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Physics Letters B

PHYSICS LETTERS B

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Bosonization of fermions coupled to topologically massive gravity

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A R T I C L E   I N F O

A B S T R A C T

We establish a duality between massive fermions coupled to topologically massive gravity (TMG) in $d = 3$ space–time dimensions and a purely gravity theory which also will turn out to be a TMG theory but with different parameters: the original graviton mass in the TMG theory coupled to fermions picks up a contribution from fermion bosonization. We obtain explicit bosonization rules for the fermionic currents and for the energy–momentum tensor showing that the identifications do not depend explicitly on the parameters of the theory. These results are the gravitational analog of the results for $2 + 1$ Abelian and non-Abelian bosonization in flat space–time.

Bosonization of fermion models is strongly connected to the existence of quantum anomalies associated with symmetries that exist at the classical level. Indeed, the well-honored fermions–boson duality in $d = 2$ space–time dimensions [1] can be reproduced within the path-integral approach once the chiral anomaly affecting the fermionic measure [2] is taken into account [3,4]. In the case of $d = 3$ fermions coupled to Abelian and non-Abelian gauge fields it is the parity anomaly that plays a central role unveiling the occurrence of a Chern–Simons term [5,6] in the bosonic dual [7–9].

Interestingly enough, when fermions are coupled to a gravitational field in odd-dimensional spaces the parity anomaly also induces a Chern–Simons term which can be written in terms of the spin connection or in terms of the Christoffel connection [10–12]. Recently this fact has been exploited to simulate the effects of crystal defects on the electronic degrees of freedom of topological insulators in condensed matter physics. Those systems exhibit physical phenomena related to parity breaking that take place in terms of Dirac fermion models in $2 + 1$ space–time dimensions in the presence of gravitational sources [13,14].

In view of the discussion above, it is natural to consider the possibility of finding a bosonization recipe connecting fermions with gravitons analogous to those arising for fermionic models coupled to gauge fields in flat space. However, contrary to the standard Fermi–Bose duality, in this case the Dirac fermions will be coupled not to background gravitational sources but to a *dy-namical* gravity theory, topologically massive gravity (TMG) theory. Dualities involving gravity theories and fermion models have recently been discussed within the gauge/gravity correspondence but in that case the gravity theory (a Vasiliev high-spin theory [15]) is defined in an AdS$_4$ bulk while the fermion dual − consisting of k fermions coupled to a $U(k)_N$ Chern–Simons term − lives in the three-dimensional boundary of AdS$_4$ [16]. In fact, since also a bosonic theory coupled to a $U(N)_k$ Chern–Simons gauge theory defined on the same boundary can be seen to be dual to Vasiliev theory in the AdS$_4$ bulk, it has been conjectured that these bosonic and fermionic theories are dual to each other in the large N limit.

The aim of this work is to establish a duality between fermions coupled to topologically massive gravity (TMG) in $d = 3$ space–time dimensions and a purely gravity theory, which also will turn out to be a TMG theory but with different parameters.

1. The action

Our model will be defined on a three-dimensional space–time manifold M^3 with local coordinates x^μ ($\mu = 0, 1, 2$) and a metric of Minkowski signature. Vierbeins (or, rather, dreibeins in $2 + 1$ dimensions) are denoted as e^a_μ where "a" is the frame index ($a = 0, 1, 2$). The spin connection ω_μ,

$$\omega_\mu = -\frac{1}{4}\omega_{ab\mu}\gamma^a\gamma^b = -\frac{i}{4}\omega_{ab\mu}\epsilon^{abc}\gamma^c \tag{1}$$

is an $SO(2, 1)$-valued 1-form on the manifold M^3 satisfying the standard relation with the dreibeins

$$\omega_{ab\mu} = e_{a\nu}e^\nu_{b;\mu} \tag{2}$$

where the semicolon refers to covariant differentiation using the Christoffel symbol.

We shall consider massive fermions coupled to topologically massive gravity (TMG) [5] with dynamics governed by the action S

$$
S[\bar{\psi}, \psi, e; s]
$$
$$
= \int d^3x \, (\det e)\bar{\psi}\left(i\gamma^\mu(\partial_\mu + \omega_\mu + s_\mu) + m\right)\psi + S_{TMG} \tag{3}
$$

where the topological massive gravity action S_{TMG} reads

$$
S_{TMG} = \frac{1}{64\pi\kappa^2 m} S_{CS}[\omega[e]] + \frac{1}{\kappa^2} S_{EH}[e, \omega[e]] \tag{4}
$$

Here $\det e$ is the determinant of the dreibein fields, and s_μ is an external source

$$
s_\mu = -\frac{1}{4} s_{ab\mu} \gamma^a \gamma^b = -\frac{i}{4} s_{ab\mu} \epsilon^{abc} \gamma^c \tag{5}
$$

which transforms covariantly under local Lorentz transformations so that $\omega + s$ transforms as a connection.

The Chern–Simons term $S_{CS}[\omega]$ reads

$$
S_{CS}[\omega] = \int d^3x \, \epsilon^{\mu\nu\alpha}\left(\omega_{\mu ab}\partial_\nu \omega_{\alpha ba} + \frac{2}{3}\omega_{\mu ab}\omega_{\nu bc}\omega_{\alpha ca}\right) \tag{6}
$$

and the Einstein–Hilbert action $S_{EH}[e, \omega[e]]$ is written as

$$
S_{EH}[e, \omega[e]] = \int d^3x \, \epsilon^{\mu\nu\alpha} e_{a\mu}\left(\partial_\nu \omega_\alpha^a - \partial_\alpha \omega_\nu^a + \epsilon^{abc}\omega_\nu^b \omega_\alpha^c\right) \tag{7}
$$

where

$$
\omega_\alpha^a = \frac{1}{2}\epsilon^{abc}\omega_{\alpha bc} \tag{8}
$$

Note that in $d = 3$ space–time dimensions the Newton constant κ^{-2} has dimensions of mass, $[\kappa^2] = m^{-1}$, in fundamental units in which the spin connection has dimension one, $[\omega_\mu] = 1$. The mass parameter m in S_{CS} is required by dimensional consistency and can be replaced by a dimensionless effective coupling constant $\kappa^2 m$ which, alternatively, can be regarded as the inverse of the level k of the Chern–Simons theory, $\kappa^2 m = 1/k$. In topologically massive gravity there is a single propagating mode with mass m and spin 2 [5].

The generating functional $Z[s]$ of the source fields s_μ is defined by

$$
Z[s] = \int \mathcal{D}\mu_\psi \mathcal{D}\mu_G \exp\left(iS[\bar{\psi}, \psi, e; s]\right) \tag{9}
$$

where $\mathcal{D}\mu_\psi$ is the appropriate fermionic path-integral measure defined in terms of Fujikawa variables [2]:

$$
\mathcal{D}\mu_\psi = \mathcal{D}\left((\det e)^{1/2}\bar{\psi}\right)\mathcal{D}\left((\det e)^{1/2}\psi\right) \tag{10}
$$

Concerning the measure of the gravitational sector $\mathcal{D}\mu_G$, one could also follow Fujikawa prescription introducing Faddeev–Popov ghosts fields and auxiliary fields arising in the anti-ghost superfield introduced in the BRST supersymmetry approach associated with general coordinate transformations [17]. In this respect one should note that in Ref. [18] it has been shown that TMG has no unitarity and ghost problems and is power-counting renormalizable. Moreover, arguments in [19] based on the existence of a functional integration measure make appear TMG to be finite as a quantum theory.

2. Bosonization

As it is well-known the $d = 3$ Dirac operator determinant arising from the fermionic path-integral in Eq. (9) can be written as the product of two factors according to its behavior under parity transformation [10,12]. The odd-parity contribution can be computed exactly leading to a Chern–Simons effective action

$$
\det\left(i\gamma^\mu(\partial_\mu + \omega_\mu + s_\mu) + m\right)_{odd} = \exp\left(\pm\frac{i}{64\pi} S_{CS}[\omega + s]\right) \tag{11}
$$

The parity-even contribution can be computed within a ∂/m approximation. Being the lowest parity-even term subleading to the parity-odd term, it will be disregarded in what follows.

Using this result we can write Eq. (9) as a purely bosonic generating functional of the form

$$
Z[s] = \int \mathcal{D}\mu_G \exp\left(\pm\frac{i}{64\pi} S_{CS}[\omega + s]\right)
$$
$$
\times \exp\left(\frac{i}{64\pi\kappa^2 m} S_{CS}[\omega] + \frac{i}{\kappa^2} S_{EH}[\omega, e]\right) \tag{12}
$$

One can calculate from the generating functional of Eq. (12) the v.e.v. of the current $J^{\mu ab}$

$$
\langle J^{\mu ab}\rangle = -\frac{1}{Z}\frac{\delta Z[S]}{\delta s_{\mu ab}}\bigg|_{s=0} \tag{13}
$$

$$
J^{\mu ab} = \frac{1}{4}\bar{\psi}\gamma^\mu[\gamma^a, \gamma^b]\psi = \frac{1}{4}e_c^\mu \bar{\psi}\gamma^c[\gamma^a, \gamma^b]\psi \tag{14}
$$

Using Eq. (12) the fermionic current v.e.v. takes form

$$
\langle J^{\mu ab}\rangle
$$
$$
= \frac{\mp 1}{64\pi}\frac{1}{Z}\int \mathcal{D}\mu_G \exp\left(\frac{i}{64\pi\kappa^2 m} S_{CS}[\omega] + \frac{i}{\kappa^2} S_{EH}[\omega[e], e]\right)
$$
$$
\times \left(\frac{\delta S_{CS}[\omega + s]}{\delta s_{\mu ab}}\right)\bigg|_{s=0}
$$
$$
= \mp\frac{1}{256\pi}\epsilon^{\mu\nu\beta}\langle\left(R_{\nu\beta ab}[\tilde{\omega}]\right)^{ba}\rangle_{bos} \tag{15}
$$

where

$$
\langle G[\omega]\rangle_{S_{bos}} \equiv \int \mathcal{D}\mu_G G[\omega] \exp\left(iS_{bos}[\omega]\right) \tag{16}
$$

Here the curvature tensor $R_{\mu\nu ab}$ with two frame and two coordinate indices is given by

$$
R_{\mu\nu ab}[\omega] = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac}\omega_{\nu cb} - \omega_{\nu ac}\omega_{\nu cb} \tag{17}
$$

and the bosonized action S_{bos} reads

$$
S_{bos} = \frac{1}{64\pi\kappa^2 m_b} S_{CS}[\omega[e]] + \frac{1}{\kappa^2} S_{EH}[e, \omega[e]] \tag{18}
$$

with

$$
m_b = \frac{m}{1 \pm \kappa^2 m} \tag{19}
$$

This result implies that the Chern–Simons level of the bosonized theory is shifted from $k = 1/(\kappa^2 m)$ to $k_b = 1/(\kappa^2 m_b) = k \pm 1$. Notice that if the level of the TMG is $k = 1$, it is possible then to have $k_b = 0$ (although formally the mass parameter m_b diverges.) In this case the parity anomaly cancels out and the effective action of the bosonized theory reduces to a pure Einstein–Hilbert action in $2 + 1$ dimensions.

We can summarize these results establishing the following bosonization recipes for action and current

$$S[\bar{\psi}, \psi, e; s] \mapsto S_{bos}[e, \omega[e]] \tag{20}$$

$$\bar{\psi}\gamma^{\mu}[\gamma^a, \gamma^b]\psi \mapsto \mp\frac{1}{256\pi}\epsilon^{\mu\nu\beta}R_{\nu\beta ab}[\omega] \tag{21}$$

We conclude that the bosonized action S_{bos} corresponds to topologically massive gravity action with a modified graviton mass given by m_b instead of the original m so that the original graviton mass picked up a contribution from fermion bosonization increasing or decreasing according to the choice of sign when regularizing determinant (11).[2]

3. The energy–momentum "tensor"

One can complete the bosonization recipe (20)–(21) by relating the energy–momentum tensor associated with the original matter-TMG model to the dual action (18). To define the topologically massive gravity energy–momentum "tensor"[3] we shall follow [22, 5] and consider that the path integral measure $D\mu_G$ is restricted to asymptotically flat space–times. We then decompose space–time metric in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{22}$$

where $\eta_{\mu\nu}$ is the flat-space–time metric with Euclidean signature and $h_{\mu\nu}$ a deviation which is not necessarily small everywhere but vanishes at infinity.

In the case of topologically massive gravity coupled to matter Einstein equations can be written as

$$G_{\alpha\beta} + \frac{1}{m}C_{\alpha\beta} = -\frac{\kappa^2}{2}T_{\alpha\beta} \tag{23}$$

where $G^{\alpha\beta}$ and $C^{\alpha\beta}$ are the Einstein and Cotton–York tensors respectively and $T_{\alpha\beta}$ is the matter (i.e. the Dirac field) energy–momentum tensor,

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R^{\gamma}{}_{\gamma}$$

$$C^{\alpha\beta} = \frac{1}{2\sqrt{g}}\left(\epsilon^{\alpha\rho\sigma}D_\rho R^{\beta}_{\sigma} + \epsilon^{\beta\rho\sigma}D_\rho R^{\alpha}_{\sigma}\right)$$

$$T_{\alpha\beta} = \frac{1}{2}(\bar{\psi}\gamma_\beta\nabla_\alpha\psi - \overline{\nabla_\alpha\psi}\gamma_\beta\psi) \tag{24}$$

Here $R_{\alpha\beta}$ is the Ricci curvature tensor ($R_{\alpha\beta} = g^{\gamma\delta}R_{\alpha\gamma\beta\delta}$), R the scalar curvature ($R = g^{\alpha\beta}R_{\alpha\beta}$) and $T_{\alpha\beta}$ the energy–momentum tensor associated with matter. We can write the previous equation as

$$G_{\alpha\beta} + \frac{1}{m}C_{\alpha\beta} + \left(G^{(1)}_{\alpha\beta} + \frac{1}{m}C^{(1)}_{\alpha\beta}\right) - \left(G^{(1)}_{\alpha\beta} + \frac{1}{m}C^{(1)}_{\alpha\beta}\right) = -\frac{\kappa^2}{2}T_{\alpha\beta}$$

where $G^{(1)}$, $C^{(1)}$ are the part of Einstein and Cotton–York tensors which are linear in $h_{\alpha\beta}$. Passing the first and third terms in the l.h.s. to the r.h.s. one can rewrite the previous equation in the form

$$G^{(1)}_{\alpha\beta} + \frac{1}{m}C^{(1)}_{\alpha\beta}$$
$$= -\left(\frac{\kappa^2}{2}T_{\alpha\beta} + \left(G_{\alpha\beta} + \frac{1}{m}C_{\alpha\beta}\right) - \left(G^{(1)}_{\alpha\beta} - \frac{1}{m}C^{(1)}_{\alpha\beta}\right)\right) \tag{25}$$

Then the quantity

$$t_{\alpha\beta} = \frac{2}{\kappa^2}\left(G_{\alpha\beta} - G^{(1)}_{\alpha\beta} + \frac{1}{m}(C_{\alpha\beta} - C^{(1)}_{\alpha\beta})\right) \tag{26}$$

can be interpreted as the energy–momentum "tensor" of the gravitational field itself and then refer to

$$\tau_{\alpha\beta} = T_{\alpha\beta} + t_{\alpha\beta} \tag{27}$$

as the total energy–momentum "tensor" of matter and gravitation.

In turn, the exact equations of motion for TMG coupled to gravity can then be written in the form

$$G^{(1)}_{\alpha\beta} + \frac{1}{m}C^{(1)}_{\alpha\beta} = -\frac{\kappa^2}{2}(T_{\alpha\beta} + t^m_{\alpha\beta}) \tag{28}$$

with $t^m_{\alpha\beta}$ the energy–momentum "tensor" of the gravitational field for TMG with graviton mass m. In the same vein, the equations of motion of the bosonized action (18) can be written as

$$G^{(1)}_{\alpha\beta} + \frac{1}{m^b}C^{(1)}_{\alpha\beta} = -\frac{\kappa^2}{2}(t^{m^b}_{\alpha\beta}) \tag{29}$$

Using the definition (26) and the fact that the two actions are equivalent, their equations of motion can be related through the bosonization recipe for the matter energy–momentum tensor

$$\frac{\kappa^2}{2}T_{\alpha\beta} \rightarrow \left(\frac{1}{m_b} - \frac{1}{m}\right)C_{\alpha\beta} = \pm\frac{\kappa^2}{64\pi}C_{\alpha\beta} \tag{30}$$

which is thus mapped onto the Cotton–York tensor of the TMG.

In summary, the complete set of bosonization recipes reads

$$S[\bar{\psi}, \psi, e, \omega[e]] \mapsto S_{bos}[e, \omega[e]] \tag{31}$$

$$\bar{\psi}\gamma^{\mu}[\gamma^a, \gamma^b]\psi \mapsto \pm\frac{1}{256\pi}\epsilon^{\mu\nu\beta}R_{\nu\beta ab}[\omega] \tag{32}$$

$$T_{\alpha\beta} = \frac{1}{2}(\bar{\psi}\gamma_\beta\nabla_\alpha\psi - \overline{\nabla_\alpha\psi}\gamma_\beta\psi) \mapsto \pm\frac{1}{32\pi}C_{\alpha\beta} \tag{33}$$

Notice the important fact that these identifications do not depend explicitly on the parameters of the theory. These results are the gravitational analog of the results of Refs. [8,9] in terms of gauge fields. As usual they should be interpreted as identities inside expectation values.

Several interesting results can be drawn form the identifications of Eqs. (18), (19), (31), (32) and (33). Let us consider the case in which $\kappa \rightarrow 0$. In this limit the TMG of Eq. (4) trivializes and the gravitational degrees of freedom reduce to diffeomorphisms of $2+1$-dimensional Minkowski space–time. In this limit the theory reduces to a theory of free massive Dirac fermions (averaged over trivial diffeomorphisms which amount to an average over boundary conditions). From the results of Eq. (18) and Eq. (19), we see that, in this limit, the bosonized theory is classical. Thus, the expectation values and correlators of the stress-tensor of the Dirac field are trivial in this regime (as it is expected since if the Dirac mass is arbitrarily large there is no energy or momentum). We note that in this limit, in which gravity acts as a probe field, in order to obtain non-trivial stress energy currents it is necessary to consider a gravity theory with torsion [13,14]. On the other hand, in the opposite limit $\kappa \rightarrow \infty$, the TMG sector of the action drops out and the original theory is a theory of massive Weyl fermions with strongly fluctuating gravitational degrees of freedom. In this limit we see that the bosonized TMG theory now is a Chern–Simons theory for the spin connection with CS level $k_b = 1$.

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[2] The double sign ambiguity is inherent to any regularization in odd-dimensional spaces as can be seen by using a ζ-function regularization where the sign depends on the choice of upper or lower half plane to close the curve where one integrates the Seeley coefficients [20]. The choice of a particular sign corresponds to a sign of the parity anomaly. In Pauli–Villars regularization this sign is determined by the sign of the mass of the Weyl fermion [6].

[3] Quotes indicate that gauge systems with spin greater than one do not possess gauge-invariant stress tensors, but only integrated Poincaré generators [21].

Acknowledgements

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