with global U(1) symmetry. The integrated expression for the Ward identity takes the form

$$\int d^3x' \, \alpha(x')\partial'_\mu \langle 0|T\{j^\mu(x')\Phi\}|0\rangle = \left(\sum_{i=1}^n Q_i\alpha(x'_i) - \sum_{j=1}^m Q_j\alpha(x'_j)\right)\langle 0|T\{\Phi\}|0\rangle, \quad (2.23)$$

where $T\{\cdots\}$ refers to time ordering of the operators inside the parenthesis, $\Phi$ are a collection of CFT operators comprising of $n$ operators with charges $Q_i$ in the 'ingoing' ($\tau < 0$) region and $m$ operators with charges $Q_j$ in 'outgoing' ($\tau > 0$) region of the boundary, and $\alpha(x')$ an arbitrary parameter. Using the relationship between creation/annihilation flat spacetime modes with operators at the boundary, the correlation function $\langle 0|T\{\Phi\}|0\rangle$ can be related with the $S$-matrix for a corresponding scattering process in the flat spacetime patch.

The following choice for $\alpha(x')$

$$\alpha(x') = \lim_{\rho\to\frac{\pi}{2}} \int d^2\hat{x}'' \, \frac{1}{4\pi} \frac{\cos^2\rho - \cos^2\tau}{(\sin\tau - \sin\rho\,\hat{x}'\cdot\hat{x}'')^2}\epsilon(\hat{x}''), \quad (2.24)$$

recovers the soft theorem, as it has the desired property of $\alpha(x')|_{\tilde{\mathcal{I}}^\pm} = \epsilon(\hat{x}')$. Hence we recover the gauge parameters as in eq. (2.21) which have no dependence on $\tau'$. The left hand side of Weinberg's soft theorem, involving the insertion of the soft photon mode, follows from the left hand side of eq. (2.23).

$$\int d^3x' \alpha(x')\partial'_\mu\langle 0|T\{j^\mu(x')\Phi\}|0\rangle = \int_{\tilde{\mathcal{I}}^\pm} d^3x'\epsilon(\hat{x}')\left[D^{z'}\langle 0|T\{j_{z'}(x')\Phi\}|0\rangle + D^{\bar{z}'}\langle 0|T\{j_{\bar{z}'}(x')\Phi\}|0\rangle\right]$$

$$(2.25)$$

The expression in eq. (2.25) can be directly associated with the soft photon insertion using eq. (2.22) for the outgoing state. In summing over all positive and negative frequency contributions, only negative frequency terms contribute in the out-state. Hence the terms in eq. (2.22) account for the insertion of a soft photon in the out-state of a given scattering process involving massless particles whose $S$-matrix results from $\langle 0|T\{\Phi\}|0\rangle$. The procedure can be carried out for soft photons inserted in the in-state, with contributions in this case coming from positive frequency modes and thus creation operators. However, by invoking the equivalence of matrix elements involving in-state and out-state soft photons insertions by $CPT$ invariance, the contributions from the in-state can be readily related to the out-state soft photon insertions.

The Weinberg soft photon theorem is recovered on considering eq. (2.24) in the right hand side of eq. (2.23). The derivation of the soft photon theorem from a CFT Ward

identity relied centrally on $\epsilon(\hat{x}')$ as in eq. (2.21), which was derived from the flat limit of HKLL reconstructed bulk gauge fields. We conclude this section with an observation of soft factors being encoded in eq. (2.20) and consider only the outgoing positive helicity mode for simplicity. The flat spacetime parametrization for null particles can be applied to particles on the flat spacetime patch and $AdS_4$ boundary, as these only depend on angles. Assuming a 'hard' massless particle with energy $E'$ and unit charge parametrized in terms of angular coordinates at the $AdS_4$ boundary, and a soft photon defined in terms of angular coordinates of the flat spacetime patch, we have

$$
\begin{aligned}
p^\mu &= \frac{E'}{1 + z'\bar{z}'} \left(1 + z'\bar{z}', z' + \bar{z}', -i(z' - \bar{z}'), 1 - z'\bar{z}'\right), \\
q^\mu &= \frac{\omega_q}{1 + z_q \bar{z}_q} \left(1 + z_q \bar{z}_q, z_q + \bar{z}_q, -i(z_q - \bar{z}_q), 1 - z_q \bar{z}_q\right) \\
\epsilon_+^\mu &= \frac{1}{\sqrt{2}} \left(\bar{z}_q, 1, -i, -\bar{z}_q\right).
\end{aligned}
\tag{2.26}
$$

We then find that eq. (2.20) is equivalent to the following expression

$$
\hat{a}_{\vec{q}}^{\text{out}(+)}(\omega_q \hat{q}) = \frac{1}{4} \int d^3 x' \, e^{i\omega_{\vec{q}} L\left(\frac{\pi}{2} - \tau'\right)} \frac{p.\epsilon_+}{p.q} \, D^{\bar{z}'} j_{\bar{z}'}^-(x'),
\tag{2.27}
$$

which involves the soft factor $\frac{p.\epsilon_+}{p.q}$ for a positive helicity soft photon. Similar expressions can be found for all other incoming and outgoing modes. This establishes that flat spacetime gauge field modes derived in terms of the boundary current contain information on the soft photon factor in soft theorems and the equivalent gauge parameter needed to derive the corresponding large gauge Ward identity.

# 3 $1/L^2$ corrections to the soft photon theorem from CFT ward identities

We will now address $1/L^2$ corrections of the soft photon theorem on $AdS_4$ spacetimes. In the following subsection, we first briefly recall the soft factor correction derived previously using classical soft theorems on $AdS_4$ black hole spacetimes [28, 29]. We will then proceed to generalize the above bulk reconstruction analysis up to $1/L^2$ corrections. The resulting expression for a perturbed soft photon mode in terms of a current can also be substituted in the Ward identity. In the last subsection, we establish that the corrected soft photon theorem resulting from the $CFT_3$ Ward identity agrees with the classical soft theorem result after expanding about a leading saddle.

## 3.1 $1/L^2$ corrected soft photon theorem from classical soft theorems

The formal derivation of soft factorization in scattering processes on asymptotically AdS spacetimes is obstructed by the absence of a globally defined $S$-matrix. This motivated our derivation of soft factors using classical soft theorems. These theorems state that the classical limit of soft photon and graviton factors may be derived from the zero frequency limit of certain classical scattering processes. More significantly, the primary requirement is that of gauge invariant observables in the case of electromagnetically mediated scattering and diffeomorphism invariance in gravitational scattering, without specifc reference to the

background geometry. This allows for the derivation of soft factors on spacetimes with a cosmological constant [28, 29, 41]. The classical scattering processes are broadly constrained to be such that the energy of the scatterer does not significantly change during the scattering process ($\Delta E_{\text{scatterer}} \ll E_{\text{scatterer}}$) and the wavelength of the emitted radiation should be greater than the large impact parameter ($\lambda_{\text{radiation}} \gg b$).

Among the scattering processes that satisfy these criteria are probe scattering processes on curved spacetimes. We accordingly considered the scattering of a probe particle on asymptotically AdS black hole spacetimes in [28, 29]. The existence of a largest length scale in the classical scattering, namely the AdS radius $L$, introduces two additional requirements in applying classical soft theorems. The first concerns the radial distance of the probe from the scatterer, which we denote by $r$. With the black hole radius $GM$, we modify the large impact parameter requirement to be $GM \ll r \ll L$. Hence the classical process is confined to a region deep in the bulk of asymptotically AdS spacetimes. The second requirement comes in the derivation of the soft limit. Since $r$ cannot take on asymptotically large values, the scattering process takes place within a finite interval of time. In addition, the frequency of massless fields on AdS spacetimes is discrete, formally preventing a zero frequency limit. We hence implement a double scaling limit, wherein $\omega \to 0$ as $L \to \infty$, while keeping $\omega L = \gamma$ constant and large.[3]

We further note that in the classical soft photon theorem derivation, we never encountered the discrete frequency of AdS. The derivation is on an asymptotically flat spacetime perturbed by small cosmological constant corrections. Hence $\omega$ and $L$ were given and we defined a suitable double scaling limit where their product is a large constant.[4]

With these assumptions, the equations for the radiative fields were derived retaining all $1/L^2$ corrections. The double scaling limit applied to the radiative fields identified $1/\gamma^2$ corrections to soft factors on asymptotically flat spacetimes. More specifically, the leading ($\omega^{-1}$) and subleading ($\ln \omega^{-1}$) soft photon and soft graviton factors were derived, each with their respective $\frac{1}{\gamma^2}$ corrections. Furthermore, the $1/L^2$ corrections of the probe particle trajectory, while present, lead to contributions at subleading order in frequency. This suggests that $1/\gamma^2$ corrected soft factors would be those for an $S$-matrix on an asymptotically flat spacetime patch within a global AdS spacetime.

In the following, we restrict ourselves to the leading soft photon factor. The inferred form of the leading soft factor for a general process involving $n$ hard particles with momenta $p^\mu_{(a)}$ and charges $Q_{(a)}$, and a single soft photon with momentum $q^\mu$ and polarization $\epsilon_\mu$ takes the form

$$S^{(0)}_{\text{em}} = S^{(0);\text{f}}_{\text{em}} + S^{(0);\text{L}}_{\text{em}}$$

$$S^{(0);\text{f}}_{\text{em}} = \sum_{a=1}^{n} Q_{(a)} \eta_{(a)} \frac{\epsilon_\mu p^\mu_{(a)}}{p_{(a)} \cdot q} \, , \tag{3.1}$$

$$S^{(0);\text{L}}_{\text{em}} = \frac{\omega^2}{4\gamma^2} \sum_{a=1}^{n} Q_{(a)} \eta_{(a)} \frac{\epsilon_\mu p^\mu_{(a)}}{p_{(a)} \cdot q} \frac{\vec{p}^2_{(a)}}{\left(p_{(a)} \cdot q\right)^2} \, , \tag{3.2}$$

---

[3]As compared to the last section, $\gamma = \omega L \approx \omega_\kappa$, but their context is different.

[4]In the AdS derivation, the discrete mode can recover continuous flat spacetime modes in the large $L$ limit by double scaling. As noted in the previous section, here we have $\omega_\kappa$ and $L$ given, while $\omega$ is defined to be the continuous flat spacetime frequency.

where $\eta_{(a)} = 1(-1)$ for outgoing (ingoing) hard particles. All indices in the above expressions are contracted with the flat spacetime metric.

As the leading soft factor is universal and holds beyond tree level, we can consider the above expression at the level of the soft photon theorem in a flat spacetime scattering process

$$\lim_{\omega \to 0} \omega \left( \langle \mathrm{out} | \hat{a}^{\mathrm{out}(+)}(\omega \hat{x}) \mathcal{S} | \mathrm{in} \rangle + n_L \langle \mathrm{out} | \hat{a}^{\mathrm{out;\,L}(+)}(\omega \hat{x}) \mathcal{S} | \mathrm{in} \rangle \right) = \left( S_{\mathrm{em}}^{(0);\mathrm{f}} + S_{\mathrm{em}}^{(0);\mathrm{L}} \right) \langle \mathrm{out} | \mathcal{S} | \mathrm{in} \rangle \tag{3.3}$$

with $\mathcal{S}$ the $S$-matrix of the scattering process and $n_L$ is an overall constant. The operators $\hat{a}^{\mathrm{out}(+)}$ and $\hat{a}^{\mathrm{out;\,L}(+)}$ are those for positive helicity soft photon modes responsible for the corresponding soft factors $S_{\mathrm{em}}^{(0);\mathrm{f}}$ and $S_{\mathrm{em}}^{(0);\mathrm{L}}$. Due to the absence of $1/L^2$ corrections of the probe particle trajectory in the derivation using classical soft theorems, we assume the soft factors are infrared divergent contributions to the uncorrected $S$-matrix on the asymptotically flat spacetime patch. This implies that $\hat{a}^{\mathrm{out}(+)}$ is the usual soft photon mode leading to the Weinberg soft factor, while $\hat{a}^{\mathrm{out;\,L}(+)}$ can be interpreted as a perturbed mode that provides the $1/L^2$ corrected soft factor $S_{\mathrm{em}}^{(0);\mathrm{L}}$. This interpretation is supported by the equivalence of the $1/L^2$ corrected soft photon theorem with a perturbed large gauge Ward identity.

We will now evaluate eq. (3.3) for the massless scattering process of interest in our paper. Assuming the parametrization of hard particles and a single soft photon as in eq. (2.26), we find the following correction to the soft photon theorem

$$\lim_{\omega_q \to 0} \frac{\sqrt{2} \omega_q n_L}{(1 + z_q \bar{z}_q)} \langle \mathrm{out} | \hat{a}_{\vec{q}}^{\mathrm{out;\,L}(+)}(\omega_q \hat{x}) \mathcal{S} | \mathrm{in} \rangle$$

$$= \frac{1}{16\gamma^2} \left[ \sum_{k=\mathrm{out}} \frac{(1 + z'\bar{z}')^2 (1 + z_q \bar{z}_q)^2}{(\bar{z}_q - \bar{z}')^2 (z_q - z')^3} Q_k - \sum_{k=\mathrm{in}} \frac{(1 + z'\bar{z}')^2 (1 + z_q \bar{z}_q)^2}{(\bar{z}_q - \bar{z}')^2 (z_q - z')^3} Q_k \right] \langle \mathrm{out} | \mathcal{S} | \mathrm{in} \rangle \tag{3.4}$$

The above result is the $1/L^2$ corrected soft photon theorem inferred from a purely classical scattering process in the bulk of AdS$_4$ spacetimes, up to an overall constant $n_L$ that cannot be fixed by classical soft theorems.

## 3.2  $1/L^2$ corrected soft photon theorem from a CFT$_3$ Ward identity

We will now consider the approach in section 2 to derive $1/L^2$ corrections to the known soft photon theorem for a $S$-matrix defined on the asymptotically flat spacetime patch in AdS$_4$. This implies that we do not consider $1/L^2$ corrections to the $L \to \infty$ limit of the global AdS$_4$ metric in eq. (2.1), nor the time ordered collection of fields $\Phi$ appearing in the Ward identity eq. (2.23). In this way, the 'hard process' remains one of the $S$-matrix on an asymptotically flat spacetime patch.[5] However, with insights from the classical soft photon

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[5]On expanding eq. (2.2) we find $ds^2 = ds_{\mathrm{flat}}^2 - \frac{r^2}{L^2} \left( dt^2 + dr^2 \right) + \mathcal{O}(L^{-4})$, with $ds_{\mathrm{flat}}^2$ the metric of the flat spacetime patch. A general scattering process on the corrected background will not be governed by a $S$-matrix. One possible way to define a $S$-matrix with such background corrections is to require it satisfy the soft graviton theorem in a process containing soft graviton modes with subleading AdS radius corrections. This analysis lies outside the scope of the present article.

theorem, we expect that a scattering process on a flat spacetime patch embedded in an AdS$_4$ spacetime admits $1/L^2$ corrections to the soft photon theorem. We systematically derive the corrected soft photon mode by expanding the integrand of eq. (2.10) up to $1/L^2$ corrections, assuming that the current remains fixed by the condition in eq. (2.4). The substitution of the bulk gauge field up to $1/L^2$ corrections in eq. (2.6) and eq. (2.7) then recovers soft photon modes as in eq. (2.22) along with its perturbation responsible for $1/L^2$ corrections to the soft photon factor.

From eqs. (2.12)–(2.15), we see that the $1/L^2$ corrections to the integrand in eq. (2.10) can come from the function $\Xi_{\kappa l}(\rho, \tau, \tau')$ and possibly modified normalizations $\tilde{\mathcal{N}}^V$ and $\tilde{\mathcal{N}}^S$ in the kernels. We first address the corrections coming from $\Xi_{\kappa l}(\rho, \tau, \tau')$ as defined in eq. (2.16). As we continue to work in the nearly flat spacetime limit by considering $L$ large, the relation between $\tau$ and $t$ remains the same as in eq. (2.1). We hence consider the same replacement of discrete frequencies $\omega_\kappa$ in AdS by $\omega L$ around large values of $\kappa$ (as in the $L \to \infty$ limit).[6]

We substitute $\omega_\kappa$ from eq. (2.19) as well as $\tau$ and $\rho$ from eq. (2.1) in eq. (2.16), and expand up to $1/L^2$ terms. The technical details behind this expansion are provided in appendix B. The final result in the vector and scalar type expressions are

$$\Xi_{\kappa l}(\rho, \tau, \tau')\Big|_{\Delta=2} = -(\pm i)^{-l} e^{i\omega t} e^{-i\omega L\left(\tau' \mp \frac{\pi}{2}\right)} \frac{r}{L} \Bigg\{ j_l(r\omega) \left(1 + \frac{1}{2\omega^2 L^2} \left(\frac{l(l+1)}{2} - \frac{(r\omega)^2}{3}\right)\right)$$
$$- \frac{1}{2\omega^2 L^2} \sqrt{\frac{\pi}{2r\omega}} \left(\frac{l(l+1)}{2} + (r\omega)^2\right) \frac{2r\omega}{3} J'_{l+\frac{1}{2}}(r\omega) \Bigg\} + \mathcal{O}\left(\frac{1}{\omega^3 L^3}\right), \tag{3.5}$$

$$\Xi_{\kappa l}(\rho, \tau, \tau')\Big|_{\Delta=1} = -(\pm i)^{-l+1} e^{i\omega t} e^{-i\omega L\left(\tau' \mp \frac{\pi}{2}\right)} (\omega L) \frac{r}{L} \Bigg\{ j_l(r\omega) \left(1 + \frac{1}{2\omega^2 L^2} \left(-\frac{l(l+1)}{2} - \frac{(r\omega)^2}{3}\right)\right)$$
$$- \frac{1}{2\omega^2 L^2} \sqrt{\frac{\pi}{2r\omega}} \left(\frac{l(l+1)}{2} + (r\omega)^2\right) \frac{2r\omega}{3} J'_{l+\frac{1}{2}}(r\omega) \Bigg\} + \mathcal{O}\left(\frac{1}{\omega^3 L^3}\right), \tag{3.6}$$

where $J_k(v)$ are Bessel functions of the first kind of order $k$ and argument $v$, while $j_l(v)$ are spherical Bessel functions defined as

$$j_l(v) = \sqrt{\frac{\pi}{2v}} J_{l+\frac{1}{2}}(v). \tag{3.7}$$

The primes on Bessel functions in eq. (3.5) and eq. (3.6) denote derivatives with respect to the argument.

The need for $1/L^2$ corrections to the normalizations in eq. (2.18) comes from requiring that perturbed soft modes of positive (negative) helicity continue being related to $D^{\bar{z}} j_{\bar{z}}^-$ ($D^z j_z^-$) current derivatives on the AdS$_4$ boundary, as discussed below eq. (2.22). If we continue to use the normalizations in eq. (2.18), we in fact get the opposite identification. As discussed in the previous section, the individual flat spacetime soft modes are further associated with the gauge parameter and soft factors of the same helicity in the boundary integrated expression over the derivatives of the current. We take this relationship between

---

[6]This approximation in the large $L$ limit is consistent with the double scaling limit $\omega \to 0$ as $L \to \infty$ with $\omega L = \gamma$ a large constant used in the derivation of $1/\gamma^2$ soft factor corrections from classical soft theorems.

modes and current components to be a constraint respected under perturbations. Up to a common shift term proportional to $\frac{1}{\omega^2 L^2}$ in both $\mathcal{N}^V$ and $\mathcal{N}^S$, this restricts the possible modifications of the corrected normalizations $\tilde{\mathcal{N}}^V$ and $\tilde{\mathcal{N}}^S$ to be either of two possibilities

$$\tilde{\mathcal{N}}^V = \mathcal{N}^V \, ; \qquad\qquad\qquad \tilde{\mathcal{N}}^S = \mathcal{N}^S \left( 1 + \frac{l(l+1)}{2\omega^2 L^2} \right) , \tag{3.8}$$

$$\tilde{\mathcal{N}}^V = \mathcal{N}^V \left( 1 - \frac{l(l+1)}{2\omega^2 L^2} \right) ; \qquad \tilde{\mathcal{N}}^S = \mathcal{N}^S \, . \tag{3.9}$$

The soft factor results we would get from these normalizations agree up to a sign. We choose eq. (3.8) in the following. We stress that the modified normalization is not motivated to satisfy a known normalization or inner product relation. Such a criteria does not exist for the perturbed modes we seek to derive about flat spacetimes. Rather, we infer this correction purely from requiring the consistency between helicity components in bulk flat spacetime modes and boundary currents is respected to $1/L^2$ corrections.

We hence find the expressions

$$\tilde{\mathcal{N}}^V \Xi_{\kappa l}(\rho,\tau,\tau')\Big|_{\Delta=2} = \frac{(\pm i)^{-l}}{4l(l+1)} e^{i\omega t} e^{-i\omega L \left( \tau' \mp \frac{\pi}{2} \right)} \frac{r}{L} \left\{ j_l(r\omega) \left( 1 + \frac{1}{2\omega^2 L^2} \left( \frac{l(l+1)}{2} - \frac{(r\omega)^2}{3} \right) \right) \right.$$
$$\left. - \frac{1}{2\omega^2 L^2} \sqrt{\frac{\pi}{2r\omega}} \left( \frac{l(l+1)}{2} + (r\omega)^2 \right) \frac{2r\omega}{3} J'_{l+\frac{1}{2}}(r\omega) \right\} + \mathcal{O}\left( \frac{1}{\omega^3 L^3} \right) , \tag{3.10}$$

$$\tilde{\mathcal{N}}^S \Xi_{\kappa l}(\rho,\tau,\tau')\Big|_{\Delta=1} = -\frac{(\pm i)^{-l}}{4l(l+1)} e^{i\omega t} e^{-i\omega L \left( \tau' \mp \frac{\pi}{2} \right)} \frac{r}{L} \left\{ j_l(r\omega) \left( 1 + \frac{1}{2\omega^2 L^2} \left( \frac{l(l+1)}{2} - \frac{(r\omega)^2}{3} \right) \right) \right.$$
$$\left. - \frac{1}{2\omega^2 L^2} \sqrt{\frac{\pi}{2r\omega}} \left( \frac{l(l+1)}{2} + (r\omega)^2 \right) \frac{2r\omega}{3} J'_{l+\frac{1}{2}}(r\omega) \right\} + \mathcal{O}\left( \frac{1}{\omega^3 L^3} \right) , \tag{3.11}$$

which can be susbtituted in eqs. (2.12)–(2.15) to find any $1/L^2$ corrected bulk gauge field component in either the ingoing $\tau < 0$ or outgoing $\tau > 0$ states. In the following, we confine ourselves to the derivation of the perturbed mode that creates a positive helicity soft photon in the outgoing state. This mode is derived from the $\hat{\mathcal{A}}_z^{\text{out}}(y)$ expression in the large $L$ limit, that takes the form

$$\hat{\mathcal{A}}_z^{\text{out}}(y) = \hat{\mathcal{A}}_z^{\text{out; f}}(y) + \hat{\mathcal{A}}_z^{\text{out;}L}(y) + \hat{\mathcal{A}}_z^{\text{out; sub}}(y) , \tag{3.12}$$

with $y$ the coordinates on the flat spacetime patch. The $\hat{\mathcal{A}}_z^{\text{out; f}}(y)$, $\hat{\mathcal{A}}_z^{\text{out; L}}(y)$ and $\hat{\mathcal{A}}_z^{\text{out; sub}}(y)$ respectively denote the flat spacetime, leading $1/L^2$ and subleading contributions, with expressions

$$\hat{\mathcal{A}}_z^{\text{out; f}}(y) = \frac{1}{4\pi} \int\limits_0^\pi d\tau' \int d\Omega' \int d\omega \, r \, j_l(r\omega)$$

$$\left[ \sum_{l,m} \frac{Y_{lm}^*(\Omega')}{-l(l+1)} \partial_z Y_{lm}(\Omega) \, (i)^{-l} e^{i\omega t} e^{-i\omega L \left( \tau' - \frac{\pi}{2} \right)} D^{\bar{z}'} j_{\bar{z}'}^+ \right.$$

$$\left. + \sum_{l,m} \frac{Y_{lm}(\Omega')}{-l(l+1)} \partial_z Y_{lm}^*(\Omega) \, (-i)^{-l} e^{-i\omega t} e^{i\omega L \left( \tau' - \frac{\pi}{2} \right)} D^{\bar{z}'} j_{\bar{z}'}^- \right] \tag{3.13}$$

$$
\hat{\mathcal{A}}_z^{\text{out; L}}(y) = \frac{1}{16\pi} \int\limits_0^\pi d\tau' \int d\Omega' \int d\omega \, \frac{r}{\omega^2 L^2} \, j_l(r\omega) l(l+1)
$$

$$
\left[ \sum_{l,m} \frac{Y_{lm}^*(\Omega')}{-l(l+1)} \partial_z Y_{lm}(\Omega) (i)^{-l} e^{i\omega t} e^{-i\omega L(\tau'-\frac{\pi}{2})} D^{\bar{z}'} j_{\bar{z}'}^+ \right.
$$

$$
\left. + \sum_{l,m} \frac{Y_{lm}(\Omega')}{-l(l+1)} \partial_z Y_{lm}^*(\Omega) (-i)^{-l} e^{-i\omega t} e^{i\omega L(\tau'-\frac{\pi}{2})} D^{\bar{z}'} j_{\bar{z}'}^- \right] \tag{3.14}
$$

$$
\hat{\mathcal{A}}_z^{\text{out; sub}}(y) = -\frac{1}{24\pi} \int\limits_0^\pi d\tau' \int d\Omega' \int d\omega \frac{r}{\omega^2 L^2} \left[ \left( \frac{l(l+1)}{2} + (r\omega)^2 \right) \sqrt{2\pi r\omega} \, J'_{l+\frac{1}{2}}(r\omega) + (r\omega)^2 j_l(r\omega) \right]
$$

$$
\left[ \sum_{l,m} \frac{Y_{lm}^*(\Omega')}{-l(l+1)} \partial_z Y_{lm}(\Omega)(i)^{-l} e^{i\omega t} e^{-i\omega L\left(\tau'-\frac{\pi}{2}\right)} D^{\bar{z}'} j_{\bar{z}'}^+ \right.
$$

$$
\left. + \sum_{l,m} \frac{Y_{lm}(\Omega')}{-l(l+1)} \partial_z Y_{lm}^*(\Omega)(-i)^{-l} e^{-i\omega t} e^{i\omega L\left(\tau'-\frac{\pi}{2}\right)} D^{\bar{z}'} j_{\bar{z}'}^- \right] + \mathcal{O}\left( \frac{1}{\omega^3 L^3} \right) \tag{3.15}
$$

On substituting eq. (3.12) in eq. (2.6) we recover corresponding outgoing positive helicity gauge field modes in a flat spacetime scattering process. The mode corresponding to the $\hat{\mathcal{A}}_z^{\text{out; f}}(y)$ contribution is the same as in eq. (2.20) and provides the mode that creates an outgoing photon, whose soft limit eq. (2.22) recovers the Weinberg soft photon theorem through the U(1) CFT$_3$ Ward identity as reviewed in the previous section.

On replacing the bulk field contribution $\hat{\mathcal{A}}_z^{\text{out; L}}(y)$ of eq. (3.14) in eq. (2.6), we find a perturbed mode in flat spacetime that we denote by $\hat{a}_{\vec{q}}^{\text{out; L}(+)}$. This mode is perturbative and it involves corrections in terms of the dimensionless parameter $1/\gamma^2 = 1/(\omega_q L)^2$. The derivation of this mode is given in appendix C with the result

$$
\hat{a}_{\vec{q}}^{\text{out; L}(+)} = \frac{1+z_q \bar{z}_q}{\sqrt{2}\omega_q} \frac{1}{32\pi\gamma^2} \int\limits_0^\pi d\tau' \int d\Omega' \int d\Omega_w \left[ \frac{(1+z'\bar{z}')^2 (1+z_w \bar{z}_w)^2}{(\bar{z}'-\bar{z}_w)^2 (z_q-z_w)^3} \right] \mathcal{D}^{\bar{z}'} j_{\bar{z}'}^- e^{i\omega_q L(\tau'-\frac{\pi}{2})} \tag{3.16}
$$

In repeating the above procedure for other ingoing and outgoing bulk field modes, we can likewise find the corresponding perturbed creation and annihilation operators on the flat spacetime patch. For instance, from the expression of $\hat{\mathcal{A}}_{\bar{z}}^{\text{out}}(y)$, we can find the perturbed negative helicity outgoing mode

$$
\hat{a}_{\vec{q}}^{\text{out; L}(-)} = \frac{1+z_q \bar{z}_q}{\sqrt{2}\omega_q} \frac{1}{32\pi\gamma^2} \int\limits_0^\pi d\tau' \int d\Omega' \int d\Omega_w \left[ \frac{(1+z'\bar{z}')^2 (1+z_w \bar{z}_w)^2}{(z'-z_w)^2 (\bar{z}_q-\bar{z}_w)^3} \right] \mathcal{D}^{z'} j_{z'}^- e^{i\omega_q L(\tau'-\frac{\pi}{2})} \tag{3.17}
$$

Apart from the inclusion of an overall factor involving $1/\gamma^2$ in these corrected modes, we draw attention to the additional integral over intermediate angles $\{w, \bar{w}\}$ in eq. (3.16) that is absent in the flat spacetime result in eq. (2.20). The appearance of intermediate angular integrals will generically be a property to all higher powers in $1/L^2$, as these terms

involve higher order derivatives of the spherical harmonics. Such terms can be expressed in terms of derivatives acting on products of Green's functions on the 2-sphere, with additional angular integrals as in eq. (3.16). The recovery of the $1/L^2$ corrected soft photon theorem in eq. (3.4) from eq. (3.16) will be considered in the following subsection.

Lastly, the bulk field contribution in eq. (3.15) (apart from the $\mathcal{O}\left(\frac{1}{\omega^3 L^3}\right)$ terms ignored in our analysis) contain terms that are subleading in frequency. More specifically, they provide $1/\gamma^2$ corrected terms with higher order $\omega_q$ contributions to the leading $\omega_q^{-1}$ soft factor in eq. (3.16). Hence the total contribution from eq. (3.15) is subleading in frequency to the leading $1/L^2$ corrected soft photon theorem.

## 3.3 Recovering the classical soft photon theorem result

In this section we recover the soft photon factor obtained from classical soft theorem in [29]. The perturbed soft photon modes can be derived by taking the soft limit, namely $\omega_q \to 0$. Taking this limit in eq. (3.16) and eq. (3.17), we find the following soft operator mode expressions in terms of the CFT$_3$ current

$$\lim_{\omega_q \to 0} \omega_q \frac{\sqrt{2}}{1 + z_q \bar{z}_q} \hat{a}_{\vec{q}}^{\mathrm{out};\, \mathrm{L}\,(+)} = \frac{1}{4} \int d^3 x' \epsilon^{\mathrm{L}}(\hat{x}') D^{\bar{z}'} j_{\bar{z}'}^{-}(x')$$

$$\lim_{\omega_q \to 0} \omega_q \frac{\sqrt{2}}{1 + z_q \bar{z}_q} \hat{a}_{\vec{q}}^{\mathrm{out};\, \mathrm{L}\,(-)} = \frac{1}{4} \int d^3 x' \epsilon^{\mathrm{L}}(\hat{x}') D^{z'} j_{z'}^{-}(x') \tag{3.18}$$

with the gauge parameter for the positive and negative helicity cases now defined as

$$\epsilon^{\mathrm{L}}(\hat{x}') = \frac{1}{8\pi\gamma^2} \int d\Omega_w \left[ \frac{(1 + z'\bar{z}')^2 (1 + z_w \bar{z}_w)^2}{(\bar{z}' - \bar{z}_w)^2 (z_q - z_w)^3} \right] \qquad (\text{+ve helicity}) \,,$$

$$\epsilon^{\mathrm{L}}(\hat{x}') = \frac{1}{8\pi\gamma^2} \int d\Omega_w \left[ \frac{(1 + z'\bar{z}')^2 (1 + z_w \bar{z}_w)^2}{(z' - z_w)^2 (\bar{z}_q - \bar{z}_w)^3} \right] \qquad (\text{-ve helicity}) \,. \tag{3.19}$$

In the above expressions, the gauge parameter involves an integration over intermediate angles and hence does not provide the result derived from classical soft theorems in eq. (3.4). If we were to evaluate the contour integral with higher order poles located at $z_q$ and $\bar{z}'$, we would find that eq. (3.16) involves a delta function relating $z_q$ with $z'$, which violates our assumption of bulk modes being derived from a fixed current on the AdS$_4$ boundary. Thus we will need to proceed differently to extract a gauge parameter expression with no dependence on intermediate angular coordinates just as in the classical soft photon theorem.

Before addressing the above point in more detail, we provide the expression for the perturbed flat spacetime soft theorem from the CFT$_3$ Ward identity. We follow the treatment in section 2 with $\alpha(x')$ in eq. (2.24) now defined in terms of $\epsilon^{\mathrm{L}}(\hat{x}'')$. Noting that the map between correlation functions of primary operators and $S$-matrix elements in the $L \to \infty$ limit is not affected by our analysis, we find that the CFT$_3$ Ward identity provides the following $1/\gamma^2$ corrected soft photon theorem due to $1/L^2$ corrections to the soft photon mode

$$\lim_{\omega_q \to 0} \frac{\sqrt{2}\omega_q}{(1 + z_q \bar{z}_q)} \langle \mathrm{out}| \hat{a}_{\vec{q}}^{\mathrm{out};\, \mathrm{L}\,(+)}(\omega_q \hat{x}) \mathcal{S}|\mathrm{in}\rangle = \left[ \sum_{k=\mathrm{out}} \epsilon^{\mathrm{L}}(x') Q_k - \sum_{k=\mathrm{in}} \epsilon^{\mathrm{L}}(x') Q_k \right] \langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle \,, \tag{3.20}$$

where we have made use of the $CPT$ invariance of matrix elements in the in-state and out-state to arrive at the result in eq. (3.20).

In order to recover the classical soft theorem result, we note a difference between the large $L$ limit in our present analysis and those for the classical soft theorem that involved isotropic coordinates. The large $L$ limit leads to the $\tau$ and $\rho$ coordinates being scaled down to the locally flat spacetime patch with respective coordinates $t$ and $r$ following eq. (2.2). However, angular separations between points on the AdS$_4$ boundary and the flat spacetime patch are not necessarily small as would be the case in isotropic coordinates. We remedy this by considering the distance $|z_q - z'| \approx \tilde{\epsilon}$ as the smallest regulated length scale, with $\{z_w, \bar{z}_w\}$ separated from either $\{z_q, \bar{z}_q\}$ or $\{z', \bar{z}'\}$ with the expansion

$$z_w = z_q + \delta e^{i\theta}, \quad z_w = z' + \delta e^{i\theta}. \tag{3.21}$$

We will not particularly distinguish the modulus $\delta$ and phase $\theta$ in the two expansions, since $\{z_q, \bar{z}_q\}$ and $\{z', \bar{z}'\}$ are considered close to one another.

Before proceeding, we make a few comments on this approximation. On the one hand, we can consider it as a means of regulating the delta function answer that would result from integrating over $\{z_w, \bar{z}_w\}$ in eq. (3.19). Another point about the approximation is that it brings us on similar footing as the choice of isotropic coordinates used in the derivation of the classical soft theorem. Lastly, by considering an expansion with $|z_q - z'|$ taken to be the smallest distance, we would expect to find a leading contribution to the gauge parameter that agrees with the classical soft photon theorem result and a remainder considered as corrections. This is because the soft photon factor should have the right divergence behaviour in the soft and collinear limits [8]. The soft limit follows from $\omega_q \to 0$, while the collinear limit involves $z' - z_q \to 0$ and $\bar{z}' - \bar{z}_q \to 0$ (where $\{z', \bar{z}'\}$ and $\{z_q, \bar{z}_q\}$ denote the angular coordinates of the hard particles and soft photon respectively). By allowing $\{z', \bar{z}'\}$ and $\{z_q, \bar{z}_q\}$ to be separated by a cut-off, with the expansion of $z_w$ as given in eq. (3.21), we can recover the collinear divergence property that is absent in the gauge parameters in eq. (3.19).

In considering eq. (3.21) with $|z_q - z'|$ as the smallest distance, it follows that the integrand of the positive helicity gauge parameter in eq. (3.19) has the leading contribution

$$\frac{(1 + z'\bar{z}')^2 (1 + z_w \bar{z}_w)^2}{(\bar{z}' - \bar{z}_w)^2 (z_q - z_w)^3} = \frac{(1 + z_q \bar{z}_q)^2 (1 + z'\bar{z}')^2}{(\bar{z}_q - \bar{z}')^2 (z_q - z')^3} \left[1 + \mathcal{O}(\delta)\right]. \tag{3.22}$$

We can formally integrate eq. (3.22) over $\{z_w, \bar{z}_w\}$. Denoting the integration over the $\mathcal{O}(\delta)$ contributions as "corrections", we find the following result on substituting eq. (3.22) in eq. (3.19) for the positive helicity gauge parameter

$$\epsilon^{\mathrm{L}}(\hat{x}') = \frac{1}{2\gamma^2} \frac{(1 + z_q \bar{z}_q)^2 (1 + z'\bar{z}')^2}{(\bar{z}_q - \bar{z}')^2 (z_q - z')^3} + \text{corrections}. \tag{3.23}$$

We now see that the leading contribution to the gauge parameter in eq. (3.23) has the right collinear divergence property when $z' - z_q \to 0$ and $\bar{z}' - \bar{z}_q \to 0$. Hence the $1/L^2$ corrected

soft photon mode in eq. (3.20) takes the form

$$
\lim_{\omega_q \to 0} \frac{\sqrt{2}\omega_q}{(1 + z_q \bar{z}_q)} \langle \text{out}| \hat{a}_{\vec{q}}^{\text{out; L}(+)}(\omega_q \hat{x}) \mathcal{S} |\text{in}\rangle
$$

$$
= \frac{1}{2\gamma^2} \left[ \sum_{k=\text{out}} \frac{(1 + z'\bar{z}')^2 (1 + z_q \bar{z}_q)^2}{(\bar{z}_q - \bar{z}')^2 (z_q - z')^3} Q_k - \sum_{k=\text{in}} \frac{(1 + z'\bar{z}')^2 (1 + z_q \bar{z}_q)^2}{(\bar{z}_q - \bar{z}')^2 (z_q - z')^3} Q_k \right] \langle \text{out}| \mathcal{S} |\text{in}\rangle
$$

$$
+ \text{ corrections} \tag{3.24}
$$

We find that the leading contribution of eq. (3.24) agrees with eq. (3.4) on choosing $n_L = \frac{1}{8}$. We recall that while classical soft theorems recover the $1/L^2$ corrected soft photon factor, there remained an overall factor of $n_L$ in the normalization of the perturbed soft photon mode. The derivation from AdS/CFT provides a resolution of this ambiguity.

The nature of the corrections in eq. (3.24) in the context of classical soft theorems remain to be better understood. It is clear that the integration over intermediate angles can also be interpreted as a sum over certain particles parametrized by the angular coordinates $\{z_w, \bar{z}_w\}$. In this way, while these contributions are present in the AdS/CFT derivation of the $1/L^2$ corrected soft photon mode, they might correspond to excitations in the context of classical soft theorems.

## 4 Discussion

In this paper, we have studied the effect of a small negative cosmological constant on the soft photon factor for scattering in a flat spacetime region centrally located within global AdS. The main result of our paper is the derivation of $1/L^2$ corrected soft photon factor observed at the null infinity of the flat region from a large $N$ boundary CFT$_3$ Ward identity. This derivation for modes in the flat patch made use of bulk gauge fields reconstructed from a U(1) boundary current via the HKLL procedure. We further noted that this result from a CFT$_3$ Ward identity, in large AdS radius limit, recovers our previous result for soft photon factor derived from the classical soft photon theorem. Our results hence provide evidence for universal 'subleading in AdS radius' corrections to soft theorems satisfied by a $S$-matrix on asymptotically flat spacetimes within a larger AdS spacetime. We note that the status of a general scattering process to $1/L^2$ and higher orders is still an open problem. Our result only addresses infrared properties of the $S$-matrix that arise from the $L \to \infty$ limit of the background spacetime and double scaling limit of bulk fields.

One aspect of the corrected soft photon mode in eq. (3.16) which distinguishes it from the flat spacetime mode in eq. (2.20) is the dependence on intermediate angles. We believe this feature holds to higher orders of AdS radius contributions as well. Through our analysis, we have noted that the HKLL kernels to order $n$ generically appear to have terms with an order $2n$ polynomial of the angular momentum mode $l$. Such terms can be expressed in terms of derivatives on the spherical harmonics with the consequence of additional Green's functions integrated over intermediate angles. Hence the inclusion of intermediate angles at $1/L^2$ appears to be a property that holds to higher orders in the expansion. On asymptotically flat spacetimes, the Weinberg soft photon factor is the leading infrared divergence coming from real soft photons, which cancel out the infrared divergences

coming from photon loop contributions to provide an IR finite scattering process. The situation on AdS spacetimes is most likely different, as the AdS radius $L$ is known to be a natural infrared regulator [35, 42] providing an exponential decay for massless particles. It is thus tempting to conjecture that the resummation of $1/\omega^n L^n$ corrections to the soft factor (for $n > 2$) leads to emitted massless particles being infrared finite. This remains a topic to explore in the future.

We also noted in section 3.3 that the agreement of this result with the classical soft photon theorem results from expanding about a leading saddle independent of intermediate angles. One way to interpret the integration over intermediate angles $\{z_w, \bar{z}_w\}$ is that they correspond to additional particles whose momenta are parametrized in terms of these coordinates. We can thus conclude that the classical soft theorem is recovered in a limit that ignores the contributions from these additional particles. While our analysis derived the $1/L^2$ corrected soft factor resulting from inserting a soft photon to a $S$-matrix in the flat spacetime patch, it will be important to consider the factorization in $1/L^2$ corrected scattering amplitudes such as those recently derived in [43, 44]. Given the universality of the leading soft factor, including $1/L^2$ corrections, this should be derivable for these amplitudes as well.

## Acknowledgments

## A  Solution of Maxwell's equations in AdS$_4$

We will be interested in solutions of Maxwell's equations in the absence of sources

$$\nabla^\mu \mathcal{F}_{\mu\nu} = 0\,, \tag{A.1}$$

where $\mathcal{F}_{\mu\nu} = \nabla_\mu \mathcal{A}_\nu - \nabla_\nu \mathcal{A}_\mu$ with $\mathcal{A}_\mu$ the bulk gauge field, $\nabla^\mu$ the covariant derivative with respect to the background. We will follow the treatment by Wald and Ishibashi [40] in deriving the classical solutions. The general metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = h_{ab}dy^a dy^b + \tilde{g}_{ij}dz^i dz^j \tag{A.2}$$

can be expressed in the global form of eq. (2.1) by choosing

$$h_{ab} = \frac{L^2}{\cos^2 \rho}\eta_{ab}$$

$$\tilde{g}_{ij}dz^i dz^j = \frac{L^2}{\cos^2 \rho}\sin^2 \rho \frac{4}{(1+z\bar{z})^2}dzd\bar{z} = 2L^2 \tan^2 \rho\, \gamma_{z\bar{z}}dzd\bar{z} \tag{A.3}$$

We carry out a vector harmonic decomposition of the Maxwell field into the following independent components

$$\mathcal{A}_i^V \, dx^i = \sum_{l,m} \Psi^{lm}(\tau, \rho) \epsilon_{ij} \partial^j Y_{lm} dz^i \,, \tag{A.4}$$

$$\mathcal{A}_\mu^S dx^\mu = \sum_{l,m} \left[ A_a^{lm}(\tau, \rho) Y_{lm} dy^a + A^{lm}(\tau, \rho) \partial_i Y_{lm} dz^i \right] \,. \tag{A.5}$$

The superscripts $V$ and $S$ respectively refer to vector and scalar type components. The scaling dimensions of the two components differ: $\Delta = 2$ in the vector case and $\Delta = 1$ in the scalar case.

From Maxwell's equations eq. (A.1) we find that the vector type component in eq. (A.4) manifestly satisfies

$$\Box \Psi^{lm} - \frac{l(l+1)}{\sin^2 \rho} \Psi^{lm} = 0 \tag{A.6}$$

The scalar type component can also be shown to satisfy a similar equation by defining the field '$\phi^{lm}$' constructed from $A_a^{lm}$ and $A^{lm}$ in the following way

$$\partial_a A^{lm} - A_a^{lm} = \epsilon_{ab} \partial^b \phi^{lm}(\tau, \rho) \,. \tag{A.7}$$

On substituting the scalar type expression eq. (A.5) in eq. (A.1), we find

$$\Box \phi^{lm} - \frac{l(l+1)}{\sin^2 \rho} \phi^{lm} = 0 \tag{A.8}$$

The solutions we need are those that satisfy the HKLL asymptotic matching condition

$$j_\mu = \lim_{\rho \to \frac{\pi}{2}} \cos^{-1} \rho \mathcal{A}_\mu \tag{A.9}$$

Thus for purely radiative solutions derived in the absence of any $j_a$ current component (no Coulombic fields), the contribution from $A_a^{lm}$ in eq. (A.7) drops out of the scalar type solution. The resulting equation eq. (A.8) simplifies to

$$\Box A^{lm} - \frac{l(l+1)}{\sin^2 \rho} A^{lm} = 0 \tag{A.10}$$

which is the same as the vector type equation eq. (A.6). We will henceforth denote $A^{lm}$ in eq. (A.10) and $\Psi^{lm}$ in eq. (A.6) commonly by $\Phi^{lm}$, with the solutions distinguished by different values of $\Delta$. The solution of the radial equation eq. (A.6) and eq. (A.10) is

$$\Phi^{lm}(\tau, \rho) = e^{\pm i \omega_\kappa \tau} \Phi^{lm}(\rho) \tag{A.11}$$

$$\text{with} \quad \Phi^{lm}(\rho) \sim \sin^{l+1} \rho \cos^{\Delta-1} \rho \, _2F_1 \left( -\kappa, \kappa + \Delta + l, \Delta - \frac{1}{2} \Big| \cos^2 \rho \right) \tag{A.12}$$

$$\text{where} \quad \kappa = \frac{\omega_\kappa - \Delta - l}{2} \tag{A.13}$$

The $\sim$ in eq. (A.12) indicates an as yet unspecified overall normalization.

A feature of the $\{z, \bar{z}\}$ coordinates is that the derivative basis simplifies considerably

$$\partial_i Y_{lm} = \partial_z Y_{lm} \quad \text{(for } i = z); \qquad\qquad \partial_i Y_{lm} = \partial_{\bar{z}} Y_{lm} \quad \text{(for } i = \bar{z}) \qquad \text{(A.14)}$$

$$\epsilon_{ij}\partial^j Y_{lm} = \partial_z Y_{lm} \quad \text{(for } i = z); \qquad \epsilon_{ij}\partial^j Y_{lm} = -\partial_{\bar{z}} Y_{lm} \quad \text{(for } i = \bar{z}) \qquad \text{(A.15)}$$

Hence the classical solutions that enter our analysis are simply

$$\mathcal{A}_z^V(\tau, \rho, \Omega) \sim \sum_{l,m} \Phi^{lm}(\tau, \rho)\Big|_{\Delta=2} \partial_z Y_{lm}(\Omega); \qquad \mathcal{A}_{\bar{z}}^V(\tau, \rho, \Omega) \sim \sum_{l,m} -\Phi^{lm}(\tau, \rho)\Big|_{\Delta=2} \partial_{\bar{z}} Y_{lm}(\Omega)$$

$$\mathcal{A}_z^S(\tau, \rho, \Omega) \sim \sum_{l,m} \Phi^{lm}(\tau, \rho)\Big|_{\Delta=1} \partial_z Y_{lm}(\Omega); \qquad \mathcal{A}_{\bar{z}}^S(\tau, \rho, \Omega) \sim \sum_{l,m} \Phi^{lm}(\tau, \rho)\Big|_{\Delta=1} \partial_{\bar{z}} Y_{lm}(\Omega)$$

These solutions, along with the general time dependence $e^{i\omega_\kappa(\tau-\tau')}$, define the function $\Xi_{\kappa l}(\rho, \tau, \tau')$ in the vector and scalar type kernels of eqs. (2.12)–(2.15).

# B $\quad 1/L^2$ corrections of the gauge field HKLL kernels

We will now describe the derivation of the $1/L^2$ corrected expressions for $\Xi_{\kappa l}(\rho, \tau, \tau')$ in eq. (3.5) and eq. (3.6), from the general expression given in eq. (2.16), which we repeat here for convenience

$$\Xi_{kl}(\rho, \tau, \tau') = e^{i\omega_\kappa(\tau-\tau')} \sin^{l+1}\rho \cos^{\Delta-1}\rho \, {}_2F_1\left(-\kappa, \kappa+\Delta+l, \Delta-\frac{1}{2}\Big| \cos^2\rho\right). \quad \text{(B.1)}$$

We will describe three intermediate steps leading to a form of $\Xi_{\kappa l}$ that we consider. The first is the transformation of $\cos^2\rho$ to $\sin^2\rho$ in the hypergeometric function argument by a linear transformation (cf. 2.4 of [45]).[7] On transforming the hypergeometric function, we find a coefficient with products of Gamma functions, some of which involve a negative argument. These can be transformed to a positive argument, and we specifically consider

$$\frac{\Gamma(-l-\frac{1}{2})}{\Gamma(-\kappa-l-\frac{1}{2})} = (-1)^{-\kappa}\frac{\Gamma(\kappa+l+\frac{3}{2})}{\Gamma(l+\frac{3}{2})}, \quad \text{(B.2)}$$

which is derived from the Euler reflection identity $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}$ (for non-integer $x$). Lastly, we replace $\omega_\kappa = \omega L$ and $\kappa = \frac{1}{2}(\omega L - \Delta - l)$, with $\kappa$ considered large.

The resulting expression for $\Xi_{\kappa l}$ is

$$\Xi_{\kappa l}(\rho, \tau, \tau') = (\pm i)^{-\Delta-l} e^{\pm i\omega L\frac{\pi}{2}} e^{i\omega L(\tau-\tau')} \times A \times B \times C$$

$$\text{where} \quad A = \tan^{l+1}\rho \cos^{\Delta+l}\rho$$

$$B = \frac{\Gamma\left(\Delta-\frac{1}{2}\right)\Gamma\left(\frac{\omega L+l-\Delta+3}{2}\right)}{\Gamma(l+\frac{3}{2})\Gamma\left(\frac{\omega L-l+\Delta-1}{2}\right)}$$

$$C = {}_2F_1\left(\frac{\Delta+l-\omega L}{2}, \frac{\Delta+l+\omega L}{2}, l+\frac{3}{2}\Big| \sin^2\rho\right) \quad \text{(B.3)}$$

The $(\pm i)^{-\Delta-l} e^{\pm i\omega L\frac{\pi}{2}}$ comes from the $(-1)^{-\kappa}$ in eq. (B.2). The $+ (-)$ sign will represent positive frequency outgoing (incoming) states in the kernels.[8]

---

[7]We consider this transformation since $1/L^2$ corrections of flat spacetime still involve Bessel functions (and their derivatives) with the argument $r\omega$. It is simpler to recover these Bessel functions from a $\sin^2\rho$ argument in the hypergeometric function.

[8]The converse convention holds for negative frequency states and follows from complex conjugation.

The $1/L^2$ corrections to the flat spacetime result from the HKLL kernels will result from expanding the above terms after substituting eq. (2.2). As $\tau$ involves a trivial rescaling, we find

$$e^{i\omega L(\tau-\tau')} = e^{i\omega t}e^{-i\omega L\tau'} ,$$

just as in the flat spacetime limit, which holds to all orders in $1/L^2$. The non-trivial expansions in $1/L^2$ come from the terms $A$, $B$ and $C$ noted above on replacing $\rho = \arctan(\frac{r\omega}{\omega L})$. Performing a Taylor expansion on $A$ gives the following result

$$A = \tan^{l+1}\rho \cos^{\Delta+l}\rho = \left(\frac{r\omega}{\omega L}\right)^{l+1}\left[1 - \frac{(l+\Delta)(r\omega)^2}{2\omega^2 L^2} + \mathcal{O}\left(\frac{1}{\omega^3 L^3}\right)\right] \qquad \text{(B.4)}$$

In the case of $B$, and specifically for the factor $\frac{\Gamma\left(\frac{\omega L+l-\Delta+3}{2}\right)}{\Gamma\left(\frac{\omega L-l+\Delta-1}{2}\right)}$ appearing within it, we can make use of the following identity for Gamma functions with a large argument $x$ (cf. 5.11.13 of [46])

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} = x^{a-b}\left(1 + \frac{1}{2}\frac{(a-b)(a+b-1)}{x}\right.$$
$$\left. + \frac{1}{12}\frac{(a-b)(a-b-1)(3(a+b-1)^2-(a-b+1))}{2x^2} + \mathcal{O}(x^{-3})\right) \quad \text{(B.5)}$$

As $\omega L$ is large, we define $x = \frac{\omega L}{2}$, $a = \frac{l+3-\Delta}{2}$ and $b = \frac{\Delta-l-1}{2}$ to find

$$\frac{\Gamma(\frac{\omega L+l-\Delta+3}{2})}{\Gamma(\frac{\omega L-l+\Delta-1}{2})} = \left(\frac{\omega L}{2}\right)^{l+2-\Delta}\left(1 - \frac{(l+1-\Delta)(l+2-\Delta)(l+3-\Delta)}{6\omega^2 L^2} + \mathcal{O}\left(\frac{1}{\omega^3 L^3}\right)\right)$$
$$\text{(B.6)}$$

Thus our expansion for the $B$ term is

$$B = \frac{\Gamma(\Delta-\frac{1}{2})}{\Gamma(l+\frac{3}{2})}\left(\frac{\omega L}{2}\right)^{l+2-\Delta}\left(1 - \frac{(l+1-\Delta)(l+2-\Delta)(l+3-\Delta)}{6\omega^2 L^2} + \mathcal{O}\left(\frac{1}{\omega^3 L^3}\right)\right)$$
$$\text{(B.7)}$$

For the $C$ term, we make use of the following expansion of the hypergeometric function in terms of Bessel functions [47]

$$\begin{aligned}
{}_2F_1\left(\lambda,\mu;\nu+1\middle|-\frac{y^2}{4\lambda\mu}\right) = \Gamma(\nu+1)\left(\frac{y}{2}\right)^{-\nu}&\left[J_\nu(y)+\frac{y^2}{8}J_{\nu+2}(y)\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\right.\\
&+\left[\frac{y^4}{128}J_{\nu+4}(y)-\frac{y^3}{24}J_{\nu+3}(y)\right]\left(\frac{1}{\lambda^2}+\frac{1}{\mu^2}\right)\\
&+\left[\frac{y^4}{64}J_{\nu+4}(y)-\frac{y^3}{8}J_{\nu+3}(y)+\frac{y^2}{8}J_{\nu+2}(y)\right]\frac{1}{\lambda\mu}\\
&\left.+\mathcal{O}(\lambda^{-3},\mu^{-3},\lambda^{-1}\mu^{-2},\cdots)\right] \qquad \text{(B.8)}
\end{aligned}$$

The leading contribution in eq. (B.8) is the relationship between the hypergeometric and Bessel functions derived by Watson [48]. The expansion in eq. (B.8) was determined through

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Watson's approach carried out to subleading order [47] and will be needed to determine the $1/L^2$ corrections of the HKLL kernels.

On comparing the expression for $C$ in eq. (B.3) with eq. (B.8), we find

$$\lambda = \frac{\Delta + l - \omega L}{2}, \qquad \mu = \frac{\Delta + l + \omega L}{2} \tag{B.9}$$

We can likewise determine $y^2$ in eq. (B.8) from $\sin^2 \rho$ in $C$. Since

$$-4\lambda\mu = \omega^2 L^2 - (\Delta + l)^2, \tag{B.10}$$

we can appropriately replace $\omega^2 L^2$ with $-4\lambda\mu$ in the expansion for $\sin^2 \rho$ to find

$$
\begin{aligned}
\sin^2 \rho &= \left(\frac{r\omega}{\omega L}\right)^2 \left(1 - \frac{r^2\omega^2}{\omega^2 L^2} + \mathcal{O}\left(\frac{1}{\omega^3 L^3}\right)\right) \\
&= \frac{r^2\omega^2}{-4\lambda\mu}\left(1 - \frac{(\Delta + l)^2 + r^2\omega^2}{\omega^2 L^2}\right) + \mathcal{O}\left(\frac{1}{\omega^3 L^3}\right) \\
&:= \frac{y^2}{-4\lambda\mu} + \mathcal{O}\left(\frac{1}{\omega^3 L^3}\right),
\end{aligned}
\tag{B.11}
$$

where in the last line of eq. (B.11) we defined

$$y = r\omega\left(1 - \frac{(\Delta + l)^2 + r^2\omega^2}{2\omega^2 L^2}\right) \tag{B.12}$$

We can hence derive the right hand side of eq. (B.8) from the given expression of $C$ in eq. (B.3). Each Bessel function appearing in the expression can be written in terms of the flat spacetime argument $r\omega$ by making use of

$$J_\nu(x + \delta x) = J_\nu(x) - \delta x J_{\nu+1}(y) + \frac{\nu}{x} J_\nu(x), \tag{B.13}$$

which can be derived from recursion relations for the Bessel functions

$$J_{\nu+1}(x) = -J'_\nu(x) + \frac{\nu}{x} J_\nu(x), \tag{B.14}$$

$$J'_{\nu+1}(x) = J_\nu(x) - \frac{\nu+1}{x} J_{\nu+1}(x). \tag{B.15}$$

In particular, on using eq. (B.12) we find that eq. (B.13) gives

$$J_\nu(y) = J_\nu(r\omega) + \frac{(\Delta + l)^2 + r^2\omega^2}{2\omega^2 L^2}\left(r\omega J_{\nu+1}(r\omega) - \nu J_\nu(r\omega)\right) \tag{B.16}$$

On replacing eq. (B.9) and eq. (B.11) in eq. (B.8), and expressing all the Bessel function arguments in terms of $r\omega$, we find the following expression for $C$

$$
\begin{aligned}
C = \Gamma\left(l + \frac{3}{2}\right)\left(\frac{2}{r\omega}\right)^{l+\frac{1}{2}}\Bigg[ & J_{l+\frac{1}{2}}(r\omega) + \frac{r\omega\left((\Delta + l)^2 + r^2\omega^2\right)}{2\omega^2 L^2} J_{l+\frac{3}{2}}(r\omega) \\
& - \frac{r^2\omega^2(l + \Delta + 1)}{2\omega^2 L^2} J_{l+\frac{5}{2}}(r\omega) + \frac{r^3\omega^3}{6\omega^2 L^2} J_{l+\frac{7}{2}}(r\omega)\Bigg]
\end{aligned}
\tag{B.17}
$$

We can now multiply eq. (B.4), eq. (B.7) and eq. (B.17) to get the expression for $\Xi_{\kappa l}(r, t; \tau')$ from eq. (B.3). Further simplifications can be performed — the first involves the use of recursion relations for Bessel functions in eq. (B.14) and eq. (B.15), which enable finding an expression involving only $J_{l+\frac{1}{2}}(r\omega)$ and its first derivative. Each $J_{l+\frac{1}{2}}(r\omega)$ can then be written in terms of Spherical Bessel functions $j_l(r\omega)$

$$j_l(r\omega) = \sqrt{\frac{\pi}{2r\omega}} J_{l+\frac{1}{2}}(r\omega) \,. \tag{B.18}$$

The other simplification that occurs is for a common expression in the two cases $\Delta = 1$ and $\Delta = 2$. We specifically have

$$\Gamma\left(\Delta - \frac{1}{2}\right) 2^{\Delta-1} \bigg|_{\Delta=2} = \sqrt{\pi} = \Gamma\left(\Delta - \frac{1}{2}\right) 2^{\Delta-1} \bigg|_{\Delta=1} \tag{B.19}$$

Following the use of Bessel function recursion relations and the substitutions mentioned above, we then find

$$\Xi_{\kappa l}(\rho, \tau, \tau') \Big|_{\Delta=2} = -(\pm i)^{-l} e^{i\omega t} e^{-i\omega L\left(\tau' \mp \frac{\pi}{2}\right)} [A \times B \times C]_{\Delta=2} \,, \tag{B.20}$$

$$\Xi_{\kappa l}(\rho, \tau, \tau') \Big|_{\Delta=1} = -(\pm i)(\pm i)^{-l} e^{i\omega t} e^{-i\omega L\left(\tau' \mp \frac{\pi}{2}\right)} [A \times B \times C]_{\Delta=1} \,, \tag{B.21}$$

with

$$[A \times B \times C]_{\Delta=2} = \frac{r}{L}\left[j_l(r\omega) + \frac{1}{2\omega^2 L^2}\left(\left(\frac{l(l+1)}{2} - \frac{(r\omega)^2}{3}\right) j_l(r\omega)\right.\right.$$
$$\left.\left. - \frac{2r\omega}{3}\sqrt{\frac{\pi}{2r\omega}}\left(\frac{l(l+1)}{2} + (r\omega)^2\right) J'_{l+\frac{1}{2}}(r\omega)\right)\right] + \mathcal{O}\left(\frac{1}{\omega^3 L^3}\right) \tag{B.22}$$

$$[A \times B \times C]_{\Delta=1} = \frac{r}{L}(\omega L)\left[j_l(r\omega) - \frac{1}{2\omega^2 L^2}\left(\left(\frac{l(l+1)}{2} + \frac{(r\omega)^2}{3}\right) j_l(r\omega)\right.\right.$$
$$\left.\left. - \frac{2r\omega}{3}\sqrt{\frac{\pi}{2r\omega}}\left(\frac{l(l+1)}{2} + (r\omega)^2\right) J'_{l+\frac{1}{2}}(r\omega)\right)\right] + \mathcal{O}\left(\frac{1}{\omega^3 L^3}\right) \tag{B.23}$$

The expressions in eq. (B.20) and eq. (B.21) are those in eq. (3.5) and eq. (3.6) respectively.

## C  Derivation of $\hat{a}_{\vec{q}}^{\text{out; L}(+)}$ and $\hat{a}_{\vec{q}}^{\text{out}(+)}$

The outgoing positive helicity photon modes result from substituting the outgoing bulk expression for $\hat{\mathcal{A}}_z^{\text{out}}(y)$ from eq. (3.12) in eq. (2.6). We define the flat spacetime mode $\hat{a}_{\vec{q}}^{\text{out}(+)}$ as that corresponding to the bulk field $\hat{\mathcal{A}}_z^{\text{out; f}}(y)$ and the $1/L^2$ corrected mode $\hat{a}_{\vec{q}}^{\text{out; L}(+)}$ as that resulting from the bulk field $\hat{\mathcal{A}}_z^{\text{out; L}}(y)$ in the following way

$$\hat{a}_{\vec{q}}^{\text{out}(+)} = \lim_{t\to\infty} i \int d^3\vec{y} (\varepsilon^{(+)\mu})^* e^{-iq\cdot y} \overleftrightarrow{\partial_0} \hat{\mathcal{A}}_z^{\text{out; f}}(y) \,, \tag{C.1}$$

$$\hat{a}_{\vec{q}}^{\text{out; L}(+)} = \lim_{t\to\infty} i \int d^3\vec{y} (\varepsilon^{(+)\mu})^* e^{-iq\cdot y} \overleftrightarrow{\partial_0} \hat{\mathcal{A}}_z^{\text{out; L}}(y) \,. \tag{C.2}$$

In both cases, we use the expressions for the polarization and plane waves given in eq. (2.9). On substituting $\hat{\mathcal{A}}_z^{\text{out; f}}(y)$ from eq. (3.13) and $\hat{\mathcal{A}}_z^{\text{out; L}}(y)$ from eq. (3.14), we then find that the expressions in eq. (C.1) and eq. (C.2) take the form

$$\hat{a}_{\vec{q}}^{\text{out}\,(+)} =$$

$$\lim_{t\to\infty} i \int r^2 dr \int d\Omega \int_0^\pi d\tau' \int d\Omega' \int d\omega \frac{1+z\bar{z}}{\sqrt{2}} \sum_{l,m,l',m'} j_{l'}(r\omega_{\vec{q}}) j_l(r\omega)$$

$$\left[ (i(\omega-\omega_q)) \frac{Y_{lm}^*(\Omega')}{-l(l+1)} Y_{l'm'}(\Omega) Y_{l'm'}^*(\Omega_q) \partial_z Y_{lm}(\Omega) (-i)^{l'}(i)^{-l} e^{i(\omega+\omega_{\vec{q}})t} e^{-i\omega L\left(\tau'-\frac{\pi}{2}\right)} D^{\bar{z}'} j_{\bar{z}'}^+ \right.$$

$$\left. + (-i(\omega+\omega_{\vec{q}})) \frac{Y_{lm}(\Omega')}{-l(l+1)} Y_{l'm'}(\Omega) Y_{l'm'}^*(\Omega_q) \partial_z Y_{lm}^*(\Omega) (-i)^{-l+l'} e^{-i(\omega-\omega_{\vec{q}})t} e^{i\omega L\left(\tau'-\frac{\pi}{2}\right)} D^{\bar{z}'} j_{\bar{z}'}^- \right]$$

$$\tag{C.3}$$

$$\hat{a}_{\vec{q}}^{\text{out; L}\,(+)} =$$

$$\lim_{t\to\infty} i \int r^2 dr \int d\Omega \int_0^\pi d\tau' \int d\Omega' \int d\omega \frac{1+z\bar{z}}{\sqrt{2}} \sum_{l,m,l',m'} j_{l'}(r\omega_{\vec{q}}) j_l(r\omega) \left( \frac{l(l+1)}{4\omega^2 L^2} \right)$$

$$\left[ (i(\omega-\omega_q)) \frac{Y_{lm}^*(\Omega')}{-l(l+1)} Y_{l'm'}(\Omega) Y_{l'm'}^*(\Omega_q) \partial_{\bar{z}} Y_{lm}(\Omega) (-i)^{l'}(i)^{-l} e^{i(\omega+\omega_{\vec{q}})t} e^{-i\omega L\left(\tau'-\frac{\pi}{2}\right)} D^{\bar{z}'} j_{\bar{z}'}^+ \right.$$

$$\left. + (-i(\omega+\omega_{\vec{q}})) \frac{Y_{lm}(\Omega')}{-l(l+1)} Y_{l'm'}(\Omega) Y_{l'm'}^*(\Omega_q) \partial_{\bar{z}} Y_{lm}^*(\Omega) (-i)^{-l+l'} e^{-i(\omega-\omega_{\vec{q}})t} e^{i\omega L\left(\tau'-\frac{\pi}{2}\right)} D^{\bar{z}'} j_{\bar{z}'}^- \right]$$

$$\tag{C.4}$$

The above expressions contain derivatives of spherical harmonics and in this regard, it is useful to introduce the Green's function $G(z,\bar{z};w,\bar{w})$ on the 2-sphere

$$G(z,\bar{z};w,\bar{w}) = \frac{1}{4\pi} \ln\left((z-w)(\bar{z}-\bar{w})\right) - \frac{1}{4\pi} \ln\left(1+z\bar{z}\right) - \frac{1}{4\pi} \ln\left(1+w\bar{w}\right). \tag{C.5}$$

From the identity

$$\partial_z \frac{1}{\bar{z}-\bar{w}} = \frac{1}{2} \delta^{(2)}(z-w) = \partial_{\bar{z}} \frac{1}{z-w}$$

we find that $G(z,\bar{z};w,\bar{w})$ satisfies the following relations

$$\partial_z \partial_{\bar{z}} G(z,\bar{z};w,\bar{w}) = \frac{1}{2} \delta^{(2)}(z-w) - \frac{\gamma_{z\bar{z}}}{8\pi}, \qquad \partial_w \partial_{\bar{w}} G(z,\bar{z};w,\bar{w}) = \frac{1}{2} \delta^{(2)}(z-w) - \frac{\gamma_{w\bar{w}}}{8\pi} \tag{C.6}$$

$$\partial_z \partial_{\bar{w}} G(z,\bar{z};w,\bar{w}) = -\frac{1}{2} \delta^{(2)}(z-w) = \partial_w \partial_{\bar{z}} G(z,\bar{z};w,\bar{w}) \tag{C.7}$$

Eq. (C.7) in particular implies the useful property

$$\partial_{z_p} \frac{Y_{lm}(\Omega_{z_p})}{-l(l+1)} = -2 \int d\Omega_w \gamma^{w\bar{w}} \partial_{\bar{w}} \partial_{z_p} G(z_p,\bar{z}_p;w,\bar{w}) \partial_w \frac{Y_{lm}(\Omega_w)}{-l(l+1)}$$

$$= \int d\Omega_w \partial_{z_p} G(z_p,\bar{z}_p;w,\bar{w}) Y_{lm}(\Omega_w), \tag{C.8}$$

where we made use of $2\gamma^{w\bar{w}} \partial_w \partial_{\bar{w}} Y_{lm}(\Omega_w) = -l(l+1) Y_{lm}(\Omega_w)$ in the second equality.

We can now use the orthogonality relations satisfied by the spherical harmonics

$$\int d\Omega Y_{lm}\left(\Omega\right)Y_{l'm'}^*\left(\Omega\right)=\delta_{ll'}\delta_{mm'} \tag{C.9}$$

$$\sum_{l,m}Y_{lm}\left(\Omega_q\right)Y_{lm}^*\left(\Omega\right)=\delta\left(\Omega_q-\Omega\right), \tag{C.10}$$

and the spherical Bessel functions

$$\int_0^\infty r^2 dr j_l(r\omega)j_l(r\omega_{\vec q})=\frac{\pi}{2\omega_{\vec q}^2}\delta(\omega-\omega_{\vec q}), \tag{C.11}$$

to simplify the integrals appearing in eq. (C.3) and eq. (C.4). In the case of eq. (C.3) we find the result

$$\hat a_{\vec q}^{\text{out}\,(+)}=\pi\frac{1+z_q\bar z_q}{\sqrt2\omega_q}\int_0^\pi d\tau'\int d\Omega'\partial_{z_q}G(z_q,z')\mathcal D^{\bar z'}j_{\bar z'}^-e^{i\omega_q L\left(\tau'-\frac{\pi}{2}\right)}$$

$$=\frac14\frac{1+z_q\bar z_q}{\sqrt2\omega_q}\int_0^\pi d\tau'\int d\Omega'\frac{1}{z_q-z'}\mathcal D^{\bar z'}j_{\bar z'}^-e^{i\omega_q L\left(\tau'-\frac{\pi}{2}\right)}, \tag{C.12}$$

which is the flat spacetime mode expression eq. (2.20) that was derived in [38].

We find that the $1/L^2$ corrected mode in eq. (C.4) simplifies to

$$\hat a_{\vec q}^{\text{out};\,L(+)}=\frac{\pi}{\omega_q^2L^2}\frac{1+z_q\bar z_q}{\sqrt2\omega_{\vec q}}\int_0^\pi d\tau'\int d\Omega'\int d\Omega_w$$

$$\partial_{z_q}\left[\left(\gamma^{z'\bar z'}\partial_{\bar z'}\partial_w G(z',\bar z';w,\bar w)\right)\left(\gamma^{w\bar w}\partial_{\bar w}\partial_{z_q}G(w,\bar w;z_q,\bar z_q)\right)\right]D^{\bar z'}j_{\bar z'}^-e^{i\omega_q L\left(\tau'-\frac{\pi}{2}\right)}$$

$$=\frac{1}{32\pi\omega_q^2L^2}\frac{1+z_q\bar z_q}{\sqrt2\omega_{\vec q}}\int_0^\pi d\tau'\int d\Omega'\int d\Omega_w\left[\frac{(1+z'\bar z')^2(1+z_w\bar z_w)^2}{(\bar z'-\bar z_w)^2(z_q-z_w)^3}\right]\mathcal D^{\bar z'}j_{\bar z'}^-e^{i\omega_q L\left(\tau'-\frac{\pi}{2}\right)} \tag{C.13}$$

which is the expression in eq. (3.16). One key difference between the flat spacetime mode in eq. (C.12) and the $1/L^2$ corrected mode in eq. (C.13) is the presence of a product of Green's function involving intermediate angles that are integrated over. This leads to the final result in the second line of eq. (C.13)

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