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A_4 modular flavour model of quark mass hierarchies close to the fixed point $\tau = i\infty$

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Abstract: We study the possibility to generate the quark mass hierarchies as well as the CKM quark mixing and CP violation without fine-tuning in a quark flavour model with modular A_4 symmetry. The quark mass hierarchies are considered in the vicinity of the fixed point $\tau = i\infty$, τ being the vacuum expectation value of the modulus. We consider first a model in which the up-type and down-type quark mass matrices M_u and M_d involve modular forms of level 3 and weights 6, 4 and 2 and each depends on four constant parameters. Two ratios of these parameters, g_u and g_d, can be sources of the CP violation. If M_u and M_d depend on the same τ, it is possible to reproduce the up-type and down-type quark mass hierarchies in the considered model for $|g_u| \sim \mathcal{O}(10)$ with all other constants being in magnitude of the same order. However, reproducing the CP violation in the quark sector is problematic. A correct description of the quark mass hierarchies, the quark mixing and CP violation is possible close to $\tau = i\infty$ with all constant being in magnitude of the same order and complex g_u and g_d, if there are two different moduli τ_u and τ_d in the up-type and down-type quark sectors. We also consider the case of M_u and M_d depending on the same τ and involving modular forms of weights 8, 4, 2 and 6, 4, 2, respectively, with M_u receiving a tiny SUSY breaking or higher dimensional operator contribution. Both the mass hierarchies of up-type and down-type quarks as well and the CKM mixing angles and CP violating phase are reproduced successfully with one complex parameter and all parameters being in magnitude of the same order. The relatively large value of $\mathrm{Im}\,\tau$, needed for describing the down-type quark mass hierarchies, is crucial for obtaining the correct up-type quark mass hierarchies.

Keywords: Flavour Symmetries, CP Violation, Quark Masses, Theories of Flavour

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1 Introduction

The idea of using modular invariance opened up a new promising direction to challenge the flavour problem of quarks and leptons [1]. The main feature of the approach is that the elements of the Yukawa coupling and fermion mass matrices in the Lagrangian of the theory are modular forms of a certain level N which are functions of a single complex scalar field τ — the modulus — and have specific transformation properties under the action of the modular group. The matter fields (supermultiplets) are assumed to transform in representations of an inhomogeneous (homogeneous) finite modular group $\Gamma_N^{(\prime)}$, while the modular forms furnish irreducible representations of the same group. For $N \leq 5$, the finite modular groups Γ_N are isomorphic to the permutation groups S_3, A_4, S_4 and A_5 (see, e.g., [2]), while the groups Γ'_N are isomorphic to the double covers of the indicated permutation groups, $S'_3 \equiv S_3$, $A'_4 \equiv T'$, S'_4 and A'_5. These discrete groups are widely used in flavour model building. The theory is assumed to possess the modular symmetry described by the finite modular group $\Gamma_N^{(\prime)}$, which plays the role of a flavour symmetry. In the simplest class of such models, the vacuum expectation value (VEV) of modulus τ is the only source of flavour symmetry breaking, such that no flavons are needed. Another very appealing feature of the proposed framework is that the VEV of τ can also be the only source of breaking of the CP symmetry [3].

There is no VEV of τ which preserves the full modular symmetry. However, as was noticed in [4] and further exploited in [5–8], there exists three fixed points of the VEV of τ in the modular group fundamental domain, which do not break the modular symmetry completely. These "symmetric" points are $\tau_{\text{sym}} = i$, $\tau_{\text{sym}} = \omega \equiv \exp(i\,2\pi/3) = -1/2 + i\sqrt{3}/2$ (the 'left cusp'), $\tau_{\text{sym}} = i\infty$, and for the theories based on the Γ_N invariance, preserve, respectively, $\mathbb{Z}_2^S$, $\mathbb{Z}_3^{ST}$, and $\mathbb{Z}_N^T$ residual symmetries. In the case of the double cover groups Γ'_N, the $\mathbb{Z}_2^S$ residual symmetry is replaced by the $\mathbb{Z}_4^S$ and there is an additional $\mathbb{Z}_2^R$ symmetry that is unbroken for any value of τ (see [9] for further details). When the flavour symmetry is fully or partially broken, the elements of the Yukawa coupling and fermion mass matrices get fixed and a certain symmetry-constrained flavour structure arises.

The approach to the flavour problem based on modular invariance has been widely explored so far primarily in the framework of supersymmetric (SUSY) theories. Within the framework of rigid ($\mathcal{N} = 1$) SUSY, modular invariance is assumed to be a property of the superpotential, whose holomorphicity restricts the number of allowed terms.

Following a bottom-up approach, phenomenologically viable "minimal" lepton flavour models based on modular symmetry, which do not include flavons, have been constructed first using the group $\Gamma_4 \simeq S_4$ [10] and $\Gamma_3 \simeq A_4$ [1] (shown to be describing correctly the available data in ref. [11]). "Non-minimal" models with flavons based on $\Gamma_2 \simeq S_3$ and $\Gamma_3 \simeq A_4$ have been proposed respectively in [12] and [13].

After these studies, the interest in the approach grew significantly and a large variety of models has been constructed and extensively studied. This includes:[1]

- lepton flavour models based on the groups $\Gamma_4 \simeq S_4$ [4, 15–17], $\Gamma_5 \simeq A_5$ [6, 18], $\Gamma_3 \simeq A_4$ [5, 8, 16, 19–21], $\Gamma_2 \simeq S_3$ [22, 23] and $\Gamma_7 \simeq \text{PSL}(2, \mathbb{Z}_7)$ [24],
- models of quark flavour [25] and of quark-lepton unification [22, 26–31],
- models with multiple moduli, considered first phenomenologically in [4, 5] and further studied, e.g., in [32–34],
- models in which the formalism of the interplay of modular and generalised CP (gCP) symmetries, developed and applied first to the lepton flavour problem in [3], is explored [8, 35–38],
- models in which current topics (dark matter, leptogenesis, LFV, etc.) are studied [39–46].

[1] A rather complete list of the articles on modular-invariant models of lepton and/or quark flavour, which appeared by December of 2022, can be found in [14]. We cite here only a representative sample.

Also the formalism of the double cover finite modular groups Γ'_N, to which top-down constructions typically lead (see, e.g., [47, 48] and references therein), has been developed and viable flavour models have been constructed for the cases of $\Gamma'_3 \simeq T'$ [49], $\Gamma'_4 \simeq S'_4$ [9, 50] and $\Gamma'_5 \simeq A'_5$ [51, 52]. Subsequently these groups have been used for flavour model building, e.g., in refs. [53–55]. The framework has been further generalised to arbitrary finite modular groups (i.e., those not described by the series $\Gamma^{(\prime)}_N$) in ref. [56]. It is hoped that the results obtained in the bottom-up modular-invariant approach to the lepton and quark flavour problems will eventually connect with top-down results (see, e.g., [57–87]), based on UV-complete theories. The problem of modulus stabilisation was also addressed in [17, 37, 88–90].

The "minimal" phenomenologically viable modular-invariant flavour models with gCP symmetry constructed so far i) of the lepton sector with massive Majorana neutrinos (12 observables) contain ≥ 6 (5) real constants + 1 phase (see, e.g., [3, 4, 54]); ii) of the quark sector (10 observables) contain ≥ 8 real parameters and one phase, while the models of lepton and quark flavours (22 observables) have ≥ 14 real parameters and one phase (see, e.g., [91]).

Reproducing correctly the mass hierarchies of quarks and charged leptons is one of the major difficulties in the flavour models. In almost all phenomenologically viable flavour models based on modular invariance constructed so far the hierarchy of the charged lepton and quark masses is obtained by severe fine-tuning of some of the constant parameters present in the models.[2] Perhaps, the only notable exceptions are the non-minimal models proposed in refs. [92–94], in which modular weights are used as Froggatt-Nielsen charges [95], and additional scalar fields of non-zero modular weights play the role of flavons.

In ref. [96] a formalism was developed that allows to construct models in which the fermion (e.g. charged-lepton and quark) mass hierarchies follow solely from the properties of the modular forms, thus avoiding the fine-tuning without the need to introduce extra scalar fields. Indeed, in [96] the authors have succeeded in constructing a viable lepton flavour model in which the charged lepton mass hierarchy is reproduced without fine-tuning of the constant parameters present in the charged lepton mass matrix.

The fermion mass matrices are strongly constrained in each of the three symmetric points (i.e., the points of residual symmetries) of the VEV of the modulus τ in the fundamental domain of the modular group discussed above [4–8, 96–99]. This fact was exploited in [96] where it was shown that fine-tuning can be avoided in the vicinity of the symmetric points $\tau = \omega$ and $\tau = i\infty$ where the charged-lepton and quark mass hierarchies can follow from the properties of the modular forms present in the corresponding fermion mass matrices rather than being determined by the values of the accompanying constants also present in the matrices.

Following the approach in [96] quark flavour models based on modular invariance have been proposed in [14, 100–103]. These studies showed, in particular, that although it is possible to reproduce without fine-tuning the up-type quark or the down-type quark mass hierarchies, reproducing both the up-type quark or down-type quark masses as well as the

[2] By fine-tuning we refer to either i) unjustified hierarchies between parameters which are introduced in a model on an equal footing and/or ii) high sensitivity of observables to model parameters.

Cabibbo, Kobayashi, Maskawa (CKM) quark mixing and CP violation in the quark sector without fine-tuning is a highly challenging problem in flavour models with modular symmetry.

More specifically, in [14] we have investigated the possibility to describe the quark mass hierarchies as well as the CKM mixing and CP violation close to symmetric point $\tau_{\rm sym} = \omega$ in a model without flavons with modular A_4 symmetry. In this model the quark doublets furnish a triplet irreducible representation of A_4, while the right-handed quark fields are (different) singlets of A_4 carrying different modular weights. According to the general results in [96], each of the up-type and down-type quark mass hierarchies can be obtained in the model without fine-tuning due to the $\mathbb{Z}_3^{ST}$ residual symmetry. The model contains eight constants, only two ratios of which, g_u and g_d, can be a source of the CP violation in addition to the VEV of the modulus, $\tau = \omega + \epsilon$, $(\epsilon)^* \neq \epsilon$, $|\epsilon| \ll 1$. This is the minimal number of parameters ensuring non-zero values of all quark masses. We have shown in [14], in particular, that in the case of real (CP-conserving) g_u and g_d and common τ (ϵ) in the up-type and down-type quark sectors, the down-type quark mass hierarchies can be reproduced without fine tuning with $|\epsilon| \cong 0.03$ and correspond approximately to $1 : |\epsilon| : |\epsilon|^2$, all other constants being of the same order in magnitude. The up-type quark mass hierarchies, which are smaller by a factor ~ 10 than the down-type ones, can be obtained with the same $|\epsilon| \cong 0.03$ but allowing $|g_u| \sim \mathcal{O}(10)$ and correspond to $1 : |\epsilon|/|g_u| : |\epsilon|^2/|g_u|^2$. In this setting the correct description of the CP violation in the quark sector represents a severe problem since it arises as a higher order correction in ϵ with $|\epsilon| \ll 1$. This problem can be alleviated in the case of complex (CP-violating) g_u and g_d with $|g_u| \sim \mathcal{O}(10)$, but the model failed to reproduce the value of the V_{cb} element of the CKM matrix. A correct non-fine-tuned description of the quark mass hierarchies, the quark mixing and CP violation is possible with all constants being of the same order in magnitude and complex g_u and g_d, if one allows different values of ϵ in the up-type and down-type quark sectors, or in a non-minimal modification of the considered model which involves one modulus and five constants in each of the two quark sectors one of which is CP violating.

The results obtained in [14] encourage us to search for alternative modular invariant models describing correctly without severe fine-tuning the quark mass hierarchies, the quark mixing and CP violation.

In this work we study the quark mass hierarchies, the CKM mixing angles and the CP violating (CPV) phase close to the fixed point $\tau_{\rm sym} = i\infty$ at which the $\mathbb{Z}_N^T$ residual symmetry holds. We investigate models based on modular A_4 symmetry ($N = 3$) since it is rather simple and the modular forms are restricted only to triplets and singlets. At $\tau_{\rm sym} = i\infty$ we have $\mathbb{Z}_3$ residual symmetry as in the case of $\tau_{\rm sym} = \omega$. However, the description of the quark mass hierarchies in the vicinity of $\tau_{\rm sym} = i\infty$ differs significantly from that in the vicinity of $\tau_{\rm sym} = \omega$ due, in particular, to the significantly larger value of $\mathrm{Im}\,\tau$.

After introducing the modular symmetry briefly in section 2, we present the general framework of generating the mass hierarchy close to $\tau = i\infty$ without fine-tuning in section 3. In section 4, we construct a "minimal" A_4 modular invariant model. We then describe how one can naturally generate hierarchical quark mass patterns in the vicinity of $\tau_{\rm sym} = i\infty$ and investigate the flavour structure of the quark mass matrices in the model. After summarizing the input quark data in section 5, we perform numerical analysis of the description of the

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quark mass hierarchies and CKM parameters by the "minimal" model. In section 6.2 we consider phenomenologically the case of existence of two different moduli τ_u and τ_d in the up-type and down-type quark sectors and perform a numerical analysis of this case. We also present in section 6.3 an alternative model in which the quark mass matrices depend on a common modulus τ, involve altogether eight constant parameters (two of which can break the CP symmetry) and the lightest u-quark mass is generated either by a tiny SUSY breaking contribution or by dimension six operators and depends on one additional real parameter. We perform also a numerical analysis of the compatibility of the model with the existing data. We summarize our results in section 7. In appendix A, the decompositions of the tensor products of the irreducible representations of A_4 are given. In appendix B, the measure of goodness of numerical fitting is presented. Appendix C contains a proof that the determinant of the up-type quark mass matrix considered in section 6.3 is zero.

2 Modular symmetry and its residual symmetries

We start by briefly reviewing the modular invariance approach to flavour. In the SUSY framework, one introduces a chiral superfield, the modulus τ, transforming non-trivially under the modular group $\Gamma \equiv \mathrm{SL}(2,\mathbb{Z})$. The group Γ is generated by the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.1}$$

obeying $S^2 = R$, $(ST)^3 = R^2 = \mathbb{1}$, and $RT = TR$. The elements γ of the modular group act on τ via fractional linear transformations,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma: \quad \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \tag{2.2}$$

while matter superfields transform as "weighted" multiplets [1, 104, 105],

$$\psi_i \to (c\tau + d)^{-k} \rho_{ij}(\gamma)\, \psi_j, \tag{2.3}$$

where $k \in \mathbb{Z}$ is the so-called modular weight[3] and $\rho(\gamma)$ is a unitary representation of Γ.

In using modular symmetry as a flavour symmetry, an integer level $N \geq 2$ is fixed and one assumes that $\rho(\gamma) = \mathbb{1}$ for elements γ of the principal congruence subgroup

$$\Gamma(N) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ (\mathrm{mod}\ N) \right\}. \tag{2.4}$$

Hence, ρ is effectively a representation of the (homogeneous) finite modular group $\Gamma'_N \equiv \Gamma \,/\, \Gamma(N) \simeq \mathrm{SL}(2,\mathbb{Z}_N)$. For $N \leq 5$, this group admits the presentation

$$\Gamma'_N = \left\langle S,\, T,\, R \mid S^2 = R,\, (ST)^3 = \mathbb{1},\, R^2 = \mathbb{1},\, RT = TR,\, T^N = \mathbb{1} \right\rangle. \tag{2.5}$$

The modulus τ acquires a VEV which is restricted to the upper half-plane and plays the role of a spurion, parameterising the breaking of modular invariance. Additional flavon

[3]While we restrict ourselves to integer k, it is also possible for weights to be fractional [47, 106–108].

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fields are not required, and we do not consider them here. Since τ does not transform under the R generator, a $\mathbb{Z}_2^R$ symmetry is preserved in such scenarios [9]. If also matter fields transform trivially under R, one may identify the matrices γ and $-\gamma$, thereby restricting oneself to the inhomogeneous modular group $\overline{\Gamma} \equiv \mathrm{PSL}(2,\mathbb{Z}) \equiv \mathrm{SL}(2,\mathbb{Z})\,/\,\mathbb{Z}_2^R$. In such a case, ρ is effectively a representation of a smaller (inhomogeneous) finite modular group $\Gamma_N \equiv \Gamma \,/\, \left\langle \Gamma(N) \cup \mathbb{Z}_2^R \right\rangle$. For $N \leq 5$, this group admits the presentation

$$\Gamma_N = \left\langle S,\, T \mid S^2 = \mathbb{1},\, (ST)^3 = \mathbb{1},\, T^N = \mathbb{1} \right\rangle . \tag{2.6}$$

In general, however, R-odd fields may be present in the theory and Γ and Γ'_N are then the relevant symmetry groups. For $N = 2, 3, 4, 5$, Γ_N, as is well known, is isomorphic to the non-Abelian discrete symmetry groups S_3, A_4, S_4, A_5 and Γ'_N is isomorphic to their respective double covers.

Finally, to understand how modular symmetry may constrain the Yukawa couplings and mass structures of a model in a predictive way, we turn to the Lagrangian — which for an $\mathcal{N} = 1$ global supersymmetric theory is given by

$$\mathcal{L} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, K(\tau, \bar{\tau}, \psi_I, \bar{\psi}_I) + \left[\int \mathrm{d}^2\theta\, W(\tau, \psi_I) + \text{h.c.} \right] . \tag{2.7}$$

Here K and W are the Kähler potential and the superpotential, respectively. The superpotential W can be expanded in powers of matter superfields ψ_I,

$$W(\tau, \psi_I) = \sum \left(Y_{I_1 \ldots I_n}(\tau)\, \psi_{I_1} \ldots \psi_{I_n} \right)_{\mathbf{1}} , \tag{2.8}$$

where one has summed over all possible field combinations and independent singlets of the finite modular group. By requiring the invariance of the superpotential under modular transformations, one finds that the field couplings $Y_{I_1 \ldots I_n}(\tau)$ have to be modular forms of level N. These are severely constrained holomorphic functions of τ, which under modular transformations obey

$$Y_{I_1 \ldots I_n}(\tau) \xrightarrow{\gamma} Y_{I_1 \ldots I_n}(\gamma\tau) = (c\tau + d)^k \rho_Y(\gamma)\, Y_{I_1 \ldots I_n}(\tau) . \tag{2.9}$$

Modular forms carry weights $k = k_{I_1} + \ldots + k_{I_n}$ and furnish unitary irreducible representations ρ_Y of the finite modular group such that $\rho_Y \otimes \rho_{I_1} \otimes \ldots \otimes \rho_{I_n} \supset \mathbf{1}$. Non-trivial modular forms of a given level exist only for $k \in \mathbb{N}$, span finite-dimensional linear spaces $\mathcal{M}_k(\Gamma(N))$, and can be arranged into multiplets of $\Gamma_N^{(\prime)}$.

The breakdown of modular symmetry is parameterised by the VEV of the modulus and there is no value of τ which preserves the full symmetry. Nevertheless, at certain so-called symmetric points $\tau = \tau_{\text{sym}}$ the modular group is only partially broken, with the unbroken generators giving rise to residual symmetries. In addition, as we have noticed, the R generator is unbroken for any value of τ, so that a $\mathbb{Z}_2^R$ symmetry is always preserved. There are only three inequivalent symmetric points (in the fundamental domain of the modular group), namely [4]:

- $\tau_{\text{sym}} = i\infty$, invariant under T, preserving $\mathbb{Z}_N^T \times \mathbb{Z}_2^R$,
- $\tau_{\text{sym}} = i$, invariant under S, preserving $\mathbb{Z}_4^S$ (recall that $S^2 = R$),
- $\tau_{\text{sym}} = \omega \equiv \exp(2\pi i/3)$, 'the left cusp', invariant under ST, preserving $\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R$.

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3 Mass hierarchy without fine-tuning close to $\tau = i\infty$

In theories where modular invariance is broken only by the VEV of modulus τ, the fermion mass matrices (in the limit of unbroken SUSY) are expressed in terms of modular forms of a given level N and a limited number of couplings in the superpotential. The flavour structure of the mass matrices is determined by the properties of the respective modular forms at the value of the VEV of τ. It can be severely constrained at the points of residual symmetries $\tau = \tau_{\text{sym}}$ which typically enforce the presence of multiple zero entries in the mass matrices due to zero values of the corresponding modular form components. As τ moves away from its symmetric value, these entries will generically become non-zero but are suppressed and thus a flavour structure arises.

This approach to fermion (charged lepton and quark) mass hierarchies was explored in, e.g., [14, 96, 97]. We present below a more detailed discussion of the approach following [96]. If ϵ parameterises the deviation of τ from a given symmetric point τ_{sym}, $|\epsilon| \ll 1$, the degree of suppression of the (residual-)symmetry-breaking entries is determined by $|\epsilon|^l$, where $l > 0$ is an integer. The integer constant l is not another free parameter of the theory. The values l can take depend on the symmetric point, i.e., on the residual symmetry. As was shown in [96], i) for $\tau_{\text{sym}} = i$, $l = 0, 1$, ii) for $\tau_{\text{sym}} = \omega$, $l = 0, 1, 2$, and iii) for $\tau_{\text{sym}} = i\infty$ of interest and Γ_N (Γ'_N) finite modular group, $N \leq 5$, l can take values $l = 0, 1, \ldots, N-1$. Thus, for $\Gamma_3 \simeq A_4$ and $\tau_{\text{sym}} = i\infty$, $l = 0, 1, 2$. For a specific would-be-zero element of the fermion mass matrix the value of l is uniquely determined by the residual symmetry and by the transformation properties of the fermion fields, associated with the entry, under the residual symmetry group [96].

Indeed, consider the modular-invariant bilinear

$$\psi_i^c \, M(\tau)_{ij} \, \psi_j \,, \tag{3.1}$$

where the superfields ψ and ψ^c transform under the modular group as[4]

$$\begin{aligned} \psi &\xrightarrow{\gamma} (c\tau + d)^{-k} \rho(\gamma) \, \psi \,, \\ \psi^c &\xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho^c(\gamma) \, \psi^c \,, \end{aligned} \tag{3.2}$$

so that each $M(\tau)_{ij}$ is a modular form of level N and weight $K \equiv k + k^c$. Modular invariance requires $M(\tau)$ to transform as

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger \,. \tag{3.3}$$

Taking τ to be close to the symmetric point, and setting γ to the residual symmetry generator, one can use this transformation rule to constrain the form of the mass matrix $M(\tau)$ [96].

At $\tau_{\text{sym}} = i\infty$ of interest we have Z_N^T residual symmetry group gnerated by the T generator of Γ_{N} (Γ'_{N}). Consider the T-diagonal representation basis for group generators

[4] Note that in the case of a Dirac bilinear ψ and ψ^c are independent fields, so in general $k^c \neq k$ and $\rho^c \neq \rho, \rho^*$.

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in which $\rho^{(c)}(T) = \mathrm{diag}(\rho_i^{(c)})$, and assume that τ is "close" to $\tau_{\mathrm{sym}} = i\infty$, i.e., that $\mathrm{Im}\,\tau$ is sufficiently large. By setting $\gamma = T$ in eq. (3.3), one finds

$$M_{ij}(T\tau) = (\rho_i^c\,\rho_j)^*\, M_{ij}(\tau)\,. \tag{3.4}$$

It is now convenient to treat the M_{ij} as functions of $\hat{q} \equiv \exp(i2\pi\tau/N)$, so that

$$\epsilon \equiv |\hat{q}| = e^{-2\pi \mathrm{Im}\tau/N} \tag{3.5}$$

parametrises the deviation of τ from the symmetric point [96]. Note that the entries $M_{ij}(\hat{q})$ depend analytically on $\hat{q}$ and that $\hat{q} \xrightarrow{T} \xi\hat{q}$, with $\xi = \exp(i\,2\pi/N)$. Thus, in terms of $\hat{q}$, eq. (3.4) reads:

$$M_{ij}(\xi\hat{q}) = (\rho_i^c\,\rho_j)^* M_{ij}(\hat{q})\,. \tag{3.6}$$

Expanding both sides in powers of $\hat{q}$, one finds:

$$\xi^n M_{ij}^{(n)}(0) = (\rho_i^c\,\rho_j)^* M_{ij}^{(n)}(0)\,, \tag{3.7}$$

where $M_{ij}^{(n)}$ denotes the n-th derivative of M_{ij} with respect to $\hat{q}$. It follows that $M_{ij}^{(n)}(0)$ can only be non-zero for values of n such that $(\rho_i^c\,\rho_j)^* = \xi^n$. In particular, the entry $M_{ij} = M_{ij}^{(0)}(0)$ is only allowed to be non-zero and be $\mathcal{O}(1)$ if $\rho_i^c\rho_j = 1$. More generally, if $(\rho_i^c\,\rho_j)^* = \xi^\ell$ with $\ell = 0, 1, 2, \ldots, N-1$,

$$M_{ij}(q) = a_0\,\hat{q}^\ell + a_1\,\hat{q}^{\ell+N} + a_2\,\hat{q}^{\ell+2N} + \ldots\,, \tag{3.8}$$

in the vicinity of the symmetric point. It crucially follows that the entry M_{ij} is expected to be $\mathcal{O}(\epsilon^\ell)$ whenever $\mathrm{Im}\,\tau$ is sufficiently large.[5] The power ℓ is uniquely determined by how the Γ_{N} (Γ'_{N}) representations of ψ_i^c and ψ_j decompose under the residual symmetry group Z_N^T [96]. In the case of A_4, as we have already indicated, ℓ can take values $\ell = 0, 1, 2$.

The discussed result allows, in principle, to obtain fermion mass hierarchies without fine-tuning.

4 A_4 modular invariant flavor model

4.1 Models of quarks

We consider next simple models of quark mass matrices in models with the level $N = 3$ modular symmetry (A_4 modular flavor symmetry). We assign the A_4 representation and the weights for the relevant chiral superfields as

- quark doublet (left-handed) $Q = ((u,d),(c,s),(t,b))$: A_4 triplet with weight 2,
- quark singlets (righ-handed) (d^c, s^c, b^c) and (u^c, c^c, t^c): A_4 singlets $(1', 1', 1')$ with weight $(4, 2, 0)$, respectively,
- up and down sector Higgs fields $H_{u,d} = H_{u,d}$: A_4 singlets 1 with weight 0,

	Q	$(d^c, s^c, b^c), (u^c, c^c, t^c)$	H_u	H_d
SU(2)	2	1	2	2
A_4	3	$(1', 1', 1')$	1	1
k	2	(4, 2, 0)	0	0

Table 1. Assignments of A_4 representations and weights in our model.

which are summarized in table 1. Then, the superpotential terms of the down-type and up-type quark masses are written for the case in table 1:

$$\begin{aligned} W_d &= \left[\alpha_d(\mathbf{Y_3^{(6)}}Q)_{1''}d^c_{1'} + \alpha'_d(\mathbf{Y_{3'}^{(6)}}Q)_{1''}d^c_{1'} + \beta_d(\mathbf{Y_3^{(4)}}Q)_{1''}s^c_{1'} + \gamma_d(\mathbf{Y_3^{(2)}}Q)_{1''}b^c_{1'}\right]H_d\,, \\ W_u &= \left[\alpha_u(\mathbf{Y_3^{(6)}}Q)_{1''}u^c_{1'} + \alpha'_u(\mathbf{Y_{3'}^{(6)}}Q)_{1''}u^c_{1'} + \beta_u(\mathbf{Y_3^{(4)}}Q)_{1''}c^c_{1'} + \gamma_u(\mathbf{Y_3^{(2)}}Q)_{1''}b^t_{1'}\right]H_u\,, \end{aligned} \tag{4.1}$$

where the subscripts of $1', 1''$ denote the A_4 representations. The decompositions of the tensor products in A_4 are given in appendix A.

We will also investigate the possibility of right-handed quark assignments being A_4 singlets $(1', 1', 1')$ with weights $(4, 2, 0)$ for (d^c, s^c, b^c) and $(6, 2, 0)$ for (u^c, c^c, t^c) in subsection 6.3. In this case modular forms of weight 8 are also involved.

4.2 Modular forms of A_4

The weight 2 triplet modular forms are given as:

$$\mathbf{Y_3^{(2)}} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \ldots \\ -6q^{1/3}(1 + 7q + 8q^2 + \ldots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \ldots) \end{pmatrix}, \tag{4.2}$$

where $q = \exp(2\pi i\tau)$. They satisfy also the constraint [1]:

$$Y_2^2 + 2Y_1Y_3 = 0\,. \tag{4.3}$$

The weight 4, 6 and 8 modular forms of interest can be expressed in terms of the weight 2 modular forms. For the weight 4 modular forms we have:

$$\begin{aligned} &\mathbf{Y_1^{(4)}} = Y_1^2 + 2Y_2Y_3\,, \qquad \mathbf{Y_{1'}^{(4)}} = Y_3^2 + 2Y_1Y_2\,, \qquad \mathbf{Y_{1''}^{(4)}} = Y_2^2 + 2Y_1Y_3 = 0\,, \\ &\mathbf{Y_3^{(4)}} = \begin{pmatrix} Y_1^{(4)} \\ Y_2^{(4)} \\ Y_3^{(4)} \end{pmatrix} = \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix}, \end{aligned} \tag{4.4}$$

where $\mathbf{Y_{1''}^{(4)}}$ vanishes due to the constraint of eq. (4.3). The expressions for the seven modular forms of weigh 6 read:

$$\begin{aligned} &\mathbf{Y_1^{(6)}} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3\,, \\ &\mathbf{Y_3^{(6)}} \equiv \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = (Y_1^2 + 2Y_2Y_3)\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}, \qquad \mathbf{Y_{3'}^{(6)}} \equiv \begin{pmatrix} Y_1'^{(6)} \\ Y_2'^{(6)} \\ Y_3'^{(6)} \end{pmatrix} = (Y_3^2 + 2Y_1Y_2)\begin{pmatrix} Y_3 \\ Y_1 \\ Y_2 \end{pmatrix}. \end{aligned} \tag{4.5}$$

[5]In practice, values of $\mathrm{Im}\,\tau \cong (2.5 - 3.0)$ prove to be sufficiently large (see [96] and further).

We give the expressions also for the nine modular forms of weight 8:

$$\mathbf{Y}^{(8)}_{\mathbf{1}} = (Y_1^2+2Y_2Y_3)^2 \,, \qquad \mathbf{Y}^{(8)}_{\mathbf{1'}} = (Y_1^2+2Y_2Y_3)(Y_3^2+2Y_1Y_2) \,, \qquad \mathbf{Y}^{(8)}_{\mathbf{1''}} = (Y_3^2+2Y_1Y_2)^2 \,,$$
$$\mathbf{Y}^{(8)}_{\mathbf{3}} \equiv \begin{pmatrix} Y_1^{(8)} \\ Y_2^{(8)} \\ Y_3^{(8)} \end{pmatrix} = (Y_1^2+2Y_2Y_3)\begin{pmatrix} Y_1^2-Y_2Y_3 \\ Y_3^2-Y_1Y_2 \\ Y_2^2-Y_1Y_3 \end{pmatrix} \,, \quad \mathbf{Y}^{(8)}_{\mathbf{3'}} \equiv \begin{pmatrix} Y_1'^{(8)} \\ Y_2'^{(8)} \\ Y_3'^{(8)} \end{pmatrix} = (Y_3^2+2Y_1Y_2)\begin{pmatrix} Y_2^2-Y_1Y_3 \\ Y_1^2-Y_2Y_3 \\ Y_3^2-Y_1Y_2 \end{pmatrix} \,. \tag{4.6}$$

We note that $\mathbf{Y}^{(8)}_{\mathbf{3}} = (Y_1^2 + 2Y_2Y_3)\mathbf{Y}^{(4)}_{\mathbf{3}}$.

At $\tau = i\infty$, the modular forms of interest take very simple forms:

$$\mathbf{Y}^{(2)}_{\mathbf{3}} = Y_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \,, \quad \mathbf{Y}^{(4)}_{\mathbf{3}} = Y_0^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \,, \quad \mathbf{Y}^{(4)}_{\mathbf{1}} = Y_0^2 \,, \quad \mathbf{Y}^{(4)}_{\mathbf{1'}} = 0 \,,$$
$$\mathbf{Y}^{(6)}_{\mathbf{3}} = Y_0^3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \,, \quad \mathbf{Y}^{(6)}_{\mathbf{3'}} = 0 \,, \quad \mathbf{Y}^{(6)}_{\mathbf{1}} = Y_0^3 \,,$$
$$\mathbf{Y}^{(8)}_{\mathbf{3}} = Y_0^4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \,, \quad \mathbf{Y}^{(8)}_{\mathbf{3'}} = 0 \,, \quad \mathbf{Y}^{(8)}_{\mathbf{1}} = Y_0^4 \,, \quad \mathbf{Y}^{(8)}_{\mathbf{1'}} = 0 \,, \quad \mathbf{Y}^{(8)}_{\mathbf{1''}} = 0 \,, \tag{4.7}$$

where we can take a normalization $Y_0 = 1$.

4.3 Modular forms "close" to $\tau = i\infty$

We present next the behavior of modular forms "close" to $\tau = i\infty$. By "close to", or "in the vicinity" of, $\tau = i\infty$ we mean values of $\mathrm{Im}\,[\tau]$ which are sufficiently large, e.g., $\mathrm{Im}\,[\tau] \sim 2.5 - 3.0$. As we will see, values of $\mathrm{Im}\,[\tau] \sim 2.5$ are also typically required to fit the quark mass and mixing data. We find first the expressions of the modular forms $Y_1(\tau)$, $Y_2(\tau)$ and $Y_3(\tau)$ in eq. (4.2) close to $\tau = i\infty$. We use the small real parameter ϵ as given in eq. (3.5), and the phase p in terms of τ as:

$$\epsilon = \exp\left(-\frac{2}{3}\pi\,\mathrm{Im}\,[\tau]\right) \,, \qquad p = \exp\left(\frac{2}{3}\pi\, i\,\mathrm{Re}\,[\tau]\right) \,, \tag{4.8}$$

where $\epsilon \ll 1$. At $\mathrm{Im}\,[\tau] = 2.5$, for example, $\epsilon \cong 5.322 \times 10^{-3}$, $\epsilon^2 \cong 2.832 \times 10^{-5}$, $\epsilon^3 \cong 1.507 \times 10^{-7}$, $\epsilon^4 \cong 8.020 \times 10^{-10}$, $\epsilon^5 \cong 4.268 \times 10^{-12}$ and $\epsilon^6 \cong 2.271 \times 10^{-14}$. The expansion parameter q in eq. (4.2) is written in terms of ϵ and p as:

$$q \equiv \exp\left(2i\pi\tau\right) = (p\,\epsilon)^3 \,. \tag{4.9}$$

For the components of triplet weight 2 modular form $\mathbf{Y}^{(2)}_{\mathbf{3}}$ we obtain in terms of the small parameter $(p\,\epsilon)$:

$$\begin{aligned} Y_1(\tau) &\simeq 1 + 12\,(p\,\epsilon)^3 + 36\,(p\,\epsilon)^6 + 12\,(p\,\epsilon)^9 + \mathcal{O}(\epsilon^{12}) \,, \\ Y_2(\tau) &\simeq -6\,(p\,\epsilon) - 42\,(p\,\epsilon)^4 - 48\,(p\,\epsilon)^7 + \mathcal{O}(\epsilon^{10}) \,, \\ Y_3(\tau) &\simeq -18\,(p\,\epsilon)^2 - 36\,(p\,\epsilon)^5 - 90\,(p\,\epsilon)^8 + \mathcal{O}(\epsilon^{11}) \,. \end{aligned} \tag{4.10}$$

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It follows from eq. (4.1) that only triplet modular forms will enter into the expressions of the down-type and up-type quark mass matrices. We use the results in eq. (4.10) and eqs. (4.4), (4.5) and (4.6) to express the higher weight triplet modular forms $Y_i^{(k)}$ in terms of $(p\,\epsilon)$. For the weight 4 triplet modular form we get:

$$\begin{aligned} Y_1^{(4)}(\tau) &\simeq 1 - 84\,(p\,\epsilon)^3 - 756\,(p\,\epsilon)^6 + \mathcal{O}(\epsilon^9) \simeq 1 - 84\,(p\,\epsilon)^3 + \mathcal{O}(10^{-11})\,, \\ Y_2^{(4)} &\simeq 6\,(p\,\epsilon) + 438\,(p\,\epsilon)^4 + 2016\,(p\,\epsilon)^7 + \mathcal{O}(\epsilon^9) \simeq 6\,(p\,\epsilon) + \mathcal{O}(10^{-7})\,, \\ Y_3^{(4)} &\simeq 54\,(p\,\epsilon)^2 + 756\,(p\,\epsilon)^5 + 3530\,(p\,\epsilon)^8 + \mathcal{O}(\epsilon^{11}) \simeq 54\,(p\,\epsilon)^2 + \mathcal{O}(10^{-9})\,. \end{aligned} \tag{4.11}$$

In the case of the two weight 6 triplet modular forms the expressions read:

$$\begin{aligned} Y_1^{(6)}(\tau) &\simeq 1 + 252\,(p\,\epsilon)^3 + 5076\,(p\,\epsilon)^6 + 41292\,(p\,\epsilon)^9 + \mathcal{O}(\epsilon^{12}) \\ &\simeq 1 + 252\,(p\,\epsilon)^3 + \mathcal{O}(10^{-10})\,, \\ Y_2^{(6)} &\simeq -6\,(p\,\epsilon) - 1482\,(p\,\epsilon)^4 - 11088\,(p\,\epsilon)^7 + \mathcal{O}(\epsilon^{10}) \simeq -6\,(p\,\epsilon) - \mathcal{O}(10^{-5})\,, \\ Y_3^{(6)} &\simeq -18\,(p\,\epsilon)^2 - 4356\,(p\,\epsilon)^5 - 47610\,(p\,\epsilon)^8 + \mathcal{O}(\epsilon^{11}) \simeq -18\,(p\,\epsilon)^2 - \mathcal{O}(10^{-8})\,, \\ Y_1^{\prime(6)}(\tau) &\simeq 216\,(p\,\epsilon)^3 - 1296\,(p\,\epsilon)^6 + 1944\,(p\,\epsilon)^9 + \mathcal{O}(\epsilon^{12}) \simeq 216\,(p\,\epsilon)^3 - \mathcal{O}(10^{-11})\,, \\ Y_2^{\prime(6)}(\tau) &\simeq -12\,(p\,\epsilon) - 40\,(p\,\epsilon)^4 + 480\,(p\,\epsilon)^7 + \mathcal{O}(\epsilon^{10}) \simeq -12\,(p\,\epsilon) - \mathcal{O}(10^{-8})\,, \\ Y_3^{\prime(6)}(\tau) &\simeq 72\,(p\,\epsilon)^2 - 72\,(p\,\epsilon)^5 - 2016\,(p\,\epsilon)^8 + \mathcal{O}(\epsilon^{11}) \simeq 72\,(p\,\epsilon)^2 + \mathcal{O}(10^{-11})\,. \end{aligned} \tag{4.12}$$

Finally we give the expansions for the two triplet weight 8 modular forms:

$$\begin{aligned} Y_1^{(8)}(\tau) &\simeq 1 + 156\,(p\,\epsilon)^3 - 18758\,(p\,\epsilon)^6 - 383864\,(p\,\epsilon)^9 + \mathcal{O}(\epsilon^{12}) \\ &\simeq 1 + 156\,(p\,\epsilon)^3 + \mathcal{O}(10^{-10})\,, \\ Y_2^{(8)} &\simeq 6\,(p\,\epsilon) + 1878\,(p\,\epsilon)^4 + 120144\,(p\,\epsilon)^7 + \mathcal{O}(\epsilon^{10}) \\ &\simeq 6\,(p\,\epsilon) + \mathcal{O}(10^{-6})\,, \\ Y_3^{(8)} &\simeq 54\,(p\,\epsilon)^2 + 13716\,(p\,\epsilon)^5 + 301610\,(p\,\epsilon)^8 + \mathcal{O}(\epsilon^{11}) \\ &\simeq 54\,(p\,\epsilon)^2 + \mathcal{O}(10^{-7})\,, \\ Y_1^{\prime(8)}(\tau) &\simeq -648\,(p\,\epsilon)^3 - 3888\,(p\,\epsilon)^6 + 17256\,(p\,\epsilon)^9 + \mathcal{O}(\epsilon^{12}) \\ &\simeq -648\,(p\,\epsilon)^3 - \mathcal{O}(10^{-10})\,, \\ Y_2^{\prime(8)}(\tau) &\simeq -12\,(p\,\epsilon) + 1104\,(p\,\epsilon)^4 + 768\,(p\,\epsilon)^7 + \mathcal{O}(\epsilon^{10}) \\ &\simeq -12\,(p\,\epsilon) + \mathcal{O}(10^{-6})\,, \\ Y_3^{\prime(8)}(\tau) &\simeq -72\,(p\,\epsilon)^2 - 4680\,(p\,\epsilon)^5 + 15840\,(p\,\epsilon)^8 + \mathcal{O}(\epsilon^{11}) \\ &\simeq -72\,(p\,\epsilon)^2 - \mathcal{O}(10^{-8})\,. \end{aligned} \tag{4.13}$$

In eqs. (4.11)–(4.13) the numerical estimates of the order of the corrections are for $\mathrm{Im}\,[\tau] = 2.5$ and we have kept the terms in the expansions which are not smaller than $\sim 10^{-5}$.

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4.4 The quark mass matrices

We obtain the quark mass matrices from the superpotential in eq. (4.1). In the right-left (RL) convention they have the form:

$$M_d = v_d \begin{pmatrix} \alpha'_d & 0 & 0 \\ 0 & \beta_d & 0 \\ 0 & 0 & \gamma_d \end{pmatrix} \begin{pmatrix} \tilde{Y}_3^{(6)} & \tilde{Y}_2^{(6)} & \tilde{Y}_1^{(6)} \\ Y_3^{(4)} & Y_2^{(4)} & Y_1^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}, \qquad M_u = v_u \begin{pmatrix} \alpha'_u & 0 & 0 \\ 0 & \beta_u & 0 \\ 0 & 0 & \gamma_u \end{pmatrix} \begin{pmatrix} \tilde{Y}_3^{(6)} & \tilde{Y}_2^{(6)} & \tilde{Y}_1^{(6)} \\ Y_3^{(4)} & Y_2^{(4)} & Y_1^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}, \tag{4.14}$$

where $Y_i^{(2)} \equiv Y_i$ and $v_{d\,(u)}$ denotes the vacuum expectation value of $H_{d\,(u)}$, and

$$\tilde{Y}_i^{(6)} = g_q Y_i^{(6)} + Y_i^{\prime(6)}, \qquad g_q \equiv \alpha_q/\alpha'_q \qquad (i = 1,\, 2,\, 3, \quad q = d,\, u)\,. \tag{4.15}$$

We take the modular invariant kinetic terms simply by[6]

$$\sum_I \frac{|\partial_\mu \psi^{(I)}|^2}{\langle -i\tau + i\bar{\tau}\rangle^{k_I}}\,, \tag{4.16}$$

where $\psi^{(I)}$ denotes a chiral superfield with weight k_I, and $\bar{\tau}$ is the anti-holomorphic modulus. Since the (anti-)holomorphic modulus is a dynamical field, it becomes a complex number after the modulus τ takes a VEV. Then, one can set $\bar{\tau} = \tau^*$.

It is important to address the transformation needed to get the kinetic terms of matter superfields in canonical form because the terms in eq. (4.16) are not canonical. Therefore, we normalize the superfields as:

$$\psi^{(I)} \to \sqrt{(2\mathrm{Im}\tau_q)^{k_I}}\, \psi^{(I)}\,. \tag{4.17}$$

The canonical form is obtained by an overall normalization, which shifts our parameters such as:

$$\begin{aligned} \alpha_q &\to \hat{\alpha}_q = \alpha_q \sqrt{(2\mathrm{Im}\tau_q)^6} = \alpha_q\,(2\mathrm{Im}\tau_q)^3, \\ \alpha'_q &\to \hat{\alpha}'_q = \alpha'_q \sqrt{(2\mathrm{Im}\tau_q)^6} = \alpha'_q\,(2\mathrm{Im}\tau_q)^3\,, \\ \beta_q &\to \hat{\beta}_q = \beta_q \sqrt{(2\mathrm{Im}\tau_q)^4} = \beta_q\,(2\mathrm{Im}\tau_q)^2\,, \\ \gamma_q &\to \hat{\gamma}_q = \gamma_q \sqrt{(2\mathrm{Im}\tau_q)^2} = \gamma_q\,(2\mathrm{Im}\tau_q)\,. \end{aligned} \tag{4.18}$$

4.5 Quark mass matrices at $\tau = i\infty$

Since $\epsilon = 0$ at $\tau = i\infty$, the modular forms $Y_2^{(k)}$ and $Y_3^{(k)}$ $(k = 2, 4, 6)$ vanish. Then, the quark mass matrices are given by:

$$M_q^{(0)} = v_q \begin{pmatrix} g_q\,\hat{\alpha}'_q & 0 & 0 \\ 0 & \hat{\beta}_q & 0 \\ 0 & 0 & \hat{\gamma}_q \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}_{\mathrm{RL}}, \qquad q = d\,, u\,. \tag{4.19}$$

[6]Possible non-minimal additions to the Kähler potential, compatible with the modular symmetry, may jeopardise the predictive power of the approach [109]. This problem is the subject of ongoing research.

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These are rank one matrices. Thus we have two massless down-type quarks and two massless up-type quarks at the fixed point $\tau = i\infty$. The matrices $M_q^{(0)\dagger} M_q^{(0)}$, $q = d,\, u$, whose eigenvalues are the squared quark masses, are given by:

$$\mathcal{M}_q^{2(0)} \equiv M_q^{(0)\dagger} M_q^{(0)} = v_q^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |g_q|^2\,|\hat{\alpha}'_q|^2 + |\hat{\beta}_q|^2 + |\hat{\gamma}_q|^2 \end{pmatrix}. \tag{4.20}$$

4.6 Quark mass matrices close to $\tau = i\infty$ and mass hierarchy

The quark mass matrices in eq. (4.19) are corrected due to the deviation from $\tau = i\infty$. By using modular forms of weight 2, 4 and 6 in eqs. (4.10), (4.11), (4.12), we obtain the deviation from $M_q^{(0)}$ given in eq. (4.19). The correction is expressed in terms of the small variable ϵ and the phase p defined in eq. (4.8) In the leading order approximation in ϵ, M_q is given by:

$$M_q = v_q \begin{pmatrix} \hat{\alpha}'_q & 0 & 0 \\ 0 & \hat{\beta}_q & 0 \\ 0 & 0 & \hat{\gamma}_q \end{pmatrix} \begin{pmatrix} 18\,(\epsilon\, p)^2 (4 - g_q) & -6\,(\epsilon\, p)(2 + g_q) & g_q \\ 54\,(\epsilon\, p)^2 & 6\,(\epsilon\, p) & 1 \\ -18\,(\epsilon\, p)^2 & -6\,(\epsilon\, p) & 1 \end{pmatrix}. \tag{4.21}$$

Correspondingly, $\mathcal{M}_q^2 \equiv M_q^\dagger M_q$ has the following structure:

$$\mathcal{M}_q^2 \sim \begin{pmatrix} \epsilon^4 & \epsilon^3 p^* & \epsilon^2 p^{*2} \\ \epsilon^3 p & \epsilon^2 & \epsilon p^* \\ \epsilon^2 p^2 & \epsilon p & 1 \end{pmatrix}. \tag{4.22}$$

Using eq. (4.21), we obtain the elements of $\mathcal{M}_q^2$ in leading order in ϵ:

$$\begin{aligned}
&\mathcal{M}_q^2[1,1] = 324\,\epsilon^4 \left[|\hat{\gamma}_q|^2 + 9|\hat{\beta}_q|^2 + |\hat{\alpha}'_q|^2 |g_q - 4|^2 \right], \\
&\mathcal{M}_q^2[2,2] = 36\,\epsilon^2 \left[|\hat{\gamma}_q|^2 + |\hat{\beta}_q|^2 + |\hat{\alpha}'_q|^2 |g_q + 2|^2 \right], \\
&\mathcal{M}_q^2[3,3] = |\hat{\gamma}_q|^2 + |\hat{\beta}_q|^2 + |\hat{\alpha}'_q|^2 |g_q|^2, \\
&\mathcal{M}_q^2[1,2] = 108\,\epsilon^3\, p^* \left[|\hat{\gamma}_q|^2 + 3|\hat{\beta}_q|^2 + |\hat{\alpha}'_q|^2 (g_q + 2)(g_q^* - 4) \right], \\
&\mathcal{M}_q^2[1,3] = -18\,\epsilon^2\, p^{*2} \left[|\hat{\gamma}_q|^2 - 3|\hat{\beta}_q|^2 + |\hat{\alpha}'_q|^2 g_q (g_q^* - 4) \right], \\
&\mathcal{M}_q^2[2,3] = -6\,\epsilon\, p^* \left[|\hat{\gamma}_q|^2 - |\hat{\beta}_q|^2 + |\hat{\alpha}'_q|^2 g_q (g_q^* + 2) \right], \\
&\mathcal{M}_q^2[2,1] = \mathcal{M}_q^2[1,2]^*, \qquad \mathcal{M}_q^2[3,1] = \mathcal{M}_q^2[1,3]^*, \qquad \mathcal{M}_q^2[3,2] = \mathcal{M}_q^2[2,3]^*,
\end{aligned} \tag{4.23}$$

where the elements are given in units of v_q^2. For the determinant of $\mathcal{M}_q^2$ we find:

$$\mathrm{Det}\,[\mathcal{M}_q^2] = m_{q_1}^2 m_{q_2}^2 m_{q_3}^2 = (12)^6 |\hat{\alpha}'_q|^2 |\hat{\beta}_q|^2 |\hat{\gamma}_q|^2\, v_q^6\, \epsilon^6, \tag{4.24}$$

with $d_1 \equiv d$, $d_2 \equiv s$, $d_3 \equiv b$, $u_1 \equiv u$, $u_2 \equiv c$ and $u_3 \equiv t$. Thus, $\mathrm{Det}\,[\mathcal{M}_q^2]$ is g_q independent. The heaviest quark mass m_{q3} can be obtained to a good approximation from $m_{q_3}^2 \cong \mathrm{Tr}(\mathcal{M}_q^2)$. We find:

$$m_{q_3}^2 \simeq (|\hat{\alpha}'_q|^2 |g_q|^2 + |\hat{\beta}_q|^2 + |\hat{\gamma}_q|^2) v_q^2\,. \tag{4.25}$$

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It is also not difficult to obtain an expression for $m_{q2}^2 m_{q3}^2$, which to a good approximation is given by the determninant of the 2-3 sector:

$$m_{q2}^2 m_{q3}^2 \simeq (12)^2 \epsilon^2 (|\hat{\alpha}'_q|^2 |\hat{\beta}_q|^2 |g_q+1|^2 + |\hat{\alpha}'_q|^2 |\hat{\gamma}_q|^2 + |\hat{\beta}_q|^2 |\hat{\gamma}_q|^2) v_q^4 \,. \tag{4.26}$$

Suppose that α'_q, β_q and γ_q, which appear in the superpotential in eq. (4.1) are in magnitude of the same order. As follows from eq. (4.18), $\hat{\alpha}'_q$, $\hat{\beta}_q$ and $\hat{\gamma}_q$ are enhanced by the factors associated with the renormalisation of the matter fields (see eqs. (4.16) and (4.17)). These factors are powers of $2\text{Im}\,\tau$, which for $\tau \to i\infty$ has a arelatively large value (in the case of interest, $\text{Im}\,\tau \sim 2.5$). After taking into account that for $|\alpha'_q| \sim |\beta_q| \sim |\gamma_q|$, as it follows from eq. (4.18), we have $|\hat{\alpha}'_q|^2 \gg |\hat{\beta}_q|^2 \gg |\hat{\gamma}_q|^2$, we get from eqs. (4.24)–(4.26):

$$\begin{aligned}
m_{q3} &\simeq \hat{\alpha}'_q \, g_q = \alpha'_q I_\tau^3 \, g_q \,, \\
m_{q2} &\simeq \frac{g_q+1}{g_q} \hat{\beta}_q (12\,\epsilon) = \frac{g_q+1}{g_q} \beta_q I_\tau^2 (12\,\epsilon) \,, \\
m_{q1} &\simeq \frac{1}{g_q} \hat{\gamma}_q (12\,\epsilon)^2 = \frac{1}{g_q} \gamma_q I_\tau (12\,\epsilon)^2 \,,
\end{aligned} \tag{4.27}$$

where $I_\tau \equiv 2\text{Im}\,\tau$ and we have assumed for simplicity that the constant g_q is real and that $g_q \gtrsim 1$. Thus, for the mass ratios we obtain:

$$m_{q3} : m_{q2} : m_{q1} \simeq I_\tau^3 g_q : (12\,\epsilon) I_\tau^2 \frac{g_q+1}{g_q} : (12\,\epsilon)^2 I_\tau \frac{1}{g_q} = I_\tau^3 g_q \left[1 : \left(\frac{12\epsilon}{I_\tau g_q} \frac{g_q+1}{g_q} \right) : \left(\frac{12\epsilon}{I_\tau g_q} \right)^2 \right] . \tag{4.28}$$

This result implies that in the case of $|\alpha'_q| \sim |\beta_q| \sim |\gamma_q|$ and $|g_q| \sim 1$ in the considered model, the quark mass hierarchies are determined by $\text{Im}\,\tau$ (we recall that $\epsilon = \exp(-2\pi \text{Im}\,\tau/3)$). The constant $|g_q|$ can play a role in obtaining the correct quark mass ratios if, e.g., $|g_q| \gg 1$.

Indeed, as we show below, both down-type and up-type quark mass hierarchies can be explained with a common τ in the down-type and up-type quark matrices with $\text{Im}\tau \sim 2.5$, by taking $|g_d| \sim 1$ and $|g_u| \sim 10$. We will explore also the alternative possibility of having two different moduli in the down-type and up-type quark mass matrices τ_d and τ_u and thus two different small parameters ϵ_d and ϵ_u, $\epsilon_d \neq \epsilon_u$.

We note finally that the unitary matrices U_q $(q = d,\, u)$, which diagonalise the matrix $M_q^\dagger M_q$ as $M_q^\dagger M_q = U_q \, \text{diag}(m_{q1}^2, m_{q2}^2, m_{q3}^2) \, U_q^\dagger$, with $u_1 \equiv u$, $u_2 \equiv c$, $u_3 \equiv t$, $d_1 \equiv d$, $d_2 \equiv s$ and $d_3 \equiv b$, form the CKM quark mixing matrix: $U_{\text{CKM}} = U_u^\dagger U_d$. We use the parametrisation of U_{CKM}, and thus the convention for three quark mixing angles and CPV phase present in U_{CKM}, θ_{12}, θ_{23}, θ_{13} and δ, proposed in [110].

5 Input data of quark masses, CKM elements

The modulus τ breaks the modular invariance by obtaining a VEV at some high mass scale. We assume this to be the GUT scale. Correspondingly, the values of the quark masses and CKM parameters at the GUT scale play the role of the observables that have to be reproduced by the considered quark flavour model. They are obtained using the

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renormalisation group (RG) equations which describe the "running" of the observables of interest from the electroweak scale, where they are measured, to the GUT scale. In the analyses which follow we adopt the numerical values of the quark Yukawa couplings at the GUT scale 2×10^{16} GeV derived in the framework of the minimal SUSY breaking scenarios with $\tan\beta = 5$ [111]:

$$\begin{aligned} \frac{y_d}{y_b} &= 9.21 \times 10^{-4}\ (1 \pm 0.111)\,, \quad \frac{y_s}{y_b} = 1.82 \times 10^{-2}\ (1 \pm 0.055)\,, \\ \frac{y_u}{y_t} &= 5.39 \times 10^{-6}\ (1 \pm 0.311)\,, \quad \frac{y_c}{y_t} = 2.80 \times 10^{-3}\ (1 \pm 0.043)\,. \end{aligned} \tag{5.1}$$

The quark masses are given as $m_q = y_q v_H$ with $v_H = 174\,\text{GeV}$. The choice of relatively small value of $\tan\beta$ allows us to avoid relatively large $\tan\beta$-enhanced threshold corrections in the RG running of the Yukawa couplings. We set these corrections to zero.

Assuming that both the ratios of the down-type and up-type quark masses, m_b, m_s, m_d and m_t, m_c, m_u, are determined in the model by the small parameter $\hat\epsilon$, which is defined in the following equations, to be compared with eq. (4.28), we have:

$$m_b : m_s : m_d \simeq 1 : \hat\epsilon : \hat\epsilon^2\,, \qquad \hat\epsilon = 0.02 \sim 0.03\,, \tag{5.2}$$

$$m_t : m_c : m_u \simeq 1 : \hat\epsilon : \hat\epsilon^2\,, \qquad \hat\epsilon = 0.002 \sim 0.003\,. \tag{5.3}$$

Thus, the required $\hat\epsilon$ for the description of the down-type and up-type quark mass hierarchies differ approximately by one order of magnitude. As indicated by eq. (4.28), this inconsistency can be "rescued" by relaxing the requirement on the constant $|g_u|$ in the up-quark sector, such as $|g_u| = \mathcal{O}(1) \to \mathcal{O}(10)$ leading to

$$m_t : m_c : m_u \simeq 1 : \frac{\hat\epsilon}{|g_u|} : \left(\frac{\hat\epsilon}{|g_u|}\right)^2\,, \tag{5.4}$$

with $\hat\epsilon = 0.02 \sim 0.03$.

The quark flavor mixing is given by the CKM matrix, which has three independent mixing angles and one CP violating phase. These mixing angles are given by the absolute values of the following three CKM elements. We take the present data on the three CKM elements in Particle Data Group (PDG) edition of Review of Particle Physics [112] as:

$$|V_{us}| = 0.22500 \pm 0.00067\,, \quad |V_{cb}| = 0.04182^{\pm 0.00085}_{-0.00074}\,, \quad |V_{ub}| = 0.00369 \pm 0.00011\,. \tag{5.5}$$

By using these values as input and $\tan\beta = 5$ we obtain the CKM mixing angles at the GUT scale of 2×10^{16} GeV [111]:

$$|V_{us}| = 0.2250\,(1 \pm 0.0032)\,, \quad |V_{cb}| = 0.0400\,(1 \pm 0.020)\,, \quad |V_{ub}| = 0.00353\,(1 \pm 0.036)\,. \tag{5.6}$$

The tree-level decays of $B \to D^{(*)}K^{(*)}$ are used as the standard candle of the CP violation. The CP violating phase of latest world average is given in PDG2022 [112] as:

$$\delta_{\rm CP} = 66.2^{\circ}{}^{+3.4^\circ}_{-3.6^\circ}\,. \tag{5.7}$$

Since the phase is almost independent of the evolution of RGE's, we refer to this value in the numerical discussions. The rephasing invariant CP violating measure $J_{\rm CP}$ [113] is also given in [112]:

$$J_{\rm CP} = 3.08^{+0.15}_{-0.13} \times 10^{-5}\,. \tag{5.8}$$

Taking into account the RG effects on the mixing angles for $\tan\beta = 5$, we have at the GUT scale 2×10^{16} GeV:

$$J_{\rm CP} = 2.80^{+0.14}_{-0.12} \times 10^{-5}\,. \tag{5.9}$$

6 Numerical analyses

In present section we show results of the numerical analyses of the considered A_4 quark flavour model. We perform a fit of the quark masses, CKM mixing angles and CP violating phase $\delta_{\rm CP}$ at the GUT scale, whose values are given in section 5.

More specifically, we perform a systematic scan restricting model parameters in the relevant ranges to reproduce the values of the observables by using a measure of goodness of the fit defined in appendix B, as shown in the following subsections. Several phenomenological possibilities are investigated, as specified below.

In the considered model we have four constant parameters in each of the two quark sectors α_q, α'_q, β_q and γ_q, $q = d,\, u$. We are interested first of all whether it is possible to describe the down-type and up-type quark mass hierarchies in terms of powers of the small parameter $\epsilon = \exp(-2\pi{\rm Im}\,\tau/3)$ avoiding fine-tuning of the constants present in the model. Thus, except possibly for $|g_u| = |\alpha_u/\alpha'_u|$, all other constants should be of the same order in magnitude, i.e., $|\beta_q|/|\alpha'_q| \sim |\gamma_q|/|\alpha'_q| \sim \mathcal{O}(1)$, $q = d,\, u$, and $|g_d| = |\alpha_d|/|\alpha'_d| \sim \mathcal{O}(1)$. In this case their influence on the strong quark mass hierarchies of interest is insignificant [96]. The minimal number of parameters in the model corresponds the case of real constants α_q, α'_q, β_q and γ_q, $q = d,\, u$. The reality of the constants can be ensured by imposing the condition of exact gCP symmetry in the model [3]. The gCP symmetry will be broken by the complex value of τ.[7] It can be broken also by some, or all, constants being complex and we will analyse also these cases.

6.1 Quark mass hierarchies with common τ in $\mathcal{M}_d$ and $\mathcal{M}_u$

In the present subsection we investigate the case that the down-type and up-type quark mass matrices depend on the same modulus τ. In what concerns the violation of the CP symmetry, there are four phenomenological possibilities, which we consider below.

6.1.1 Real g_d and g_u

In this case the CP symmetry is violated only by the VEV of the modulus τ. A tentative sample set for the fitting of the values of the observables at the GUT scale given in section 5 and the results obtained are presented in tables 2 and 3. For the expansion parameter we get $\epsilon \simeq 5.8 \times 10^{-3}$.

[7] In order for the gCP symmetry to be broken the value of τ should not lie on the border of the fundamental domain of the modular group and ${\rm Re}(\tau) \neq 0$ [96].

τ	$\frac{\beta_d}{\alpha'_d}$	$\frac{\gamma_d}{\alpha'_d}$	g_d	$\frac{\beta_u}{\alpha'_u}$	$\frac{\gamma_u}{\alpha'_u}$	g_u
$-0.493 + i\,2.459$	3.33	0.98	-0.70	2.43	1.07	-11.8

Table 2. Values of the constant parameters obtained in the fit of the quark mass ratios and of CKM mixing angles. See text for details.

	$\frac{m_s}{m_b}\times 10^2$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_u}{m_t}\times 10^6$	$\|V_{us}\|$	$\|V_{cb}\|$	$\|V_{ub}\|$	$J_{\rm CP}$
Fit	1.56	6.43	2.65	1.67	0.2246	0.0787	0.00366	2.0×10^{-10}
Exp	1.82	9.21	2.80	5.39	0.2250	0.0400	0.00353	2.8×10^{-5}
$1\,\sigma$	± 0.10	± 1.02	± 0.12	± 1.68	± 0.0007	± 0.0008	± 0.00013	$^{+0.14}_{-0.12}\times 10^{-5}$

Table 3. Results of the fit of the quark mass ratios and CKM mixing angles. $J_{\rm CP}$ factor is output. 'Exp' denotes the respective values at the GUT scale, including 1σ errors.

τ	$\frac{\beta_d}{\alpha'_d}$	$\frac{\gamma_d}{\alpha'_d}$	$\|g_d\|$	$\arg\,[g_d]$	$\frac{\beta_u}{\alpha'_u}$	$\frac{\gamma_u}{\alpha'_u}$	g_u
$-0.0920 + i\,2.394$	3.47	1.27	0.88	161°	1.85	1.15	-10.54

Table 4. Values of the constant parameters obtained in the fit of the quark mass ratios, CKM mixing angles and of the CPV phase $\delta_{\rm CP}$ with complex g_d. See text for details.

The CP violating measure $J_{\rm CP}$ is much smaller than the observed one. As g_q $(q = d,\, u)$, are real, the CP violating phase is generated by $\mathrm{Re}\,\tau$. This contribution is strongly suppressed for the same reason it is suppressed in the quark flavour model with A_4 modular symmetry with real g_d and g_u, considered in the vicinity of the symmetric point $\tau = \omega$ in [14]. The cause of the suppression is analysed in detail in section 5 of ref. [14] and we will not repeat the analysis here. We only note that, as it follows from the discussion in [14], the suppression of interest might not take place if some of the constants present in the down-type and up-type quark mass matrices M_d and M_u are complex and CP-violating and/or if M_d and M_u depend on two different CP-violating moduli τ_d and τ_u, respectively, with $\tau_d \neq \tau_u$. In what follows we explore phenomenologically these possibilities.

6.1.2 Complex g_d (g_u)

We assume first that g_d is complex, while g_u is real. We present a sample set of the results of the fitting of the input data in tables 4 and 5.

The results collected in table 5 show that in this case it is possible to reproduce the observed value of the CPV phase $\delta_{\rm CP}$. However, owing to the fact that the $|V_{\rm cb}|$ and $|V_{\rm ub}|$ elements of the CKM matrix are larger respectively by the factors of 1.9 and 1.2 than the central values of these observables at the GUT scale, the $J_{\rm CP}$ factor is also larger by more than a factor of 2 than the value that should be reproduced.

We have investigated also the case of complex g_d and real g_u. We find that in this case it is impossible to reproduce the observed value of the CPV phase $\delta_{\rm CP}$ and of the

	$\frac{m_s}{m_b}\times 10^2$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_u}{m_t}\times 10^6$	$\lvert V_{us}\rvert$	$\lvert V_{cb}\rvert$	$\lvert V_{ub}\rvert$	$\lvert J_{\rm CP}\rvert$	$\delta_{\rm CP}$
Fit	1.56	8.87	2.65	3.16	0.2263	0.0774	0.00436	6.7×10^{-5}	$64.3°$
Exp $1\,\sigma$	1.82 ± 0.10	9.21 ± 1.02	2.80 ± 0.12	5.39 ± 1.68	0.2250 ± 0.0007	0.0400 ± 0.0008	0.00353 ± 0.00013	2.8×10^{-5} ${}^{+0.14}_{-0.12}\times 10^{-5}$	$66.2°$ ${}^{+3.4°}_{-3.6°}$

Table 5. Results of the fit of the quark mass ratios, CKM mixing angles, $\delta_{\rm CP}$ and $J_{\rm CP}$ with complex ϵ and g_d. 'Exp' denotes the values of the observables at the GUT scale, including 1σ errors, quoted in eqs. (5.1), (5.6), (5.7) and (5.9) and obtained from the measured ones.

τ	$\frac{\beta_d}{\alpha'_d}$	$\frac{\gamma_d}{\alpha'_d}$	$\lvert g_d\rvert$	$\arg\,[g_d]$	$\frac{\beta_u}{\alpha'_u}$	$\frac{\gamma_u}{\alpha'_u}$	$\lvert g_u\rvert$	$\arg\,[g_u]$
$-0.3862+i\,2.4132$	3.60	1.04	0.86	$161.2°$	2.02	1.34	10.4	$205.6°$

Table 6. Values of the constant parameters obtained in the fit of the quark mass ratios, CKM mixing angles and of the CPV phase $\delta_{\rm CP}$ with complex g_d and g_u. See text for details.

	$\frac{m_s}{m_b}\times 10^2$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_u}{m_t}\times 10^6$	$\lvert V_{us}\rvert$	$\lvert V_{cb}\rvert$	$\lvert V_{ub}\rvert$	$\lvert J_{\rm CP}\rvert$	$\delta_{\rm CP}$
Fit	1.52	6.59	2.81	3.37	0.2233	0.076	0.00400	5.8×10^{-5}	$60.6°$
Exp $1\,\sigma$	1.82 ± 0.10	9.21 ± 1.02	2.80 ± 0.12	5.39 ± 1.68	0.2250 ± 0.0007	0.0400 ± 0.0008	0.00353 ± 0.00013	2.8×10^{-5} ${}^{+0.14}_{-0.12}\times 10^{-5}$	$66.2°$ ${}^{+3.4°}_{-3.6°}$

Table 7. Results of the fit of the quark mass ratios, CKM mixing angles, $\delta_{\rm CP}$ and $J_{\rm CP}$ with complex g_d and g_u. 'Exp' denotes the values of the observables at the GUT scale, including 1σ errors, quoted in eqs. (5.1), (5.6), (5.7) and (5.9) and obtained from the measured ones.

$J_{\rm CP}$ factor: the values we obtain for these observables are too small. The problem with reproducing the value of $|V_{cb}|$ also persists.

6.1.3 Complex g_d and g_u

We consider further the case of both g_d and g_u being complex. We present a sample set of the results of the fitting in tables 6 and 7.

We find that, as in the case of complex g_d and real g_u, it is possible to reproduce the observed value of the CPV phase $\delta_{\rm CP}$ but the value of the $J_{\rm CP}$ factor is larger by more than a factor of 2 than the correct value. (see table 7). This can be traced again to the larger than the GUT scale values of $|V_{\rm cb}|$ and $|V_{\rm ub}|$ obtained in the fit. We did not attempt to improve the description of $|V_{\rm cb}|$ and $|V_{\rm ub}|$ by varying $\tan\beta$ and taking into account the threshold correction effetcs in the RG evolution of these two observables. Accounting for these effects warrants an independent comprehensive study.

It follows from the results reported in the preceding subsections that it is possible to reproduce the down-type and up-type quark mass hierarchies in the considered model with $|\beta_q|/|\alpha_q| \sim |\gamma_q|/|\alpha_q| \sim \mathcal{O}(1)$ $(q = d,\, u)$, $|g_d| = |\alpha_d|/|\alpha'_d| \sim \mathcal{O}(1)$ and $|g_u| \sim \mathcal{O}(10)$ when the down-type and up-type quark mass matrices depend on the same modulus τ. However,

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τ_d	τ_u	$\frac{\beta_d}{\alpha'_d}$	$\frac{\gamma_d}{\alpha'_d}$	g_d	$\frac{\beta_u}{\alpha'_u}$	$\frac{\gamma_u}{\alpha'_u}$	g_u
$0.4234 + i\,2.4672$	$-0.4528 + i\,3.5211$	3.57	1.32	-0.703	1.70	0.82	-0.774

Table 8. Values of the constant parameters obtained in the fit of the quark mass ratios, CKM mixing angles and of the CPV phase $\delta_{\rm CP}$ in the case of two moduli τ_d and τ_u.

	$\frac{m_s}{m_b}\times 10^2$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_u}{m_t}\times 10^6$	$\lvert V_{us}\rvert$	$\lvert V_{cb}\rvert$	$\lvert V_{ub}\rvert$	$\lvert J_{\rm CP}\rvert$	$\delta_{\rm CP}$
Fit	1.53	7.90	2.51	10.27	0.2238	0.0467	0.00342	7.7×10^{-6}	12.8°
Exp $1\,\sigma$	1.82 ± 0.10	9.21 ± 1.02	2.80 ± 0.12	5.39 ± 1.68	0.2250 ± 0.0007	0.0400 ± 0.0008	0.00353 ± 0.00013	2.8×10^{-5} $^{+0.14}_{-0.12}\times 10^{-5}$	66.2° $^{+3.4^\circ}_{-3.6^\circ}$

Table 9. Results of the fits of the quark mass ratios, CKM mixing angles, $J_{\rm CP}$ and $\delta_{\rm CP}$ in the vicinity of two different moduli in the down-quark and up-quark sectors, τ_d and τ_u, $\tau_d \neq \tau_u$. 'Exp' denotes the values of the observables at the GUT scale, including 1σ error, quoted in eqs. (5.1), (5.6), (5.7) and (5.9) and obtained from the measured ones.

reproducing the CP violation in the quark sector is problematic. In certain cases this is due to the fact that the values of $|V_{ub}|$ and/or $|V_{cb}|$ resulting from the fit are larger than the GUT scale values of these observables.

Thus, in the next subsection we investigate phenomenologically also the possibility of having two different moduli in the down-type and up-type quark mass matrices τ_d and τ_u and thus two different small parameters ϵ_d and ϵ_u, $\epsilon_d \neq \epsilon_u$.[8] We consider the cases of real CP-conserving g_d and g_u constants as well as complex CP-violating g_d and g_u constants. In both cases the two moduli τ_d and τ_u have CP violating VEVs.

6.2 Two moduli τ_d and τ_u

6.2.1 Real g_d and g_u

We present a sample set of the results of the fitting in this case in tables 8 and 9.

It follows from table 9 that the CP violation in the quark sector cannot be reproduced: the value of the CPV phase $\delta_{\rm CP}$ is approximately by a factor of 5 smaller than the observed value at the GUT scale. Correspondingly, the $J_{\rm CP}$ factor is also smaller than the observed value.

6.2.2 Complex g_d and g_u

We consider next the case of both constants g_d and g_u being complex. A sample set of the results of the fitting in this case is presented in tables 10 and 11.

As follows from tables 10 and 11, all quark observables can be successfully reproduced with all constant being in magnitude of the same order. The magnitude of the measure of

[8]This possibility may be realised in modular invariant flavour models with multiple moduli, such as models based on simplectic modular symmetry [38], or, e.g., in models based on $A_4 \times A_4$ modular symmetry. Constructing such a model is beyond the scope of the present study.

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τ_d	τ_u	$\frac{\beta_d}{\alpha'_d}$	$\frac{\gamma_d}{\alpha'_d}$	$\|g_d\|$	$\arg[g_d]$	$\frac{\beta_u}{\alpha'_u}$	$\frac{\gamma_u}{\alpha'_u}$	$\|g_u\|$	$\arg[g_u]$
$-0.3087 + i\,2.4012$	$0.1231 + i\,3.8110$	4.60	1.33	0.80	198.5°	3.00	2.10	1.22	218.3°

Table 10. Values of the constant parameters obtained in the fit of the quark mass ratios, CKM mixing angles and of the CPV phase $\delta_{\rm CP}$ in the case of two moduli τ_d and τ_u.

	$\frac{m_s}{m_b}\times 10^2$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_u}{m_t}\times 10^6$	$\|V_{us}\|$	$\|V_{cb}\|$	$\|V_{ub}\|$	$\|J_{\rm CP}\|$	$\delta_{\rm CP}$
Fit	1.76	7.74	2.84	5.61	0.2251	0.0414	0.00337	2.83×10^{-5}	67.7°
Exp $1\,\sigma$	1.82 ± 0.10	7.93 ± 1.02	2.82 ± 0.12	6.55 ± 1.68	0.2250 ± 0.0007	0.0400 ± 0.0008	0.00353 ± 0.00013	2.8×10^{-5} $^{+0.14}_{-0.12}\times 10^{-5}$	66.2° $^{+3.4°}_{-3.6°}$

Table 11. Results of the fits of the quark mass ratios, CKM mixing angles, $J_{\rm CP}$ and $\delta_{\rm CP}$ in the vicinity of two different moduli in the down-quark and up-quark sectors, τ_d and τ_u, $\tau_d \neq \tau_u$. 'Exp' denotes the values of the observables at the GUT scale, including 1σ error, quoted in eqs. (5.1), (5.6), (5.7) and (5.9) and obtained from the measured ones.

	Q	(d^c, s^c, b^c)	(u^c, c^c, t^c)	H_u	H_d
SU(2)	2	1		2	2
A_4	3	$(1', 1', 1')$	$(1', 1', 1')$	1	1
k	2	(4, 2, 0)	(6, 2, 0)	0	0

Table 12. Assignments of A_4 representations and weights in our model.

goodness of the fitting $N\sigma$, which is defined in appendix B, is $N\sigma = 2.8$. The number of parameters is rather large — this successful model has altogether 14 real parameters (10 real constants and 4 phases). In the next subsection we consider alternative models with a common modulus τ in the down-type and up-type quark mass matrices, with smaller number of parameters.

6.3 Alternative model with a common modulus τ

Finally, we discuss an alternative model. In this model we introduce weight 8 modular forms in addition to the weights 4 and 6 ones in an attempt to get a correct description of the observed three CKM mixing angles and CP violating phase with one modulus τ. The model is obtained from the considered one by replacing in table 1 the weights (6, 2 ,0) of the right-handed quarks (u^c, c^c, t^c). The weights are same ones (4, 2 ,0) for right-handed quarks (d^c, s^c, b^c). These assigments are summarized in table 12. Then, the quark mass matrices are easily obtained as:

$$M_d = v_d \begin{pmatrix} \hat\alpha'_d & 0 & 0 \\ 0 & \hat\beta_d & 0 \\ 0 & 0 & \hat\gamma_d \end{pmatrix} \begin{pmatrix} \tilde Y_3^{(6)} & \tilde Y_2^{(6)} & \tilde Y_1^{(6)} \\ \tilde Y_3^{(4)} & \tilde Y_2^{(4)} & \tilde Y_1^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}, \quad M_u = v_u \begin{pmatrix} \hat\alpha'_u & 0 & 0 \\ 0 & \hat\beta_u & 0 \\ 0 & 0 & \hat\gamma_u \end{pmatrix} \begin{pmatrix} \tilde Y_3^{(8)} & \tilde Y_2^{(8)} & \tilde Y_1^{(8)} \\ \tilde Y_3^{(4)} & \tilde Y_2^{(4)} & \tilde Y_1^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}, \qquad (6.1)$$

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where

$$\tilde{Y}_i^{(6)} = g_d Y_i^{(6)} + Y_i^{'(6)}\,, \quad \tilde{Y}_i^{(8)} = f_u Y_i^{(8)} + Y_i^{'(8)}\,, \quad g_d \equiv \alpha_d/\alpha_d' \quad f_u \equiv \alpha_u/\alpha_u'\,. \tag{6.2}$$

In order to get the canonical form of the kinetic term, as discussed in eq. (4.17), the couplings are shifted by overall normalizations as follows:

$$\begin{aligned}
&\alpha_u \to \hat{\alpha}_u = \alpha_u \sqrt{(2\mathrm{Im}\tau)^8} = \alpha_u\,(2\mathrm{Im}\tau)^4, \quad &&\alpha_u' \to \hat{\alpha}_u' = \alpha_u' \sqrt{(2\mathrm{Im}\tau)^8} = \alpha_u'\,(2\mathrm{Im}\tau)^4\,,\\
&\beta_u \to \hat{\beta}_u = \beta_u \sqrt{(2\mathrm{Im}\tau)^4} = \beta_u\,(2\mathrm{Im}\tau)^2\,, \quad &&\gamma_u \to \hat{\gamma}_u = \gamma_u \sqrt{(2\mathrm{Im}\tau)^2} = \gamma_u\,(2\mathrm{Im}\tau)\,,\\
&\alpha_d \to \hat{\alpha}_d = \alpha_d \sqrt{(2\mathrm{Im}\tau)^6} = \alpha_d\,(2\mathrm{Im}\tau)^3, \quad &&\alpha_d' \to \hat{\alpha}_d' = \alpha_d' \sqrt{(2\mathrm{Im}\tau)^6} = \alpha_d'\,(2\mathrm{Im}\tau)^3\,,\\
&\beta_d \to \hat{\beta}_d = \beta_d \sqrt{(2\mathrm{Im}\tau)^4} = \beta_d\,(2\mathrm{Im}\tau)^2\,, \quad &&\gamma_d \to \hat{\gamma}_d = \gamma_d \sqrt{(2\mathrm{Im}\tau)^2} = \gamma_d\,(2\mathrm{Im}\tau)\,.
\end{aligned} \tag{6.3}$$

In the considered model we have two parameters g_d and f_u in addition to α_q', β_q and γ_q. The elements of the matrix $M_d^\dagger M_d$ are given in eq. (4.23). We give below the elements of $(M_u^\dagger M_u)_{ij} \equiv \mathcal{M}_u^2[i,j]$ in units of v_u^2 to leading order in ϵ:

$$\begin{aligned}
\mathcal{M}_u^2[1,1] &= 324\,\epsilon^4 \left[|\hat{\gamma}_u|^2 + 9|\hat{\beta}_u|^2 + |\hat{\alpha}_u'|^2|3f_u - 4|^2\right]\,,\\
\mathcal{M}_u^2[2,2] &= 36\,\epsilon^2 \left[|\hat{\gamma}_u|^2 + |\hat{\beta}_u|^2 + |\hat{\alpha}_u'|^2|f_u - 2|^2\right]\,,\\
\mathcal{M}_u^2[3,3] &= |\hat{\gamma}_u|^2 + |\hat{\beta}_u|^2 + |\hat{\alpha}_u'|^2|f_u|^2\,,\\
\mathcal{M}_u^2[1,2] &= 108\,\epsilon^3\,p^* \left[|\hat{\gamma}_u|^2 + 3|\hat{\beta}_u|^2 + |\hat{\alpha}_u'|^2(f_u-2)(3f_u^*-4)\right]\,,\\
\mathcal{M}_u^2[1,3] &= 18\,\epsilon^2\,p^{*2} \left[-|\hat{\gamma}_u|^2 + 3|\hat{\beta}_u|^2 + |\hat{\alpha}_u'|^2\,f_u(3f_u^*-4)\right]\,,\\
\mathcal{M}_u^2[2,3] &= 6\,\epsilon\,p^* \left[-|\hat{\gamma}_u|^2 + |\hat{\beta}_u|^2 + |\hat{\alpha}_u'|^2\,f_u(f_u^*-2)\right]\,,\\
\mathcal{M}_u^2[2,1] &= \mathcal{M}_u^2[1,2]^*\,, \qquad \mathcal{M}_u^2[3,1] = \mathcal{M}_u^2[1,3]^*\,, \qquad \mathcal{M}_u^2[3,2] = \mathcal{M}_u^2[2,3]^*\,.
\end{aligned} \tag{6.4}$$

Using these results we obtain to leading order in ϵ:

$$m_t^2 \simeq \mathrm{Tr}\,(\mathcal{M}_{\mathrm{u}}^2) \simeq v_u^2(|\hat{\alpha}_u'|^2\,|f_u|^2 + |\hat{\beta}_u|^2 + |\hat{\gamma}_u|^2)\,, \tag{6.5}$$

and, from the determinant of the submatric of the 2-3 sector,

$$m_c^2 m_t^2 \simeq (12)^2\,v_u^4\,\epsilon^2(|\hat{\alpha}_u'|^2\,|\hat{\beta}_u|^2 + (1-\mathrm{Re}(f_u))^2\,|\hat{\alpha}_u'|^2\,|\hat{\gamma}_u|^2 + |\hat{\beta}_u|^2\,|\hat{\gamma}_u|^2)\,. \tag{6.6}$$

We find also that the determinant of $\mathcal{M}_u^2$ vanishes:

$$\mathrm{Det}\,[\mathcal{M}_u^2] = 0\,. \tag{6.7}$$

As shown in appendix C, the vanishing of Det $[\mathcal{M}_u^2]$ is exact: it follows from the expression of M_u in eq. (6.1) and is a consequence of the properties of the modular forms given in eq. (4.3) and the relation $\mathbf{Y_3^{(8)}} = (Y_1^2 + 2Y_2Y_3)\mathbf{Y_3^{(4)}}$. As a consequence, M_u is a rank two matrix and the lightest quark, u-quark, is massless. According to [112], given the results of various estimates of the value of m_u (lattice calculations, chiral perturbation theory, etc.), $m_u = 0$ seems very unlikely and strongly disfavored. However, there exist mechanisms

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which generate the requisite tiny u-quark mass without affecting the other predictions of the model.

The u-quark mass $m_u \neq 0$ could be generated by supersymmetry breaking [97]. If supersymmetry is broken by some F-term, the Yukawa couplings are corrected by terms of the order of F/Λ^2, where F is the supersymmetry breaking expectation value with dimension of mass square and Λ is the SUSY breaking messenger scale [13, 97].

As a second possibility of generating $m_u \neq 0$, one can consider the dimension six operators [97]:

$$(u^c Q H_u)(H_u H_d)\,, \qquad (c^c Q H_u)(H_u H_d)\,, \qquad (t^c Q H_u)(H_u H_d)\,, \tag{6.8}$$

whose Wilson coefficient should be a modular form of the appropriate weight. In order for this mechanism to work, the weight assignments in table 12 should be modified. The following conditions have to be fulfilled:

$$k_Q = 2 - k_{Hd}\,, \quad k_{u^c} = 6 + k_{Hd} - k_{Hu}\,, \quad k_{c^c} = 2 + k_{Hd} - k_{Hu} \quad k_{t^c} = k_{Hd} - k_{Hu}\,, \tag{6.9}$$

with the additional constraint

$$k_{Hd} + k_{Hu} \neq 0\,, \tag{6.10}$$

where k_{Hd} and k_{Hu} denote weights of H_d and H_u, respectively. The conditions in eq. (6.9) ensure that the superpotentials discussed in the previous subsections have weight zero, and the one in eq. (6.10) implies that the operator of eq. (6.8) has different weight from the corresponding renormalisable Yukawa term, so that it couples to an independent modular form multiplet with weight $8+k_{Hd}+k_{Hu}$, $4+k_{Hd}+k_{Hu}$, $2+k_{Hd}+k_{Hu}$. These terms make the resulting up-type quark mass matrix of rank three. Such a mechanism generates tiny quark mass $m_u \sim v_d v_u^2/\Lambda^2$, where Λ is the scale at which the operator in eq. (6.8) is generated.

Adding the tiny corrections due to the SUSY breaking or the dimension six operator discussed above, the up-type quark mass matrix M_u is modified as follows:

$$M_u = v_u \begin{pmatrix} \hat{\alpha}'_u & 0 & 0 \\ 0 & \hat{\beta}_u & 0 \\ 0 & 0 & \hat{\gamma}_u \end{pmatrix} \begin{pmatrix} \tilde{Y}_3^{(8)}(1+C_{u1}) & \tilde{Y}_2^{(8)} & \tilde{Y}_1^{(8)} \\ \tilde{Y}_3^{(4)}(1+C_{u2}) & \tilde{Y}_2^{(4)} & \tilde{Y}_1^{(4)} \\ Y_3^{(2)}(1+C_{u3}) & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}, \tag{6.11}$$

where the mass correction terms appear in the first column. Their contribution are negligibly small in the second and third columns for τ close to $\tau = i\infty$ (e.g., $\mathrm{Im}\tau \sim 2.5$). The contribution to the down-type quark mass matrix is negligible as well. Unless $C_{u1} = C_{u2} = C_{u3}$, the determinant of $\mathcal{M}_u^2$ is non-vanishing in this case:

$$\mathrm{Det}[\mathcal{M}_u^2] = (6)^6 \hat{\alpha}_q'^2 \hat{\beta}_q^2 \hat{\gamma}_q^2 v_q^6\, \epsilon^6\, |C_u|^2\,, \tag{6.12}$$

where

$$C_u = 3 f_u\,(C_{u1} - C_{u2}) + (-4C_{u1} + 3C_{u2} + C_{u3})\,. \tag{6.13}$$

Thus, Det $[\mathcal{M}_q^2]$ depends on $|C_u|^2$. Indeed, as shown numerically below, the observed u-quark mass is reproduced for $|C_u| \sim 0.1$ and $|f_u| = \mathcal{O}(1)$. The contributions of $|C_{ui}| \sim 0.1$ ($i = 1, 2, 3$) to the quark mixing angles and the CPV phase are negligible because they are much smaller than the magnitudes of modular forms $\tilde{Y}_3^{(k)}$.

In the considered model the up-type quark mass hierarchy differs from that discussed after eq. (4.26) in subsection 4.6. Suppose that $|\alpha'_u|$, $|\beta_u|$ and $|\gamma_u|$ are of the same order. Bringing the kinetic terms to their canonical forms leads to normalisation factors of α'_u, β_u and γ_u which differ from those in eq. (4.18) since the weights of the right-handed up-type quarks are different in the considered alternative model ($k_{u^c} = 6$, $k_{c^c} = 2$, $k_{y^c} = 0$). After taking into account that due to the renormalisation factors $|\hat{\alpha}'_u| \gg |\hat{\beta}_u| \gg |\hat{\gamma}_u|$ and assuming for simplicity that f_u is real and $f_u \gtrsim 1$ we have:

$$m_t \simeq \hat{\alpha}'_u f_u = \alpha'_q I_\tau^4 f_u, \quad m_c \simeq \hat{\beta}_u \frac{1}{f_u}(12\,\epsilon) = \beta_q I_\tau^2 \frac{1}{f_u}(12\,\epsilon), \quad m_u \simeq 6^3 \hat{\gamma}_u \epsilon^2 |C_u| = 6^3 \gamma_u I_\tau \epsilon^2 |C_u|, \tag{6.14}$$

where as before $I_\tau = 2\mathrm{Im}\,\tau$. Thus, in the case of $|\alpha'_u| \sim |\beta_u| \sim |\gamma_u|$ for the mass ratios we get:

$$\begin{aligned} m_u : m_c : m_t &\simeq I_\tau^4 f_u : 12 I_\tau^2 \frac{1}{f_u}\epsilon : 6^3 I_\tau \epsilon^2 |C_u| \\ &= \left[1 : \left(\frac{12\epsilon}{I_\tau f_u}\frac{1}{I_\tau f_u}\right) : \frac{3}{2}\left(\frac{12\epsilon}{I_\tau f_u}\frac{1}{I_\tau f_u}\right)^2 f_u^3 I_\tau |C_u|\right] I_\tau^4 f_u\,. \end{aligned} \tag{6.15}$$

The observed up-type quark mass hierarchies are reproduced for $\mathrm{Im}\,\tau \sim 2.5$ and small $|C_u|$ with $f_u = \mathcal{O}(1)$. For example, we get $m_u/m_t = 5 \times 10^{-6}$ if $\mathrm{Im}\,\tau = 2.5$ and $|C_u/f_u| = 0.1$ in eq. (6.15). The ratios of the up-type quark masses should be compared with the down-type quark mass ratios:

$$1 : \frac{m_s}{m_b} : \frac{m_d}{m_b} \simeq 1 : \left(\frac{12\epsilon}{I_\tau g_d}\right) : \left(\frac{12\epsilon}{I_\tau g_d}\right)^2 . \tag{6.16}$$

We note that the relatively large value of $\mathrm{Im}\,\tau$ together with $|C_u| \sim 0.1$ gives the requisite stronger hierarchies of up-type quarks masses as compared to the down-type quark mass hierarchies even if $f_u = \mathcal{O}(1)$.

In what follows we show samples of results of the numerical analysis of the model. We take C_{u1} to be real and neglect C_{u2} and C_{u3} for simplicity.

6.3.1 Real g_d and f_u

In the case of real g_d and f_u the model contains 11 real parameters. A tentative sample set of fitting of the values of the observables at the GUT scale, given in section 5, is shown in tables 13 and 14.

Since g_d and f_u are real and thus CP-conserving, the value of the CP violating measure J_{CP} is much smaller than the value of observed one at the GUT scale. The reason is practically identical to that discussed in [14]. By introducing an additional source of the CP violation in the form of complex g_d or/and f_u, the quality of the fit is improved considerably, as is shown below.

τ	$\frac{\beta_d}{\alpha'_d}$	$\frac{\gamma_d}{\alpha'_d}$	g_d	$\frac{\beta_u}{\alpha'_u}$	$\frac{\gamma_u}{\alpha'_u}$	f_u	C_{u1}
$-0.3416 + i\,2.3818$	3.89	1.12	−0.66	1.92	3.45	−1.78	−0.069

Table 13. Values of the constant parameters obtained in the fit of the quark mass ratios and of CKM mixing angles. See text for details.

	$\frac{m_s}{m_b}\times10^2$	$\frac{m_d}{m_b}\times10^4$	$\frac{m_c}{m_t}\times10^3$	$\frac{m_u}{m_t}\times10^6$	$\lvert V_{us}\rvert$	$\lvert V_{cb}\rvert$	$\lvert V_{ub}\rvert$	$J_{\rm CP}$
Fit	2.09	8.90	3.14	6.60	0.2267	0.0296	0.00382	7.0×10^{-9}
Exp $1\,\sigma$	1.82 ±0.10	9.21 ±1.02	2.80 ±0.12	5.39 ±1.68	0.2250 ±0.0007	0.0400 ±0.0008	0.00353 ±0.00013	2.8×10^{-5} $^{+0.14}_{-0.12}\times10^{-5}$

Table 14. Results of the fit of the quark mass ratios and CKM mixing angles. $J_{\rm CP}$ factor is output. 'Exp' denotes the respective values at the GUT scale, including 1σ errors.

τ	$\frac{\beta_d}{\alpha'_d}$	$\frac{\gamma_d}{\alpha'_d}$	g_d	$\frac{\beta_u}{\alpha'_u}$	$\frac{\gamma_u}{\alpha'_u}$	$\lvert f_u\rvert$	$\arg\,[f_u]$	C_{u1}
$-0.3952 + i\,2.4039$	3.82	1.17	−0.677	1.72	3.21	1.68	127.3°	−0.07147

Table 15. Values of the constant parameters obtained in the fit of the quark mass ratios, CKM mixing angles and of the CPV phase $\delta_{\rm CP}$ with real C_{u1}, real g_d and complex f_u. See text for details.

	$\frac{m_s}{m_b}\times10^2$	$\frac{m_d}{m_b}\times10^4$	$\frac{m_c}{m_t}\times10^3$	$\frac{m_u}{m_t}\times10^6$	$\lvert V_{us}\rvert$	$\lvert V_{cb}\rvert$	$\lvert V_{ub}\rvert$	$\lvert J_{\rm CP}\rvert$	$\delta_{\rm CP}$
Fit	1.89	8.78	2.81	5.52	0.2251	0.0390	0.00364	2.94×10^{-5}	70.7°
Exp $1\,\sigma$	1.82 ±0.10	9.21 ±1.02	2.80 ±0.12	5.39 ±1.68	0.2250 ±0.0007	0.0400 ±0.0008	0.00353 ±0.00013	2.8×10^{-5} $^{+0.14}_{-0.12}\times10^{-5}$	66.2° $^{+3.4^\circ}_{-3.6^\circ}$

Table 16. Results of the fit of the quark mass ratios, CKM mixing angles, $\delta_{\rm CP}$ and $J_{\rm CP}$ with real g_d and complex f_u. 'Exp' denotes the values of the observables at the GUT scale, including 1σ errors, quoted in eqs. (5.1), (5.6), (5.7) and (5.9) and obtained from the measured ones.

6.3.2 Real g_d and complex f_u

This is the case with minimum number of parameters in which we can possibly have sufficiently large violation of the CP symmetry in the quark sector. The CP violation is generated by the complex parameter f_u and the modulus τ. The total number of parameters is 12 (10 real constants and 2 phases). The numerical results of the fitting are presented in tables 15 and 16.

As follows from tables 15 and 16 all quark observables can be successfully reproduced with all constant being in magnitude of the same order. The magnitude of the measure of goodness of the fit $N\sigma$ is $N\sigma = 2.0$. We emphasise that both the mass hierarchies of down-type and up-type quarks are reproduced by the common modulus τ with one complex parameter f_u, which is of order one in magnitude.

τ	$\frac{\beta_d}{\alpha'_d}$	$\frac{\gamma_d}{\alpha'_d}$	$\lvert g_d \rvert$	$\arg[g_d]$	$\frac{\beta_u}{\alpha'_u}$	$\frac{\gamma_u}{\alpha'_u}$	$\lvert f_u \rvert$	$\arg[f_u]$	C_{u1}
$-0.3629 + i\,2.4090$	3.82	1.06	0.68	$177.7°$	1.60	3.28	1.61	$122.5°$	-0.0707

Table 17. Values of the constant parameters obtained in the fit of the quark mass ratios, CKM mixing angles and of the CPV phase $\delta_{\rm CP}$ with real C_{u1}, and complex g_d and f_u. See text for details.

	$\frac{m_s}{m_b}\times 10^2$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_u}{m_t}\times 10^6$	$\lvert V_{us} \rvert$	$\lvert V_{cb} \rvert$	$\lvert V_{ub} \rvert$	$\lvert J_{\rm CP} \rvert$	$\delta_{\rm CP}$
Fit	1.86	7.77	2.86	5.29	0.2251	0.0398	0.00357	2.87×10^{-5}	$67.1°$
Exp $1\,\sigma$	1.82 ± 0.10	9.21 ± 1.02	2.80 ± 0.12	5.39 ± 1.68	0.2250 ± 0.0007	0.0400 ± 0.0008	0.00353 ± 0.00013	2.8×10^{-5} $^{+0.14}_{-0.12}\times 10^{-5}$	$66.2°$ $^{+3.4°}_{-3.6°}$

Table 18. Results of the fit of the quark mass ratios, CKM mixing angles, $\delta_{\rm CP}$ and $J_{\rm CP}$ with complex g_d and f_u. 'Exp' denotes the values of the observables at the GUT scale, including 1σ errors, quoted in eqs. (5.1), (5.6), (5.7) and (5.9) and obtained from the measured ones.

6.3.3 Complex g_d and f_u

The case of the complex g_d with real f_u does not give us a fitting result with $N\sigma$ smaller than $N\sigma = 5$, so, we omit the discussion of this case.

In the case with both complex g_d and f_u the number of parameters is altogether 13. The sources of the CP violation in this case are the phases of the modulus τ and of the constants g_d and f_u. The numerical results are presented in tables 17 and 18.

As it follows from tables 17 and 18 all quark observables can be successfully reproduced with all constant being in magnitude of the same order. The magnitude of the measure of goodness of the fitting $N\sigma$ is $N\sigma = 1.58$.

7 Summary

We have studied the quark mass hierarchies as well as the CKM quark mixing and CP violation without fine-tuning in a quark flavour model with modular A_4 symmetry. The quark mass hierarchies are considered close to the fixed point $\tau = i\infty$, τ being the VEV of the modulus. In the considered A_4 model the three left-handed quark doublets $Q = ((u,d),(c,s),(t,b))$ are assumed to furnish a triplet irreducible representation of A_4 and to carry weight 2, while the three right-handed down-type quark and up-type quark singlets are supposed to be the A_4 singlets $(\mathbf{1}',\mathbf{1}',\mathbf{1}')$ carrying weights $(4,2,0)$, respectively. The down-type and up-type quark mass matrices in the model, M_d and M_u, involve modular forms of level 3 and weights 6, 4 and 2, and each contains four constants, only two ratios of which, g_d and g_u, can be a source of the CP violation in addition to the VEV of the modulus, τ. If M_d and M_u depend on the same τ, it is possible to reproduce the down-type and up-type quark mass hierarchies in the considered model for $|g_u| \sim \mathcal{O}(10)$ with all other constants being in magnitude of the same order. However, reproducing the CP violation in the quark sector is problematic. This is due to the fact that the values of $|V_{ub}|$ and/or

$|V_{cb}|$ elements of the CKM matrix resulting from the fit are larger than the values of these observables. We have shown that a correct description of the quark mass hierarchies, the quark mixing and CP violation is possible close to $\tau = i\infty$ with all constant being in magnitude of the same order and complex g_d and g_u, if there are two different moduli τ_d and τ_u in the down-type and up-type quark sectors.

We have considered also the case when M_d and M_u depend on the same τ and involving modular forms of weights 6, 4, 2 and 8, 4, 2, respectively, with M_u receiving a tiny SUSY breaking or higher dimensional operator contribution as well. Both the mass hierarchies of down-type and up-type quarks as well and the CKM mixing angles and the CP violating phase are reproduced successfully with one (or two) complex parameter(s) and all other parameters being in magnitude of the same order. The relatively large value of $\text{Im}\,\tau$, needed for describing the down-quark mass hierarchies, is crucial for obtaining the correct up-type quark mass hierarchies.

We would like to note that if all theoretical uncertainties (due to the scale of SUSY breaking, threshold corrections etc.), which are not included in our analyses, are taken into account, the numerical conclusions may change. Then, it is not excluded that the models we have found to fail to describe correctly the data on certain observable ($|V_{cb}|$, for example) might actually be successful in describing the corresponding observable.

Finally we comment on the number of parameters of our models. The minimum number of parameters is 12 (10 real constants and 2 phases) as shown in subsection 6.3.1. Although it is larger than the 10 quark observables, the model is valuable in revealing the possible origin of the down-type and up-type quark mass hierarchies.

The results of our study show that describing correctly without severe fine-tuning the quark mass hierarchies, the quark mixing and CP violation in the quark sector is remarkably challenging within the modular invariance approach to the quark flavour problem.

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A Tensor product of A_4 group

We take the generators of A_4 group for the triplet as follows:

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{A.1}$$

where $\omega = e^{i\frac{2}{3}\pi}$. In this basis, the multiplication rules are:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}}$$
$$= (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus (a_3b_3 + a_1b_2 + a_2b_1)_{\mathbf{1}'} \oplus (a_2b_2 + a_1b_3 + a_3b_1)_{\mathbf{1}''}$$
$$\oplus \frac{1}{3}\begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix}_{\mathbf{3}}, \tag{A.2}$$

$$\mathbf{1}\otimes\mathbf{1} = \mathbf{1}\,, \qquad \mathbf{1}'\otimes\mathbf{1}' = \mathbf{1}''\,, \qquad \mathbf{1}''\otimes\mathbf{1}'' = \mathbf{1}'\,, \qquad \mathbf{1}'\otimes\mathbf{1}'' = \mathbf{1}\,,$$
$$\mathbf{1}' \otimes \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} = \begin{pmatrix} a_3 \\ a_1 \\ a_2 \end{pmatrix}_{\mathbf{3}}, \qquad \mathbf{1}'' \otimes \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} = \begin{pmatrix} a_2 \\ a_3 \\ a_1 \end{pmatrix}_{\mathbf{3}}, \tag{A.3}$$

where

$$S(\mathbf{1}') = 1\,, \qquad S(\mathbf{1}'') = 1, \qquad T(\mathbf{1}') = \omega\,, \qquad T(\mathbf{1}'') = \omega^2. \tag{A.4}$$

Further details can be found in the reviews [114–116].

B A measure of fit

As a measure of goodness of fit, we use the sum of one-dimensional $\Delta\chi^2$ for eight observable quantities $q_j = (m_d/m_b,\ m_s/m_b,\ m_u/m_t,\ m_c/m_t,\ |V_{us}|,\ |V_{cb}|,\ |V_{ub}|,\ \delta_{\rm CP})$. By employing the Gaussian approximation, we difine $N\sigma \equiv \sqrt{\Delta\chi^2}$, where

$$\Delta\chi^2 = \sum_j \left(\frac{q_j - q_{j,\text{best fit}}}{\sigma_j}\right)^2 . \tag{B.1}$$

C Massless quark in the mass matrix with weight $(6, 2, 0)$

Consider the mass matrix by only replacing in table 1 the weights $(6, 2, 0)$ of the right-handed quarks. Then, the mass matrix is obtained easily as:

$$M_q = v_a \begin{pmatrix} \alpha'_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} \tilde{Y}_3^{(8)} & \tilde{Y}_2^{(8)} & \tilde{Y}_1^{(8)} \\ Y_3^{(4)} & Y_2^{(4)} & Y_1^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}, \tag{C.1}$$

where

$$\tilde{Y}_i^{(8)} = f_q Y_i^{(8)} + Y_i^{\prime(8)}, \qquad f_q \equiv \alpha_a/\alpha'_a\,. \tag{C.2}$$

After putting $Y_i^{(4)}$, $Y_i^{(8)}$ and $Y_i^{\prime(8)}$ in eqs. (4.4) and (4.6), we can estimate the determinant of M_q as:

$$\begin{aligned} \text{Det}[M_q] &= v_q\alpha'_q\beta_q\gamma_q \\ &\times (Y_1+Y_2+Y_3)(Y_2^2+2Y_1Y_3)(2Y_1Y_2+Y_3^2)(Y_1^2+Y_2^2+Y_3^2-Y_1Y_2-Y_2Y_3-Y_1Y_3) = 0\,, \end{aligned} \tag{C.3}$$

because of $(Y_2^2 + 2Y_1Y_3) = 0$ as seen in eq. (4.3). We can also check that the matrix is rank 2. This implies that the mass matrix predicts $m_q = 0$.

References

[1] F. Feruglio, *Are neutrino masses modular forms?*, in *From My Vast Repertoire...: Guido Altarelli's Legacy*, A. Levy et al. eds., World Scientific (2019), p. 227–266 [DOI:10.1142/9789813238053_0012] [arXiv:1706.08749] [INSPIRE].

[2] R. de Adelhart Toorop, F. Feruglio and C. Hagedorn, *Finite Modular Groups and Lepton Mixing*, *Nucl. Phys. B* **858** (2012) 437 [arXiv:1112.1340] [INSPIRE].

[3] P.P. Novichkov, J.T. Penedo, S.T. Petcov and A.V. Titov, *Generalised CP Symmetry in Modular-Invariant Models of Flavour*, *JHEP* **07** (2019) 165 [arXiv:1905.11970] [INSPIRE].

[4] P.P. Novichkov, J.T. Penedo, S.T. Petcov and A.V. Titov, *Modular S_4 models of lepton masses and mixing*, *JHEP* **04** (2019) 005 [arXiv:1811.04933] [INSPIRE].

[5] P.P. Novichkov, S.T. Petcov and M. Tanimoto, *Trimaximal Neutrino Mixing from Modular A4 Invariance with Residual Symmetries*, *Phys. Lett. B* **793** (2019) 247 [arXiv:1812.11289] [INSPIRE].

[6] P.P. Novichkov, J.T. Penedo, S.T. Petcov and A.V. Titov, *Modular A_5 symmetry for flavour model building*, *JHEP* **04** (2019) 174 [arXiv:1812.02158] [INSPIRE].

[7] H. Okada and M. Tanimoto, *Modular invariant flavor model of A_4 and hierarchical structures at nearby fixed points*, *Phys. Rev. D* **103** (2021) 015005 [arXiv:2009.14242] [INSPIRE].

[8] H. Okada and M. Tanimoto, *Spontaneous CP violation by modulus τ in A_4 model of lepton flavors*, *JHEP* **03** (2021) 010 [arXiv:2012.01688] [INSPIRE].

[9] P.P. Novichkov, J.T. Penedo and S.T. Petcov, *Double cover of modular S_4 for flavour model building*, *Nucl. Phys. B* **963** (2021) 115301 [arXiv:2006.03058] [INSPIRE].

[10] J.T. Penedo and S.T. Petcov, *Lepton Masses and Mixing from Modular S_4 Symmetry*, *Nucl. Phys. B* **939** (2019) 292 [arXiv:1806.11040] [INSPIRE].

[11] T. Kobayashi et al., *Modular A_4 invariance and neutrino mixing*, *JHEP* **11** (2018) 196 [arXiv:1808.03012] [INSPIRE].

[12] T. Kobayashi, K. Tanaka and T.H. Tatsuishi, *Neutrino mixing from finite modular groups*, *Phys. Rev. D* **98** (2018) 016004 [arXiv:1803.10391] [INSPIRE].

[13] J.C. Criado and F. Feruglio, *Modular Invariance Faces Precision Neutrino Data*, *SciPost Phys.* **5** (2018) 042 [arXiv:1807.01125] [INSPIRE].

[14] S.T. Petcov and M. Tanimoto, *A_4 modular flavour model of quark mass hierarchies close to the fixed point $\tau = \omega$*, *Eur. Phys. J. C* **83** (2023) 579 [arXiv:2212.13336] [INSPIRE].

[15] T. Kobayashi et al., *New A_4 lepton flavor model from S_4 modular symmetry*, *JHEP* **02** (2020) 097 [arXiv:1907.09141] [INSPIRE].

[16] G.-J. Ding, S.F. King, X.-G. Liu and J.-N. Lu, *Modular S_4 and A_4 symmetries and their fixed points: new predictive examples of lepton mixing*, *JHEP* **12** (2019) 030 [arXiv:1910.03460] [INSPIRE].

[17] T. Kobayashi et al., *A_4 lepton flavor model and modulus stabilization from S_4 modular symmetry*, *Phys. Rev. D* **100** (2019) 115045 [*Erratum ibid.* **101** (2020) 039904] [arXiv:1909.05139] [INSPIRE].

[18] G.-J. Ding, S.F. King and X.-G. Liu, *Neutrino mass and mixing with A_5 modular symmetry*, *Phys. Rev. D* **100** (2019) 115005 [arXiv:1903.12588] [INSPIRE].

[19] G.-J. Ding, S.F. King and X.-G. Liu, *Modular A_4 symmetry models of neutrinos and charged leptons*, *JHEP* **09** (2019) 074 [arXiv:1907.11714] [INSPIRE].

[20] D. Zhang, *A modular A_4 symmetry realization of two-zero textures of the Majorana neutrino mass matrix*, *Nucl. Phys. B* **952** (2020) 114935 [arXiv:1910.07869] [INSPIRE].

[21] T. Kobayashi, T. Nomura and T. Shimomura, *Type II seesaw models with modular A_4 symmetry*, *Phys. Rev. D* **102** (2020) 035019 [arXiv:1912.00637] [INSPIRE].

[22] T. Kobayashi et al., *Modular S_3-invariant flavor model in* SU(5) *grand unified theory*, *PTEP* **2020** (2020) 053B05 [arXiv:1906.10341] [INSPIRE].

[23] H. Okada and Y. Orikasa, *Modular S_3 symmetric radiative seesaw model*, *Phys. Rev. D* **100** (2019) 115037 [arXiv:1907.04716] [INSPIRE].

[24] G.-J. Ding, S.F. King, C.-C. Li and Y.-L. Zhou, *Modular Invariant Models of Leptons at Level 7*, *JHEP* **08** (2020) 164 [arXiv:2004.12662] [INSPIRE].

[25] H. Okada and M. Tanimoto, *CP violation of quarks in A_4 modular invariance*, *Phys. Lett. B* **791** (2019) 54 [arXiv:1812.09677] [INSPIRE].

[26] T. Kobayashi et al., *Finite modular subgroups for fermion mass matrices and baryon/lepton number violation*, *Phys. Lett. B* **794** (2019) 114 [arXiv:1812.11072] [INSPIRE].

[27] F.J. de Anda, S.F. King and E. Perdomo, SU(5) *grand unified theory with A_4 modular symmetry*, *Phys. Rev. D* **101** (2020) 015028 [arXiv:1812.05620] [INSPIRE].

[28] H. Okada and M. Tanimoto, *Towards unification of quark and lepton flavors in A_4 modular invariance*, *Eur. Phys. J. C* **81** (2021) 52 [arXiv:1905.13421] [INSPIRE].

[29] J.-N. Lu, X.-G. Liu and G.-J. Ding, *Modular symmetry origin of texture zeros and quark lepton unification*, *Phys. Rev. D* **101** (2020) 115020 [arXiv:1912.07573] [INSPIRE].

[30] H. Okada and M. Tanimoto, *Quark and lepton flavors with common modulus τ in A_4 modular symmetry*, *Phys. Dark Univ.* **40** (2023) 101204 [arXiv:2005.00775] [INSPIRE].

[31] P. Chen, G.-J. Ding and S.F. King, SU(5) *GUTs with A_4 modular symmetry*, *JHEP* **04** (2021) 239 [arXiv:2101.12724] [INSPIRE].

[32] I. de Medeiros Varzielas, S.F. King and Y.-L. Zhou, *Multiple modular symmetries as the origin of flavor*, *Phys. Rev. D* **101** (2020) 055033 [arXiv:1906.02208] [INSPIRE].

[33] S.F. King and Y.-L. Zhou, *Trimaximal TM_1 mixing with two modular S_4 groups*, *Phys. Rev. D* **101** (2020) 015001 [arXiv:1908.02770] [INSPIRE].

[34] G.-J. Ding, F. Feruglio and X.-G. Liu, *Automorphic Forms and Fermion Masses*, *JHEP* **01** (2021) 037 [arXiv:2010.07952] [INSPIRE].

[35] C.-Y. Yao, J.-N. Lu and G.-J. Ding, *Modular Invariant A_4 Models for Quarks and Leptons with Generalized CP Symmetry*, *JHEP* **05** (2021) 102 [arXiv:2012.13390] [INSPIRE].

[36] X. Wang and S. Zhou, *Explicit perturbations to the stabilizer $\tau = i$ of modular A_5' symmetry and leptonic CP violation*, *JHEP* **07** (2021) 093 [arXiv:2102.04358] [INSPIRE].

[37] T. Kobayashi et al., *CP violation in modular invariant flavor models*, *Phys. Rev. D* **101** (2020) 055046 [arXiv:1910.11553] [INSPIRE].

[38] G.-J. Ding, F. Feruglio and X.-G. Liu, *CP symmetry and symplectic modular invariance*, *SciPost Phys.* **10** (2021) 133 [arXiv:2102.06716] [INSPIRE].

[39] T. Nomura and H. Okada, *A modular A_4 symmetric model of dark matter and neutrino*, *Phys. Lett. B* **797** (2019) 134799 [arXiv:1904.03937] [INSPIRE].

[40] T. Asaka, Y. Heo, T.H. Tatsuishi and T. Yoshida, *Modular A_4 invariance and leptogenesis*, *JHEP* **01** (2020) 144 [arXiv:1909.06520] [INSPIRE].

[41] T. Asaka, Y. Heo and T. Yoshida, *Lepton flavor model with modular A_4 symmetry in large volume limit*, *Phys. Lett. B* **811** (2020) 135956 [arXiv:2009.12120] [INSPIRE].

[42] T. Kobayashi, T. Shimomura and M. Tanimoto, *Soft supersymmetry breaking terms and lepton flavor violations in modular flavor models*, *Phys. Lett. B* **819** (2021) 136452 [arXiv:2102.10425] [INSPIRE].

[43] H. Okada, Y. Shimizu, M. Tanimoto and T. Yoshida, *Modulus τ linking leptonic CP violation to baryon asymmetry in A_4 modular invariant flavor model*, *JHEP* **07** (2021) 184 [arXiv:2105.14292] [INSPIRE].

[44] M. Tanimoto and K. Yamamoto, *Electron EDM arising from modulus τ in the supersymmetric modular invariant flavor models*, *JHEP* **10** (2021) 183 [arXiv:2106.10919] [INSPIRE].

[45] T. Kobayashi, H. Otsuka, M. Tanimoto and K. Yamamoto, *Lepton flavor violation, lepton $(g-2)_{\mu,e}$ and electron EDM in the modular symmetry*, *JHEP* **08** (2022) 013 [arXiv:2204.12325] [INSPIRE].

[46] F. Feruglio, A. Strumia and A. Titov, *Modular invariance and the QCD angle*, *JHEP* **07** (2023) 027 [arXiv:2305.08908] [INSPIRE].

[47] H.P. Nilles, S. Ramos-Sánchez and P.K.S. Vaudrevange, *Eclectic Flavor Groups*, *JHEP* **02** (2020) 045 [arXiv:2001.01736] [INSPIRE].

[48] S. Kikuchi et al., *Modular symmetry by orbifolding magnetized $T^2 \times T^2$: realization of double cover of Γ_N*, *JHEP* **11** (2020) 101 [arXiv:2007.06188] [INSPIRE].

[49] X.-G. Liu and G.-J. Ding, *Neutrino Masses and Mixing from Double Covering of Finite Modular Groups*, *JHEP* **08** (2019) 134 [arXiv:1907.01488] [INSPIRE].

[50] X.-G. Liu, C.-Y. Yao and G.-J. Ding, *Modular invariant quark and lepton models in double covering of S_4 modular group*, *Phys. Rev. D* **103** (2021) 056013 [arXiv:2006.10722] [INSPIRE].

[51] X. Wang, B. Yu and S. Zhou, *Double covering of the modular A_5 group and lepton flavor mixing in the minimal seesaw model*, *Phys. Rev. D* **103** (2021) 076005 [arXiv:2010.10159] [INSPIRE].

[52] C.-Y. Yao, X.-G. Liu and G.-J. Ding, *Fermion masses and mixing from the double cover and metaplectic cover of the A_5 modular group*, *Phys. Rev. D* **103** (2021) 095013 [arXiv:2011.03501] [INSPIRE].

[53] H. Okada and Y. Orikasa, *Lepton mass matrix from double covering of A_4 modular flavor symmetry*, *Chin. Phys. C* **46** (2022) 123108 [arXiv:2206.12629] [INSPIRE].

[54] G.-J. Ding, X.-G. Liu and C.-Y. Yao, *A minimal modular invariant neutrino model*, *JHEP* **01** (2023) 125 [arXiv:2211.04546] [INSPIRE].

[55] G.-J. Ding, F.R. Joaquim and J.-N. Lu, *Texture-zero patterns of lepton mass matrices from modular symmetry*, *JHEP* **03** (2023) 141 [arXiv:2211.08136] [INSPIRE].

[56] X.-G. Liu and G.-J. Ding, *Modular flavor symmetry and vector-valued modular forms*, *JHEP* **03** (2022) 123 [arXiv:2112.14761] [INSPIRE].

[57] T. Kobayashi et al., *Modular symmetry and non-Abelian discrete flavor symmetries in string compactification*, *Phys. Rev. D* **97** (2018) 116002 [arXiv:1804.06644] [INSPIRE].

[58] T. Kobayashi and S. Tamba, *Modular forms of finite modular subgroups from magnetized D-brane models*, *Phys. Rev. D* **99** (2019) 046001 [arXiv:1811.11384] [INSPIRE].

[59] A. Baur, H.P. Nilles, A. Trautner and P.K.S. Vaudrevange, *Unification of Flavor, CP, and Modular Symmetries*, *Phys. Lett. B* **795** (2019) 7 [arXiv:1901.03251] [INSPIRE].

[60] Y. Kariyazono et al., *Modular symmetry anomaly in magnetic flux compactification*, *Phys. Rev. D* **100** (2019) 045014 [arXiv:1904.07546] [INSPIRE].

[61] A. Baur, H.P. Nilles, A. Trautner and P.K.S. Vaudrevange, *A String Theory of Flavor and CP*, *Nucl. Phys. B* **947** (2019) 114737 [arXiv:1908.00805] [INSPIRE].

[62] T. Kobayashi and H. Otsuka, *Classification of discrete modular symmetries in Type IIB flux vacua*, *Phys. Rev. D* **101** (2020) 106017 [arXiv:2001.07972] [INSPIRE].

[63] H. Abe, T. Kobayashi, S. Uemura and J. Yamamoto, *Loop Fayet-Iliopoulos terms in T^2/Z_2 models: Instability and moduli stabilization*, *Phys. Rev. D* **102** (2020) 045005 [arXiv:2003.03512] [INSPIRE].

[64] H. Ohki, S. Uemura and R. Watanabe, *Modular flavor symmetry on a magnetized torus*, *Phys. Rev. D* **102** (2020) 085008 [arXiv:2003.04174] [INSPIRE].

[65] T. Kobayashi and H. Otsuka, *Challenge for spontaneous CP violation in Type IIB orientifolds with fluxes*, *Phys. Rev. D* **102** (2020) 026004 [arXiv:2004.04518] [INSPIRE].

[66] H.P. Nilles, S. Ramos-Sánchez and P.K.S. Vaudrevange, *Lessons from eclectic flavor symmetries*, *Nucl. Phys. B* **957** (2020) 115098 [arXiv:2004.05200] [INSPIRE].

[67] S. Kikuchi et al., *Revisiting modular symmetry in magnetized torus and orbifold compactifications*, *Phys. Rev. D* **102** (2020) 105010 [arXiv:2005.12642] [INSPIRE].

[68] H.P. Nilles, S. Ramos-Sánchez and P.K.S. Vaudrevange, *Eclectic flavor scheme from ten-dimensional string theory — I. Basic results*, *Phys. Lett. B* **808** (2020) 135615 [arXiv:2006.03059] [INSPIRE].

[69] A. Baur et al., *The eclectic flavor symmetry of the $\mathbb{Z}_2$ orbifold*, *JHEP* **02** (2021) 018 [arXiv:2008.07534] [INSPIRE].

[70] K. Ishiguro, T. Kobayashi and H. Otsuka, *Spontaneous CP violation and symplectic modular symmetry in Calabi-Yau compactifications*, *Nucl. Phys. B* **973** (2021) 115598 [arXiv:2010.10782] [INSPIRE].

[71] H.P. Nilles, S. Ramos-Sánchez and P.K.S. Vaudrevange, *Eclectic flavor scheme from ten-dimensional string theory — II detailed technical analysis*, *Nucl. Phys. B* **966** (2021) 115367 [arXiv:2010.13798] [INSPIRE].

[72] K. Hoshiya et al., *Classification of three-generation models by orbifolding magnetized $T^2 \times T^2$*, *PTEP* **2021** (2021) 033B05 [arXiv:2012.00751] [INSPIRE].

[73] A. Baur et al., *Siegel modular flavor group and CP from string theory*, Phys. Lett. B **816** (2021) 136176 [arXiv:2012.09586] [INSPIRE].

[74] A. Baur et al., *Top-down anatomy of flavor symmetry breakdown*, Phys. Rev. D **105** (2022) 055018 [arXiv:2112.06940] [INSPIRE].

[75] H.P. Nilles, S. Ramos-Sánchez, A. Trautner and P.K.S. Vaudrevange, *Orbifolds from $Sp(4, Z)$ and their modular symmetries*, Nucl. Phys. B **971** (2021) 115534 [arXiv:2105.08078] [INSPIRE].

[76] S. Kikuchi, T. Kobayashi and H. Uchida, *Modular flavor symmetries of three-generation modes on magnetized toroidal orbifolds*, Phys. Rev. D **104** (2021) 065008 [arXiv:2101.00826] [INSPIRE].

[77] C.-C. Li, X.-G. Liu and G.-J. Ding, *Modular symmetry at level 6 and a new route towards finite modular groups*, JHEP **10** (2021) 238 [arXiv:2108.02181] [INSPIRE].

[78] T. Kobayashi and H. Otsuka, *On stringy origin of minimal flavor violation*, Eur. Phys. J. C **82** (2022) 25 [arXiv:2108.02700] [INSPIRE].

[79] T. Kobayashi, H. Otsuka, M. Tanimoto and K. Yamamoto, *Modular symmetry in the SMEFT*, Phys. Rev. D **105** (2022) 055022 [arXiv:2112.00493] [INSPIRE].

[80] K. Hoshiya, S. Kikuchi, T. Kobayashi and H. Uchida, *Quark and lepton flavor structure in magnetized orbifold models at residual modular symmetric points*, Phys. Rev. D **106** (2022) 115003 [arXiv:2209.07249] [INSPIRE].

[81] T. Kobayashi et al., *Generalized matter parities from finite modular symmetries*, PTEP **2023** (2023) 043B02 [arXiv:2207.14014] [INSPIRE].

[82] S. Kikuchi, T. Kobayashi, M. Tanimoto and H. Uchida, *Texture zeros of quark mass matrices at fixed point $\tau = \omega$ in modular flavor symmetry*, Eur. Phys. J. C **83** (2023) 591 [arXiv:2207.04609] [INSPIRE].

[83] S. Kikuchi, T. Kobayashi, M. Tanimoto and H. Uchida, *Mass matrices with CP phase in modular flavor symmetry*, PTEP **2022** (2022) 113B07 [arXiv:2206.08538] [INSPIRE].

[84] S. Kikuchi et al., *Modular symmetry of soft supersymmetry breaking terms*, PTEP **2022** (2022) 123B02 [arXiv:2203.14667] [INSPIRE].

[85] S. Kikuchi, T. Kobayashi, Y. Ogawa and H. Uchida, *Yukawa textures in modular symmetric vacuum of magnetized orbifold models*, PTEP **2022** (2022) 033B10 [arXiv:2112.01680] [INSPIRE].

[86] A. Baur et al., *Completing the eclectic flavor scheme of the $\mathbb{Z}_2$ orbifold*, JHEP **06** (2021) 110 [arXiv:2104.03981] [INSPIRE].

[87] A. Baur et al., *The first string-derived eclectic flavor model with realistic phenomenology*, JHEP **09** (2022) 224 [arXiv:2207.10677] [INSPIRE].

[88] K. Ishiguro, T. Kobayashi and H. Otsuka, *Landscape of Modular Symmetric Flavor Models*, JHEP **03** (2021) 161 [arXiv:2011.09154] [INSPIRE].

[89] P.P. Novichkov, J.T. Penedo and S.T. Petcov, *Modular flavour symmetries and modulus stabilisation*, JHEP **03** (2022) 149 [arXiv:2201.02020] [INSPIRE].

[90] V. Knapp-Perez et al., *Matter matters in moduli fixing and modular flavor symmetries*, arXiv:2304.14437 [INSPIRE].

JHEP08(2023)086

[91] B.-Y. Qu, X.-G. Liu, P.-T. Chen and G.-J. Ding, *Flavor mixing and CP violation from the interplay of an S_4 modular group and a generalized CP symmetry*, Phys. Rev. D **104** (2021) 076001 [arXiv:2106.11659] [INSPIRE].

[92] J.C. Criado, F. Feruglio and S.J.D. King, *Modular Invariant Models of Lepton Masses at Levels 4 and 5*, JHEP **02** (2020) 001 [arXiv:1908.11867] [INSPIRE].

[93] S.J.D. King and S.F. King, *Fermion mass hierarchies from modular symmetry*, JHEP **09** (2020) 043 [arXiv:2002.00969] [INSPIRE].

[94] H. Kuranaga, H. Ohki and S. Uemura, *Modular origin of mass hierarchy: Froggatt-Nielsen like mechanism*, JHEP **07** (2021) 068 [arXiv:2105.06237] [INSPIRE].

[95] C.D. Froggatt and H.B. Nielsen, *Hierarchy of Quark Masses, Cabibbo Angles and CP Violation*, Nucl. Phys. B **147** (1979) 277 [INSPIRE].

[96] P.P. Novichkov, J.T. Penedo and S.T. Petcov, *Fermion mass hierarchies, large lepton mixing and residual modular symmetries*, JHEP **04** (2021) 206 [arXiv:2102.07488] [INSPIRE].

[97] F. Feruglio, V. Gherardi, A. Romanino and A. Titov, *Modular invariant dynamics and fermion mass hierarchies around $\tau = i$*, JHEP **05** (2021) 242 [arXiv:2101.08718] [INSPIRE].

[98] F. Feruglio, *Universal Predictions of Modular Invariant Flavor Models near the Self-Dual Point*, Phys. Rev. Lett. **130** (2023) 101801 [arXiv:2211.00659] [INSPIRE].

[99] F. Feruglio, *Fermion masses, critical behavior and universality*, JHEP **03** (2023) 236 [arXiv:2302.11580] [INSPIRE].

[100] S. Kikuchi et al., *Quark hierarchical structures in modular symmetric flavor models at level 6*, Phys. Rev. D **107** (2023) 055014 [arXiv:2301.03737] [INSPIRE].

[101] Y. Abe, T. Higaki, J. Kawamura and T. Kobayashi, *Quark masses and CKM hierarchies from S_4' modular flavor symmetry*, arXiv:2301.07439 [INSPIRE].

[102] S. Kikuchi et al., *Quark mass hierarchies and CP violation in $A_4 \times A_4 \times A_4$ modular symmetric flavor models*, JHEP **07** (2023) 134 [arXiv:2302.03326] [INSPIRE].

[103] Y. Abe, T. Higaki, J. Kawamura and T. Kobayashi, *Quark and lepton hierarchies from S_4' modular flavor symmetry*, Phys. Lett. B **842** (2023) 137977 [arXiv:2302.11183] [INSPIRE].

[104] S. Ferrara, D. Lust, A.D. Shapere and S. Theisen, *Modular Invariance in Supersymmetric Field Theories*, Phys. Lett. B **225** (1989) 363 [INSPIRE].

[105] S. Ferrara, D. Lust and S. Theisen, *Target Space Modular Invariance and Low-Energy Couplings in Orbifold Compactifications*, Phys. Lett. B **233** (1989) 147 [INSPIRE].

[106] L.J. Dixon, V. Kaplunovsky and J. Louis, *On Effective Field Theories Describing* (2, 2) *Vacua of the Heterotic String*, Nucl. Phys. B **329** (1990) 27 [INSPIRE].

[107] L.E. Ibanez and D. Lust, *Duality anomaly cancellation, minimal string unification and the effective low-energy Lagrangian of 4-D strings*, Nucl. Phys. B **382** (1992) 305 [hep-th/9202046] [INSPIRE].

[108] Y. Olguín-Trejo and S. Ramos-Sánchez, *Kähler potential of heterotic orbifolds with multiple Kähler moduli*, J. Phys. Conf. Ser. **912** (2017) 012029 [arXiv:1707.09966] [INSPIRE].

[109] M.-C. Chen, S. Ramos-Sánchez and M. Ratz, *A note on the predictions of models with modular flavor symmetries*, Phys. Lett. B **801** (2020) 135153 [arXiv:1909.06910] [INSPIRE].

JHEP08(2023)086

[110] Particle Data Group collaboration, *Review of Particle Physics*, *Phys. Rev. D* **98** (2018) 030001 [inSPIRE].

[111] S. Antusch and V. Maurer, *Running quark and lepton parameters at various scales*, *JHEP* **11** (2013) 115 [arXiv:1306.6879] [inSPIRE].

[112] Particle Data Group collaboration, *Review of Particle Physics*, *PTEP* **2022** (2022) 083C01 [inSPIRE].

[113] C. Jarlskog, *Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Nonconservation*, *Phys. Rev. Lett.* **55** (1985) 1039 [inSPIRE].

[114] H. Ishimori et al., *Non-Abelian Discrete Symmetries in Particle Physics*, *Prog. Theor. Phys. Suppl.* **183** (2010) 1 [arXiv:1003.3552] [inSPIRE].

[115] H. Ishimori et al., *An introduction to non-Abelian discrete symmetries for particle physicists*, Springer, *Lect. Notes Phys.* **858** (2012) 1 [inSPIRE].

[116] T. Kobayashi et al., *An Introduction to Non-Abelian Discrete Symmetries for Particle Physicists*, Springer, *Lect. Notes Phys.* **995** (2022) 1 [inSPIRE].

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