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Fermion masses and mixing in $SU(5) \times D_4 \times U(1)$ model

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Abstract

We propose a supersymmetric $SU(5) \times G_f$ GUT model with flavor symmetry $G_f = D_4 \times U(1)$ providing a good description of fermion masses and mixing. The model has twenty eight free parameters, eighteen are fixed to produce approximative experimental values of the physical parameters in the quark and charged lepton sectors. In the neutrino sector, the TBM matrix is generated at leading order through type I seesaw mechanism, and the deviation from TBM studied to reconcile with the phenomenological values of the mixing angles. Other features in the charged sector such as Georgi–Jarlskog relations and CKM mixing matrix are also studied.

1. Introduction

Standard Model (SM) of elementary particle physics is a great achievement of modern quantum physics; but despite this success basic questions still remain without answer; one of them concerns the origin of the three generations of fermions, quark–lepton masses and mixing angles. Although the SM is sufficient to describe the masses of charged leptons and quarks, neutrinos $(\nu_i)_{i=1,2,3}$ are considered as massless particles in this model which is in conflict with observations. Indeed, neutrino oscillation experiments have shown that they have very tiny masses m_i

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Table 1
The global fit values for the squared-mass differences Δm_{ij}^2 and mixing angles θ_{ij} as reported by Ref. [6]. NH and IH stand for normal and inverted hierarchies respectively.

Parameters	Best fit$^{(+1\sigma,+2\sigma,+3\sigma)}_{(-1\sigma,-2\sigma,-3\sigma)}$(NH)	Best fit$^{(+1\sigma,+2\sigma,+3\sigma)}_{(-1\sigma,-2\sigma,-3\sigma)}$(IH)
$\Delta m_{21}^2\left[10^{-5}\ \mathrm{eV}^2\right]$	$7.60^{(+0.19,+0.39,+0.58)}_{(-0.18,-0.34,-0.49)}$	$7.60^{(+0.19,+0.39,+0.58)}_{(-0.18,-0.34,-0.49)}$
$\left\|\Delta m_{31}^2\left[10^{-3}\ \mathrm{eV}^2\right]\right\|$	$2.48^{(+0.05,+0.11,+0.17)}_{(-0.07,-0.13,-0.18)}$	$-2.38^{(+0.05,+0.10,+0.16)}_{(-0.06,-0.12,-0.18)}$
$\sin^2\theta_{12}$	$0.323^{(+0.016,+0.034,+0.052)}_{(-0.016,-0.031,-0.045)}$	$0.323^{(+0.016,+0.034,+0.052)}_{(-0.016,-0.031,-0.045)}$
$\sin^2\theta_{23}$	$0.567^{(+0.032,+0.056,+0.076)}_{(-0.124,-0.153,-0.174)}$	$0.573^{(+0.025,+0.048,+0.067)}_{(-0.039,-0.138,-0.170)}$
$\sin^2\theta_{31}$	$0.0226^{(+0.0012,+0.0024,+0.0036)}_{(-0.0012,-0.0024,-0.0036)}$	$0.0229^{(+0.0012,+0.0023,+0.0036)}_{(-0.0012,-0.0024,-0.0036)}$

and that the different flavors are mixed with some mixing angles θ_{ij}. The PMNS matrix which describe the mixing in the lepton sector contains two large angles θ_{12} and θ_{23} consistent with tribimaximal mixing matrix (TBM) [1], and a vanishing angle θ_{13} which is in disagreement with the recent neutrino experiments[1] [2–5]. The measurements of the mixing angles and the squared-mass differences was reported by several global fits of neutrino data [6–8]; see Table 1. This mixing together with the non-zero neutrino mass might be the best evidence of physics beyond the standard model; in this context, many models have been proposed in recent years, and Supersymmetric Grand Unified Theories (SUSY-GUTs) are one of the most appealing extension of the SM unifying three forces of nature in a single gauge symmetry group [9–11]. These quantum field theories contain naturally the right-handed neutrino needed to generate light masses for neutrinos through the seesaw mechanism. Moreover, particles are unified into different representations of the GUT groups; for instance, in $SO(10)$ GUT model [11], all the fermions including the right-handed neutrino belong to the 16-dimensional spinor representation of $SO(10)$, and in $SU(5)$ GUT model, all the matter fits into two irreducible representations, the conjugate five $F = \bar{\mathbf{5}}$ and the ten $T = \mathbf{10}$ [10]. In addition, extending GUT models with flavor symmetries might be the key to understand the flavor structure; indeed many flavor symmetries have been suggested in GUT models, in particular, the non-abelian discrete alternating A_4 and symmetric S_4 groups are widely studied in the literature. These discrete groups have been used in many papers to realize the TBM matrix [15], and used recently to accommodate a non-zero reactor angle [16–20], and lately, the models studied in Refs. [21,22] provided successfully the masses for all fermions and the mixing in the charged and chargeless sectors including spontaneous CP violation. In addition, there are many other non-abelian discrete groups proposed as family symmetry with the $SU(5)$ GUT group; for example the $SU(5) \times T'$ model [23], and the $SU(5) \times \Delta(96)$ model [24]. As for the flavor models based on $SO(10)$ gauge group, we refer for instance to the $SO(10) \times A_4$ model [25], $SO(10) \times S_4$ model [26], $SO(10) \times PSL(2,7)$ model [27], and $SO(10) \times \Delta(27)$ model [28].

In this paper, we propose a supersymmetric $SU(5) \times G_f$ GUT model with flavor symmetry $G_f = D_4 \times U(1)$ providing a good description of fermion masses; and leading as well to neutrino mixing properties agreeing with known results. The model has *twenty eight free parameters* in which we need to fix eighteen in order to produce the approximative experimental

[1] In addition to the TBM matrix approximation, similar mixing matrices with vanishing θ_{13} have been proposed such as Bimaximal (BM) [12], Golden-Ratio (GR) [13] and Democratic [14] mixing pattern.

values of the physical parameters in the quark and lepton sectors as given by Tables (5.2)–(5.3) and Tables (5.5)–(5.9). To fix ideas, let us comment rapidly some key points of this G_f based construction and some motivations behind the choice of the discrete D_4 dihedral symmetry.

First, notice that the discrete flavor D_4 symmetry is the finite dihedral group; and, like the alternating A_4, it is also a non-abelian subgroup of the symmetric S_4 with particular properties. It has 5 irreducible representations: four singlets $\mathbf{1}_{p,q}$ with indices $p, q = \pm 1$; and one doublet $\mathbf{2}_{0,0}$ offering therefore several pictures to engineer hierarchy among the three generations of matter; for example by accommodating one generation in a given $\mathbf{1}_{p,q}$ representation, while the two others in the $\mathbf{2}_{0,0}$ doublet. Another example is to treat the three generations in quite similar manner by accommodating them in 1-dimensional representations $\mathbf{1}_{p_i,q_i}$ but with different characters. Recall that the order of D_4—which is 8—is linked to the sum of the squared dimensions of its five irreducible representations $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_5$ like $8 = 1^2_{+,+} + 1^2_{+,-} + 1^2_{-,+} + 1^2_{-,-} + 2^2_{0,0}$; the four representations $\boldsymbol{R}_i \equiv \mathbf{1}_{p,q}$ and the fifth $\boldsymbol{R}_5 = \mathbf{2}_{0,0}$ are indexed by the characters $\chi(\alpha), \chi(\beta)$ of the two non-commuting generators α and β of the dihedral D_4; a remarkable feature of discrete group theory allowing to distinguish the four D_4 singlets in a natural way.

Besides particularities of its singlet representations as well as its similarity with the popular alternating A_4 group; our interest into a flavor invariance $G_f \supset D_4$ has been also motivated from other reasons; in particular by the wish to complete partial results in supersymmetric GUTs which aren't embedded in brane picture of F-theory compactification along the line of [33]; and also by special features of the dihedral group. The discrete D_4 symmetry has been considered as flavor symmetry in several models to study the mixing in the lepton sector, see for instance [29–31], and one of its interesting properties is that it predicts the $\mu - \tau$ symmetry in a natural way as noticed by Grimus and Lavoura (GL) [29]. It was considered also in heterotic orbifold model building [32], as well as in constructing viable MSSM-like prototypes in F-theory [33]. But to our knowledge, the dihedral group D_4 was never used as a flavor symmetry in GUT models which doesn't descend from string compactification; this lack will be completed in present study.

To build the supersymmetric model $SU(5) \times D_4 \times U(1)_f$, we need building blocks of the construction and their couplings; in particular the chiral superfields Φ_i of the prototype; their quantum numbers under flavor symmetry and their superpotential $W(\Phi)$. After identifying the $SU(5)$ superfield spectrum with appropriate D_4 quantum numbers, we introduce an additional global $U(1)_f$ symmetry which will make our model quasi-realistic—$U(1)_f \equiv U(1)$. As we will show; this extra continuous symmetry is needed to control the superpotential in the quark- and lepton-sectors, and also to prevent dangerous operators that mediate rapid proton decay. Our $SU(5) \times D_4 \times U(1)$ model involves, in addition to the usual $SU(5)$ superfield spectrum collected in Tables (2.7)–(2.8), *eleven* flavon superfields carrying quantum numbers under the flavor symmetry $D_4 \times U(1)$ as given by (2.13)–(2.14); these flavon superfields will play an important role in obtaining the appropriate masses for the quarks and leptons. Moreover, we have *twenty eight* free parameters—*fifteen* Yukawa coupling constants, *eleven* flavon VEVs, the 45-dimensional Higgs *VEV* and the cutoff scale Λ—where we fix *eighteen* of them; *eight* in the quark and charged lepton sectors and *ten* in the neutrino sector. We end this study by performing a numerical study, where we use the experimental values of $\sin\theta_{ij}$ and Δm_{ij} to make predictions concerning numerical estimations of the parameters obtained in the neutrino sector.

The paper is organized as follows. In section 2, we present the superfield content of the $SU(5)$ model as well as a superfield spectrum containing flavons superfields in D_4 representations. Then, we assign $U(1)$ charges to all the superfields of the model. In section 3, we first study the neutrino mass matrix and its diagonalization with the TBM matrix; then we study the deviation of the TBM matrix by introducing extra flavon superfields, and we make a numerical study to fix the parameters of the neutrino sector. In section 4, we study the mass matrix of the up quark sector and we make a comment concerning the scale of the flavon VEVs derived from the experimental values of the quark up masses; then, we analyse the down quarks–charged leptons sector by calculating their mass matrices as well as the mixing matrix of the quarks. In section 5, we give our conclusion and numerical results. In Appendix A, we give all the higher dimensional operators yielding to the rapid proton decay which are forbidden by the $U(1)$ symmetry. In Appendix B, we give useful tools and details on D_4 tensor products.

2. $SU(5)$ model with $D_4 \times U(1)$ flavor symmetry

In this section, we first describe the chiral superfields content of the supersymmetric $SU(5)$ GUT model; then we extend this model by implementing the D_4 flavor symmetry accompanied with extra flavon superfields which are gauge singlets. This extension is further stretched with a flavor symmetry $U(1)$ needed to exclude unwanted couplings.

2.1. Superfields in $SU(5)$ model

In this subsection, we review briefly the building blocks of the usual supersymmetric $SU(5)$-GUT model that contain the minimal supersymmetric model (MSSM) quarks and leptons as well as the right-handed neutrino; we also use this description to fix some notations and conventions. We will focus mainly on the chiral superfields of the model and the invariant superpotential; the Kahler sector of the model involving as well gauge superfields is understood the presentation. The chiral sector of $SU(5)$ model has two kinds of building blocks: matter and Higgs; they are as follows

• *Matter superfields*

In supersymmetric $SU(5)$-GUT, each family $\mathcal{F}$ of quarks Q (with colors r, b, g) and leptons L fits nicely into a reducible $SU(5)$ representation involving the leading irreducible $\mathbf{1}$, $\bar{\mathbf{5}}$, $\mathbf{10}$. In superspace language, left-handed fermions are described by chiral superfields $F_i \equiv \bar{\mathbf{5}}_i$ and $T_i \equiv \mathbf{10}_i$; the right-handed neutrinos are also described by chiral superfields but living in $SU(5)$ singlets $N_i \equiv \mathbf{1}_i$. The index $i = 1, 2, 3$ refers to the three possible generations of matter $\mathcal{F}_i = \{F_i, T_i, N_i\}$; for example the first family $\mathcal{F}_1$, the constituents of F_1 and T_1 are explicitly as follows [35]

$$F_1 = \begin{pmatrix} d_r^c \\ d_b^c \\ d_g^c \\ e^- \\ -\nu_e \end{pmatrix} \quad , \quad T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_g^c & -u_b^c & u_r & d_r \\ -u_g^c & 0 & u_r^c & u_b & d_b \\ u_b^c & -u_r^c & 0 & u_g & d_g \\ -u_r & -u_b & -u_g & 0 & e^c \\ -d_r & -d_b & -d_g & -e^c & 0 \end{pmatrix} \tag{2.1}$$

• *Higgs superfields*

We distinguish several kinds of $SU(5)$-GUT Higgs superfields; in particular the H_5, $H_{\bar{5}}$, H_{24} and the $H_{\overline{45}}$. The chiral superfields $H_5 = 5_{H_u}$ and $H_{\bar{5}} = \overline{5}_{H_d}$ are respectively the analogue of two light Higgs doublet superfields H_u and H_d of the MSSM; in general the MSSM Higgs doublet H_d is a combination of the $H_{\bar{5}}$ Higgs with the 45-dimensional Higgs denoted by $H_{\overline{45}}$. This extra Higgs superfield will also used later on in order to distinguish the down quarks masses from the leptons masses.

The $SU(5)$ GUT symmetry is broken down to the standard model symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y$ by the VEV of the adjoint Higgs H_{24}. This is done by choosing $\langle H_{24}\rangle$ along the following particular Cartan direction in the Lie algebra of $SU(5)$

$$\langle H_{24}\rangle = \sqrt{\frac{2}{15}}\upsilon_{24}\begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & -\frac{3}{2} & \\ & & & & \frac{3}{2} \end{pmatrix} \tag{2.2}$$

so the $SU(5)$ fields are given in standard model terms as

$$\begin{aligned} 10_M &\to (3,2)_{\frac{1}{3}} + (\overline{3},1)_{\frac{-4}{3}} + (1,1)_2 \\ \overline{5}_M &\to (1,2)_{-1} + (\overline{3},1)_{\frac{2}{3}} \\ 5_{H_u} &\to (1,2)_1 + (3,1)_{\frac{-2}{3}} \\ \overline{5}_{H_d} &\to (1,2)_{-1} + (\overline{3},1)_{\frac{2}{3}} \end{aligned} \tag{2.3}$$

and

$$24 \to (8,1)_0 + (1,3)_0 + (1,1)_0 + (3,2)_{\frac{-5}{3}} + \left(\overline{3},2\right)_{\frac{5}{3}} \tag{2.4}$$

as well as

$$45 \to (8,2)_1 + \left(\overline{6},1\right)_{\frac{-2}{3}} + (3,3)_{\frac{-2}{3}} + \left(\overline{3},2\right)_{\frac{-7}{3}} + (3,1)_{\frac{-1}{3}} + \left(\overline{3},1\right)_{\frac{8}{3}} + (1,2)_1 \tag{2.5}$$

In what follows we describe our extension of supersymmetric $SU(5)$-GUT by a global flavor symmetry G_f which is given $D_4 \times U(1)_f$, the product of the finite discrete Dihedral group and the $U(1)_f$ global continuous phase.

2.2. Implementing D_4 flavor symmetry

Here, we present our extension of the supersymmetric $SU(5)$ GUT model by the flavor symmetry D_4, details of the Dihedral group D_4 are provided in Appendix B. First, we give the D_4-quantum numbers of the superfields of usual SUSY $SU(5)$ matter; then we describe the needed extra matter required by dihedral flavor symmetry.

In the usual $SU(5)$ model reviewed in previous subsection, the matter and Higgs superfields are as collected in first line of Tables (2.7)–(2.8); they are unified in the $SU(5)$ representations with link to MSSM as

$$\begin{aligned} 10_m &= (u^c, e^c, Q_L) \quad , \quad & 5_{H_u} &= (\Delta_u, H_u) \\ \overline{5}_m &= (d^c, L) \quad , \quad & \overline{5}_{H_d} &= (\Delta_d, H_d) \end{aligned} \tag{2.6}$$

The three generations of 10^i_m and $\overline{5}^i_m$ are denoted as T_i and F_i respectively, the three right-handed neutrinos denoted as N_i are singlets under $SU(5)$; and the two GUT Higgses denoted as H_5 and $H_{\overline{5}}$ like 5_{H_u} and $\overline{5}_{H_d}$.

In our extension with a D_4 flavor symmetry, we have a larger set of chiral superfields that can be organized into two basic subsets: (a) the usual $SU(5)$ matter and Higgs superfields; but carrying as well quantum numbers under D_4; and (b) an extra subset of chiral superfields required by D_4 flavor invariance; they are as described below.

a) Matter and Higgs sectors in $SU(5) \times D_4$

The superfield content of this sector is same as the $SU(5)$ matter and Higgs superfields; but with extra quantum numbers under D_4 flavor invariance as given here below

Matter	T_1	T_2	T_3	F_1	$F_{2,3}$	N_1	$N_{2,3}$
$SU(5)$	10_m^1	10_m^2	10_m^3	$\overline{5}_m^1$	$\overline{5}_m^{2,3}$	1_ν^1	$1_\nu^{2,3}$
D_4	$1_{+,-}$	$1_{+,-}$	$1_{+,+}$	$1_{+,-}$	$2_{0,0}$	$1_{+,+}$	$2_{0,0}$

(2.7)

and

Higgs	H_5	$H_{\overline{5}}$	$H_{\overline{45}}$
$SU(5)$	5_{H_u}	$\overline{5}_{H_d}$	$\overline{45}_H$
D_4	$1_{+,-}$	$1_{+,+}$	$1_{+,-}$

(2.8)

The matter superfields 10_m^i of the three generations $i = 1, 2, 3$ are assigned into the D_4 representations $1_{+,-}$, $1_{+,-}$ and $1_{+,+}$ respectively; while the $\overline{5}_m^i$ matter superfields are assigned into the D_4 singlet $1_{+,-}$ and the D_4 doublet $2_{0,0}$. The right-handed neutrino N_1 sits in the D_4 trivial singlet $1_{+,+}$, and the two $N_{2,3}$ sit together in the D_4 doublet $2_{0,0}$. The GUT Higgses H_5, $H_{\overline{5}}$ and $H_{\overline{45}}$ are put in different D_4 singlets; $1_{+,-}$, $1_{+,+}$ and $1_{+,-}$ respectively.

b) Flavon sector

In addition to the $SU(5)$ superfields of (2.7)–(2.8), the $SU(5) \times D_4$ model has eleven flavon chiral superfields namely four doublets and seven singlets; they transform as singlets under gauge group $SU(5)$, but carry charges under D_4 flavor symmetry as follows

Flavons	Γ	Ω	$\digamma$	ϕ	φ	η	χ	σ	ρ	ρ'	ζ
$SU(5)$	1	1	1	1	1	1	1	1	1	1	1
D_4	$1_{+,-}$	$1_{+,-}$	$1_{+,-}$	$2_{0,0}$	$2_{0,0}$	$1_{+,+}$	$2_{0,0}$	$2_{0,0}$	$1_{+,-}$	$1_{-,-}$	$1_{+,+}$

(2.9)

These flavon superfields couple to the matter and Higgs superfields of the model. The above quantum numbers are required by the building of the chiral superpotential $W_{SU_5 \times D_4}$ of the supersymmetric model. This complex superpotential is a superspace density which, after performing superspace integration, leads to a space time lagrangian density $\mathcal{L}_{SU_5 \times D_4}$ describing matter couplings through Higgs and flavons. The typical form of $\mathcal{L}_{SU_5 \times D_4}$ is given by

$$\mathcal{L}_{SU_5 \times D_4} = \int d^2\theta \, W_{SU_5 \times D_4}(\Phi_1, \ldots) + hc \tag{2.10}$$

where the generic Φ_i's stand for the chiral superfields of Tables (2.7)–(2.9). This superpotential involves several free coupling parameters to be studied in forthcoming sections. The flavons in Table (2.9) have been required by D_4 invariance; they are briefly commented below:

(i) *Neutrinos couplings*

Invariant neutrinos superpotential $W_{SU_5 \times D_4}(N, ..)$ under D_4 flavor symmetry requires in turns the flavons η, χ, ρ, ρ', ζ, σ:

- the flavon η and χ are needed to produce the TBM matrix in the neutrino mass matrix.
- the flavons ρ, ρ', ζ and σ are added to generate the deviation from TBM matrix.

(ii) *Quarks and charged leptons superpotentials*
Flavor symmetry invariant superpotentials $W_{SU_5 \times D_4}(T, F, ..)$ involving quarks and charged leptons require the flavon superfields Γ, Ω, $\digamma$, ϕ, φ with quantum numbers as listed in (2.9) for the following purposes:

- the three flavons Ω, Γ and $\digamma$ contribute to the up-, charm- and top-quark masses respectively.
- the two flavons Γ and Ω are also needed by down quarks/charged leptons in order to generate masses for the first two families.
- the flavon ϕ is required by down quarks/charged leptons in order to produce the mass of the third family.
- the flavon φ is needed for two goals: first to contribute to the mass of the first two generations of down quarks/charged leptons together with the flavon singlets Γ and Ω; and second to couple to the 45-dimensional Higgs $H_{\overline{45}}$ in order to distinguish between the down quarks and charged leptons mass matrices.

2.3. *Need of $U(1)_f$ symmetry*

In order to engineer a semi-realistic model, we need additional flavor symmetries; in our D_4 based proposal, we found that we have to add an abelian $U(1)$ symmetry to fully control the couplings of $SU(5) \times D_4$ model for reasons such as the ones given below:

(i) *Eliminate unwanted couplings*
The global $U(1)$ symmetry is necessary to eliminate unwanted couplings and to produce the observed mass hierarchies, it makes the model quasi-realistic for the two following things:

- first to control the superpotential of the quark and lepton sectors in the $SU(5) \times D_4$ model; for example the flavon $\digamma$, transforming as $1_{+,-}$, is used to generate a heavy mass for the top quark; but the two other flavons Γ and Ω share the same D_4 representation $1_{+,-}$ and so can couple quark and lepton superfields in a D_4 invariant manner. These coupling cannot be dropped out without imposing an extra constraint; moreover, the three flavons could be mixed in the operators of each family of the Yukawa up type; so they could affect the top quark mass, and consequently risking to lose the mass hierarchy between the top and the up, charm quarks. This issue is handled by accommodating the flavons which possess the same D_4 representation in different $U(1)$ representations as in Table (2.13).
- second, the $U(1)$ charge assignments are chosen to produce the TBM as well as its deviation to get a non-zero reactor angle in the neutrino sector which will be discussed in section 3.

(ii) *Avoid rapid proton decay*
The $U(1)$ flavor symmetry is also needed to forbid the operators yielding to rapid proton decay such as the couplings of type $10_m.\overline{5}_m.\overline{5}_m$. The $SU(5) \times D_4$ model have several invariant operators of this type and of other types which will be discussed in Appendix A; they are prevented by the extra global $U(1)$ symmetry with charge assignments as in the following tables:

* *families*

matter	T_1	T_2	T_3	F_1	$F_{2,3}$	N_1	$N_{2,3}$
$SU(5)$	10_m^3	10_m^1	10_m^1	$\overline{5}_m^1$	$\overline{5}_m^{2,3}$	1_ν^1	$1_\nu^{2,3}$
D_4	$1_{+,-}$	$1_{+,-}$	$1_{+,+}$	$1_{+,-}$	$2_{0,0}$	$1_{+,+}$	$2_{0,0}$
$U(1)$	12	7	−27	14	14	−6	−6

(2.11)

* *Higgs*

Higgs	H_5	$H_{\overline{5}}$	$H_{\overline{45}}$
$SU(5)$	5_{H_u}	$\overline{5}_{H_d}$	$\overline{45}_H$
D_4	$1_{+,-}$	$1_{+,+}$	$1_{+,-}$
$U(1)$	−8	11	10

(2.12)

* *flavons*

flavons	Γ	Ω	F	ϕ	φ
$SU(5)$	1	1	1	1	1
D_4	$1_{+,-}$	$1_{+,-}$	$1_{+,-}$	$2_{0,0}$	$2_{0,0}$
$U(1)$	−6	−16	62	2	−31

(2.13)

flavons	η	χ	σ	ρ	ρ'	ζ
$SU(5)$	1	1	1	1	1	1
D_4	$1_{+,+}$	$2_{0,0}$	$2_{0,0}$	$1_{+,-}$	$1_{-,-}$	$1_{+,+}$
$U(1)$	12	12	−24	−24	−24	36

(2.14)

3. Neutrino sector in $SU(5) \times D_4 \times U(1)$ model

In this section, we first study the mass matrices of Dirac and Majorana neutrinos; then we use the seesaw type I to get a neutrino mass matrix compatible with TBM as a leading approximation. Next, we study the deviation from TBM by adding new flavons. Notice that the right-handed neutrinos are $SU(5)$ singlets, thus the light neutrino masses are only generated through type-I seesaw mechanism [34]

$$m_\nu = m_D M_R^{-1} m_D^T \tag{3.1}$$

where the m_D and M_R are the Dirac and the Majorana mass matrices respectively.

3.1. Neutrino mass matrix and tribimaximal mixing

We begin by considering Dirac mass matrix involving left- and right-handed neutrinos; and turn after to calculate the Majorana masses.

3.1.1. Dirac neutrinos

The Dirac mass matrix couples the left-handed neutrinos in the $(F_i)_{i=1,2,3}$ to the right-handed ones $(N_i)_{i=1,2,3}$ living in different representations of $SU(5) \times G_f$ with flavor symmetry $G_f =$

$D_4 \times U(1)$. As described in section 2, the F_1 lives in the non-trivial D_4 singlet $1_{+,-}$ while F_2 and F_3 live together in the D_4 doublet $2_{0,0}$; they have the same $U(1)$ charge $q_{F_i} = 14$. The right-handed neutrinos have different quantum numbers under D_4; the N_1 lives in the D_4 representation $1_{+,+}$ while N_2 and N_3 live together in the D_4 doublet $2_{0,0}$; they have the same $U(1)$ charge $q_{N_i} = -6$. The chiral superpotential $W_D(F, N, H)$ for neutrino Yukawa couplings respecting gauge invariance and flavor $D_4 \times U(1)$ symmetry is given by

$$W_D = \lambda_1 N_1 F_1 H_5 + \lambda_2 N_{2,3} F_{2,3} H_5 \tag{3.2}$$

where λ_1 and λ_2 are Yukawa coupling constants. Using the tensor product of D_4 irreducible representations given in Eqs. (B.4)–(B.5) and denoting the Higgs by H_u, the superpotential (3.2) become

$$W_D = \lambda_1 H_u(\nu_e L_e) + \lambda_2 H_u \left(\nu_\mu L_\mu + \nu_\tau L_\tau\right) \tag{3.3}$$

When the Higgs doublet develop its VEV as usual $\langle H_u \rangle = \upsilon_u$, we get the Dirac mass matrix of neutrinos

$$m_D = \upsilon_u \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \tag{3.4}$$

3.1.2. Majorana neutrinos

A Majorana mass matrix couples the three right-handed neutrinos N_i to themselves; this mass matrix is obtained from the superpotential $W_M(N, ...)$ respecting gauge invariance and flavor symmetry of the model. Using Tables (2.11)–(2.14), one can check that this chiral superpotential is given by

$$W_M = \lambda_3 N_1 N_1 \eta + \lambda_4 N_{2,3} N_{2,3} \eta + \lambda_5 N_1 N_{2,3} \chi \tag{3.5}$$

In this expression, we have added the third term involving the flavon χ to satisfy the TBM conditions and to generate appropriate masses for the neutrinos. This term—which is at the renormalizable level—will contribute to the entries (12) and (13) in the Majorana mass matrix. By using the multiplication rule of D_4 representations, the superpotential W_M develops into

$$W_M = \lambda_3 (\nu_1 \nu_1) \eta + \lambda_4 (\nu_2 \nu_3 + \nu_3 \nu_2) \eta + \lambda_5 \nu_1 (\nu_2 \chi_2 + \nu_3 \chi_1) \tag{3.6}$$

and by taking the VEVs of the flavons χ and η as

$$\langle \chi_1 \rangle = \langle \chi_2 \rangle = \upsilon_\chi, \qquad \langle \eta \rangle = \upsilon_\eta$$

we find the Majorana neutrino mass matrix M_R as follows

$$M_R = \begin{pmatrix} \lambda_3 \upsilon_\eta & \lambda_5 \upsilon_\chi & \lambda_5 \upsilon_\chi \\ \lambda_5 \upsilon_\chi & 0 & \lambda_4 \upsilon_\eta \\ \lambda_5 \upsilon_\chi & \lambda_4 \upsilon_\eta & 0 \end{pmatrix} \tag{3.7}$$

The light neutrino mass matrix is obtained using type I seesaw mechanism formula $m_\nu = m_D M_R^{-1} m_D^T$, and we find

$$m_\nu = \upsilon_u^2 \begin{pmatrix} \frac{\lambda_1^2 \lambda_4 \upsilon_\eta}{\lambda_3 \lambda_4 \upsilon_\eta^2 - 2\lambda_5^2 \upsilon_\chi^2} & -\frac{\lambda_1 \lambda_2 \lambda_5 \upsilon_\chi}{\lambda_3 \lambda_4 \upsilon_\eta^2 - 2\lambda_5^2 \upsilon_\chi^2} & -\frac{\lambda_1 \lambda_2 \lambda_5 \upsilon_\chi}{\lambda_3 \lambda_4 \upsilon_\eta^2 - 2\lambda_5^2 \upsilon_\chi^2} \\ -\frac{\lambda_1 \lambda_2 \lambda_5 \upsilon_\chi}{\lambda_3 \lambda_4 \upsilon_\eta^2 - 2\lambda_5^2 \upsilon_\chi^2} & \frac{\lambda_2^2 \lambda_5^2 \upsilon_\chi^2}{\lambda_3 \lambda_4^2 \upsilon_\eta^3 - 2\lambda_4 \lambda_5^2 \upsilon_\eta \upsilon_\chi^2} & -\frac{\lambda_2^2 \left(\lambda_5^2 \upsilon_\chi^2 - \lambda_3 \lambda_4 \upsilon_\eta^2\right)}{\lambda_3 \lambda_4^2 \upsilon_\eta^3 - 2\lambda_4 \lambda_5^2 \upsilon_\eta \upsilon_\chi^2} \\ -\frac{\lambda_1 \lambda_2 \lambda_4 \lambda_5 \upsilon_\eta \upsilon_\chi}{\lambda_3 \lambda_4^2 \upsilon_\eta^3 - 2\lambda_4 \lambda_5^2 \upsilon_\eta \upsilon_\chi^2} & -\frac{\lambda_2^2 \left(\lambda_5^2 \upsilon_\chi^2 - \lambda_3 \lambda_4 \upsilon_\eta^2\right)}{\lambda_3 \lambda_4^2 \upsilon_\eta^3 - 2\lambda_4 \lambda_5^2 \upsilon_\eta \upsilon_\chi^2} & \frac{\lambda_2^2 \lambda_5^2 \upsilon_\chi^2}{\lambda_3 \lambda_4^2 \upsilon_\eta^3 - 2\lambda_4 \lambda_5^2 \upsilon_\eta \upsilon_\chi^2} \end{pmatrix} \tag{3.8}$$

this form of m_ν can realize the TBM matrix by adopting the following

$$\begin{aligned} \lambda_1 &= \lambda_2 \\ \lambda_4 \upsilon_\eta &= \lambda_3 \upsilon_\eta + \lambda_5 \upsilon_\chi \end{aligned} \tag{3.9}$$

so the above mass matrix m_ν is diagonalized as $M_\nu = U^T m_\nu U = \text{diag}(m_1, m_2, m_3)$ with the TBM matrix U given by

$$U = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{3.10}$$

It predicts the mixing angles as follows

$$\sin^2\theta_{12} = \frac{1}{3}, \qquad \sin^2\theta_{23} = \frac{1}{2}, \qquad \sin^2\theta_{13} = 0 \tag{3.11}$$

the eigen-masses are

$$m_1 = \frac{\lambda_1^2 \upsilon_u^2}{\lambda_3 \upsilon_\eta - \lambda_5 \upsilon_\chi}, \qquad m_2 = \frac{\lambda_1^2 \upsilon_u^2}{\lambda_3 \upsilon_\eta + 2\lambda_5 \upsilon_\chi}, \qquad m_3 = -\frac{\lambda_1^2 \upsilon_u^2}{\lambda_3 \upsilon_\eta + \lambda_5 \upsilon_\chi} \tag{3.12}$$

which yield to a non-vanishing solar and the atmospheric mass-squared differences Δm_{21}^2 and Δm_{31}^2.

3.2. *Deviation of mixing angles θ_{13} and θ_{23}*

In this subsection we study the deviation from TBM matrix which consists of breaking the μ–τ symmetry in the neutrino mass matrix in order to reconcile the reactor angle θ_{13} with the global fit data in Table 1. Recently, the deviation from TBM using additional flavons has been extensively studied in the literature and there are two matrix perturbations that allow for a suitable deviation of the mixing angles (for deviation by using non-trivial singlets, see for example Ref. [36]), they are:

$$\delta M_{33}^{12} = \varepsilon \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \delta M_{22}^{13} = \varepsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.13}$$

where the indices (12), (33), (13) and (22) are the elements that should be perturbed in the neutrino matrix to deviate from TBM and ε is the deviation parameter.

Using the flavon superfields σ, ζ, ρ and ρ' of Table (2.14), we see that we can perform a symmetric perturbation of the superpotential (3.5) that induces a deviation of the Majorana neutrino mass matrix M_R of Eq. (3.7). Thus, the additional higher dimensional operators that respect the symmetries of the model are as follows:

$$\delta W_M = \frac{1}{\Lambda}\left(\lambda_6 N_1 N_{2,3} \sigma\zeta + \lambda_7 N_{2,3} N_{2,3} \rho\zeta + \lambda_8 N_{2,3} N_{2,3} \rho'\zeta\right) \tag{3.14}$$

The invariance of δW_M may be explicitly exhibited by using the D_4 representation language,

$$\begin{aligned} N_1 N_{2,3} \sigma\zeta &\sim 1_{+,+} \otimes 2_{0,0} \otimes 2_{0,0} \otimes 1_{+,+} \\ N_{2,3} N_{2,3} \rho\zeta &\sim 2_{0,0} \otimes 2_{0,0} \otimes 1_{+,-} \otimes 1_{+,+} \\ N_{2,3} N_{2,3} \rho'\zeta &\sim 2_{0,0} \otimes 2_{0,0} \otimes 1_{-,-} \otimes 1_{+,+} \end{aligned} \tag{3.15}$$

Hence, to obtain the desired D_4 invariant, the tensor product between the D_4 doublets should be $1_{+,+}$ for the first term, $1_{+,-}$ for the second term and $1_{-,-}$ for the last term. Thus, we obtain

$$\delta W_M = \frac{1}{\Lambda}\left(\lambda_6(\nu_1\nu_3)\sigma\zeta + \lambda_7\left(\nu_2\nu_2+\nu_3\nu_3\right)\rho\zeta + \lambda_8\left(\nu_2\nu_2-\nu_3\nu_3\right)\rho'\zeta\right) \tag{3.16}$$

Assuming that

$$\lambda_7 = \lambda_8, \qquad \lambda_6 = 2\lambda_7 \tag{3.17}$$

and if we choose the VEVs of the flavons as

$$\langle\rho\rangle = \langle\rho'\rangle = \langle\sigma\rangle, \qquad \text{with} \qquad \langle\sigma\rangle = (\upsilon_\sigma, 0)^T \tag{3.18}$$

we get the second matrix perturbation in Eq. (3.13)

$$\delta M = \Lambda\begin{pmatrix} 0 & 0 & \varepsilon \\ 0 & \varepsilon & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}, \qquad \text{with} \qquad \varepsilon = \lambda_6\frac{\langle\zeta\rangle\langle\sigma\rangle}{\Lambda^2} \tag{3.19}$$

With this correction, the previous Majorana neutrino mass matrix M_R gets deformed as

$$M'_R = \Lambda\begin{pmatrix} \frac{\lambda_3\upsilon_\eta}{\Lambda} & \frac{\lambda_5\upsilon_\chi}{\Lambda} & \frac{\lambda_5\upsilon_\chi}{\Lambda}+\varepsilon \\ \frac{\lambda_5\upsilon_\chi}{\Lambda} & \varepsilon & \frac{\lambda_3\upsilon_\eta+\lambda_5\upsilon_\chi}{\Lambda} \\ \frac{\lambda_5\upsilon_\chi}{\Lambda}+\varepsilon & \frac{\lambda_3\upsilon_\eta+\lambda_5\upsilon_\chi}{\Lambda} & 0 \end{pmatrix} \tag{3.20}$$

In order to extract the mixing matrix and the neutrino masses, we will parameterize M'_R in the following way

$$\begin{aligned} a &= \tfrac{\lambda_3\upsilon_\eta}{\Lambda} \\ c &= \tfrac{\lambda_5\upsilon_\chi}{\Lambda} \end{aligned} \tag{3.21}$$

which leads to

$$M'_R = \Lambda\begin{pmatrix} a & c & c+\varepsilon \\ c & \varepsilon & a+c \\ c+\varepsilon & a+c & 0 \end{pmatrix} \tag{3.22}$$

Notice that since the Dirac mass matrix m_D is diagonal (see Eq. (3.4)), it does not affect the correction induced in the Majorana matrix M'_R, and by using type I seesaw mechanism formula $m_\nu^{eff} = m_D M_R'^{-1} m_D^T$, we obtain the new neutrino mass matrix with elements given explicitly as

$$\begin{aligned} m_{11} &= \frac{m_0}{k}\left(a^2+2ac+c^2\right) \\ m_{22} &= \frac{m_0}{k}(c^2+2c\varepsilon+\varepsilon^2) \\ m_{33} &= -\frac{m_0}{k}(a\varepsilon-c^2) \\ m_{12} = m_{21} &= -\frac{m_0}{k}\left(a\varepsilon+c\varepsilon+ac+c^2\right) \\ m_{13} = m_{31} &= \frac{m_0}{k}\left(c\varepsilon-ac-c^2+\varepsilon^2\right) \\ m_{23} = m_{32} &= -\frac{m_0}{k}(-a^2-ac+c^2+\varepsilon c) \end{aligned} \tag{3.23}$$

where $k = a^3 + 2a^2c - ac^2 - 2ac\varepsilon - 2c^3 - c^2\varepsilon + 2c\varepsilon^2 + \varepsilon^3$ and $m_0 = \frac{\lambda_1^2 \upsilon_u^2}{\Lambda}$. This is a symmetric matrix that can be diagonalized by a similarity transformation like $m_\nu^{diag} = \tilde{U}^T m_\nu^{eff} \tilde{U}$. The system of eigenvectors and eigenvalues can be computed perturbatively; we find up to order $O(\varepsilon^2)$, the unitary matrix $\tilde{U}$ which diagonalize the neutrino mass matrix m_ν^{eff} given in terms of its eigenvectors as

$$\tilde{U} = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & -\frac{\varepsilon}{2a\sqrt{2}} \\ \frac{1}{\sqrt{6}} + \frac{3\varepsilon}{4a\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} + \frac{\varepsilon}{4a\sqrt{2}} \\ \frac{1}{\sqrt{6}} - \frac{3\varepsilon}{4a\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{\varepsilon}{4a\sqrt{2}} \end{pmatrix} + O\left(\varepsilon^2\right) \tag{3.24}$$

consequently, the reactor and atmospheric angles develops into

$$\sin\theta_{13} = \left|\frac{\varepsilon}{2a\sqrt{2}}\right|, \qquad \sin\theta_{23} = \left|\frac{\varepsilon}{4a\sqrt{2}} - \frac{1}{\sqrt{2}}\right| \tag{3.25}$$

while the solar angle θ_{12} maintain its TBM value; $\sin\theta_{12} = \frac{1}{\sqrt{3}}$. It is easy to check that the matrix $\tilde{U}$ coincides with the TBM matrix in the limit $\varepsilon \to 0$. As for the eigenvalues of m_ν^{eff}, they read up to order $O(\varepsilon^2)$,

$$\begin{aligned} m_1 &= \frac{m_0}{-c+\sqrt{a^2-a\varepsilon+\varepsilon^2}} \\ m_2 &= \frac{m_0}{\varepsilon+a+2c} \\ m_3 &= -\frac{m_0}{c+\sqrt{a^2-a\varepsilon+\varepsilon^2}} \end{aligned} \tag{3.26}$$

Using these masses, we calculate the solar and the atmospheric mass-squared differences

$$\begin{aligned} \Delta m_{sol}^2 = \Delta m_{21}^2 &= -4\frac{m_0^2\left(3a\varepsilon+3c\varepsilon+6ac+3c^2\right)}{4(a-c)(a+2c)\left(a\varepsilon-4c\varepsilon+ac+a^2-2c^2\right)} \\ \Delta m_{atm}^2 = \Delta m_{31}^2 &= 2m_0^2\frac{c(\varepsilon-2a)}{\left(a^2-c^2\right)\left(-2a\varepsilon+a^2-c^2\right)} \end{aligned} \tag{3.27}$$

Since the parameters a and c contribute to the tiny mass of neutrinos (see Eq. (3.26)), the VEVs υ_η and υ_χ should be small and close to the cutoff $\upsilon_\eta, \upsilon_\chi \lesssim \Lambda$ which means that

$$|a| \lesssim 1, \qquad |c| \lesssim 1 \tag{3.28}$$

3.2.1. Fixing a for allowed $\sin\theta_{ij}$

Focusing on relations in Eq. (3.25), we fix the parameter of deviation ε in the range of $O(\frac{1}{10})$, and we use the experimental values of $\sin\theta_{ij}$ given in Table 1; then, we plot in Fig. 1 $\sin\theta_{23}$ as a function of $\sin\theta_{13}$ in terms of the ratio $\frac{\varepsilon}{a}$ induced by the VEV of the singlet η. The values of the ratio $\frac{\varepsilon}{a}$ that are compatible with both $\sin\theta_{13}$ and $\sin\theta_{23}$ are shown in the left panel (right panel) of Fig. 1 within their 3σ allowed range for the normal hierarchy (inverted hierarchy) case; see Table 1. We observe that for the left panel, the mixing angles θ_{13} and θ_{23} vary within the acceptable 3σ ranges

$$\begin{aligned} 0.138 \lesssim \sin\theta_{13} \lesssim 0.161 \\ 0.626 \lesssim \sin\theta_{23} \lesssim 0.638 \end{aligned} \tag{3.29}$$

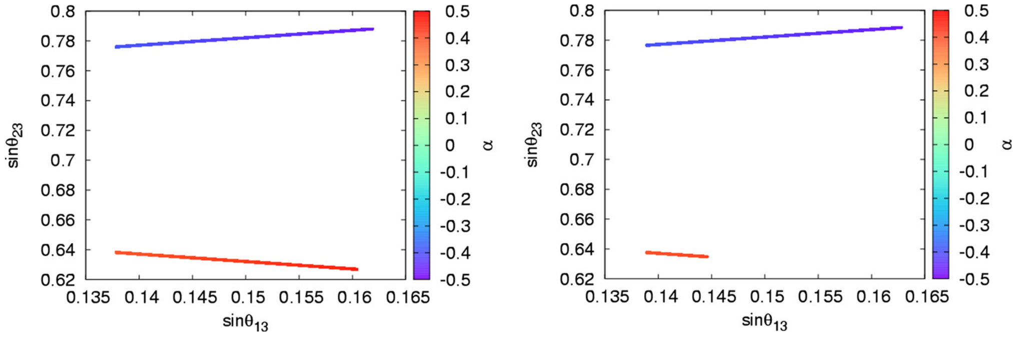

Fig. 1. Left: $\sin\theta_{23}$ as a function of $\sin\theta_{13}$ with the relative parameter $\frac{\varepsilon}{a}$ shown in the palette. Right: The same variation as in the left panel but for inverted hierarchy. (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

for the orange line which corresponds to

$$0.38 \lesssim \frac{\varepsilon}{a} \lesssim 0.45 \tag{3.30}$$

and

$$\begin{aligned} 0.138 &\lesssim \sin\theta_{13} \lesssim 0.162 \\ 0.776 &\lesssim \sin\theta_{23} \lesssim 0.788 \end{aligned} \tag{3.31}$$

for the blue line which corresponds to

$$-0.45 \lesssim \frac{\varepsilon}{a} \lesssim -0.38 \tag{3.32}$$

As for the right panel of Fig. 1, the mixing angles θ_{13} and θ_{23} vary within the acceptable 3σ ranges

$$\begin{aligned} 0.139 &\lesssim \sin\theta_{13} \lesssim 0.144 \\ 0.634 &\lesssim \sin\theta_{23} \lesssim 0.637 \end{aligned} \tag{3.33}$$

for the orange line which corresponds to

$$0.39 \lesssim \frac{\varepsilon}{a} \lesssim 0.41 \tag{3.34}$$

and

$$\begin{aligned} 0.139 &\lesssim \sin\theta_{13} \lesssim 0.163 \\ 0.776 &\lesssim \sin\theta_{23} \lesssim 0.788 \end{aligned} \tag{3.35}$$

for the blue line which corresponds to

$$-0.46 \lesssim \frac{\varepsilon}{a} \lesssim -0.39 \tag{3.36}$$

In order to get estimations of the parameter a, we plot in the left panel in Fig. 2 $\sin\theta_{13}$ as a function of ε with the parameter a shown in the palette on the right while $\sin\theta_{23}$ is considered as an input parameter to get the value of the parameter a compatible with both mixing angles. We observe that the values of $\sin\theta_{13}$ in the interval $[0.138, 0.162]$ for ε of $O(\frac{1}{10})$ corresponds to

$$-0.25 \lesssim a \lesssim -0.0007 \tag{3.37}$$

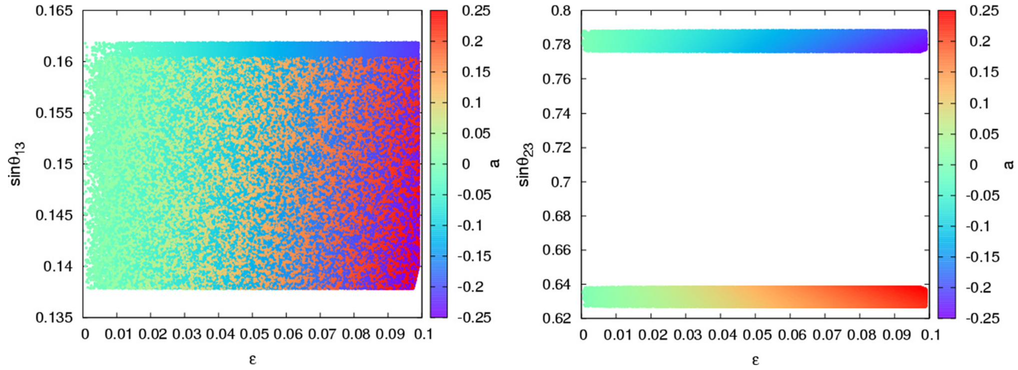

Fig. 2. Left: $\sin\theta_{13}$ as a function of ε with the relative parameter a shown in the palette. Right: The same as in the left panel but for $\sin\theta_{23}$ instead of $\sin\theta_{13}$.

while for values of $\sin\theta_{13}$ in the interval $[0.138, 0.161]$, we have

$$0.0003 \lesssim a \lesssim 0.25 \tag{3.38}$$

Normally, the left panel in Fig. 2 is sufficient to obtain the allowed ranges of the parameter a because the intervals obtained in Eqs. (3.37)–(3.38) are compatible with both mixing angles θ_{13} and θ_{23}, but the allowed range of the parameter a in the left panel provide us only the allowed values of $\sin\theta_{13}$. To extract the allowed ranges of $\sin\theta_{23}$ that are compatible with the ranges of the parameter a obtained in Eqs. (3.37)–(3.38), we plot in the right panel of Fig. 2 $\sin\theta_{23}$ as a function of ε with the parameter a shown in the palette on the right while $\sin\theta_{13}$ is considered as an input parameter. We observe that the values of $\sin\theta_{23}$ in the interval $[0.776, 0.788]$ corresponds to the range of the parameter a given in Eq. (3.37)

$$-0.25 \lesssim a \lesssim -0.0002 \tag{3.39}$$

while for values of $\sin\theta_{23}$ in the interval $[0.626, 0.638]$, we have the range of a given in Eq. (3.38).

$$0.0004 \lesssim a \lesssim 0.25 \tag{3.40}$$

3.2.2. *Fixing c for allowed Δm_{ij}*

To fix the parameter c, we consider the second relation in Eq. (3.27) where we have two unknown parameters (namely m_0 and c). Thus, we plot in Fig. 3 Δm_{31} as a function of m_0 with the parameter c presented in the palette on the right. In the left panel of Fig. 3, Δm_{31} vary within its 3σ allowed range for the normal hierarchy case; see Table 1. For the rest of the parameters of Eq. (3.27), we have earlier fixed the parameter ε in the range of $O(\frac{1}{10})$, and from Eqs. (3.37), (3.38), (3.39), and (3.40) we have fixed the parameter a in the interval $[-0.25 : 0.25]$. We also have restricted the parameter c in the range $[-1 : 1]$ in Eq. (3.28). Gathering all these restrictions, we observe from the color palette in the left panel of Fig. 3 that c can take any value in the range $[-1 : 1]$—except the zero value which is easy to notice from the second relation in Eq. (3.27)). One can also see that for the 3σ allowed range of Δm_{31}, the values of c close to zero—presented by the green light color—corresponds to the values of m_0 close to zero, and as m_0 increases—say $m_0 \gtrsim 0.03$ eV—the parameter c vary from large negative (blue–purple colors) to large positive values (orange–red colors).

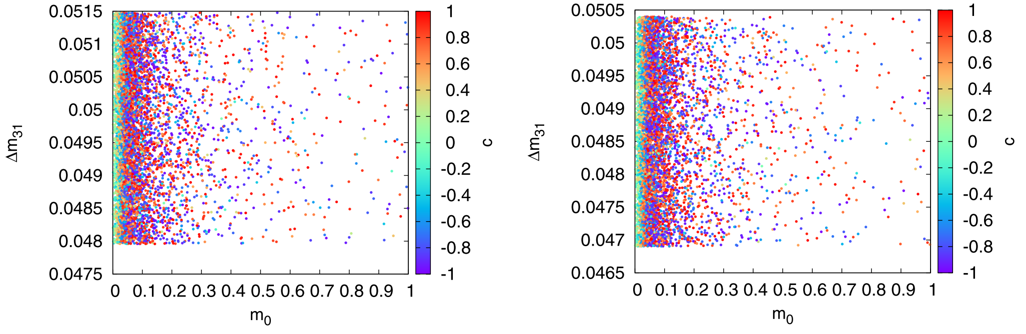

Fig. 3. Left: Δm_{31} [eV] as a function of m_0 [eV] with the parameter c presented in the palette on the right for normal hierarchy. Right: same variation in the left panel but for inverted hierarchy. (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

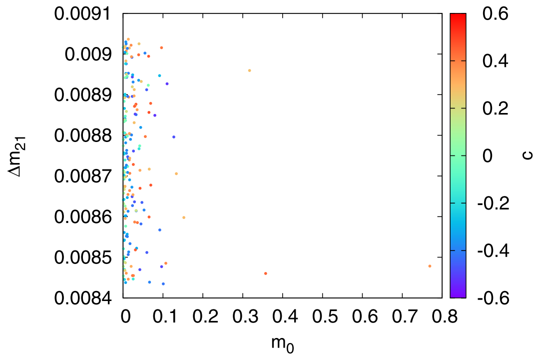

Fig. 4. Δm_{21} [eV] as a function of m_0 [eV] with the parameter c presented in the palette on the right.

For the right panel, Δm_{31} vary within its 3σ allowed range for the inverted hierarchy case, and the parameters m_0 and c vary in the same ranges as in the left panel. One can see approximately the same distribution of colors as in the left panel, and the only difference is the 3σ allowed range of Δm_{31}. Now we consider the first relation in Eq. (3.27) to get the allowed ranges of m_0 and c in the case of Δm_{21}, thus, we plot in Fig. 4 Δm_{21} as a function of m_0 with the parameter c presented in the palette on the right. We observe that range of the parameter c is reduced to

$$-0.6 \lesssim c \lesssim 0.6 \tag{3.41}$$

and the range of the parameter m_0 is reduced to

$$0.00018 \text{ eV} \lesssim m_0 \lesssim 0.77 \text{ eV} \tag{3.42}$$

4. Charged fermions in $SU(5) \times D_4 \times U(1)$ model

In this section we give the invariant operators under $SU(5) \times D_4 \times U(1)$ that determine the mass matrices of the up-, down-quarks and the charged leptons. Moreover we add operator which contain the 45-dimensional Higgs in order to avoid the bad relation between the down quarks and the leptons $Y_d = Y_e^T$ predicted in the GUT scale. Recall that the mass matrices of the quarks and charged leptons can be embedded in the Yukawa couplings given by

$$10_M.10_M.5_{H_u} \supset Q_L u^c H_u \tag{4.1}$$

for the up-quarks type, and

$$10_M.\bar{5}_M.\bar{5}_{H_d} \supset Q_L d^c H_d + L e^c H_d \tag{4.2}$$

for the down-quarks and charged leptons.

4.1. Up quark sector

We start with the mass matrix of the up quark which originate from the up-type Yukawa couplings $10.10.5 \equiv T.T.H_u$. The leading order (LO) $D_4 \times U(1)$ invariant superpotential giving rise to the mass matrix of the up quarks reads

$$W_{up} = \frac{y_1}{\Lambda} T_1 T_1 \Omega H_5 + \frac{y_2}{\Lambda} T_2 T_2 \Gamma H_5 + \frac{y_3}{\Lambda} T_3 T_3 Ғ H_5 \tag{4.3}$$

where y_1, y_2 and y_3 are the Yukawa coupling constants and Λ is the cutoff scale of the model. The superpotential W_{up} decompose into the SM Yukawa couplings as follows

$$W_{up} = \frac{y_1}{\Lambda} \left(Q_{L_1} u^c\right) \Omega H_u + \frac{y_2}{\Lambda} \left(Q_{L_2} c^c\right) \Gamma H_u + \frac{y_3}{\Lambda} \left(Q_{L_3} t^c\right) Ғ H_u \tag{4.4}$$

When the flavon develop their VEVs as

$$\langle \Omega \rangle = \upsilon_\Omega, \qquad \langle \Gamma \rangle = \upsilon_\Gamma, \qquad \langle Ғ \rangle = \upsilon_Ғ \tag{4.5}$$

and the Higgs as usual $\langle H_u \rangle = \upsilon_u$, this leads to a diagonal up quark mass matrix given by

$$M_{up} = \upsilon_u \begin{pmatrix} \frac{y_1}{\Lambda}\upsilon_\Omega & 0 & 0 \\ 0 & \frac{y_2}{\Lambda}\upsilon_\Gamma & 0 \\ 0 & 0 & \frac{y_3}{\Lambda}\upsilon_Ғ \end{pmatrix} \tag{4.6}$$

where the eigen-masses are

$$m_u = \upsilon_u \frac{y_1 \upsilon_\Omega}{\Lambda}, \qquad m_c = \upsilon_u \frac{y_2 \upsilon_\Gamma}{\Lambda}, \qquad m_t = \upsilon_u \frac{y_3 \upsilon_Ғ}{\Lambda} \tag{4.7}$$

By using the experimental values of the up quark, the charm quark and the top quark masses as given by the Particle Data Group [39] namely $m_u \simeq 2.3$ MeV, $m_c \simeq 1.275$ GeV and $m_t \simeq 173.21$ GeV, and by taking the VEV $\upsilon_u \approx 174$ GeV we obtain the following constraints

$$\begin{aligned} y_1 \upsilon_\Omega &\approx 1.32 \times 10^{-5} \Lambda \\ y_2 \upsilon_\Gamma &\approx 7.32 \times 10^{-3} \Lambda \\ y_3 \upsilon_Ғ &\approx 0.995 \Lambda \end{aligned} \tag{4.8}$$

Notice that if we assume the coupling constant $y_3 \approx O(1)$, the VEV $\upsilon_Ғ$ should be close to the cutoff scale Λ in order to accomodate the numerical value of the top quark mass.

4.2. Down quark and charged lepton sector

The $D_4 \times U(1)$ invariant superpotential generating the masses of the down quarks and charged leptons is given by

$$W_{e,d} = \frac{y_4}{\Lambda^2} T_2 F_1 \Omega \Omega H_{\bar{5}} + \frac{y_5}{\Lambda^2} T_1 \left(F_{2,3} \varphi\right) \Gamma H_{\bar{5}} + \frac{y_6}{\Lambda} T_3 \left(F_{2,3} \phi\right) H_{\bar{5}} \tag{4.9}$$

where y_4, y_5 and y_6 are the Yukawa coupling constants associated to the down quarks and charged leptons sector. The masses of the down quarks and charged leptons are generated from the same down-type Yukawa couplings (namely $10_M.\overline{5}_M.\overline{5}_{H_d}$) leading to the GUT mass relations

$$m_b = m_\tau \qquad , \qquad m_s = m_\mu \qquad , \qquad m_d = m_e \tag{4.10}$$

which are acceptable for the third generation at the GUT scale but fails for the first and second generations due to their inconstancy with the experimental values; so the alternative relations which are much closer to the present data are the well known Georgie–Jarskog (GJ) [37] formulas given by

$$m_b = m_\tau \qquad , \qquad m_d = 3m_e \qquad , \qquad 3m_s = m_\mu \tag{4.11}$$

These relations may be predicted by allowing additional couplings to the Higgs field that belongs to the 45-dimensional representation of $SU(5)$. The Higgs $H_{\overline{45}}$ couple to operators $T_i F_i$ and lead to different mass matrices of the down quarks and the charged leptons. Moreover, in additional to the GJ formulas, several relations between the down quarks and charged leptons are possible by considering Higgses that belong to different $SU(5)$ representations [38]. In order to reproduce the difference between the charged lepton mass and the down type quark mass in our model, we introduce the 45-dimensional Higgs denoted as $H_{\overline{45}}$ which transform as non-trivial singlet under D_4 flavor symmetry (namely $H_{\overline{45}} \sim 1_{+,-}$) as well as carrying the $U(1)$ charge $q_{U(1)} = 10$, this Higgs is antisymmetric and satisfy the following relations

$$\begin{aligned} (H_{\overline{45}})^{ab}_c &= -(H_{\overline{45}})^{ba}_c \;,\; (H_{\overline{45}})^{ab}_a = 0 \\ \left\langle (H_{\overline{45}})^{i5}_i \right\rangle &= \quad \upsilon_{45} \quad , \quad i = 1,2,3 \\ \left\langle (H_{\overline{45}})^{45}_4 \right\rangle &= \quad -3\upsilon_{45} \end{aligned} \tag{4.12}$$

With respect to the invariance under $SU(5) \times D_4 \times U(1)$ symmetry model, the $H_{\overline{45}}$ Higgs can only combine with the operator given by

$$W^{45}_{e,d} = \frac{y_7}{\Lambda} T_2 \left(F_{2,3}\varphi\right) H_{\overline{45}} \tag{4.13}$$

Thus, the total superpotential of the down quarks and charged leptons reads as

$$W_{e,d} = \frac{y_4}{\Lambda^2} T_2 F_1 \Omega\Omega H_{\overline{5}} + \frac{y_5}{\Lambda^2} T_1 \left(F_{2,3}\varphi\right) \Gamma H_{\overline{5}} + \frac{y_6}{\Lambda} T_3 \left(F_{2,3}\phi\right) H_{\overline{5}} + \frac{y_7}{\Lambda} T_2 \left(F_{2,3}\varphi\right) H_{\overline{45}} \tag{4.14}$$

which becomes after performing tensor product under D_4 as

$$\begin{aligned} W_{e,d} = {} & \frac{y_4}{\Lambda^2} T_2 F_1 \Omega\Omega H_{\overline{5}} + \frac{y_5}{\Lambda^2} T_1 \left(F_2\varphi_2 + F_3\varphi_1\right) \Gamma H_{\overline{5}} + \frac{y_6}{\Lambda} T_3 \left(F_2\phi_2 + F_3\phi_1\right) H_{\overline{5}} \\ & + \frac{y_7}{\Lambda} T_2 \left(F_2\varphi_2 + F_3\varphi_1\right) H_{\overline{45}} \end{aligned} \tag{4.15}$$

• *Down mass matrix*

Using Eq. (4.14), the $D_4 \times U(1)$ invariant superpotential of the down quarks in terms of the SM Yukawa couplings reads

$$\begin{aligned} W_d = {} & \frac{y_4}{\Lambda^2} \left(Q_{L_2} d^c\right) \Omega\Omega H_d + \frac{y_5}{\Lambda^2} Q_{L_1} \left(s^c\varphi_2 + b^c\varphi_1\right) \Gamma H_d + \frac{y_6}{\Lambda} Q_{L_3} \left(s^c\phi_2 + b^c\phi_1\right) H_d \\ & + \frac{y_7}{\Lambda} Q_{L_2} \left(s^c\varphi_2 + b^c\varphi_1\right) h_{\overline{45}} \end{aligned} \tag{4.16}$$

where H_d and $h_{\overline{45}}$, are the doublet components of the $SU(5)$ Higgses $H_{\overline{5}}$ and $H_{\overline{45}}$ respectively. Taking the VEVs of the H_d as usual—$\langle H_d \rangle = \upsilon_d$—and the flavons Γ and Ω as in Eq. (4.5), and assuming the VEVs of ϕ and φ as

$$\langle \phi \rangle = \left(\upsilon_\phi, 0\right)^T \qquad , \qquad \langle \varphi \rangle = \left(0, \upsilon_\varphi\right)^T \tag{4.17}$$

the mass matrix of the down quarks is given by

$$M_d = \begin{pmatrix} 0 & \upsilon_d \alpha & 0 \\ \upsilon_d \beta & h & 0 \\ 0 & 0 & \upsilon_d \delta \end{pmatrix} \tag{4.18}$$

where

$$\beta = y_4 \frac{\upsilon_\Omega^2}{\Lambda^2}, \qquad \alpha = y_5 \frac{\upsilon_\Gamma \upsilon_\varphi}{\Lambda^2}, \qquad \delta = y_6 \frac{\upsilon_\phi}{\Lambda}, \qquad h = y_7 \frac{\upsilon_{45} \upsilon_\varphi}{\Lambda} \tag{4.19}$$

• *Leptons mass matrix*

Using Eq. (4.14), the $D_4 \times U(1)$ invariant superpotential of the charged leptons in terms of the SM Yukawa couplings reads

$$\begin{aligned} W_e = & \frac{y_4}{\Lambda^2} \left(L_1 \mu^c\right) \Omega \Omega H_d + \frac{y_5}{\Lambda^2} \left(L_2 \varphi_2 + L_3 \varphi_1\right) e^c \Gamma H_d \\ & + \frac{y_6}{\Lambda} \left(L_2 \phi_2 + L_3 \phi_1\right) \tau^c H_d - 3 \frac{y_7}{\Lambda} \left(L_2 \varphi_2 + L_3 \varphi_1\right) \mu^c h_{\overline{45}} \end{aligned} \tag{4.20}$$

As the flavons VEVs are the same as in the down sector, we find the following charged leptons mass matrix

$$M_e = \begin{pmatrix} 0 & \upsilon_d \beta & 0 \\ \upsilon_d \alpha & -3h & 0 \\ 0 & 0 & \upsilon_d \delta \end{pmatrix} \tag{4.21}$$

where α, β and δ are the same as in Eq. (4.19). Recall that the Higgs $H_{\overline{45}}$ contribute to the element 2–2 for both down quark and charged lepton mass matrices with the factor -3 in M_e to differentiate between the two sectors, this factor is an $SU(5)$ Clebsch–Gordan coefficient which come from the properties of the Higgs $H_{\overline{45}}$ given in Eq. (4.12). Diagonalizing the mass matrices M_d and M_e, the down-type quark masses are given by

$$\begin{aligned} m_d &= \left| \frac{1}{2} h - \frac{1}{2} \sqrt{h^2 + 4\upsilon_d^2 \alpha\beta} \right| = \left| \frac{y_4 y_5}{y_7} \frac{\upsilon_d^2 \upsilon_\Gamma \upsilon_\Omega^2}{\upsilon_{45} \Lambda^3} \right| \\ m_s &= \left| \frac{1}{2} h + \frac{1}{2} \sqrt{h^2 + 4\upsilon_d^2 \alpha\beta} \right| = \left| y_7 \frac{\upsilon_{45} \upsilon_\varphi}{\Lambda} + \frac{y_4 y_5}{y_7} \frac{\upsilon_d^2 \upsilon_\Gamma \upsilon_\Omega^2}{\upsilon_{45} \Lambda^3} \right| \\ m_b &= \left| \upsilon_d \delta \right| = \left| y_6 \upsilon_d \frac{\upsilon_\phi}{\Lambda} \right| \end{aligned} \tag{4.22}$$

while for the charged leptons masses, we find

$$\begin{aligned} m_e &= \left| -\frac{3}{2} h + \frac{1}{2} \sqrt{9h^2 + 4\upsilon_d^2 \alpha\beta} \right| = \left| \frac{y_4 y_5}{3 y_7} \frac{\upsilon_d^2 \upsilon_\Gamma \upsilon_\Omega^2}{\upsilon_{45} \Lambda^3} \right| \\ m_\mu &= \left| -\frac{3}{2} h - \frac{1}{2} \sqrt{9h^2 + 4\upsilon_d^2 \alpha\beta} \right| = \left| 3 y_7 \frac{\upsilon_{45} \upsilon_\varphi}{\Lambda} + \frac{y_4 y_5}{3 y_7} \frac{\upsilon_d^2 \upsilon_\Gamma \upsilon_\Omega^2}{\upsilon_{45} \Lambda^3} \right| \end{aligned} \tag{4.23}$$

$$m_\tau = |\upsilon_d \delta| = \left| y_6 \upsilon_d \frac{\upsilon_\phi}{\Lambda} \right|$$

Thus, the masses of the quarks and charged leptons of the first and the second family are successfully differentiated by 45-dimensional Higgs $H_{\overline{45}}$, and the GJ relations are guaranteed if we assume

$$h \gg \upsilon_d \alpha \approx \upsilon_d \beta \tag{4.24}$$

To get the experimental values of down quark masses taking into account the GJ relation between the down quarks and charged leptons, we take several estimations of the mass parameters in (4.22). Taking into consideration the estimations assumed in the up quark sector (see Eq. (4.8)), to reach the numerical values of the down, strange and bottom quark masses as given by the Particle Data Group [39], namely $m_d \simeq 4.8$ MeV, $m_s \simeq 95$ MeV and $m_b \simeq 4.66$ GeV, we assume that $\upsilon_d \approx 174$ GeV and

$$\begin{aligned} \frac{y_4 y_5}{y_7 y_2 y_1^2} \frac{1}{\upsilon_{45}} &= 12.45 \times 10^4 \text{ GeV}^{-1} \\ y_7 \upsilon_{45} \upsilon_\varphi &= 90.2\Lambda \text{ MeV} \\ y_6 \upsilon_\phi &= 2.67 \times 10^{-2} \Lambda \end{aligned} \tag{4.25}$$

4.3. Quark mixing matrix

Regarding the mixing matrix of the quark sector, the unitary matrix that diagonalizing the up quark mass matrix is the identity matrix $U_{Up} = I_{id}$ since the up quark matrix obtained is diagonal (4.6), in the other hand, the down quark mass matrix (4.18) is diagonalized by the unitary matrix

$$U_{Down} = \begin{pmatrix} \frac{-h-\sqrt{h^2+4\alpha\beta\upsilon_d^2}}{\beta\upsilon_d\sqrt{4+\left(\frac{h+\sqrt{h^2+4\alpha\beta\upsilon_d^2}}{\beta\upsilon_d}\right)^2}} & \frac{-h+\sqrt{h^2+4\alpha\beta\upsilon_d^2}}{\beta\upsilon_d\sqrt{4+\left(\frac{h-\sqrt{h^2+4\alpha\beta\upsilon_d^2}}{\beta\upsilon_d}\right)^2}} & 0 \\ \frac{2}{\sqrt{4+\left(\frac{h+\sqrt{h^2+4\alpha\beta\upsilon_d^2}}{\beta\upsilon_d}\right)^2}} & \frac{2}{\sqrt{4+\left(\frac{h-\sqrt{h^2+4\alpha\beta\upsilon_d^2}}{\beta\upsilon_d}\right)^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.26}$$

and consequently the total mixing matrix for the quark sector is given by

$$|U_Q| = \left|U_{Up}^\dagger U_{Down}\right| = \begin{pmatrix} \frac{\left|-h-\sqrt{h^2+4\alpha\beta\upsilon_d^2}\right|}{\left|\beta\upsilon_d\sqrt{4+\left(\frac{h+\sqrt{h^2+4\alpha\beta\upsilon_d^2}}{\beta\upsilon_d}\right)^2}\right|} & \frac{\left|-h+\sqrt{h^2+4\alpha\beta\upsilon_d^2}\right|}{\left|\beta\upsilon_d\sqrt{4+\left(\frac{h-\sqrt{h^2+4\alpha\beta\upsilon_d^2}}{\beta\upsilon_d}\right)^2}\right|} & 0 \\ \frac{2}{\left|\sqrt{4+\left(\frac{h+\sqrt{h^2+4\alpha\beta\upsilon_d^2}}{\beta\upsilon_d}\right)^2}\right|} & \frac{2}{\left|\sqrt{4+\left(\frac{h-\sqrt{h^2+4\alpha\beta\upsilon_d^2}}{\beta\upsilon_d}\right)^2}\right|} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.27}$$

Using the estimations in Eqs. (4.8)–(4.25) and assuming

$$\alpha \approx \beta \simeq 12.6 \times 10^{-5} \tag{4.28}$$

we obtain the total quark mixing matrix as follows

$$\left|U_Q\right| = \left|U_{Up}^{\dagger} U_{Down}\right| = \begin{pmatrix} 0.9743 & 0.225 & 0 \\ 0.225 & 0.9743 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.29}$$

which are reasonably close to the experimental values—$\left|U_Q\right| \sim |U_{CKM}|$, especially the elements $|U_{ud}|$, $|U_{us}|$, $|U_{cd}|$ and $|U_{cs}|$, while the zero mixing elements predicted in (4.29), have non-zero but small values comparing to the observed values given by [39]

$$|U_{CKM}| = \begin{pmatrix} 0.97427 & 0.22536 & 0.00355 \\ 0.22522 & 0.97433 & 0.0414 \\ 0.00886 & 0.0405 & 0.99914 \end{pmatrix} \tag{4.30}$$

We end this section by noticing that spontaneous breaking of discrete symmetry leads in general to cosmological domain walls [40]. To avoid this problem, various scenarios have been proposed, the most common ones are either based on inflation ideas [43] or by using explicit symmetry breaking which is used in several models such as the minimally extended supersymmetric standard model (NMSSM) and string theory inspired prototypes [41,42]. The inflation based scenario might be a nice solution of domain walls problem for GUT models provided the inflationary scale is big; say around $\mathcal{O}(10^{16})$ GeV [43]; at this scale, the topological defects are formed before the end of inflation. This is the case in our SUSY GUT model where the discrete symmetry D_4 is broken by the flavon superfields getting their VEVs at the GUT scale, and consequently the domain walls are inflated away. Notice by the way that the greatest danger of domain walls arises for broken symmetry at lower scale as topological defects may occur after the inflationary stage. For example, in the model proposed in Ref. [44] with superpotential $W(X)$ having Z_{n+3} as discrete symmetry, the domain walls problem occurs in the degenerate minima of $W(X)$; and it has been suggested that the annihilation of such walls as due to a small deformation of the superpotential that breaks explicitly Z_{n+3} symmetry. This idea is realized by adding to $W(X)$ a small deformation term $\delta W = \alpha X$ linear in the chiral superfield X which breaks Z_{n+3} symmetry explicitly, for further details see [44].

5. Conclusion and numerical results

In this paper we have constructed a supersymmetric $SU(5) \times D_4 \times U(1)_f$ GUT model providing a good description of quarks and leptons mass hierarchies and neutrino mixing properties. Besides the bosonic gauge field degrees of freedom and their superpartners described by vector superfields V valued in the Lie algebra of $SU(5)$, the supersymmetric GUT model has also chiral superfields $\{\Phi\}$ that play a basic role in this construction; they can be classified into three kinds as follows:

(a) matter sector described by the generation superfields (T_i, F_i, N_i) carrying quantum numbers under the gauge symmetry as $T_i \sim \mathbf{10}_i$, $F_i \sim \bar{\mathbf{5}}_i$ and $N_i \sim \mathbf{1}_i$; but also under the flavor symmetry $G_f = D_4 \times U(1)_f$ as in (2.7)–(2.8).

(b) Higgs sector described by the superfields $(H_5, H_{\bar{5}}, H_{\overline{45}})$ transform under the gauge symmetry as $H_5 \sim \mathbf{5}_H$, $H_{\bar{5}} \sim \bar{\mathbf{5}}_H$ and $H_{\overline{45}} \sim \overline{\mathbf{45}}_H$; and they carry as well non-trivial quantum number under $G_f = D_4 \times U(1)_f$ as in (2.8)–(2.12).

(c) Flavons sector described by *eleven* chiral superfields; they are scalars under $SU(5)$ gauge invariance; but distinguished by quantum numbers under flavor symmetry $G_f = D_4 \times U(1)_f$ as shown on Tables (2.13)–(2.14).

The invariant chiral superpotential $W(\Phi)$ of the model has *twenty eight free parameters* in which we need to fix eighteen in order to produce the approximative experimental values of the physical parameters in the quark and lepton sectors as given by tables reported below; see Tables (5.2)–(5.3) and Tables (5.5)–(5.9). The total superpotential $W(\Phi) = W_{ch} + W_{chs}$ of the model has a contribution W_{ch} coming from the charged sector and another W_{chs} from the chargeless sector; they are as follows

$$\begin{aligned} W_{ch} &= W_{up} + W_{e,d} \\ W_{chs} &= W_D + W_M + \delta W_M \end{aligned} \tag{5.1}$$

where the superpotentials W_{up} and $W_{e,d}$ of the charged sector are given in Eqs. (4.3)–(4.15) and the superpotentials of the chargeless sector W_D, W_M and δW_M are given in Eqs. (3.2), (3.5) and (3.14).

Notice that the role of the discrete D_4 dihedral group factor in the flavor symmetry G_f may be compared with the role of the alternating group A_4 used in other $SU(5)$ based GUT models building; see for instance [21]. Here D_4 has been motivated by its natural description of μ–τ symmetry as well as by the wish to complete partial results in supersymmetric GUTs. The extra continuous global $U(1)_f$ invariance is necessary to control the superpotential $W(\Phi)$ of the GUT model and also to forbid higher dimensional operators that yields to rapid proton decay.

Among the key steps of this work, we mention the following ones: First, we have required a scale difference among the VEVs of the flavons Γ, Ω and $\digamma$ to fulfill the hierarchy among the three generations of up quarks. We then allowed for the presence of the flavon superfields φ and ϕ along with the flavons Γ and Ω used in the up sector, and the 45-dimensional Higgs in the down quarks–charged leptons sector in order to reconcile with the GJ relations which allow to distinguish between the two sectors. Next, we have studied the neutrino sector where the effective light neutrino mass matrix arise at LO through the type I seesaw mechanism; and by using the D_4 representation properties, the Dirac mass matrix was found diagonal thus allowing the Majorana mass matrix to control the TBM matrix. Finally, in order to generate a non-zero reactor angle, we have added four extra flavon superfields to induce the deviation from TBM pattern.

We end this study by giving comments and a summary of the numerical results obtained in the charged and chargeless fermion sectors. As noticed before, our model involves in total twenty eight free parameters in which we need to fix eighteen to produce the approximative experimental values of the physical parameters in the quark and lepton sectors.

5.1. Numerical results

First we give numerical results for the chargeless sector; see Tables (5.2) and (5.3); then we turn to give numerical estimations of flavon VEVs that lead to masses of the quarks and charged leptons; see Tables (5.5)–(5.9).

5.1.1. Neutrino sector

The neutrino sector in our model involves *fourteen free parameters* in which we have *fixed ten* parameters to reproduce the experimental values of the physical parameters in the allowed

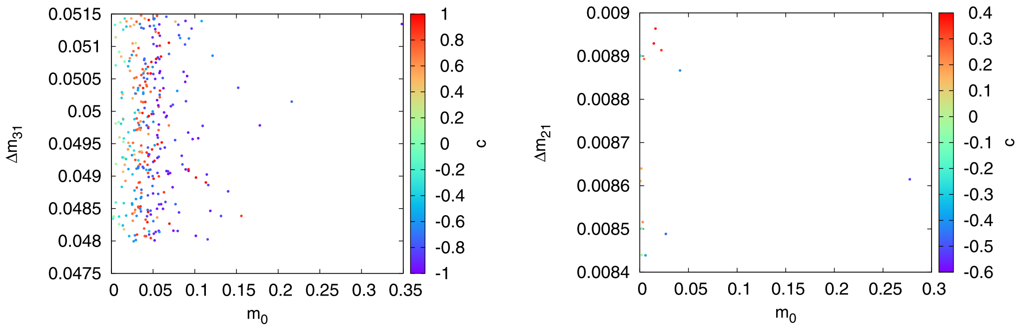

Fig. 5. Left: Δm_{31} [eV] as a function of m_0 [eV] and the parameter c presented in the palette on the right for NH with the parameters a, ε, $\sin\theta_{13}$ and $\sin\theta_{23}$ as inputs. Right: same variation in the left panel but for Δm_{21} [eV].

ranges. To produce the TBM pattern in the neutrino mass matrix as well as generating the non-zero reactor angle θ_{13}, we have *fixed six* parameters by imposing the constraints in Eqs. (3.9), (3.17), (3.18). The *four remaining* parameters to fix (namely ε, a, c and m_0), come from the parameterizations used in Eqs. (3.19)–(3.21). These four parameters are successfully confined to produce the physical parameters Δm_{ij} and $\sin\theta_{ij}$ in the neutrino sector.

As we have mentioned in section 3, the parameter of deviation ε is fixed in the range $[0 : 0.1]$, while the parameter a is fixed as in Eqs. (3.38)–(3.39). In the other hand, the remaining two parameters c and m_0 are fixed using the 3σ allowed ranges of Δm_{31} and Δm_{21} (see Figs. 3–4).

As a final comment, notice that more precise ranges of the parameters c and m_0 may be obtained if we consider their compatibility with the mixing angles $\sin\theta_{13}$ and $\sin\theta_{23}$. We distinguish two cases as follows:

i) m_0 and c for allowed Δm_{31}, $\sin\theta_{13}$ and $\sin\theta_{23}$

We plot in the left panel of Fig. 5 Δm_{31} as a function of m_0, with c presented in the palette on the right, while the 3σ allowed ranges of $\sin\theta_{13}$ and $\sin\theta_{23}$ are included as input parameters. This inclusion of the mixing angles has reduced the allowed values of m_0 and c as can be seen in the left panel of Fig. 5. Since Δm_{31}, $\sin\theta_{13}$ and $\sin\theta_{23}$ depend also on the parameters a and ε, their values get also restricted. To summarize, we take few examples of the allowed values of a and ε that are compatible with the mixing angles $\sin\theta_{13}$ and $\sin\theta_{23}$ and the parameters c, m_0 and Δm_{31} as shown in the left panel of Fig. 5 (see Table (5.2)).

Free parameters				Observables		
ε	a	c	m_0 [eV]	$\sin\theta_{13}$	$\sin\theta_{23}$	Δm_{31} [eV]
0.0647	0.149	−0.732	0.0434	0.153	0.630	0.0484
0.0906	0.214	−0.951	0.0542	0.149	0.632	0.0495
0.0801	−0.199	0.819	0.0350	0.142	0.778	0.0505
0.0566	−0.142	0.903	0.0493	0.140	0.777	0.0492

(5.2)

ii) m_0 and c for allowed Δm_{21}, $\sin\theta_{13}$ and $\sin\theta_{23}$

We plot in the right panel of Fig. 5 the same as in the left panel but for Δm_{21} instead of Δm_{31}; hence, we repeat the same study as in the previous case, and we take a few examples of the allowed values of a and ε that are compatible with the mixing angles $\sin\theta_{13}$ and $\sin\theta_{23}$ and the parameters c, m_0 and Δm_{31} as shown in the right panel of Fig. 5 (see Table (5.3)).

Free parameters				Observables		
ε	a	c	m_0 [eV]	$\sin\theta_{13}$	$\sin\theta_{23}$	Δm_{21} [eV]
0.0958	−0.215	0.284	0.0062	0.157	0.785	0.00860
0.0969	−0.240	0.387	0.0143	0.142	0.778	0.00892
0.0779	0.193	−0.244	0.00179	0.142	0.635	0.00877
0.0824	0.207	−0.443	0.0222	0.140	0.636	0.00899

(5.3)

5.1.2. Quarks and charged leptons sectors

The quarks and charged leptons mass matrices in (4.6), (4.18), (4.21) involve in total *fourteen free parameters* that we collect hereafter

$$\begin{array}{ccccccc} y_1, & y_2, & y_3, & y_4, & y_5, & y_6, & y_7 \\ \upsilon_\Omega, & \upsilon_\Gamma, & \upsilon_F, & \upsilon_{45}, & \upsilon_\phi, & \alpha, & \beta \end{array} \tag{5.4}$$

From these free parameters we need to fix eight of them in order to reproduce the phenomenological charged fermion masses by taking into account the GJ relations as well as the quark mixing matrix. The choice of the parameters is done in three steps as follows:

- In the up quark sector, we have fixed three parameters as in Eq. (4.8) to generate the phenomenological masses of the three up-type quarks. To have masses agreeing with experimental values taken from Ref. [39]

Observables	Model values	Experimental values
m_u	2.3 MeV	$2.3^{+0.7}_{-0.5}$ MeV
m_c	1.275 GeV	1.275 ± 0.025 GeV
m_t	173.21 GeV	$173.21 \pm 0.51 \pm 0.71$ GeV

(5.5)

  we need to fix the VEVs of the flavons Ω, Γ and F as follows

$$\begin{aligned} y_1 \frac{\upsilon_\Omega}{\Lambda} &\approx 1.32 \times 10^{-5} \\ y_2 \frac{\upsilon_\Gamma}{\Lambda} &\approx 7.32 \times 10^{-3} \\ y_3 \frac{\upsilon_F}{\Lambda} &\approx 0.995 \end{aligned} \tag{5.6}$$

- In the down quarks–charged leptons sector, besides Eq. (4.8) used in the up-quark sector, we have fixed four parameters as in Eqs. (4.24)–(4.25) to establish the numerical masses of the down quarks. To ensure the values

Down quarks	Model values	Experimental values
m_d	4.8 MeV	$4.8^{+0.5}_{-0.3}$ MeV
m_s	95 MeV	95 ± 5 MeV
m_b	4.66 GeV	4.66 ± 0.03 GeV

(5.7)

  we have used the following

$$\begin{aligned}
\frac{y_4 y_5}{y_7 y_2 y_1^2}\frac{1}{\upsilon_{45}} &\approx 12.45\times 10^4\ \mathrm{GeV}^{-1}\\
y_7\frac{\upsilon_{45}\upsilon_{\varphi}}{\Lambda} &\approx 90.2\ \mathrm{MeV}\\
y_6\frac{\upsilon_{\phi}}{\Lambda} &\approx 2.67\times 10^{-2}\\
\alpha &\approx \beta\\
h &\gg \upsilon_d\alpha
\end{aligned} \tag{5.8}$$

- In addition to Eqs. (4.8), (4.24), (4.25) used to generate the phenomenological masses of the charged fermions, we have also imposed $\alpha \approx \beta \simeq 12.6\times 10^{-5}$ fixing one more parameter of the GUT model. This choice allowed us to obtain approximately the experimental values of the CKM elements $|U_{ij}|$ collected in following table

Observables	Model values	Experimental values
$\lvert U_{ud}\rvert$	0.9743	0.97427 ± 0.00014
$\lvert U_{us}\rvert$	0.225	0.22536 ± 0.00061
$\lvert U_{cd}\rvert$	0.225	0.22522 ± 0.00061
$\lvert U_{cs}\rvert$	0.9743	0.97343 ± 0.00015
$\lvert U_{ub}\rvert$	0	0.00413 ± 0.000049
$\lvert U_{cb}\rvert$	0	0.0414 ± 0.0012
$\lvert U_{tb}\rvert$	1	0.99914 ± 0.00005
$\lvert U_{ts}\rvert$	0	$0.0405^{+0.0011}_{-0.0012}$
$\lvert U_{td}\rvert$	0	$0.00886^{+0.00033}_{-0.00032}$

(5.9)

Appendix A. Proton decay in $SU(5)\times D_4\times U(1)$ model

In this appendix we provide a discussion concerning the proton decay in our model $SU(5)\times D_4\times U(1)$; it is organized into two sub-subsections: the first part concerns the usual 4 and 5 dimensional operators yielding to fast proton decay. The second part deals with those 7 and 8 operators induced by integrating out the colored Higgs triplets Δ_u and Δ_d from the superpotential (4.3), (4.16).

A.1. Four and five dim operators leading to proton decay

We start by recalling that in $SU(5)$ based GUT models, there are several baryon number violating terms leading to nucleon decay. The present experimental bounds come from Super-Kamiokand where the lower limit of lifetime for $p\to e^+\pi^0$ is $\tau(p\to e^+\pi^0) > 1.4\times 10^{34}$ years and the lifetime limit for $p\to \nu K^+$ is obtained as 5.9×10^{33} years [45]. In supersymmetric SU(5) model, the dangerous proton decay terms arise from the dimension 4 and dimension 5 operators which have the form

$$\begin{aligned}
10_M.\bar{5}_M.\bar{5}_M &\to \lambda_{QLd}(Q_L L d^c) + \lambda_{udd}\left(u^c d^c d^c\right) + \lambda_{ell}(e^c LL)\\
10_M.10_M.10_M.\bar{5}_M &\to \lambda_{QQQL}(Q_L Q_L Q_L L) + \lambda_{uude}(u^c u^c d^c e^c)
\end{aligned} \tag{A.1}$$

Regarding the dimension 4 operators $10_M.\bar{5}_M.\bar{5}_M$, interaction processes involving violating lepton number term $(Q_{Li}L_jd_k^c)$ and the violating baryon number $\left(u_1^cd_1^cd_k^c\right)$ lead to rapid proton decay with family indices as $i=1,2$; $j=1,2$ and $k=2,3$. As we mentioned in subsection 2.2, the matter superfields $T_i=10_m^i$ are assigned into the D_4 representations $1_{+,-}$, $1_{+,-}$ and $1_{+,+}$ respectively; while the $F_i=\overline{5}_m^i$ matter superfields are hosted by the D_4 singlet $1_{+,-}$ and the D_4 doublet $2_{0,0}$. Therefore, the dimension 4 operators yielding to proton decay in $SU(5)\times D_4\times U(1)$ model are given by

$$\begin{array}{ll} T_1.F_1.F_{2,3} \ , & T_2.F_1.F_{2,3} \\ T_1.F_{2,3}.F_{2,3} \ , & T_2.F_{2,3}.F_{2,3} \end{array} \tag{A.2}$$

The operator couplings in the first row of (A.2) are forbidden by D_4 discrete symmetry while those of the second row are permitted. This feature may be exhibited by taking the tensor products of D_4 representations. For $T_1.F_1.F_{2,3}$ and $T_2.F_1.F_{2,3}$ we have $1_{+,-}\otimes 1_{+,-}\otimes 2_{0,0}$ behaving as a doublet. The undesired couplings $T_1.F_{2,3}.F_{2,3}$ and $T_2.F_{2,3}.F_{2,3}$ are eliminated by the global $U(1)$ symmetry (see Table (A.4)). As for the dimension 5 operators $10_M.10_M.10_M.\bar{5}_M$ which are given in the second line in Eq. (A.1) are generically generated via color triplet Higgsino exchange [48]. For instance, the following dimension 5 operators lead to rapid proton decay

$$T_1.T_1.T_3.F_{2,3} \ , \ T_1.T_1.T_2.F_{2,3} \ , \ T_1.T_1.T_2.F_1 \tag{A.3}$$

The first two couplings in Eq. (A.3) are excluded by the D_4 symmetry while the third one is invariant under D_4, but is ruled out by the global $U(1)$ symmetry since its charge is $q_{U(1)}=45$ and therefore is absent. The dimension 4 and 5 operators leading to rapid proton decay and suppressed by D_4 symmetry and global U(1) are listed in the following table:

4- and 5-dim operators	D_4 invariance	$U(1)$
$T_1.F_1.F_{2,3}$	No	40
$T_2.F_1.F_{2,3}$	No	35
$T_1.F_{2,3}.F_{2,3}$	Yes	40
$T_2.F_{2,3}.F_{2,3}$	Yes	35
$T_1.T_1.T_3.F_{2,3}$	No	11
$T_1.T_1.T_2.F_{2,3}$	No	45
$T_1.T_1.T_2.F_1$	Yes	45

(A.4)

Notice that in our $SU(5)\times D_4\times U(1)$ model, there are also operators with dimension[2] equal to 6 involving flavon superfields as

$$T_1.F_1.F_{2,3}.\sigma.\Omega \ , \ T_1.F_{2,3}.F_{2,3}.\rho.\Omega \ , \ T_1.F_{2,3}.F_{2,3}.\rho'.\Omega$$

and may lead to rapid nucleon decay; but can be eliminated by the usual R-parity [53]; this discrete symmetry is known to avoid all renormalizable baryon and lepton number violating operators such as $T_i.F_j.F_k$ in SUSY models. Concerning operators of dimension 5 (A.3), their couplings with the flavon superfields to form operators of dimension 6 are forbidden by the

[2] The 6-dimension operators are the highest dimensional couplings used in our model (except for the operators in Eq. (A.8) derived from the Yukawa superpotential), thus, we restrict our discussion concerning the higher couplings leading to fast proton decay to the 6 dimensional operators.

global $U(1)$ symmetry. Finally, notice that the MSSM μ-term $\mu H_u H_d$ coming from the coupling between the $SU(5)$ Higgses 5_{H_u} and $\overline{5}_{H_d}$ is forbidden by the D_4 discrete symmetry

A.2. More on proton decay suppression

Here we first examine the seven and eight dimensional couplings inherited from W_{up} and W_d superpotentials given by Eqs. (4.3), (4.16); these couplings are mediated by colored Higgsino triplet Δ and are relevant to nucleon decay after including the dressing procedure [54]. Then, we discuss the effect of the dressing through the exchange of charged winos $\tilde{w}^{\pm}$ and higgsinos $\tilde{h}^{\pm}$.

• *Operators mediated by colored Higgsino triplet*
First, recall that the minimal supersymmetric $SU(5)$ GUT in the low scale SUSY suffers from several problems; and has been ruled out as it predicts a fast proton decay arising from the operators of dimension five which are mediated by colored Higgsino triplet Δ; these operators come from the Yukawa superpotential; see for instance [46,47]. In Ref. [47], after examining the RGEs for the gauge couplings at one loop, the mass of colored Higgs triplet is found to be $M_\Delta \leq 3.6 \times 10^{15}$ GeV which is less than the limit $M_\Delta \geq 7.6 \times 10^{16}$ GeV required to ensure the proton stability.

In our $SU(5) \times D_4 \times U(1)$ model, the operators mediated by the colored Higgsino triplet are inherited from the superpotentials W_{up} and W_d in Eqs. (4.3)–(4.16). These superpotentials, which have the same structure as homologue considered in [21], read in terms of colored Higgs triplets $\Delta_u \in H_5$ and $\Delta_d \in H_{\overline{5}}$ as follows

$$\begin{aligned} W'_{up} &= \frac{y_1}{\Lambda}[Q_{L_1}Q_{L_1} + u^c e^c]\Omega\Delta_u + \frac{y_2}{\Lambda}[Q_{L_2}Q_{L_2} + c^c\mu^c]\Gamma\Delta_u \\ &+ \frac{y_3}{\Lambda}[Q_{L_3}Q_{L_3} + t^c\tau^c]F\Delta_u \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} W'_d &= \frac{y_4}{\Lambda^2}\left[Q_{L_2}L_1 + c^c d^c\right]\Omega\Omega\Delta_d + \frac{y_5}{\Lambda^2}\left[Q_{L_1}L_2\varphi_2 + u^c s^c\varphi_2\right]\Gamma\Delta_d \\ &+ \frac{y_6}{\Lambda}\left[Q_{L_3}L_3\phi_1 + t^c b^c\phi_1\right]\Delta_d \end{aligned} \tag{A.6}$$

Integrating out Δ_u and Δ_d in Eqs. (A.5)–(A.6), the remaining operators relevant for nucleon decay are of dimension seven and eight as follows

$$\begin{aligned} W_{7,8} = \frac{1}{M_\Delta}[&\frac{y_1 y_4}{\Lambda^3}\left(Q_{L_1}Q_{L_1}Q_{L_2}L_1 + u^c e^c c^c d^c\right)\Omega^3 \\ &+ \frac{y_1 y_6}{\Lambda^2}\left(Q_{L_1}Q_{L_1}Q_{L_3}L_3 + u^c e^c t^c b^c\right)\Omega\phi_1 \\ &+ \frac{y_2 y_5}{\Lambda^3}\left(Q_{L_2}Q_{L_2}Q_{L_1}L_2 + c^c\mu^c u^c s^c\right)\varphi_2\Gamma^2 \\ &+ \frac{y_3 y_5}{\Lambda^3}\left(Q_{L_3}Q_{L_3}Q_{L_1}L_2 + t^c\tau^c u^c s^c\right)\varphi_2\Gamma F] \end{aligned} \tag{A.7}$$

where M_Δ is the colored Higgs triplet mass which is expected to be at the GUT scale; say $\mathcal{O}\left(10^{16}\right)$. Notice that it is known in GUT literature that the Higgsino mediated proton decay is strongly associated with the so called "doublet–triplet splitting" (DTS) problem [50] on how the Higgs triplets Δ_u and Δ_d acquire GUT-scale masses M_Δ while leaving their doublet partners H_u and H_d with only weak-scale masses. Several ways have been proposed to resolve this problem such as: (*i*) tuning the parameters in the Higgs superpotential, see for instance [49]; (*ii*) using the Missing Partner Mechanism [51]; or (*ii*) using Double Missing Partner Mechanism [52]. In the present paper, the doublet–triplet splitting problem might be circumvented by using the Missing Partner Mechanism which is considered as the most used solution of DTS. The general idea of this mechanism relies on giving the colored Higgs triplet M_Δ a mass involving additional Higgses sitting the 50, $\overline{50}$ and 75 representations of $SU(5)$. We will not develop this issue here; we refer to literature where several papers using this approach have addressed this question; see for instance Ref. [21].

Returning to eqs. (A.7), the higher dimensional couplings in $W_{7,8}$ may be exhibited by using the superfield assignments of $SU(5) \times D_4 \times U(1)$ model; we have

$$\begin{aligned}
&\frac{1}{M_\Delta}\frac{y_1 y_4}{\Lambda^3}\left[T_1.T_1.T_2.F_1.\Omega^3\right] \\
&\frac{1}{M_\Delta}\frac{y_1 y_6}{\Lambda^2}\left[T_1.T_1.T_3.F_{2,3}.\Omega.\phi\right] \\
&\frac{1}{M_\Delta}\frac{y_2 y_5}{\Lambda^3}\left[T_2.T_2.T_1.F_{2,3}.\varphi.\Gamma^2\right] \\
&\frac{1}{M_\Delta}\frac{y_3 y_5}{\Lambda^3}\left[T_3.T_3.T_1.F_{2,3}.\varphi.\Gamma.\digamma\right]
\end{aligned} \tag{A.8}$$

By using Eqs. (4.8)–(4.25), it is clear that all the operators in the list (A.8) are highly suppressed by a factor proportional to

$$\frac{1}{M_\Delta}\frac{y_1 y_4}{\Lambda^3}\langle\Omega\rangle^3 \tag{A.9}$$

for the first coupling; and

$$\frac{1}{M_\Delta}\frac{y_1 y_6}{\Lambda^2}\langle\Omega\rangle\langle\phi\rangle \tag{A.10}$$

for the second coupling; and

$$\frac{1}{M_\Delta}\frac{y_2 y_5}{\Lambda^3}\langle\varphi\rangle\langle\Gamma\rangle^2 \tag{A.11}$$

for the third coupling; and finally

$$\frac{1}{M_\Delta}\frac{y_3 y_5}{\Lambda^3}\langle\varphi\rangle\langle\Gamma\rangle\langle\digamma\rangle \tag{A.12}$$

for the last coupling. Assuming the Yukawa couplings $y_i \approx O(1)$, the first coupling (A.9) is suppressed by $\frac{2.3}{M_\Delta \times 10^{15}}$ which is of order of 10^{-31} GeV^{-1}; while the suppression of the remaining couplings (A.10)–(A.12) are of order 10^{-23} GeV^{-1}, 10^{-24} GeV^{-1} and 10^{-22} GeV^{-1} respectively. In what follows, we turn to study the contribution to proton decay coming from dressing diagrams with winos and higgsinos mediators.

- *Dressing by higgsinos and winos exchange*

The dressing of dimension five proton decay operators via the exchange of gluino, charginos and neutralinos concerns the processus $qq \to \tilde{l}\tilde{q}$; and consists of converting the sleptons $\tilde{l}$ and $\tilde{q}$ squarks into leptons l' and quarks q'. In order for these operators to be relevant to proton decay, the bosons need to be transformed to fermions by a loop diagram through the gluino, neutralino,

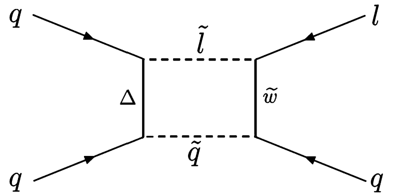

Fig. 6. Dimension 5 operator diagram mediated by the colored Higgs triplet Δ. The superparticles (dashed lines) are transformed in particles via wino exchange. A similar diagram with higgsino exchange and others can also drawn; see appendix C of Ref. [56].

charginos dressing procedure; this leads to four-fermion interactions $qqql$ and $u^c u^c d^c e^c$ with baryon and lepton violating dimension six operators [55]. In SUSY SU(5) models, the dressing of the dimension five operators is studied in the limit where the dominant contribution to the $qqql$ operator comes from a diagram with charged wino dressing while the dominant contribution to the $u^c u^c d^c e^c$ operator arises from a charged higgsino dressing as illustrated in Fig. 6; see for instance Ref. [56] and the references therein.

In our $SU(5) \times D_4 \times U(1)$ model, the dressing of the operators $QQQL$ and $u^c u^c d^c e^c$ of (A.7) involves charged winos and higgsinos and an effective coupling depending on the flavon field VEVs. For clarity, we split the superpotential $W_{7,8}$ as the sum of two parts

$$W_{7,8} = W_{7,8}^L + W_{7,8}^R \tag{A.13}$$

where the part $W_{7,8}^L$ contains the operators of type $QQQL$ coupled to flavons as follows

$$\begin{aligned} W_{7,8}^L = \frac{1}{M_\Delta}[&\frac{y_1 y_4}{\Lambda^3}\left(Q_{L_1} Q_{L_1} Q_{L_2} L_1\right)\Omega^3 + \frac{y_1 y_6}{\Lambda^2}\left(Q_{L_1} Q_{L_1} Q_{L_3} L_3\right)\Omega\phi_1 \\ &+ \frac{y_2 y_5}{\Lambda^3}\left(Q_{L_2} Q_{L_2} Q_{L_1} L_2\right)\varphi_2\Gamma^2 + \frac{y_3 y_5}{\Lambda^3}\left(Q_{L_3} Q_{L_3} Q_{L_1} L_2\right)\varphi_2\Gamma F] \end{aligned} \tag{A.14}$$

and the $W_{7,8}^R$ part contains the operators of type $u^c u^c d^c e^c$ like

$$\begin{aligned} W_{7,8}^R = \frac{1}{M_\Delta}[&\frac{y_1 y_4}{\Lambda^3}\left(u^c e^c c^c d^c\right)\Omega^3 + \frac{y_1 y_6}{\Lambda^2}\left(u^c e^c t^c b^c\right)\Omega\phi_1 + \frac{y_2 y_5}{\Lambda^3}\left(c^c \mu^c u^c s^c\right)\varphi_2\Gamma^2 \\ &+ \frac{y_3 y_5}{\Lambda^3}\left(t^c \tau^c u^c s^c\right)\varphi_2\Gamma F] \end{aligned} \tag{A.15}$$

The two first operators in Eq. (A.14) are dressed by the charged winos and are significant for the decay mode $p \to K^+\overline{\nu}$; this wino dressing contributes to the amplitude of nucleon decay with a factor proportional to

$$\frac{1}{M_\Delta}\frac{y_1 y_4}{\Lambda^3}\langle\Omega\rangle^3\left(\frac{m_{\tilde{w}}}{m_{\tilde{l}_1} m_{\tilde{q}_2}}\right) \tag{A.16}$$

for the first operator and

$$\frac{1}{M_\Delta}\frac{y_1 y_6}{\Lambda^2}\langle\Omega\rangle\langle\phi_1\rangle\left(\frac{m_{\tilde{w}}}{m_{\tilde{l}_3} m_{\tilde{q}_3}}\right)$$

for the second. The $m_{\tilde{w}}$ is the wino mass and $m_{\tilde{l}}$ and $m_{\tilde{q}}$ are the slepton and the squark masses respectively. If we take the masses of the sfermions and the charged winos as in Murayama and Pierce paper [47] ($m_{sf} \sim \mathcal{O}(1\text{ TeV})$ and $m_{\tilde{w}} \in [100, 400]$ GeV), these extra contributions from the ratio of the winos and superparticle masses are small and enhance the suppression of the factors in Eqs. (A.9)–(A.10).

Regarding the first operator in Eq. (A.15) which is dressed by charged higgsino is relevant to the same mode $p \to K^+\overline{\nu}$, its contribution to the amplitude of nucleon decay is proportional to

$$\frac{1}{M_\Delta}\frac{y_1 y_4}{\Lambda^3}\langle\Omega\rangle^3\left(\frac{m_{\tilde{h}}}{m_{\tilde{e}}m_{\tilde{c}}}\right) \tag{A.17}$$

where $m_{\tilde{h}}$ is the charged higgsino mass and $m_{\tilde{e}}$ and $m_{\tilde{c}}$ are the masses of the selectron and the scharm respectively. Following [47], the mass of higgsino varies in the range $m_{\tilde{h}} \in [100, 1000]$ GeV; thus the ratio of the higgsino and the superparticle masses is also small and the contribution from charged higgsino dressing that arise in Eq. (A.17) is also highly suppressed.

Appendix B. Dihedral group D_4

The Dihedral group D_4 is a non-abelian discrete symmetry group generated by two non-commuting generators a and b obeying to the conditions $a^4 = b^2 = e$; they have the 4×4 matrix realization

$$a = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad , \quad b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{B.1}$$

The D_4 discrete group consists of eight elements which could be classified in the five conjugacy classes as

$$C_1 : \{e\}, \qquad C_2 : \left\{a, a^3\right\}, \qquad C_3 : \left\{a^2\right\}, \qquad C_4 : \left\{b, a^2 b\right\}, \qquad C_5 : \left\{ab, a^3 b\right\} \tag{B.2}$$

It has five irreducible representations; four singlets $1_{+,+}$, $1_{+,-}$, $1_{-,+}$ and $1_{-,-}$, and one doublet $2_{0,0}$ where the sub-indices on the representations refer to their characters under the two generators a and b as in the table

χ_{ij}	e	a	b
$\chi_{1_{+,+}}$	$+1$	$+1$	$+1$
$\chi_{1_{+,-}}$	$+1$	-1	$+1$
$\chi_{1_{-,+}}$	$+1$	$+1$	-1
$\chi_{1_{-,-}}$	$+1$	-1	-1
$\chi_{2_{0,0}}$	2	0	0

(B.3)

The Kronecker product of two doublets $2_x = (x_1, x_2)^T$ and $2_y = (y_1, y_2)^T$ in the D_4 group is given by

$$2_x \times 2_y = 1_{+1,+1} + 1_{+1,-1} + 1_{-1,+1} + 1_{-1,-1}\ , \tag{B.4}$$

where

$$\begin{aligned} 1_{+1,+1} &= x_1 y_2 + x_2 y_1, \\ 1_{+1,-1} &= x_1 y_1 + x_2 y_2, \\ 1_{-1,+1} &= x_1 y_2 - x_2 y_1, \\ 1_{-1,-1} &= x_1 y_1 - x_2 y_2, \end{aligned} \tag{B.5}$$

and the singlets product are

$$1_{\alpha,\beta} \times 1_{\gamma,\delta} = 1_{\alpha\gamma,\beta\delta} \quad \text{with } \alpha, \gamma, \beta, \delta = \pm\,. \tag{B.6}$$

For more details on the D_4 Dihedral group see for instance Ref. [57].

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