Holographic superconductor induced by charge density waves

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Understanding the role of charge density wave (CDW) in high-temperature superconductivity is a longstanding challenge in condensed matter physics. We construct a holographic superconductor model in which the $U(1)$ symmetry is spontaneously broken only due to the presence of CDWs, rather than previously known free charges with constant density. Below the critical temperature of superconductivity, CDW phase and superconducting phase coexist, which is also justified by the numerical results of optical conductivity. The competitive and cooperative relations between CDW phase and superconducting phase are observed. This work supports the opinion that the appearance of pseudo-gap in CDW phase promotes the prepairing of electrons as well as holes such that the formation of superconductivity benefits from the presence of CDW.

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I. INTRODUCTION

In the past decade, AdS/CMT duality [1–3] has been becoming a powerful tool for understanding the fundamental problems in strongly coupled system since the seminal work of holographic superconductor [4,5]. Considerable efforts have been triggered to understand the mechanism of high-temperature superconductivity from the holographic point of view. The key insight in this approach is that the AdS background may become unstable due to the presence of negative mass terms such that the $U(1)$ gauge symmetry is spontaneously broken below the critical temperature. A charged scalar condensates and works as the order parameter to characterize the superconducting phase in the boundary theory, following the standard holographic dictionary. In parallel, holographic models of charge density waves (CDWs) have been constructed by spontaneously breaking the translational symmetry in the bulk [6] and its fundamental features as a metal-insulator transition have been observed in [7]. The strongly coupled nature of the above systems has been signaled by the energy gap or pseudo-gap in the optical conductivity which are much larger than that predicted by BCS theory [5,7].

High-temperature superconductor exhibits very abundant phase structure, which involves the interplay between CDW phase and superconducting phase. Disclosing their relation is crucial for understanding the mechanism of high-temperature superconductivity. For cuprate oxides that is usually treated as a doped Mott insulator, the interplay between the CDW phase, also known as the pseudogap phase, and the superconducting phase has been under debate for decades (For instance, see [8] for review). The coexistence and the competition between the CDW phases and the superconducting phases are well known [9–15]. Specifically, the competitive relation is signaled by the suppression of T_c by the CDW phase [16,17], and the competition of their order parameters [18,19]. Superconducting phase can grab charge carriers from CDW phases [19]. On the other hand, recent experiments reveal a novel cooperative relationship between them as well. This is signaled by the positive correlation between their critical temperatures [20–22]. The CDW assists the superconductivity through phonons which serve as a "glue" between electrons to form a Cooper pair [22]. Moreover, the CDW is argued to cause the superconductivity under certain conditions [23,24]. Nonetheless, at present a complete theoretical interpretation on their relations has not been emerged in condensed matter physics. One challenge comes from the strongly coupled nature of the high temperature superconductivity and the CDW phases (Recent experiments suggest that the mechanism of CDW phase in superconductors may involve

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strongly correlated physics [25–27]). This fact renders holographic duality theory an effective weapon for attacking them.

High temperature superconductivity is related to both free charge carriers and CDW that usually come from doping a Mott insulator. To reveal the role of CDW solely in high temperature superconductivity, one good strategy is to separate it from free carriers in the compound. This is a hard task because typical high temperature superconductors are produced in the presence of both free charge carriers and CDW. In this paper, we construct a holographic superconductor model in the presence of CDWs only. Previously, all the holographic superconductor models are built in the presence of free charges with constant density [28,29], including some models with CDW [30,31].[1] For the first time, our model demonstrates that the superconductivity may form from the preexiting CDW phase by a first order phase transition. This may provide a direct evidence to the argument that the CDW state or the pseudo-gap state, is the precursor to the superconducting states [32–36], which has also been verified by the experiment on Nernst effect [37]. Moreover, below the transition temperature, our model is characterized by the coexistence of the CDW phase and superconducting phase, which has been observed in many superconducting materials as discovered in [9–12]. Furthermore, in our model the CDW phase can not only compete with superconducting phase, but also cooperate with superconducting phase under certain conditions, similar to the phenomena observed in [22]. Specifically, the competitive relation is manifestly disclosed by the order parameters. Above the critical temperature T_c of superconductivity, the CDW order parameter increases with the decrease of the temperature. However, when the temperature drops below T_c, the order parameter of CDW decreases while that of superconductivity increases, which exactly coincides with the phenomena observed in [18,19]. On the other hand, the positive correlation between their critical temperatures above a critical momentum mode suggests a cooperative relationship between them, which also coincides with experiment [22]. In addition, the system in our model is neutral in the sense that the average value of charge density over a period vanishes, implying that the charge excitations consist of both electron pairs and hole pairs. The hole superconductivity as a promising candidate for high temperature superconductivity was previously investigated in [38–40].

II. THE HOLOGRAPHIC SETUP

In this section, we build the holographic model in the framework of Einstein-Maxwell-Dilaton theory. The action reads as,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}[R - \frac{1}{2}(\nabla\Phi)^2 - V(\Phi) - \frac{1}{4}Z_A(\Phi)F^2 - \frac{1}{4}Z_B(\Phi)G^2 - \frac{1}{2}Z_{AB}(\Phi)FG - |(\nabla - ieB)\Psi|^2 - m_v^2\Psi\Psi^*], \tag{1}$$

where $F = dA$, $G = dB$, $Z_A(\Phi) = 1 - \frac{\beta}{2}L^2\Phi^2$, $Z_B(\Phi) = 1$, $Z_{AB}(\Phi) = \frac{\gamma}{\sqrt{2}}L\Phi$, $V(\Phi) = -\frac{1}{L^2} + \frac{1}{2}m_s^2\Phi^2$. A and B are both $U(1)$ gauge fields, among which we treat B as the Maxwell field dual to the electrical currents. For clarity, we denote the symmetry groups related to A and B fields as $U_A(1)$ and $U_B(1)$, respectively. The real dilaton field Φ is viewed as the order parameter of translational symmetry breaking, while the other complex field Ψ, charged only under the field B, is viewed as the order parameter of $U_B(1)$ gauge symmetry breaking. This two gauge field formalism, dual to a two-current system that ubiquitously exist in nature [41], is proposed for two-fold reasons. First, the doping is crucial for the emergence of the CDW, while in holographic theories, the CDW can emerge in the presence of an additional gauge field [6,7]. Therefore, we introduce the additional gauge field A to bring to the system the doping effect. Note that the CDW can also exist with only a single gauge field, such as the Einstein-Maxwell-dilaton model or a class of models [6] containing Chern-Simons terms [42]. It has been suggested in [43] that A can be dual to a conserved spin current. Second, the formalism with multiple gauge fields emerges in the high-order correction of the gravity theory, and is naturally introduced in this holographic model. We also stress that $Z_{AB}(\Phi)$ plays a vital role in making the AdS-RN black hole become unstable to form spatially modulated modes[6,7].

In the absence of the complex field Ψ, the holographic CDW as well as its optical conductivity has previously been obtained within the above framework in [7]. After the spontaneous breaking of translational symmetry, the background with CDW becomes spatially modulated. Without loss of generality, throughout this paper we set the AdS radius $l^2 = 6L^2 = 1/4$, the masses of the dilaton and the condensation $m_s^2 = m_v^2 = -2/l^2 = -8$, the coupling constant $\beta = -130$ and $\gamma = 16.6$, then the maximal critical temperature for translational symmetry breaking is $T_{\mathrm{CDW}} \approx 0.071\mu_A$ with the critical momentum mode $k_c \approx 0.335\mu_A$, as illustrated in Fig. 1, where μ_A, the boundary value of time component of the gauge field A, is the chemical potential of doping charge of the dual field theory that is adopted as the scaling unit of the system [43]. In addition, the large N limit of holographic duality requires that $\frac{l^2}{\kappa^2} \gg 1$.

The charge density can be read off from the near boundary expansion $B_t = -\rho(x)z + O(z^2)$ with

[1] Even in the probe limit where the background is Schwarzschild-AdS black hole, accompanying the condensation of scalar field, free charges with constant density are perturbatively generated as well.

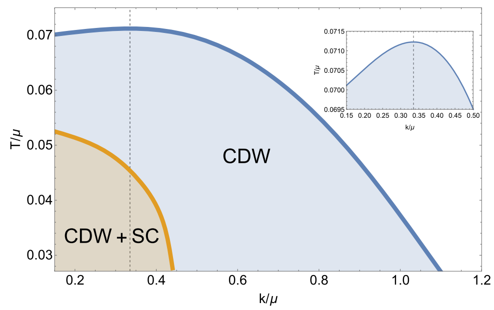

FIG. 1. The phase diagram in the plane of $(T/\mu_A, k/\mu_A)$. The blue curve denotes the critical temperature of CDW namely T_{CDW} and the orange curve denotes the critical temperature of superconductivity, namely T_c. The inset zooms in the critical point of the CDW curve.

$$\rho(x) = \rho_0 + \rho_1 \cos(k_c x) + \cdots + \rho_n \cos(n k_c x) \quad (2)$$

One remarkable feature of this model is that all the even-order coefficients of the charge density vanish, namely, $\rho(x) = \rho_1 \cos(k_c x) + \rho_3 \cos(3k_c x) + \cdots$, which has been justified by numerical analysis in [7]. It means that the dual boundary theory contains only CDWs, without free charges, namely, $\rho_0 = 0$. Thanks to this feature, the Peierls phase transition as a typical metal-insulator transition can be manifestly observed [7]. This advantage persists even after the superconducting phase transition, as we will describe below.

III. THE INSTABILITY AND THE CRITICAL TEMPERATURE OF SUPERCONDUCTIVITY

In this section, we examine the instability of striped black brane with CDW and evaluate the critical temperature T_c for the condensation of Ψ. At this stage we may set Ψ as $\eta e^{i\theta}$. The $U_B(1)$ symmetry breaking we considered is global, therefore θ could be an arbitrary constant. Here we set $\theta = 0$ for simplicity.

As pointed out in [44,45], whether the background is stable or not can be treated as a positive self-adjoint eigenvalue problem for e^2. Thus we rewrite the equation of motion for η as,

$$(\nabla^2 - m_v^2)\eta = e^2 B^2 \eta. \quad (3)$$

The key condition for a nonzero η solution is that $B^2 = g^{tt} B_t B_t \neq 0$. In all the previous holographic models, this condition is guaranteed by the presence of nonzero charge density ρ_0. Here, only CDW presents and we wonder if it would induce the instability of the background. The affirmative answer can be obtained by numerics. We search for non-zero solutions of η by changing the value of charge e (See Appendix B). Here we plot the phase diagram in Fig. 1 with $e = 4$.[2] First, the superconducting phase can only exist in the presence of CDW. This shows that CDW plays a crucial role in the formation of the superconducting phase. Another important phenomenon is the correlation between the critical temperature of CDW phase and superconducting phase. For $k < k_c$, T_{CDW} is negatively correlated with T_c (competitive relation); while for $k > k_c$, T_{CDW} is positively correlated with T_c (cooperative relation). This relation is similar to results of very recent experiments [22].

IV. NUMERICAL SOLUTIONS FOR BACKGROUND WITH SUPERCONDUCTIVITY

Next, we numerically solve the full holographic system. Adopting the ansatz in Appendix B, the resultant equations of motion consist of nine partial differential equations with respect to x and z, which can be numerically solved with Einstein-DeTurck method [46,47]. Especially, the near boundary expansion of η is,

$$\eta = z\eta_1(x) + z^2\eta_2(x) + O(z^3). \quad (4)$$

According to the holographic dictionary, each one of the expansion coefficients $\eta_1(x), \eta_2(x)$ can be regarded as the source, and the other one is regarded as the expected value. Here, we treat $\eta_1(x), \eta_2(x)$ as the source and the expectation, respectively. The $U_B(1)$ symmetry is expected to be broken spontaneously, hence we set $\eta_1(x) = 0$. Below T_c, a nontrivial solution of the condensation term $\eta_2(x)$ can be expanded as,

$$\eta_2(x) = \eta_2^{(0)} + \eta_2^{(1)} \cos(k_c x) + \cdots + \eta_2^{(n)} \cos(n k_c x). \quad (5)$$

Numerically, we find that only even orders survive.[3] In Fig. 2, we plot the constant term $\eta_2^{(0)}$ of the condensation vs T at $k = k_c$. The condensation saturates in low temperature region, while in the vicinity of critical temperature, the condensation becomes a multivalue function of the temperature. This feature clearly indicates a first-order phase transition, which can also be justified with the analysis of the free energy. We found that, the branch of solution with higher condensation has lower free energy, that is favored by thermodynamics.

Next, we study the behavior of charge density and CDW with temperature after the superconducting phase transition. Surprisingly, we find that all the even orders of the charge density in Fourier expansion still vanish after condensation. Namely, it takes the form as

[2]For other values of e, the phase diagram is qualitatively the same, except that the coexisting region becomes larger or smaller, which can also be justified in Appendix B.

[3]Mathematically we also find the other branch of solutions with odd orders only. But these solutions are not physical, because η_2, as the modulus of the condensation, must be positive-definite.

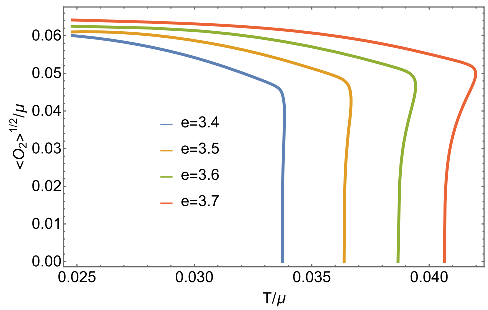

FIG. 2. The condensation of the charged scalar field for different values of e at $k = k_c$.

$\rho(x) = \rho_1 \cos(k_c x) + \rho_3 \cos(3k_c x) + \cdots$. This means that the superconducting phase forms from CDWs, not from free charges.

The near boundary expansion of the CDW order parameter is $\Phi = \phi_1(x)\mathfrak{z}^2 + O(\mathfrak{z}^3)$, and we numerically find ϕ_1 behaves as $\phi_1^{(1)} \cos(k_c x) + \phi_1^{(3)} \cos(3k_c x) + \cdots$. In Fig. 3, we plot ρ_1 and $\phi_1^{(1)}$ as functions of the temperature with $e = 4$. The left plot shows that after superconducting phase transition, the charge density becomes larger in comparison with that in the absence of superconducting phase transition. While the right plot shows that the order parameter of CDW is always smaller than that in the absence of superconducting phases. This evidently indicates that the CDW is suppressed by the presence of the superconductivity.

The physics behind these phenomena can be understood as follows. First, during all the process, no free charges involve in but only CDWs present in the charge density. We conclude that the superconductivity is induced by CDWs, rather than the free charge. This fact can also be justified by Eq. (3). Moreover, the increase of charge density after superconducting phase transition indicates that the periodic charge density consists of both normal carriers and superconducting carriers. Consequently, the carrier pairs of superconducting phase must contain both charge pairs and hole pairs, because the net charge is zero. This unconventional mechanism of carrier pairing, such as hole superconductivity, is a promising candidate for the mechanism of high temperature superconductivity [38–40].

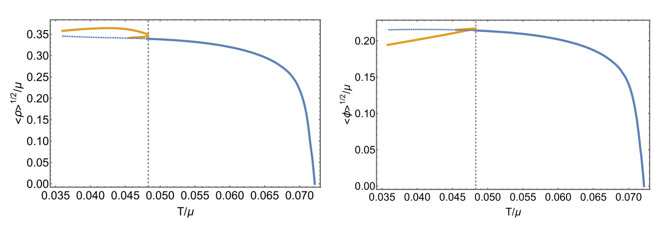

FIG. 3. The charge density ρ_1 and order parameter $\phi_1^{(1)}$ as functions of the temperature during the course of condensation. The vertical dotted line denotes the location of $T_c \approx 0.048\mu_A$. Below T_c, the yellow curve is plotted with superconducting condensation, while the dotted curve is plotted without superconducting condensation.

They beat the traditional BCS theory in explaining the high frequency absorption behavior and high transition temperature of superconductivity.

V. THE OPTICAL CONDUCTIVITY

To justify that the above condensation due to $U(1)$ symmetry breaking gives rise to a superconducting state indeed, we study the optical conductivity by linear perturbations (see Appendix C). Here we only consider the linear response of B field along x direction, namely δB_x. Expanding δB_x with $\delta B_x = (1 + j_x(x)z + \cdots)e^{-i\omega t}$. According to Ohm's law, the optical conductivity can be read off as,

$$\sigma(\omega) = 4\frac{j_x^{(0)}}{i\omega}, \tag{6}$$

where $j_x^{(0)}$ is the leading term of the Fourier expansion.

We plot the optical conductivity at temperatures close to T_c in Fig. 4. Above T_c, the appearance of the pseudo-gap in the real part shows that the system is in the insulating phase, as revealed in [7], and the imaginary part of the conductivity converges to zero. We stress that the dc conductivity here is finite and approaches to zero as the temperature goes down from T_{CDW}, which results from the vanishing average charge density. No charge is coupled with the goldstone modes in our model, hence no delta function in dc conductivity, which is in contrast to the cases in [48–51] where the goldstone modes carry nonzero charges. However, below T_c, the imaginary part exhibits a power law $\mathrm{Im}\sigma \sim 1/\omega$ in low frequency region, implying that a delta function arises at $\omega = 0$ in the real part of the conductivity by Kramers-Kronig relations. It indicates that the system undergoes a phase transition from insulating phase to superconducting phase indeed.

In low frequency region, we fit the optical conductivity by adding a pole to the damped harmonic oscillators with Lorentz resonance, namely

$$\sigma(\omega) = i\frac{K_s}{\omega} + \frac{K_1\tau_1}{1 - i\omega\tau(1 - \omega_{01}^2/\omega^2)} + \frac{K_2\tau_2}{1 - i\omega\tau(1 - \omega_{02}^2/\omega^2)}, \tag{7}$$

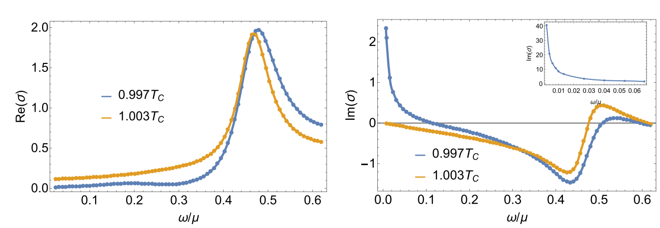

FIG. 4. The optical conductivity before and after the superconducting phase transition. The left plot is the real part of the optical conductivity and the right plot is the imaginary part.

TABLE I. The fitted parameters for optical conductivity.

T/T_c	K_s/μ_A	K_1/μ_A	$\tau_1\mu_A$	ω_{01}/μ_A	K_2/μ_A	$\tau_1\mu_A$	ω_{02}/μ_A
1.007	0	0.136	12.240	0.468	3.279	0.302	1.794
0.997	0.020	0.159	9.947	0.483	2.893	0.431	1.457

where K_s is proportional to the number of superfluid density, and K_i is proportional to the number density of CDW, while τ_i is the relaxation time, and ω_{0i} is the average resonance frequency ($i = 1, 2$). The last two terms in (7) have been widely studied in the analysis of CDW optical response experiments [52]. The fitted parameters are listed in Table I. From this table, one can see that the superfluid density becomes nonzero below the critical temperature T_c.

In the end, we remark that the energy gap of the superconductor is almost the same as the pseudogap of CDW, $\Delta_{\rm CDW} \approx \Delta_c \approx 9.041 T_c$, both of which are much larger than the BCS value. The above phase transition implies that the appearance of pseudogap can promote the prepairing of carriers and support the cooperative relation between the pseudogap phase and the superconducting phase. Similar phenomenon was previously observed in a holographic model with novel insulator as well [53].

VI. DISCUSSION

We have constructed a holographic model in which the role of CDW during the phase transition of superconductivity has been clearly disclosed. First, the $U_B(1)$ gauge symmetry of in the bulk is spontaneously broken due to the presence of CDW only, indicating that the superconductivity can form from the preexisting CDW phase by the first-order phase transition. Second, below the critical temperature T_c, the system is characterized by the coexistence of the CDW phase and the superconducting phase. It is found that CDW phase and superconducting phase can both cooperate and compete with each other. Furthermore, the system remains neutral during the phase transition, implying that the charge excitations are composed of both electron pairs and hole pairs, beyond the traditional Cooper pairs. It also supports the opinion that the pseudo-gap in CDW phase promotes the preformed pairs of carriers such that the formation of superconductivity benefits from the presence of CDW. Our work has provided a novel understanding on the complicated relationship between CDW and superconducting phases from the holographic point of view, and may shed light on the mechanism of high temperature superconductivity.

The most desirable work next is to further investigate the features of superconductivity induced by CDW, and compare them with those induced by free charges, in particular, their responses to external magnetic field or disorder effects. Moreover, it is intriguing to investigate the relation between CDW and superconductivity in more complicated environment, since the practical high temperature superconductors are doped compounds with free carriers. To this end, one may turn on the chemical potential and the free charges with constant density such that the black hole background is charged under the field B as well. In this situation, the superconductivity may be induced by both CDW and free charges, analogous to the materials in experiments. Furthermore, the order parameters of CDW and superconductor are not directly coupled in current model. We definitely can introduce the coupling of these order parameters to explore the complicated relations between CDW and superconductors. Last but not the least, one may treat field A as the doping and consider its impacts on the relations between CDW and superconductors.

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APPENDIX A: THE EQUATIONS OF MOTION

In this Appendix, we derive the equations of motion for our model. We rewrite the $U_B(1)$ charged complex scalar field Ψ as $\eta e^{i\theta}$, where $\eta > 0$ is a real scalar field and θ is a Stückelberg field, then the action can be cast as,

$$\begin{aligned} S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[& R - \frac{1}{2}(\nabla\Phi)^2 - V(\Phi) \\ & - \frac{1}{4}Z_A(\Phi)F^2 - \frac{1}{4}Z_B(\Phi)G^2 - \frac{1}{2}Z_{AB}(\Phi)FG \\ & - (\nabla\eta)^2 - m_v^2\eta^2 - \eta^2(\nabla\theta - eB)^2\Big]. \end{aligned} \tag{A1}$$

The equations of motion for all fields can be written as,

$$\begin{aligned} R_{\mu\nu} - T^{\Phi}_{\mu\nu} - T^{A}_{\mu\nu} - T^{B}_{\mu\nu} - T^{AB}_{\mu\nu} - T^{\eta}_{\mu\nu} - T^{\theta}_{\mu\nu} &= 0, \\ \nabla^2\Phi - \frac{1}{4}Z_A' F^2 - \frac{1}{4}Z_B' G^2 - \frac{1}{2}{Z_{AB}}' FG - V' &= 0, \\ \nabla^2\eta - m_v^2\eta - (\nabla\theta - eB)^2\eta &= 0, \\ \nabla_\mu(Z_A F^{\mu\nu} + Z_{AB}G^{\mu\nu}) &= 0, \\ \nabla_\mu(Z_B G^{\mu\nu} + Z_{AB}F^{\mu\nu}) + 2e\eta^2(\nabla^\nu\theta - eB^\nu) &= 0, \\ \nabla_\mu(\eta^2(\nabla^\mu\theta - eB^{\mu_A})) &= 0, \end{aligned} \tag{A2}$$

where

$$
\begin{aligned}
T^{\Phi}_{\mu\nu} &= \frac{1}{2}\nabla_\mu\Phi\nabla_\nu\Phi + \frac{1}{2}Vg_{\mu\nu},\\
T^{A}_{\mu\nu} &= \frac{Z_A}{2}\left(F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F^2\right),\\
T^{B}_{\mu\nu} &= \frac{Z_B}{2}\left(G_{\mu\rho}G^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}G^2\right),\\
T^{AB}_{\mu\nu} &= Z_{AB}\left(F_{(\mu|\rho|}G^{\rho}_{\nu)} - \frac{1}{4}g_{\mu\nu}FG\right),\\
T^{\eta}_{\mu\nu} &= \nabla_\mu\eta\nabla_\nu\eta + e^2\eta^2B_\mu B_\nu + \frac{1}{2}m_v^2\eta^2g_{\mu\nu},\\
T^{\theta}_{\mu\nu} &= \eta^2(\nabla_\mu\theta\nabla_\nu\theta - 2eB_{(\mu}\nabla_{\nu)}\theta),
\end{aligned}
\tag{A3}
$$

and the prime denotes the derivative with respect to Φ.

APPENDIX B: THE NUMERICAL ANALYSIS OF THE BACKGROUND

We adopt the following ansatz for the background with CDW,

$$
\begin{aligned}
ds^2 &= \frac{1}{z^2}\Big[-(1-z)p(z)Qdt^2 + \frac{Sdz^2}{(1-z)p(z)}\\
&\quad + Vdy^2 + T(dx + z^2Udz)^2\Big],\\
A &= \mu_A(1-z)\psi dt, \qquad B = (1-z)\chi dt,\\
\Phi &= z\phi, \qquad \eta = z\zeta, \qquad \theta = 0,
\end{aligned}
\tag{B1}
$$

where $p(z) = 4(1 + z + z^2 - \frac{\mu^2 z^3}{16})$ and μ_A is the chemical potential mismatch in the dual field theory. Q, S, V, T, U, ψ, χ, ϕ, ζ are functions of x and z. In this coordinate system, the black hole horizon locates at $z = 1$ and the AdS boundary at $z = 0$. Thanks to the Einstein-DeTurck method, the Hawking temperature of the black hole with stripes is simply given by $T/\mu_A = (48 - \mu_A^2)/(16\pi\mu_A)$. Notice that if we set $Q = S = V = T = \psi = 1$ and $U = \chi = \phi = \zeta = 0$, then the background goes back to the AdS-RN black hole.

First, we compute the critical temperature T_c by turning on η in probe limit and solving (4). We plot the relation between the charge e and T_c in Fig. 5. It indicates that below the critical temperature T_c, the background with CDW becomes unstable and the complex scalar hair starts to condensate. Moreover, the condensation become easier for larger charge, and for smaller momentum modes.

Next, we numerically solve the equations of motion (A2) with the ansatz (B1). We apply Einstein-DeTurck method to fix the coordinates with appropriate boundary conditions [46]. In addition, we impose the regular boundary condition on the horizon, and require an asymptotic AdS_4 on the boundary. With the equations of motion and boundary conditions, we obtain the numerical solutions by the pseudospectral method and Newton-Raphson iteration method [7,44,47]. As an example, Fig. 6 shows the solutions for scalar field ϕ and the charged scalar field ζ at the temperature $T = 0.988T_c$ with $e = 4$ and $k = k_c$.

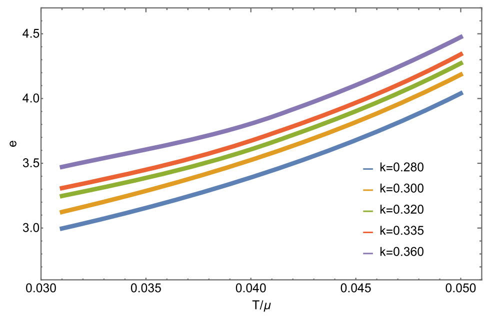

FIG. 5. The relation between the charge e and the critical temperature T_c for the condensate of the charged scalar field, where for each curve the momentum mode k of CDW is fixed.

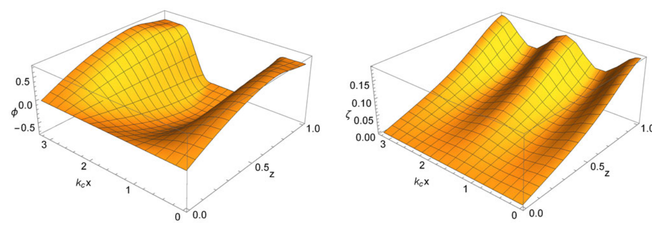

FIG. 6. The solution of the scalar ϕ and the charged scalar field ζ for $T = 0.988T_c$, $e = 4$ and $k = k_c$.

APPENDIX C: THE NUMERICAL ANALYSIS OF THE LINEAR PERTURBATIONS

Consider the following linear perturbation,

$$
\begin{aligned}
g_{\mu\nu} &= \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \qquad A_\mu = \bar{A}_\mu + \delta A_\mu, \qquad B_\mu = \bar{B}_\mu + \delta B_\mu,\\
\Phi &= \bar{\Phi} + \delta\Phi, \qquad \zeta = \bar{\zeta} + \delta\zeta, \qquad \theta = \bar{\theta} + \delta\theta.
\end{aligned}
\tag{C1}
$$

We mark the unperturbated quantities with an overbar in the above formula. Each perturbation oscillates with time as $e^{-i\omega t}$. In order to extract the optical conductivity along x direction, we only turn on B_x and keep $B_y = 0$.

Note also that we must turn on the Stückelberg field θ for self-consistency. To fix the solutions, we adopt the Donder gauge condition and Lorentz gauge condition,

$$
\bar{\nabla}^\mu \hat{h}_{\mu\nu} = 0, \qquad \bar{\nabla}^\mu \delta A_\mu = 0, \qquad \bar{\nabla}^\mu \delta B_\mu = 0, \tag{C2}
$$

where $\hat{h}_{\mu\nu} = \delta g_{\mu\nu} - \delta g\bar{g}_{\mu\nu}/2$. As a result, we obtain 21 linear equations of motion for the perturbations. We impose the ingoing boundary conditions on the horizon. Then the whole perturbation system can be solved by pseudospectral method.

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