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Ward-constrained melonic renormalization group flow

Vincent Lahoche [a], Dine Ousmane Samary [b,a,*]

[a] Commissariat à l'Énergie Atomique (CEA, LIST), 8 Avenue de la Vauve, 91120 Palaiseau, France

[b] International Chair in Mathematical Physics and Applications (ICMPA-UNESCO Chair), University of Abomey-Calavi, 072B.P.50, Cotonou, Republic of Benin

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ABSTRACT

In recent years, interesting investigations of the nonperturbative renormalization group equations for tensorial group field theories have been performed in the truncation method, while completely discarding the Ward identities from their analysis. In this letter, in continuation of our recent series of papers, we present a new framework of the investigation, namely, the effective vertex expansion, allowing us to consider infinite sectors rather than finite-dimensional subspaces of the full theory space. We focus on the ultraviolet behavior and provide a new and complete description of the renormalization group flow constrained with Ward identities.

1. Introduction

Applying the Functional renormalization group (FRG) to tensor models (TM) and group field theory (GFT) has been the subject of intense works in recent years because of its close relationship with the fluctuation problem in quantum gravity phenomena [1–19]. Despite the difficulties related to the nonlocal behavior of the interactions and combinatorics, several classes of new techniques have helped to think about using the FRG for tensorial group field theory (TGFT) [1–4,13]. First insights have been gained from the nonperturbative Wetterich equation, in particular, by an investigation of the melonic leading order interactions with a new method called effective vertex expansion (EVE) [1–4]. EVE is a new approach to the FRG by improving the truncation and by appropriately choosing a regulator. As such, it will certainly become a promising way of investigating the nonperturbative field theory. Many phase transitions which are identified near the fixed points are shown to be non-physical due to the violation of the Ward identities (WI) [2]. The WI is an additional constraint on the flow and therefore should not be overlooked in the study of the renormalization group. In the symmetric phase, we discovered, even though no physical fixed point may be observed, the possible existence of first-order phase transition in the reduced subspace of theory space (see [1]). The complete definition of this new phase transition remains to be rigorously probed.

A very useful concept for the study of the FRG and, in particular, for the phase transitions, is the coarse-grained free energy or effective average action Γ_k. The k dependence of this quantity is due to the regulator r_k where k ranges from IR to UV. The nontrivial form of the Ward identity for the TGFT with nontrivial propagator in the functional actions is not just a consequence of the regulator r_k but rather is due to the violation of the kinetic term under $\mathrm{U}(N)$ symmetry. Let us remark that for standard gauge-invariant theories like QED see [22] the regulator generally breaks the explicit invariance of the kinetic term and leads to a new nontrivial Ward identity that depends on the regulator r_k, which becomes trivial in the IR. This is not the case for TGFT models for which the kinetic term intrinsically violates the $\mathrm{U}(N)$ symmetry. Therefore the appearance of the regulator generalizes the definition of the theory but does not add any new information concerning the shape of Ward's identities. To be more precise, in the deep IR – in which the influence of the regulator disappears –, the Ward identities are still nontrivial due to the nontrivial propagator of the theory. Finally, the WI appears, like the flow equations themselves, as a formal consequence of the quantum model, and has to be taken into account on the same footing as the flow equations. As the Wetterich equation describes the k variation of Γ_k, the Ward identity describes the momentum dependence of the same quantity. Ignoring these new dynamics related to the Ward identity would be a serious lack in the study of the FRG.

In this present letter, the FRG is studied with a new alternative way by considering together both dynamical aspects of the average effective action. The first is dictated by the Wetterich flow equation [20,21] and the second by the Ward identity [1]. The fun-

* Corresponding author.
E-mail addresses: vincent.lahoche@cea.fr (V. Lahoche), dine.ousmanesamary@cipma.uac.bj (D. Ousmane Samary).

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damental effort in this work is to pool these two dynamics in a system of flow equation inside the constrained subspace. We mean by constrained subspace the theory space on which these two dynamics are compatible and we denote this space by $\mathcal{E}_{\mathcal{C}}$. We derive the melonic constraint flow by merging these two dynamics equations in the physical subspace $\mathcal{E}_{\mathcal{C}}$ of theory space. Note that the study of phase transitions is deeply related to the classification of all possible universality classes of the exact overlap of the critical exponents. This universality is broken by the Ward constraint driven by the EVE in the same way by the truncation method, such that the method proposed in this article may be used for the generalization to any other interaction of higher rank.

The paper is organized as follows. In section 2 we provide in detail useful ingredients for the description of the FRG to TGFT. In section 3, the EVE is derived to provide the FRG with a new alternative way without crude truncation. Note that the expression *"without crude truncation" does not mean that we have not made an approximation in our approach. The originality of the EVE method, in contrast to the other ones used in TFTs, is to keep the entire sector with infinite dimension rather than finite-dimensional domains of the full theory space*. The corresponding flow equations which improve the truncation method are given. Section 4 describes our new proposal to merge the Wetterich equation and the Ward identity in the physical melonic phase space $\mathcal{E}_{\mathcal{C}}$ of theory space. In the last section 5 we give our conclusion.

2. Preliminaries

A *group field* φ is a field, complex or real, defined over d–copies of a group manifold G rather than on spacetime:

$$\varphi : \mathrm{G}^d \to \mathbb{R}, \mathbb{C}\,. \tag{1}$$

Standard choices (to make contact with physics) are SU(2) and SO(4) [10–12]. In this paper, we focus only on the non-local aspects of the interactions, and consider the Abelian version of the theory, setting $\mathrm{G} = \mathrm{U}(1)$. For this choice, the field may be equivalently described on the Fourier dual group $\mathbb{Z}^d$ by a *tensor field* $T : \mathbb{Z}^d \to \mathbb{C}$. We consider a theory for two complex fields φ and $\bar{\varphi}$, requiring two complex tensors fields T and $\bar{T}$. The allowed configurations are then constrained by the choice of the action, completing the definition of the GFT. At the classical level, for free fields, we choose the familiar form:

$$S_{\text{kin}}[T,\bar{T}] := \sum_{\vec{p}\in\mathbb{Z}^d} \bar{T}_{p_1\cdots p_d}\left(\vec{p}^{\,2}+m^2\right) T_{p_1\cdots p_d}\,, \tag{2}$$

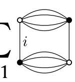
with the standard notation $\vec{p}^{\,2} := \sum_i p_i^2$, $\vec{p} := (p_1, \cdots, p_d)$. For the rest of this paper, we use the short notation $T_{\vec{p}} \equiv T_{p_1\cdots p_d}$. The equation (2) defines the bare propagator $C^{-1}(\vec{p}\,) := \vec{p}^{\,2} + m^2$. Among the natural transformations that we can consider for a pair of complex tensor fields, the unitary transformations play an important role. They provide the principle that allows to build the interactions, which are chosen to be invariant under such a transformation. Denoting by N the *size* of the tensor field, restricting the domain of the indices p_i into the window $[\![-N, N]\!]$, we require invariance with respect to independent transformations along each of the d indices of the tensors:

$$T'_{p_1\cdots p_d} = \sum_{\vec{q}\in\mathbb{Z}^d}\left[\prod_{i=1}^{d} U^{(i)}_{p_i q_i}\right] T_{q_1\cdots q_d}\,, \tag{3}$$

with $U^{(i)}(U^{(i)})^\dagger = \text{id}$. Define $\mathbb{U}(N)$ as the set of unitary symmetries of size N, a transformation for tensors is then a set of d independent elements of $\mathbb{U}(N)$, $\mathcal{U} := (U_1, \cdots, U_d) \in \mathbb{U}(N)^d$, one per index of the tensor fields. The unitary symmetries admitting an inductive limit for arbitrary large N, we will implicitly consider the limit $N \to \infty$ in the rest of this paper. We call *bubble* all the invariant interactions which cannot be factorized into two or more smaller bubbles. Observe that, because the transformations are independent, the bubbles are not local in the usual sense over the group manifold G^d. However, locality does not make sense without physical content. In standard field theory, for instance, or in physics in general, locality is defined by the way according to which the fields or particles interact together and, as for tensors, this choice reflects invariance with respect to some transformations like translations and rotations. With this respect, the transformation rule (3) defines both the nature of the field (a tensor) and the corresponding locality principle. To summarize:

Definition 1. Any interaction bubble is said to be local. By extension, any function expanded as a sum of bubbles will be local.

This locality principle called *traciality* in the literature has some good properties of the usual ones. In particular, it allows to define local counter-terms and to follow the standard renormalization procedure for interacting quantum fields with UV divergences. In this paper, we focus on the quartic melonic model in rank $d = 5$, described by the classical interaction:

$$S_{\text{int}}[T,\bar{T}] = g\sum_{i=1}^{d}$$  $, \tag{4}$

g denoting the coupling constant and where we adopted the standard graphical convention [23] to picture the interaction bubble as d-colored bipartite regular connected graphs. The black (resp. white) nodes corresponding to T (resp. $\bar{T}$) fields, and the colored edges fixing the contractions of their indices. Note that, because we contract indices of the same color between T and $\bar{T}$ fields, the unitary symmetry is ensured by construction. The model that we consider has been shown to be *just renormalizable* is the usual sense, that is to say, all the UV divergences can be subtracted with a finite set of counter-terms for mass, coupling and field strength. From now, we will consider m^2 and g as the bare couplings and their counter-terms δm^2, δg, and we introduce explicitly the wave function renormalization Z replacing the propagator C^{-1} by

$$C^{-1}(\vec{p}\,) = Z\vec{p}^{\,2} + m^2\,. \tag{5}$$

The equations (2) and (4) define the classical model, without fluctuations. We quantize it using the path integral formulation, and define the partition function by integrating over all configurations, weighted by e^{-S}:

$$\mathcal{Z}(J,\bar{J}) := \int dTd\bar{T}\, e^{-S[T,\bar{T}]+\langle\bar{J},T\rangle+\langle\bar{T},J\rangle}\,, \tag{6}$$

the sources being tensor fields themselves i.e. $J, \bar{J} : \mathbb{Z}^d \to \mathbb{C}$ and $\langle\bar{J}, T\rangle := \sum_{\vec{p}} \bar{J}_{\vec{p}}\, T_{\vec{p}}$. Note that the quantization procedure provides a canonical definition of UV and IR regimes. The UV theory corresponding to the classical action $S = S_{\text{kin}} + S_{\text{int}}$ whereas the IR theory corresponds to the standard effective action defined as the Legendre transform of the free energy $\mathcal{W} := \ln(\mathcal{Z}(J,\bar{J}))$.

Renormalization in standard field theory allows us to subtract divergences, and it has been shown that quantum GFT can be renormalized in the usual sense [24,25]. Concerning the quantization procedure, moreover, the renormalization group allows describing quantum effects *scale by scale*, through more and more effective models, defining a path from UV to IR by integrating out fluctuations of increasing size.

Recognizing this path from UV to IR as an element of the quantization procedure itself, we substitute to the global quantum description (6) a set of models $\{\mathcal{Z}_k\}$ indexed by a referent scale k. This scale defines what is UV, and integrated out and what is IR, and frozen out from the long distance physics. The set of scales may be discrete or continuous, and in this paper we choose a continuous description $k \in [0, \Lambda]$ for some fundamental UV cut-off Λ. There are several ways to build what we call functional renormalization group. We focus on the Wetterich-Morris approach [20,21], $\mathcal{Z}_k(J, \bar{J})$ being defined as:

$$\mathcal{Z}_k(J, \bar{J}) := \int dT d\bar{T}\, e^{-S_k[T,\bar{T}]+\langle \bar{J}, T\rangle + \langle \bar{T}, J\rangle}\,, \tag{7}$$

with: $S_k[T,\bar{T}] := S[T,\bar{T}] + \sum_{\vec{p}} \bar{T}_{\vec{p}}\, r_k(\vec{p}^{\,2}) T_{\vec{p}}$. The momentum-dependent mass term $r_k(\vec{p}^{\,2})$ called *regulator* vanishes for UV fluctuations $\vec{p}^{\,2} \gg k^2$ and becomes very large for the IR ones $\vec{p}^{\,2} \ll k^2$. Some additional properties for $r_k(\vec{p}^{\,2})$ may be found in standard references [29,30]. Without explicit mentions, we focus on the Litim's modified regulator:

$$r_k(\vec{p}^{\,2}) := Z(k)(k^2 - \vec{p}^{\,2})\theta(k^2 - \vec{p}^{\,2})\,, \tag{8}$$

where θ designates the Heaviside step function and $Z(k)$ is the running wave function strength. The renormalization group flow equation, describing the trajectory of the RG flow into the full theory space is the so called Wetterich equation [20,21], which for our model reads:

$$\frac{\partial}{\partial k}\Gamma_k = \sum_{\vec{p}} \frac{\partial r_k}{\partial k}(\vec{p}\,)\left(\Gamma_k^{(2)} + r_k\right)^{-1}_{\vec{p}\,\vec{p}}\,, \tag{9}$$

where $(\Gamma_k^{(2)})_{\vec{p}\,\vec{p}\,'}$ is the second derivative of the *average effective action* Γ_k with respect to the classical fields M and $\bar{M}$:

$$\left(\Gamma_k^{(2)}\right)_{\vec{p}\,\vec{p}\,'} = \frac{\partial^2 \Gamma_k}{\partial M_{\vec{p}}\, \partial \bar{M}_{\vec{p}\,'}}\,, \tag{10}$$

where $M_{\vec{p}} = \partial \mathcal{W}_k / \partial \bar{J}_{\vec{p}}$, $\bar{M}_{\vec{p}} = \partial \mathcal{W}_k / \partial J_{\vec{p}}$ and:

$$\Gamma_k[M, \bar{M}] + \sum_{\vec{p}} \bar{M}_{\vec{p}}\, r_k(\vec{p}^{\,2}) M_{\vec{p}} := \langle \bar{M}, J\rangle + \langle \bar{J}, M\rangle - \mathcal{W}_k(M, \bar{M})\,, \tag{11}$$

with $\mathcal{W}_k = \ln(\mathcal{Z}_k)$.

The flow equation (9) is a consequence of the variation of the propagator, indeed

$$\frac{\partial r_k}{\partial k} = \frac{\partial C_k^{-1}}{\partial k}\,, \tag{12}$$

for the *effective covariance* $C_k^{-1} := C^{-1} + r_k$. But the propagator has other source of variability. In particular, it is not invariant with respect to the unitary symmetry of the classical interactions (4). Focusing on an infinitesimal transformation: $\delta_1 := (\mathrm{id} + \epsilon, \mathrm{id}, \cdots, \mathrm{id})$ acting non-trivially only on the color 1 for some infinitesimal anti-hermitian transformations ϵ, the transformation rule for the propagator follows the Lie bracket:

$$\mathcal{L}_{\delta_1} C_k^{-1} = [C_k^{-1}, \epsilon]\,. \tag{13}$$

The source terms are non invariant as well. However, due to the translation invariance of the Lebesgue measure $dT d\bar{T}$ involved in the path integral (7), we must have $\mathcal{L}_{\delta_1}\mathcal{Z}_k = 0$. Translating this invariance at the first order in ϵ provides a non-trivial *Ward-Takahashi identity* for the quantum model:

Theorem 1. ***(Ward identity.)*** *The non-invariance of the kinetic action with respect to unitary symmetry induces non-trivial relations between* $\Gamma^{(n)}$ *and* $\Gamma^{(n+2)}$ *for all n, summarized as:*

$$\sum_{\vec{p}_\perp, \vec{p}\,'_\perp}{}' \Bigg\{ \left[C_k^{-1}(\vec{p}) - C_k^{-1}(\vec{p}\,')\right]\left[\frac{\partial^2 \mathcal{W}_k}{\partial \bar{J}_{\vec{p}\,'}\, \partial J_{\vec{p}}} + \bar{M}_{\vec{p}} M_{\vec{p}\,'}\right] - \bar{J}_{\vec{p}}\, M_{\vec{p}\,'} + J_{\vec{p}\,'} \bar{M}_{\vec{p}} \Bigg\} = 0\,. \tag{14}$$

where $\sum'_{\vec{p}_\perp, \vec{p}\,'_\perp} := \sum_{\vec{p}_\perp, \vec{p}\,'_\perp} \delta_{\vec{p}\,\vec{p}\,'_\perp}$.

In this statement, we introduced the notations $\vec{p}_\perp := (p_2, \cdots, p_d) \in \mathbb{Z}^{d-1}$ and $\delta_{\vec{p}\,\vec{p}\,'_\perp} = \prod_{j\neq 1} \delta_{p_j\, p'_j}$. Equations (9) and (14) are two formal consequences of the path integral (7), coming both from the non-trivial variations of the propagator. Therefore, there are no reason to treat these two equations separately. This formal proximity is highlighted in their expanded forms, comparing equations (23)–(24) and (29)–(30). Instead of a set of partition functions, the quantum model may be alternatively defined as an (infinite) set of effective vertices $\mathcal{Z}_k \sim \{\Gamma_k^{(n)}\} =: \mathfrak{h}_k$. RG equations dictate how to move from $\mathfrak{h}_k \underset{\text{RG}}{\to} \mathfrak{h}_{k+\delta k}$ whereas Ward identities dictate how to move in the momentum space, along $\mathfrak{h}_k$.

3. Effective vertex expansion

This section essentially summarizes the state of the art in [1–4]. The exact RG equation cannot be solved except for very special cases. The main difficulty is that the Wetterich equation (9) splits as an infinite hierarchical system, the derivative of $\Gamma^{(n)}$ involving $\Gamma^{(n+2)}$, and so one. Appropriate approximation schemes are then required to extract information on the exact solutions. The effective vertex expansion (EVE) is a recent technique allowing to build an approximation considering infinite sectors rather than crude truncations on the full theory space. We focus on the *melonic sector*, which takes into account all the divergences of the model and then dominate the flow in the UV. One should remember that melonic diagrams are defined as diagrams with an optimal degree of divergence. Fixing some fundamental cut-off Λ, we consider the domain $1 \ll k \ll \Lambda$, equally far from the deep UV and the deep IR regime. At this place, the flow is dominated by the renormalized couplings, which have positive or zero *flow dimension* (see [3]). We recall that the flow dimension reflects the behavior of the RG flow of the corresponding quantity, and discriminates between relevant, marginal and irrelevant couplings just like standard dimension in quantum field theory.[1] Because our theory is just-renormalizable, one has necessarily $[m^2] = 2$ and $[g] = 0$, denoting as $[X]$ the flow dimension of X.

The basic strategy of the EVE is to close the hierarchical system coming from (9) using the analytic properties of the effective vertex functions[2] and the rigid structure of the melonic diagrams. More precisely, the EVE expresses all the melonic effective vertices $\Gamma^{(n)}$ having negative flow dimension (that is for $n > 4$) in terms of effective vertices with positive or null flow dimension, that is $\Gamma^{(2)}$ and $\Gamma^{(4)}$, and their flow is entirely driven by just-renormalizable couplings. As recalled, in this way we keep the entirety of the melonic sector and the full momentum dependence of the effective vertices.

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[1] For ordinary quantum field theory, the dimension is fixed by the background itself. Without background, this is the behavior of the RG flow which fixes the canonical dimension.

[2] Melonic diagrams may be easily counted as “trees”, and the (renormalized) melonic perturbation series is easy to sum.

We work in the *symmetric phase*, i.e. in the interior of the domain where the vacuum $M = \bar{M} = 0$ makes sense. This condition ensures that effective vertices with an odd number of external points, or not the same number of black and white external nodes are discarded from the analysis. These ones being called *assorted functions*. Moreover, due to the momentum conservation along the boundaries of faces, $\Gamma_k^{(2)}$ must be diagonal:

$$\Gamma^{(2)}_{k,\,\vec{p}\,\vec{q}} = \Gamma^{(2)}_k(\vec{p}\,)\delta_{\vec{p}\,\vec{q}}\,. \tag{15}$$

We denote as G_k the effective 2-point function $G_k^{-1} := \Gamma_k^{(2)} + r_k$.

The main assumption of the EVE approach is the existence of a finite analyticity domain for the leading order effective vertex functions, in which they may be identified with the resumed perturbative series. For the melonic vertex function, the existence of such an analytic domain is ensured, melons can be mapped as trees and easily summed. Moreover, these resumed functions satisfy the Ward-Takahashi identities, written without additional assumption than the cancellation of odd and assorted effective vertices. Then we will restrict ourselves in the symmetric phase i.e. Then we will restrict ourselves in the symmetric phase i.e. the case where the entire cover the perturbative domain. The symmetric phase corresponds to the case where the two-point correlation functions are symmetric i.e. $G_{\vec{p}\vec{q}} \propto \delta_{\vec{p}\vec{q}}$.

Among the properties of the melonic diagrams, we recall the following statement:

Proposition 1. *Let $\mathcal{G}_N$ be a $2N$-point 1PI melonic diagrams built with more than one vertex for a purely quartic melonic model. We call external vertices the vertices hooked to at least one external edge of $\mathcal{G}_N$ has :*

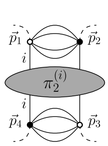
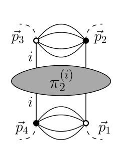
- *Two external edges per external vertices, sharing $d-1$ external faces of length one.*
- *N external faces of the same color running through the interior of the diagram.*

Due to this proposition, the melonic effective vertices $\Gamma_k^{(n)}$ decompose as d functions $\Gamma_k^{(n),i}$, labeled with a color index i:

$$\Gamma^{(n)}_{k,\,\vec{p}_1,\cdots,\vec{p}_n} = \sum_{i=1}^{d} \Gamma^{(n),i}_{k,\,\vec{p}_1,\cdots,\vec{p}_n}\,. \tag{16}$$

The Feynman diagrams contributing to the perturbative expansion of $\Gamma^{(n,i)}_{k,\,\vec{p}_1,\cdots,\vec{p}_n}$ fix the relations between the different indices. For $n = 4$ for instance, we get, from Proposition 1:

$$\Gamma^{(4),i}_{\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4} = \text{[diagram]} + \text{[diagram]}\,, \tag{17}$$

where the half dotted edges correspond to the amputated external propagators, and the reduced vertex functions $\pi_2^{(i)} : \mathbb{Z}^2 \to \mathbb{R}$ denotes the sum of the interiors of the graphs, excluding the external nodes and the colored edges hooked to them. In the same way, we expect that the melonic effective vertex $\Gamma^{(6),i}_{\text{melo}\,\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4,\vec{p}_5,\vec{p}_6}$ is completely determined by a reduced effective vertex $\pi_3^{(i)} : \mathbb{Z}^3 \to \mathbb{R}$ hooked to a boundary configuration such as:

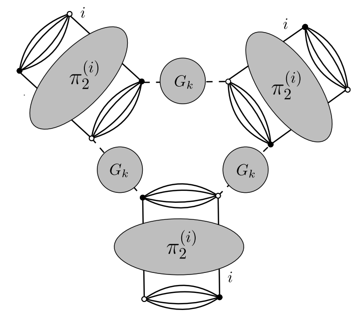
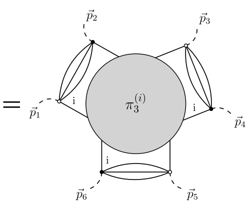


Fig. 1. Internal structure of the 1PI 6-points graphs.

$$\Gamma^{(6),i}_{\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4,\vec{p}_5,\vec{p}_6} = \text{[diagram]} + \text{perm}\,, \tag{18}$$



and so one for $\Gamma^{(n),i}_{k,\,\vec{p}_1,\cdots,\vec{p}_n}$, involving the reduced vertex $\pi_n^{(i)} : \mathbb{Z}^n \to \mathbb{R}$. In the last expression, perm denote the permutation of the external edges like in (17). The reduced vertices $\pi_2^{(i)}$ can be formally resumed as a geometric series [1–3]:

$$\begin{aligned}\pi^{(1)}_{2,pp} &= 2\left(g - 2g^2\mathcal{A}_{2,p} + 4g^3(\mathcal{A}_{2,p})^2 - \cdots\right)\\ &= \frac{2g}{1+2g\mathcal{A}_{2,p}}\,,\end{aligned} \tag{19}$$

where $\pi^{(1)}_{2,pp}$ is the diagonal element of the matrix $\pi_2^{(1)}$ and :

$$\mathcal{A}_{n,p} := \sum_{\vec{p}} G_k^n(\vec{p})\delta_{p\,p_1}\,. \tag{20}$$

The reduced vertex $\pi^{(1)}_{2,pp}$ depends implicitly on k, and the renormalization conditions defining the *renormalized coupling* g_r are such that:

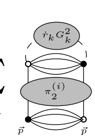
$$\pi^{(i)}_{2,00}|_{k=0} =: 2g_r\,. \tag{21}$$

For arbitrary k, the zero-momentum value of the reduced vertex defines the effective coupling for the local quartic melonic interaction: $\pi^{(i)}_{2,00} =: 2g(k)$. The explicit expression for $\pi_3^{(1)}$ may be investigated from the Proposition 1. The constraint over the boundaries and the recursive definition of melonic diagram enforce the internal structure pictured on Fig. 1 [see Lahoche-Samary]. Explicitly:

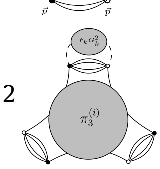
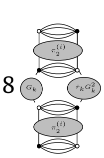
$$\pi^{(i)}_{3,ppp} = (\pi^{(i)}_{2,pp})^3\,\mathcal{A}_{3,p}\,. \tag{22}$$

The two orientations of the external effective vertices being taken into account in the definition of $\pi^{(i)}_{2,pp}$. Expanding the exact flow equation (9) and keeping only the relevant contraction for large k, one gets the following relevant contributions for $\dot{\Gamma}_k^{(2)}$ and $\dot{\Gamma}_k^{(4)}$:

$$\dot{\Gamma}_k^{(2)} = -\sum_{i=1}^{d} \text{[diagram]} \tag{23}$$



$$\dot{\Gamma}_k^{(4),i} = -2\,\text{[diagram]} + 8\,\text{[diagram]} \tag{24}$$

where $\dot{X} := k\partial X/\partial k$. The computation requires the explicit expression of $\Gamma_k^{(2)}$. In the melonic sector, the self-energy obeys a closed equation, reputed difficult to solve. We approximate the exact solution by considering only the first term in the derivative expansion in the interior of the windows of momenta allowed by $\dot{r}_k$:

$$\Gamma_k^{(2)}(\vec{p}\,) := Z(k)\vec{p}^{\,2} + m^2(k)\,, \tag{25}$$

where $Z(k) := \partial\Gamma_k^{(2)}/\partial p_1^2(\vec{0})$ and $m^2(k) := \Gamma_k^{(2)}(\vec{0})$ are both renormalized and effective field strength and mass. From the definition (8), and with some calculation (see [3]), we obtain the following statement:

Proposition 2. *In the UV domain $1 \ll k \ll \Lambda$ and in the symmetric phase, the leading order flow equations for relevant and marginal local couplings are given by:*

$$\begin{cases} \beta_m &= -(2+\eta)\bar{m}^2 - 10\bar{g}\,\frac{\pi^2}{(1+\bar{m}^2)^2}\,\left(1+\frac{\eta}{6}\right)\,, \\ \beta_g &= -2\eta\bar{g} + 4\bar{g}^2\,\frac{\pi^2}{(1+\bar{m}^2)^3}\,\left(1+\frac{\eta}{6}\right)\Big[1 \\ & \qquad -\frac{1}{2}\pi^2\bar{g}\left(\frac{1}{(1+\bar{m}^2)^2}+\left(1+\frac{1}{1+\bar{m}^2}\right)\right)\Big]. \end{cases} \tag{26}$$

With:

$$\eta = 4\bar{g}\pi^2\frac{(1+\bar{m}^2)^2 - \frac{1}{2}\bar{g}\pi^2(2+\bar{m}^2)}{(1+\bar{m}^2)^2\Omega(\bar{g},\bar{m}^2) + \frac{(2+\bar{m}^2)}{3}\bar{g}^2\pi^4}\,, \tag{27}$$

and

$$\Omega(\bar{m}^2,\bar{g}) := (\bar{m}^2+1)^2 - \pi^2\bar{g}\,. \tag{28}$$

In this proposition, $\beta_g := \dot{\bar{g}}$, $\beta_m := \dot{\bar{m}}^2$ and the effective-renormalized mass and couplings are defined as: $\bar{g} := Z^{-2}(k)g(k)$ and $\bar{m}^2 := Z^{-1}(k)k^{-2}m^2(k)$. For the computation, note that we made use of the approximation (25) only for absolutely convergent quantities, and into the windows of momenta allowed by $\dot{r}_k$. As pointed out in [1–3], taking into account the full momentum-dependence of the effective vertex $\pi_2^{(i)}$ in (19) drastically modifies the expression of the anomalous dimension η with respect to crude truncations. In particular, the singularity line discussed in [2] disappears below the singularity $\bar{m}^2 = -1$. Moreover, because all the effective melonic vertices only depend on $\bar{m}^2$ and $\bar{g}$, any fixed point for the system (26) is a global fixed point for the melonic sector. The system (26) admits a fixed point given by $p := (\bar{g}_*; \bar{m}_*^2) \approx (0.003; -0.55)$.

4. The melonic constrained flow

To close the hierarchical system derived from (9) and to obtain the autonomous set (26), we made use of the explicit expressions (19) and (22). In this derivation, we mentioned the Ward identity but they do not contribute explicitly. In this section, we take into account their contribution and show that they introduce a strong constraint over the RG trajectories.

Deriving successively the Ward identity (14) with respect to external sources, and setting $J = \bar{J} = 0$ at the end of the computation, we get the two following relations involving $\Gamma_k^{(4)}$ and $\Gamma_k^{(6)}$ (see [2])

$$\pi_{2,00}^{(1)}\,\mathcal{L}_{2,k} = -\frac{\partial}{\partial p_1^2}\left(\Gamma_k^{(2)}(\vec{p}\,) - Z\vec{p}^{\,2}\right)\Big|_{\vec{p}=\vec{0}}\,, \tag{29}$$

$$2\left(\pi_{3,00}^{(1)}\,\mathcal{L}_{2,k} - (\pi_{2,00}^{(1)})^2\,\mathcal{L}_{3,k}\right) = -\frac{d}{dp_1^2}\pi_{2,p_1p_1}^{(1)}\big|_{p_1=0}\,, \tag{30}$$

where:

$$\mathcal{L}_{n,k} := \sum_{\vec{p}_\perp}\left(Z + \frac{\partial r_k}{\partial p_1^2}(\vec{p}_\perp)\right)G_k^n(\vec{p}_\perp)\,. \tag{31}$$

It can be easily checked that the structure equations (19) and (22) satisfy the second Ward identity (30) see [1–3] and also [26–28]. In the same way, the first Ward identity (29) has been checked to be compatible with the equation (19) and the melonic closed equation for the 2-point function. However, the last condition does not exhaust the information contained in (19). Indeed, with the same level of approximation as for the computation of the flow equations (26), the first Ward identity can be translated locally as a constraint between beta functions (see [3]):

$$\mathcal{C} := \beta_g + \eta\bar{g}\,\frac{\Omega(\bar{g},\bar{m}^2)}{(1+\bar{m}^2)^2} - \frac{2\pi^2\bar{g}^2}{(1+\bar{m}^2)^3}\beta_m = 0\,. \tag{32}$$

Generally, the solutions of the system (26) do not satisfy the constraint $\mathcal{C} = 0$. We call *physical melonic phase space* and denote as $\mathcal{E}_\mathcal{C}$ the subspace of the melonic theory space satisfying $\mathcal{C} = 0$. An attempt to describe this space has been provided in [1]. In particular, we showed that there are no global fixed point of (26) which satisfy the constraint $\mathcal{C} = 0$.

In the description of the physical flow over $\mathcal{E}_\mathcal{C}$ provided in [1], we substituted the explicit expressions of β_g, β_m and η, translating the relations between velocities as a complicated constraint on the couplings $\bar{g}$ and $\bar{m}^2$. Solving this constraint, we build a systematic projection of the RG trajectories. Beyond the fact that this strategy is difficult to extend for renormalized models involving higher-order interactions, even for the quartic melonic model some difficulties appear, as multi-branch phenomenon [1]. In this section, we provide an alternative description that simplifies the description of $\mathcal{E}_\mathcal{C}$ and which can be easily extended for a model with higher-order interactions. Substituting the flow equations (26) into the constraint (32), we implicitly impose the conservation of the relation (22) between $\pi_2^{(i)}$ and $\pi_3^{(i)}$ on $\mathcal{E}_\mathcal{C}$. We propose to relax this constraint, fixing $\pi_3^{(i)}$ by the flow itself. Our procedure is the following.

(1) We keep β_m and fix β_g from the equation (32):

$$\begin{cases} \beta_m &= -(2+\eta)\bar{m}^2 - \frac{10\pi^2\bar{g}}{(1+\bar{m}^2)^2}\,\left(1+\frac{\eta}{6}\right)\,, \\ \beta_g &= -\eta\bar{g}\,\frac{\Omega(\bar{g},\bar{m}^2)}{(1+\bar{m}^2)^2} + \frac{2\pi^2\bar{g}^2}{(1+\bar{m}^2)^3}\beta_m\,. \end{cases} \tag{33}$$

(2) We fix $\pi_{3,00}^{(i)}$ dynamically from the flow equation (24):

$$\beta_g = -2\eta\bar{g} - \frac{1}{2}\bar{\pi}_3^{(1)}\frac{\pi^2}{(1+\bar{m}^2)^2}\left(1+\frac{\eta}{6}\right) + 4\bar{g}^2\,\frac{\pi^2}{(1+\bar{m}^2)^3}\left(1+\frac{\eta}{6}\right) \tag{34}$$

(3) We compute $\frac{d}{dp_1^2}\pi_{2,00}^{(i)}$ from equation (30), and finally deduce the anomalous dimension η. The computation requires the sums $\mathcal{L}_{2,k}$ and $\mathcal{L}_{3,k}$. Following [1–3], $\mathcal{L}_{3,k}$ may be computed using the approximation (25), but not $\mathcal{L}_{2,k}$ which has a vanishing power counting. However, $\mathcal{L}_{2,k}$ may be expressed in term of $Z(k)$ and $g(k)$ from equation (29). Indeed, setting $k = 0$ and fixing the renormalization condition such that $Z(k=0) = 1$,[3] we get that, in the continuum limit $\Lambda \to \infty$, $Z \to 0$. Consequently (29) reduces to $-2g(k)\mathcal{L}_{2,k} = Z(k)$, and from (30):

[3] This condition may be refined, see [3], but this point has no consequence on our discussion.

$$\frac{d}{dp_1^2}\pi_{2,00}^{(1)} = \left(Z(k)\frac{\pi_{3,00}^{(1)}}{g(k)} + 2(\pi_{2,00}^{(1)})^2\,\mathcal{L}_{3,k} \right). \tag{35}$$

Computing $\mathcal{L}_{3,k}$, one gets straightforwardly, in the continuum limit:

$$\mathcal{L}_{3,k} = -\frac{1}{2Z^2(k)k^2}\frac{\pi^2}{(1+\bar{m}^2)^3}\,. \tag{36}$$

Then, from equations (35), (34) and from the flow equation (23), it is easy to get the explicit expression of η on $\mathcal{E}_\mathcal{C}$, replacing the expression (27):

$$\eta = 4\pi^2\bar{g}\frac{(1+\bar{m}^2)^3 + 9\pi^2\bar{g}}{(1+\bar{m}^2)^5 - \Omega'(\bar{g},\bar{m}^2)}\,, \tag{37}$$

with:

$$\Omega'(\bar{g},\bar{m}^2) := \pi^2\bar{g}(1+\bar{m}^2)^3\left(1 - \frac{8}{3(1+\bar{m}^2)^2}\right) + 6\pi^4\bar{g}^2 \tag{38}$$

Note that the two equations (30) and (32) are satisfied by construction. Moreover, the hierarchy remains closed. Indeed, $\pi_{3,00}^{(i)}$ being fixed, we may compute $\dot{\pi}_{3,00}^{(i)}$ and to equal with the corresponding flow equation provided by (9). Then it's obvious that we fix $\pi_{4,00}^{(i)}$.

The system (33) completed with the new anomalous dimension (37) both describe the physical RG flow over $\mathcal{E}_\mathcal{C}$. Note that setting $\bar{m}^2 \to 0$ and keeping only the first order contributions in $\bar{g}$, we get:

$$\beta_g \approx -\eta\bar{g}\,, \quad \beta_m \approx -2\bar{m}^2\,, \quad \eta \approx 4\pi^2\bar{g}\,, \tag{39}$$

recovering the well known asymptotic freedom. As expected, the same result may be obtained from the unconstrained system (26), or from a direct perturbative computation. As a result, the physical space $\mathcal{E}_\mathcal{C}$ is connected to the Gaussian fixed point $(\bar{g},\bar{m}^2) = (0,0)$. The flow equation has essentially two sources of singularities. The first one for $\bar{m}^2 = -1$ due to the symmetric phase restriction, and the second one due to the denominator of η, $\mathrm{den}(\bar{g},\bar{m}^2) := (1+\bar{m}^2)^5 - \Omega'$. From a direct inspection, the Gaussian fixed point is into the region den > 0, and the relevant investigated region have to satisfy $\bar{m}^{>} - 1$ and den > 0. Numerical investigations show that there are no global fixed points over $\mathcal{E}_\mathcal{C}$ for the global fixed point. Indeed, we get three non-Gaussian fixed points: $p_1 \approx (1.25, -0.13)$, $p_2 \approx (-9, 6.6)$ and $p_3 \approx (-0.9, 0.0006)$. The two last ones are in the region den < 0, and therefore disconnected from the Gaussian fixed point. For p_1 however den$(p_1) > 0$. This fixed point has zero anomalous dimension $\eta(p_1) = 0$ and two relevant directions; with critical exponents $(\theta_1, \theta_2) \approx (-4.4, -0.3)$ and eigendirections :

$$\mathbf{v}_1 \approx (-1, 0; -0.1)\,, \qquad \mathbf{v}_2 \approx (0.9, -0.2)\,. \tag{40}$$

Figs. 2a and 2b describe respectively the behavior constrained RG flow for $\bar{g} \leq 0$ and $\bar{g} \geq 0$ from a numerical integration.

In contrast with standard analysis based on truncation or unconstrained FRG method like the EVE expansion, there is no global fixed point in the region 2b. Recalling that all the RG trajectories are oriented from IR to UV, we recognize the Gaussian fixed point as en UV attractor for $\bar{g}(k) > 0$, with a very clear large river effect. All the trajectories reach the mainstream corresponding to the red line and finally go to the Gaussian Fixed point. Reversing the arrows, we see that all the trajectories split into two types: The ones going to a region with negative mass and the others, reaching a region with positive mass.

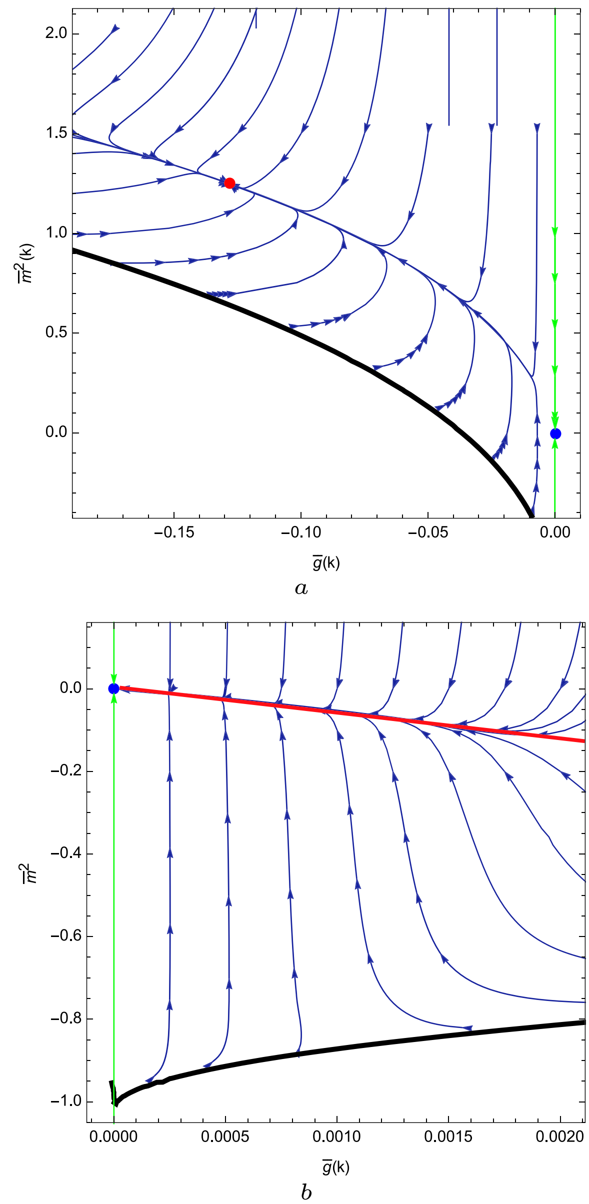


Fig. 2. The numerical renormalization group flow around the Gaussian fixed point, for $\bar{g}(g) < 0$ (a) and $\bar{g}(k) > 0$ (b). The blue point on both sides corresponds to the Gaussian fixed point, whereas the red point on (a) corresponds to the fixed point p_1. The black line corresponds to the singularity den $= 0$, and the green line corresponds to eigendirections of the Gaussian fixed point. Regular trajectories are pictured in blue.

This splitting scenario uncontrolled by a fixed point (except the Gaussian one) is reminiscent of a first-order phase transition rather than a second-order one, as frequently suggested (see [1–19] and [31] about first-order phase transition). Note that there are the presence of the black singularity line on both sides (a) and (b). In the purely EVE expansion, this singularity has been avoidable, being displaced under the singularity $\bar{m}^2 = -1$ from its original position coming from truncations. The resurgence of this singularity is understood as the mark of a significant limitation of our construction, focused on the symmetric phase. Going beyond the symmetric phase and other approximations like (25), and investigating the nature of the transition are works in progress.

On the left hand side (Fig. 2b), for $\bar{g}(k) < 0$, the scenario is repeated. The non-Gaussian fixed point p_1 behaves like an attractor, with very similar characteristics like the Gaussian fixed point. We have the mainstream on both sides of the fixed points, and all the trajectories reach the stream before to go on p_1. The integral curve of the Eigen directions for the Gaussian fixed point (in green) separates the flow. Any trajectory on the side $\bar{g}(k) >$ cannot reach the region $\bar{g}(k) <$ and so one. As a result, at least into the investigated region of the phase space, the two regions are disconnected from the RG flow. As a consequence, requiring the coupling to be positive, to ensure integrability of the partition function, it is tempting to view the region 2b as a formal artifact and to keep only the region 2b for physical investigation on the considered model.

5. Conclusion

In this paper, we provided a short presentation of an improved version of the standard EVE method, allowing to build an approximation of the exact renormalization group flow sector by sector for a tensorial group field theory, and taking into account systematically the constraint coming from Ward identities along with the flow. The resulting effective equation has a single non-trivial fixed point with zero anomalous dimension, positive mass, negative effective coupling, and two purely attractive eigendirections. This fixed point, which has similar characteristics as the Gaussian one has been discarded, because it belongs to the region with negative coupling and any trajectories starting from the positive region can reach the negative one. In particular, the melonic constrained flow has no Wilson-Fisher type fixed points, following the results of our previous works [1–4], and therefore no second-order phase transition may be identified. Our final landscape is a Gaussian fixed point with a repulsive streamline in the IR.

Remark that the EVE method, in contrast with the truncation method, cuts "smoothly" into the full phase space and selects "sectors" (that is, infinite sets of observables) rather than a finite-dimensional subset of interactions. The results of fixed points are similar to truncations in the melonic sector. However, the methods differ in their philosophy. With the EVE method, the phase space is built of an infinite set of interactions, parametrized with a finite set of couplings, and the full momentum dependence of the effective vertices is taken into account. Moreover, some singularities occurring in the truncation method disappear in the EVE. In that sense, EVE extends maximally the domain of investigation of the phase space. However, the formalism seems to be less flexible than truncations. In particular, it is less difficult to nest "sectors with sectors". Remark also that, from the renormalization group, there is no legitimacy to focus only on the melonic sector because other non melonic renormalizable sectors exist. We have already discussed such sectors in a previous contribution [3], and showed how to nest different renormalizable sectors to increase the precision of the approximation. The melonic diagrams provide the first renormalizable interactions in the potential development of the field, and we can expect them to construct a relatively good approximation of the flow, not so far from the Gaussian fixed point. A more accurate approximation would require more work, which goes far beyond the scope of this letter.

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