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Mass perturbation theory in the 2-flavor Schwinger model with opposite masses with a review of the background

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Abstract: I discuss the 2-flavor Schwinger model with $\theta = 0$ and small equal and opposite fermion masses (or $\theta = \pi$ with equal masses). The massless model has an unparticle sector with unbroken conformal symmetry. I argue that this special mass term modifies the conformal sector without breaking the conformal symmetry. I show in detail how mass-perturbation-theory works for correlators of flavor-diagonal fermion scalar bilinears. The result provides quantitative evidence that the theory has no mass gap for small non-zero fermion masses. The massive fermions are bound into conformally invariant unparticle stuff. I show how the long-distance conformal symmetry is maintained when small fermion masses are turned on and calculate the relevant scaling dimensions for small mass. I calculate the corrections to the 2- and 4-point functions of the fermion-bilinear scalars to leading order in perturbation theory in the fermion mass and describe a straightforward procedure to extend the calculation to all higher scalar correlators. I hope that this model is a useful and non-trivial example of unparticle physics, a sector with unbroken conformal symmetry coupled to interacting massive particles, in which we can analyze the particle physics in a consistent approximation.

Keywords: Field Theories in Lower Dimensions, Scale and Conformal Symmetries, Integrable Field Theories

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1 Introduction

The massless 2-flavor Schwinger model is an unparticle theory[1] in 1+1 dimensions with a free massive scalar and a conformal sector that survives at low energy. In a previous paper, [2], I discussed the massive 2-flavor Schwinger model, resolving some puzzles posed many years ago by Coleman. [3] Part of the resolution was a conjecture that in the model with equal and opposite fermion masses at $\theta = 0$ (or equal masses at $\theta = \pi$), small fermion masses do not break the conformal symmetry of the long-distance sector of the model even though the massive scalar has nontrivial interactions.[2] Thus I argued that the massive fermions are bound into conformally invariant unparticle stuff. In this paper, I describe some quantitative evidence for this wild-sounding conjecture by finding the correlation functions of the flavor-diagonal fermion-bilinear scalar conformal operators. I find that the mass term does not break the conformal symmetry, but modifies it and I calculate the non-trivial scaling dimensions of the unparticle stuff in perturbation theory in the fermion mass parameter. I introduce tools that make these calculations easier and discuss some of the calculations in detail.

While I focus on the long-distance conformal theory in this paper, my primary interest is in the particle physics of the full model. I hope that it is an example of unparticle physics, a conformal sector interacting with massive particles without breaking the conformal symmetry, with a well-defined procedure for the calculation of physical quantities. Though the physics is still very simple, it is non-trivial and we can calculate. The resulting theory may be an interesting laboratory for studying the particle physics of interacting unparticle theories.

[1] See for example, [1].

[2] For simplicity of presentation in this paper, we will keep $\theta = 0$.

2 The Schwinger model

The Lagrangian of the n-flavor Schwinger model is

$$\mathcal{L} = \left(\sum_{j=1}^{n} \overline{\psi}_j \left(i\,\partial\!\!\!/ - eA\!\!\!/\right)\psi_j\right) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \sum_{j=1}^{n}\mu_j\overline{\psi}_j\psi_j \tag{2.1}$$

I begin by discussing $\mu_j = 0$ and consider the mass term in section 4.[3, 4] The massless model has a classical $\mathrm{U}(n)\times\mathrm{U}(n)$ chiral symmetry acting on the right- and left-moving fermion fields,

$$\psi_{j1} \equiv \frac{1+\gamma^5}{2}\psi_j \to R_{jk}\,\psi_{k1} \quad \psi_{j2} \equiv \frac{1-\gamma^5}{2}\psi_j \to L_{jk}\,\psi_{k2} \tag{2.2}$$

It is broken by the anomaly down to $\mathrm{SU}(n)\times\mathrm{SU}(n)\times\mathrm{U}(1)$.

In Lorenz gauge, $\partial_\mu A^\mu = 0$, we can write

$$A^\mu = \epsilon^{\mu\nu}\partial_\nu\mathcal{A}/m \tag{2.3}$$

where

$$m^2 = n\,e^2/\pi \tag{2.4}$$

At this point, it is not obvious why we should choose m this way but we will see that the answer is the chiral U(1) anomaly. Then the Lagrangian is

$$\mathcal{L} = \left(\sum_{j=1}^{n}\left(i\overline{\psi}_j\,\partial\!\!\!/\,\psi_j - e\overline{\psi}_j\gamma_\mu\psi_j\epsilon^{\mu\nu}\partial_\nu\mathcal{A}/m\right)\right) + \frac{1}{2m^2}\mathcal{A}\,\Box^2\mathcal{A} \tag{2.5}$$

If we change the fermionic variables to

$$\Psi_j = e^{ie\mathcal{A}\gamma^5/m}\psi_j = e^{i(\pi/n)^{1/2}\mathcal{A}\gamma^5}\psi_j \tag{2.6}$$

the fermions become free and the Lagrangian becomes

$$\mathcal{L} = \left(\sum_{j=1}^{n} i\overline{\Psi}_j\,\partial\!\!\!/\,\Psi_j\right) + \frac{1}{2m^2}\mathcal{A}\,\Box^2\mathcal{A} - \frac{1}{2}\partial_\mu\mathcal{A}\partial^\mu\mathcal{A} \tag{2.7}$$

The last term is the effect of the anomaly. It is worth recalling how this works in more detail. The redefinition (2.6) is an axial U(1) transformation — $\partial^\mu\mathcal{A}$ has axial-vector couplings because

$$\gamma_\mu\epsilon^{\mu\nu}\partial_\nu\mathcal{A} = \gamma_\mu\gamma^5\partial^\mu\mathcal{A} \tag{2.8}$$

[3] Some of ideas in this paper are related to the analysis of diagonal color models in 1+1 [4]. See also [5–8].

[4] My conventions are: $g^{00} = -g^{11} = 1$, $\epsilon^{01} = -\epsilon^{10} = -\epsilon_{01} = \epsilon_{10} = 1$. From the defining properties $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ and $\gamma^5 = -\frac{1}{2}\epsilon_{\mu\nu}\gamma^\mu\gamma^\nu$, it follows that $\gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu$ and $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \epsilon^{\mu\nu}\gamma^5$, and we will use the representation $\gamma^0 = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\gamma^1 = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, $\gamma^5 = \gamma^0\gamma^1 = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. Then in the massless theory, the Dirac components ψ_1 and ψ_2 describe right-moving and left-moving fermions, respectively.

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and an axial transformation induces a change in the Lagrangian because of the chiral U(1) anomaly. The effect from an infinitesimal axial transformation is proportional to the 2D anomaly of the axial U(1) current,

$$\partial_\mu j_5^\mu = -n\,\frac{e}{\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu = -n\,\frac{e}{\pi}\Box\mathcal{A}/m \tag{2.9}$$

$$\begin{aligned}\frac{d}{d\alpha}&\left(\sum_{j=1}^{n}\overline{e^{ie\alpha\mathcal{A}\gamma^5/m}\psi_j}\,\gamma_\mu\Big(i\partial^\mu - e\,\epsilon^{\mu\nu}\partial_\nu(1-\alpha)\mathcal{A}/m\Big)\,e^{ie\alpha\mathcal{A}\gamma^5/m}\psi_j\right)\\ &= -\frac{n\,e^2}{m^2\pi}\mathcal{A}(1-\alpha)\Box\mathcal{A}\end{aligned} \tag{2.10}$$

Integrating (2.10) from $\alpha = 0$ to 1 gives

$$\begin{aligned}\left(\sum_{j=1}^{n}\overline{\psi}_j\,\gamma_\mu\Big(i\partial^\mu - e\,\epsilon^{\mu\nu}\partial_\nu\mathcal{A}/m\Big)\,\psi_j\right) &= \left(\sum_{j=1}^{n} i\overline{\Psi}_j\,\partial\!\!\!/\,\Psi_j\right) + \frac{n\,e^2}{2m^2\pi}\mathcal{A}\Box\mathcal{A}\\ &= i\overline{\Psi}\,\partial\!\!\!/\,\Psi - \frac{1}{2}\partial_\mu\mathcal{A}\partial^\mu\mathcal{A}\end{aligned} \tag{2.11}$$

where Ψ is given by (2.6). This is why we chose m the way we did in (2.3) and (2.4).

Focusing on $\mathcal{A}$ in (2.7), we can replace it with somewhat more normal looking fields as follows.

$$\frac{1}{2m^2}\mathcal{A}\,\Box^2\mathcal{A} - \frac{1}{2}\partial_\mu\mathcal{A}\partial^\mu\mathcal{A} \to -\frac{m^2}{2}\mathcal{B}^2 + \mathcal{B}\Box\mathcal{A} - \frac{1}{2}\partial_\mu\mathcal{A}\partial^\mu\mathcal{A} \tag{2.12}$$

$$= -\frac{m^2}{2}\mathcal{B}^2 + \frac{1}{2}\partial_\mu\mathcal{B}\partial^\mu\mathcal{B} - \frac{1}{2}\partial_\mu\mathcal{C}\partial^\mu\mathcal{C} \tag{2.13}$$

where

$$\mathcal{C} = \mathcal{A} + \mathcal{B} \tag{2.14}$$

so $\mathcal{B}$ is a massive free field and $\mathcal{C}$ is a massless ghost and the Lagrangian becomes

$$\mathcal{L} = \left(\sum_{j=1}^{n} i\overline{\Psi}_j\,\partial\!\!\!/\,\Psi_j\right) - \frac{m^2}{2}\mathcal{B}^2 + \frac{1}{2}\partial_\mu\mathcal{B}\partial^\mu\mathcal{B} - \frac{1}{2}\partial_\mu\mathcal{C}\partial^\mu\mathcal{C} \tag{2.15}$$

Thus for gauge invariant correlators of local fields, the result of summing the perturbation theory to all orders can be found simply by making the following replacements:[5]

$$A^\mu = \epsilon^{\mu\nu}\partial_\nu(\mathcal{B}-\mathcal{C})/m \tag{2.16}$$

$$F^{01} = \partial_\mu\partial^\mu(\mathcal{B}-\mathcal{C})/m \tag{2.17}$$

$$\psi_j = e^{-i(\pi/n)^{1/2}\,(\mathcal{C}-\mathcal{B})\gamma^5}\,\Psi_j \tag{2.18}$$

with $m = e\sqrt{n/\pi}$ from (2.4) and using the free-field Lagrangian, (2.15).

We will be particularly concerned with flavor-diagonal fermion-bilinear scalar operators.

$$O_j = \psi_{j1}^*\psi_{j2} = e^{2i(\pi/n)^{1/2}\,(\mathcal{C}-\mathcal{B})}\,\Psi_{j1}^*\Psi_{j2} \tag{2.19}$$

[5]This argument appears in [9].

The free-fermion bilinears (2.19) have the remarkable property of "bosonization." [10, 11] For us, what this means is that any non-zero correlator of the O_js and O_j^*s can be calculated in terms of the massive scalar field, $\mathcal{B}$, the ghost $\mathcal{C}$, and free canonically normalized massless "scalar fields", $\mathcal{D}_j$ with the replacement

$$O_j \to \frac{\xi m}{2\pi}\, e^{2i(\pi/n)^{1/2}\,(\mathcal{C}-\mathcal{B})}\, e^{2i\pi^{1/2}\,\mathcal{D}_j} \tag{2.20}$$

where

$$\xi \equiv e^{\gamma_E}/2 \quad \text{where } \gamma_E \text{ is Euler's constant.} \tag{2.21}$$

Note that in perturbation theory, the only non-zero correlators are those with equal numbers of O_js and O_j^*s for each j. But there are important non-perturbative effects, again related to the anomaly.

The non-perturbative effects are particularly simple in the 1-flavor model, where (2.18) gives

$$\psi = e^{-i\pi^{1/2}\,(\mathcal{C}-\mathcal{B})\gamma^5}\,\Psi \tag{2.22}$$

and there is only one conjugate pair of scalar fermion bilinears

$$O_1 = \psi_1^*\psi_2 \to e^{2i\pi^{1/2}\,(\mathcal{C}-\mathcal{B})}\Psi_1^*\Psi_2 \to \frac{\xi m}{2\pi}\, e^{2i\pi^{1/2}\,(\mathcal{C}-\mathcal{B})}\, e^{2i\pi^{1/2}\,\mathcal{D}} \tag{2.23}$$

Now the effects of the bosonization field $\mathcal{D}$ are exactly canceled by the effects of the ghost $\mathcal{C}$ and (2.23) is

$$O_1 = \psi_1^*\psi_2 \to \frac{\xi m}{2\pi}\, e^{-2i\pi^{1/2}\,\mathcal{B}} \tag{2.24}$$

Because the $\mathcal{B}$ field is massive, this means that the effects of the O operators on one another are exponentially suppressed at distances larger that $1/m$. But then if we have any combination of O_1 and O_1^* fields in some region of space, we can look at their correlator with a conjugate set in a distant region. We can then calculate the correlator perturbatively using (2.24) and as the distance between the regions goes to infinity, the result factors into a product of correlators in the separate regions. Cluster decomposition then implies that we can calculate the correlator of any combination of O_1s and O_1^*s using (2.24) up to a phase factor

$$O_1 \to e^{i\theta}\,\frac{\xi m}{2\pi}\, e^{-2i\pi^{1/2}\,\mathcal{B}} \tag{2.25}$$

This implies, among other things, that

$$\langle 0|O_1|0\rangle = e^{i\theta}\,\frac{\xi m}{2\pi} \tag{2.26}$$

so O_1 has a VEV that breaks the chiral symmetry. One can think of the massless field $\mathcal{D}$ as the Goldstone boson of the spontaneously broken chiral symmetry, but it is unphysical because its effects are completely canceled by the ghost field $\mathcal{C}$.[6] If we add a fermion mass, the $\mathcal{B}$ field is no longer free and in addition to the physical fermion mass, the parameter θ in (2.25) becomes the physical θ-parameter.[7]

[6] See [12].
[7] See for example, [3].

3 Two flavors

The 2-flavor model has a non-Abelian chiral symmetry, but we will again be mostly concerned with the physics of the flavor diagonal fermion-bilinear scalars that carry the chiral T_3 symmetry

$$\psi_{11} \to e^{i\phi}\psi_{11} \quad \psi_{21} \to e^{-i\phi}\psi_{21} \quad \psi_{12} \to e^{-i\phi}\psi_{12} \quad \psi_{22} \to e^{i\phi}\psi_{22} \tag{3.1}$$

and the chiral U(1) symmetry

$$\psi_{11} \to e^{i\phi}\psi_{11} \quad \psi_{21} \to e^{i\phi}\psi_{21} \quad \psi_{12} \to e^{-i\phi}\psi_{12} \quad \psi_{22} \to e^{-i\phi}\psi_{22} \tag{3.2}$$

Now with two flavors, we can again write the (flavor-diagonal) fermion bilinears in bosonized form

$$O_j \equiv \psi_{j1}^*\psi_{j2} = e^{i\sqrt{2\pi}\,(\mathcal{C}-\mathcal{B})}\Psi_{j2}^*\Psi_{j1} = \frac{\xi m}{2\pi}\,e^{i\sqrt{2\pi}\,(\mathcal{C}-\mathcal{B})}\,e^{2i\pi^{1/2}\mathcal{D}_j} \tag{3.3}$$

and when we calculate any correlator that is non-zero in perturbation theory, and thus allowed by the perturbatively conserved chiral symmetries (3.1) and (3.2), standard bosonization arguments imply that (3.3) gives the result of summing the perturbation theory to all orders.

The massless scalar fields $\mathcal{D}_1$ and $\mathcal{D}_2$ and the ghost field $\mathcal{C}$ do not make physical sense in isolation because of infrared divergences, [13] but their exponentials in (3.3) generate conformally invariant correlators in the theory at long distances and the scale is fixed by the mass m. This is unparticle stuff with no particle interpretation. [1] For example

$$\langle 0|T\,O_1(x)\,O_1^*(y)|0\rangle = \frac{(\xi m)}{(2\pi)^2}\exp\left[K_0\left(m\sqrt{-(x-y)^2+i\epsilon}\right)\right]\left(-(x-y)^2+i\epsilon\right)^{-1/2} \tag{3.4}$$

where K_0 is related to the scalar propagator[8]

$$K_0\left(m\sqrt{-x^2+i\epsilon}\right) = 2\pi i\int\frac{d^2p}{(2\pi)^2}\,\frac{e^{-ipx}}{p^2-m^2+i\epsilon} \tag{3.5}$$

Eq. (3.4) implies that the fermion bilinears have scaling dimension $1/2$ rather than their naive engineering dimension 1. For $n > 2$, the fermion bilinears have scaling dimension $1-1/n$. This is zero for $n = 1$ which is why there is no conformal sector at all in the Schwinger model.

We can rewrite (3.3) as

$$\begin{aligned}
O_1 &= \psi_{11}^*\psi_{12} = \frac{\xi m}{2\pi}\,e^{i\sqrt{2\pi}\,(\mathcal{C}-\mathcal{B})}\,e^{i\sqrt{2\pi}\,(\mathcal{D}_++\mathcal{D}_-)} = \frac{\xi m}{2\pi}\,e^{i\sqrt{2\pi}\,(\mathcal{C}+\mathcal{D}_+-\mathcal{B}+\mathcal{D}_-)}\\
O_2 &= \psi_{21}^*\psi_{22} = \frac{\xi m}{2\pi}\,e^{i\sqrt{2\pi}\,(\mathcal{C}-\mathcal{B})}\,e^{i\sqrt{2\pi}\,(\mathcal{D}_+-\mathcal{D}_-)} = \frac{\xi m}{2\pi}\,e^{i\sqrt{2\pi}\,(\mathcal{C}+\mathcal{D}_+-\mathcal{B}-\mathcal{D}_-)}\\
O_1^* &= \psi_{12}^*\psi_{11} = \frac{\xi m}{2\pi}\,e^{-i\sqrt{2\pi}\,(\mathcal{C}-\mathcal{B})}\,e^{-i\sqrt{2\pi}\,(\mathcal{D}_++\mathcal{D}_-)} = \frac{\xi m}{2\pi}\,e^{-i\sqrt{2\pi}\,(\mathcal{C}+\mathcal{D}_+-\mathcal{B}+\mathcal{D}_-)}\\
O_2^* &= \psi_{22}^*\psi_{21} = \frac{\xi m}{2\pi}\,e^{-i\sqrt{2\pi}\,(\mathcal{C}-\mathcal{B})}\,e^{-i\sqrt{2\pi}\,(\mathcal{D}_+-\mathcal{D}_-)} = \frac{\xi m}{2\pi}\,e^{-i\sqrt{2\pi}\,(\mathcal{C}+\mathcal{D}_+-\mathcal{B}-\mathcal{D}_-)}
\end{aligned} \tag{3.6}$$

[8] Note that there is no arbitrariness here because these composite operators do not require multiplicative renormalization for $\mu_j = 0$ so the position-space correlators are well-defined for non-zero separation. A subtractive renormalization is required for the 2-point function at zero separation and is needed to define the Fourier transforms.

where

$$\mathcal{D}_\pm = \frac{1}{\sqrt{2}}\,(\mathcal{D}_1 \pm \mathcal{D}_2) \tag{3.7}$$

Now $\mathcal{D}_+$ transforms like a Goldstone boson associated with spontaneous breaking of the chiral U(1) and as in the $n = 1$ model its effects in gauge invariant matrix elements are completely canceled by the ghost field $\mathcal{C}$. Thus we are tempted to write

$$\begin{aligned}
O_1 &\equiv \psi_{11}^*\psi_{12} \to \frac{\xi m}{2\pi}\, e^{-i\sqrt{2\pi}\,(\mathcal{B}-\mathcal{D}_-)} \qquad & O_2 &\equiv \psi_{21}^*\psi_{22} \to \frac{\xi m}{2\pi}\, e^{-i\sqrt{2\pi}\,(\mathcal{B}+\mathcal{D}_-)} \\
O_1^* &\equiv \psi_{12}^*\psi_{11} \to \frac{\xi m}{2\pi}\, e^{-i\sqrt{2\pi}\,(-\mathcal{B}+\mathcal{D}_-)} \qquad & O_2^* &\equiv \psi_{22}^*\psi_{21} \to \frac{\xi m}{2\pi}\, e^{-i\sqrt{2\pi}\,(-\mathcal{B}-\mathcal{D}_-)}
\end{aligned} \tag{3.8}$$

But (3.3), (3.6), and (3.8) cannot be right in general because they would imply VEVs for O_1 and O_2, and their conjugates, breaking the chiral T_3 symmetry spontaneously. This cannot happen in 1+1 dimensions [13] .

But (3.8) is nevertheless a very useful shorthand because we can show using cluster decomposition that it gives the correct matrix elements for the correlators that are not forbidden by the conserved chiral T_3 symmetry, (3.1), up to the arbitrary angle θ. To understand this, note that we know that (3.8) works for perturbatively allowed correlators and consider two similar looking correlators

$$\langle 0|T\, O_1(x)\, O_2(0)\, O_1^*(y)\, O_2^*(y+z)|0\rangle \tag{3.9}$$

and

$$\langle 0|T\, O_1(x)\, O_2^*(0)\, O_1^*(y)\, O_2(y+z)|0\rangle \tag{3.10}$$

$$\text{for } -(y)^2 \to \infty \text{ with } x^2 \text{ and } z^2 \text{ fixed}$$

Cluster decomposition requires that in the limit (3), (3.9) factorizes into

$$\langle 0|T\, O_1(x)\, O_2(0)|0\rangle\ \langle 0|T\, O_1^*(y)\, O_2^*(y+z)|0\rangle \tag{3.11}$$

When we calculate (3.9) using (3.8), the exponentials of K_0 in the terms that involve y all go to 1 and the power-law terms are

$$\frac{(-y^2)^{1/2}\,\left(-(y+z-x)^2\right)^{1/2}}{\left(-(y-x)^2\right)^{1/2}\left(-(y+z)^2\right)^{1/2}} \to 1 \tag{3.12}$$

and we must conclude that the two factors in (3.11) are non-zero and the non-zero result is given by their calculation from (3.8) up an arbitrary phase $e^{i\theta}$.

But for (3.10) the power law terms are

$$\left(-y^2\right)^{-1/2}\left(-(y+z-x)^2\right)^{-1/2}\left(-(y-x)^2\right)^{-1/2}\left(-(y+z)^2\right)^{-1/2} \to 0 \tag{3.13}$$

consistent with the fact that expectation values in the separate factors vanish because of chiral T_3 conservation.

Similar considerations apply to all correlators and we can use (3.8) to calculate all correlators with zero chiral T_3 up to a single arbitrary phase that we will set to 1. When we add a mass term in the next section, this means that will keep $\theta = 0$.

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So for example

$$\langle 0|T\, O_1(x)\, O_2(y)|0\rangle = \frac{(\xi m)}{(2\pi)^2} \exp\left[-K_0\left(m\sqrt{-(x-y)^2+i\epsilon}\right)\right] \left(-(x-y)^2+i\epsilon\right)^{-1/2} \tag{3.14}$$

The dictionary for computing the non-zero correlators of the exponentials is standard. Between each pair of operators we include the terms

$$\langle 0|e^{is_1\sqrt{2\pi}\,\mathcal{B}(x)}\, e^{is_2\sqrt{2\pi}\,\mathcal{B}(y)}|0\rangle \to \exp\left[-s_1 s_2 K_0\left(m\sqrt{-(x-y)^2+i\epsilon}\right)\right] \tag{3.15}$$

$$\langle 0|e^{is_1\sqrt{2\pi}\,\mathcal{D}_-(x)}\, e^{is_2\sqrt{2\pi}\,\mathcal{D}_-(y)}|0\rangle \to (\xi m)^{s_1 s_2} \left(-(x-y)^2+i\epsilon\right)^{s_1 s_2/2} \tag{3.16}$$

So for example, in (3.14), one factor of $\xi m/(2\pi)$ comes from each of the two operators in the correlator, a $1/(\xi m)$ comes from (3.16).

Note that the results of this dictionary are identical to those of [2] but there they were derived in a much more complicated way by first perturbatively evaluating correlators involving operators with zero dimension and then using cluster decomposition to isolate the nonperturbative contributions. The dictionary, (3.8), (3.15), and (3.16), does all this automatically as long as we only apply it to the non-zero correlators with zero chiral T_3.

Notice also that a parity transformation interchanges

$$O_1 \leftrightarrow O_1^* \quad \text{and} \quad O_2 \leftrightarrow O_2^* \tag{3.17}$$

so the $\mathcal{B}$ and $\mathcal{D}_-$ fields in (3.8) are pseudo-scalars.

4 Conformal coalescence, parity, and ± mass

Notice that in (3.8), the pair of operators O_1 and O_2^* (and similarly the conjugate pair O_1^* and O_2) have the same dependence on $\mathcal{D}_-$. We are also interested in the parity, so we define the operators

$$\begin{aligned}
O_{+-} &= \frac{1}{2}\left(\frac{\xi m}{2\pi}\right)\left(O_1 - O_2^* + O_1^* - O_2\right)\\
O_{--} &= \frac{1}{2i}\left(\frac{\xi m}{2\pi}\right)\left(O_1 - O_2^* - O_1^* + O_2\right)\\
O_{++} &= \frac{1}{2}\left(\frac{\xi m}{2\pi}\right)\left(O_1 + O_2^* + O_1^* + O_2\right)\\
O_{-+} &= \frac{1}{2i}\left(\frac{\xi m}{2\pi}\right)\left(O_1 + O_2^* - O_1^* - O_2\right)
\end{aligned} \tag{4.1}$$

where the first subscript gives the parity and the second subscript controls the low-energy behavior. Every term in the expansion of the O_{+-} and O_{--} operators contains at least one massive $\mathcal{B}$. Thus we expect these operators to disappear from the low-energy theory and we expect the O_{++} and O_{-+} operators to simplify. At low energies we expect (always with $\theta = 0$)

$$\begin{aligned}
O_{+-} &= i\left(\frac{\xi m}{2\pi}\right)\sin\!\left(\sqrt{2\pi}\,\mathcal{B}\right)\,\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-}-e^{-i\sqrt{2\pi}\,\mathcal{D}_-}\right) \\
&\to 0 && (4.2) \\
O_{--} &= \left(\frac{\xi m}{2\pi}\right)\sin\!\left(\sqrt{2\pi}\,\mathcal{B}\right)\,\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-}+e^{-i\sqrt{2\pi}\,\mathcal{D}_-}\right) \\
&\to 0 && (4.3) \\
O_{++} &= \left(\frac{\xi m}{2\pi}\right)\cos\!\left(\sqrt{2\pi}\,\mathcal{B}\right)\,\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-}+e^{-i\sqrt{2\pi}\,\mathcal{D}_-}\right) \\
&\to \left(\frac{\xi m}{2\pi}\right)\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-}+e^{-i\sqrt{2\pi}\,\mathcal{D}_-}\right) && (4.4) \\
O_{-+} &= -i\left(\frac{\xi m}{2\pi}\right)\cos\!\left(\sqrt{2\pi}\,\mathcal{B}\right)\,\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-}-e^{-i\sqrt{2\pi}\,\mathcal{D}_-}\right) \\
&\to -i\left(\frac{\xi m}{2\pi}\right)\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-}-e^{-i\sqrt{2\pi}\,\mathcal{D}_-}\right) && (4.5)
\end{aligned}$$

Note that it looks like we can combine the exponentials of $\pm i\sqrt{2\pi}\,\mathcal{D}_-$ into sines and cosines, but this would actually be a mistake because these exponentials carry opposite values of the chiral T_3 and must be treated separately in correlators.

The disappearance of the $O_{\pm -}$ fields in the low-energy limit was discussed (in a more complicated way) in [4] and [2] and called "conformal coalescence".

The low-energy limits of the non-zero 2-point functions of these fields for zero fermion mass are exactly as expected from the low-energy forms in (4.5).

$$\begin{aligned}
\langle 0|T\,O_{+-}(x)\,O_{+-}(y)|0\rangle_0 &= \langle 0|T\,O_{--}(x)\,O_{--}(y)|0\rangle_0 \\
&= 2\frac{(\xi m)}{(2\pi)^2}\sinh\left[K_0\left(m\sqrt{-(x-y)^2+i\epsilon}\right)\right]\,\left(-(x-y)^2+i\epsilon\right)^{-1/2} \\
&\to 0 && (4.6) \\
\langle 0|T\,O_{++}(x)\,O_{++}(y)|0\rangle_0 &= \langle 0|T\,O_{-+}(x)\,O_{-+}(y)|0\rangle_0 \\
&= 2\frac{(\xi m)}{(2\pi)^2}\cosh\left[K_0\left(m\sqrt{-(x-y)^2+i\epsilon}\right)\right]\,\left(-(x-y)^2+i\epsilon\right)^{-1/2} \\
&\to 2\frac{(\xi m)}{(2\pi)^2}\,\left(-(x-y)^2+i\epsilon\right)^{-1/2} && (4.7)
\end{aligned}$$

Thus we can calculate the low-energy correlators directly using the low-energy forms.

We will use this to investigate the effect of a VERY SPECIAL fermion mass term. We add to the Lagrangian (2.1) (for $n = 2$) the fermion mass term

$$\delta\mathcal{L} = -\mu\left(\overline{\psi}_1\psi_1-\overline{\psi}_2\psi_2\right) = -2\mu\,O_{+-} = 2i\mu\left(\frac{\xi m}{2\pi}\right)\sin\!\left(\sqrt{2\pi}\,\mathcal{B}\right)\,\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-}-e^{-i\sqrt{2\pi}\,\mathcal{D}_-}\right) \tag{4.8}$$

with equal and opposite masses for the fermions at $\theta = 0$. In [2], I briefly discussed the consequences of a mass term like (4.8) proportional to O_{+-}. Here I will expand on this and calculate the matching of such a mass term onto the low-energy conformal theory in a perturbation expansion in the fermion mass. Normally, one might expect a mass to produce a mass gap, eliminating the low-energy conformal sector and breaking the conformal stuff into ordinary particles. If this happens, perturbation theory in the mass the parameter would be plagued by infrared divergences. But in this case, because of the special properties of the O_{+-} operator, the matching occurs at the scale m and the matching contribution involves only short distance physics.[9] I will argue that this modifies the conformal symmetry without breaking it while producing non-trivial interactions for the massive scalar and the unparticle stuff.

The leading contribution at low energies is the second order term obtained by integrating out the $\mathcal{B}$, using

$$\langle 0|\mathcal{B}(z_1)\,\mathcal{B}(z_2)|0\rangle \to -\frac{i}{m^2}\,\delta(z_1-z_2) \tag{4.9}$$

which gives an effective interaction

$$\left(\frac{\xi^2\mu^2}{\pi}\right)\left(e^{2i\sqrt{2\pi}\,\mathcal{D}_-}+e^{-2i\sqrt{2\pi}\,\mathcal{D}_-}-2\right) \tag{4.10}$$

The fermion mass term (4.8) breaks the chiral symmetry but not parity so operators with different parity do not mix. Thus we are interested in the diagonal correlators in the low-energy effective theory below the m scale,

$$\begin{aligned}&\langle 0|T\,O_{\pm+}(x)\,O_{\pm+}(y)|0\rangle_\mu \\ &\quad=\pm\left(\frac{\xi m}{2\pi}\right)^2\langle 0|T\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-(x)}\pm e^{-i\sqrt{2\pi}\,\mathcal{D}_-(x)}\right)\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-(y)}\pm e^{-i\sqrt{2\pi}\,\mathcal{D}_-(y)}\right)|0\rangle_\mu\end{aligned} \tag{4.11}$$

The first-order term in μ^2 is

$$\begin{aligned}&\pm i\left(\frac{\xi m}{2\pi}\right)^2\left(\frac{\xi^2\mu^2}{\pi}\right)\int d^2z \\ &\qquad\times\langle 0|\,T\,\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-(x)}\pm e^{-i\sqrt{2\pi}\,\mathcal{D}_-(x)}\right)\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-(y)}\pm e^{-i\sqrt{2\pi}\,\mathcal{D}_-(y)}\right) \\ &\qquad\times\left(e^{2i\sqrt{2\pi}\,\mathcal{D}_-(z)}+e^{-2i\sqrt{2\pi}\,\mathcal{D}_-(z)}-2\right)|0\rangle_0\end{aligned} \tag{4.12}$$

We can now evaluate this by looking for the terms with chiral $T_3=0$.

The third term in the third set of parentheses is not interesting. Because it doesn't depend on z, it is just a vacuum energy contribution (which is the same for both the O_{++} and O_{-+} correlators as it must be) and so we can ignore it. But the first and second terms give non-trivial contributions - both the same so they add with the result

$$\pm\, i\mu^2\,\frac{\xi}{2m\pi^3}\int d^2z\,\frac{\sqrt{-(x-y)^2+i\epsilon}}{\left(-(x-z)^2+i\epsilon\right)\left(-(y-z)^2+i\epsilon\right)} \tag{4.13}$$

Thus we need the integral

$$\int d^2z\,\frac{1}{\left(-(x-z)^2+i\epsilon\right)\left(-(y-z)^2+i\epsilon\right)} \tag{4.14}$$

[9] See section 3 of [14].

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Similar integrals in momentum space are very familiar but here the roles of UV and IR divergences are reversed! There are short-distance singularities at $z = x$ and $z = y$ as expected, because the non-zero mass term requires regularization and is multiplicatively renormalized.[10] But there is no large z infrared divergence so the expansion in powers of μ makes sense. We can combine denominators as usual to get

$$\int_0^1 d\alpha \int d^2z \, \frac{1}{\left(-z^2 + \alpha(1-\alpha)(-(x-y)^2 + i\epsilon)\right)^2} \tag{4.15}$$

Wick rotation is now $z^0 \to -iz^2$ and the integral becomes the Euclidean integral

$$-\, i \int_0^1 d\alpha \int d^2z \, \frac{1}{\left(z^2 + \alpha(1-\alpha)(-(x-y)^2)\right)^2} \tag{4.16}$$

One way to deal with the short distance singularities is to use dimensional regularization. Because we are computing a matching contribution onto the long-distance theory for distances larger than $1/m$, it is appropriate to choose the dimensional scale to be the matching scale of order m. Then our integral becomes (in dimension $2+\eta$)

$$\begin{aligned}
&-\, i\, m^\eta \int_0^1 d\alpha \int d^{2+\eta}z \, \frac{\sqrt{-(x-y)^2}}{\left(z^2 + \alpha(1-\alpha)(-(x-y)^2)\right)^2} \\
&\quad = -i\, m^\eta \frac{2\pi^{1+\eta/2}}{\Gamma(1+\eta/2)} \int_0^1 d\alpha \int_0^\infty dz \, z^{1-\eta} \, \frac{\sqrt{-(x-y)^2}}{\left(z^2 + \alpha(1-\alpha)(-(x-y)^2)\right)^2} \\
&\quad = -i\, m^\eta \frac{2\pi^{1+\eta/2}}{\Gamma(1+\eta/2)} \, \frac{\pi\eta}{4\sin(\pi\eta/2)} \int_0^1 d\alpha \, \frac{\sqrt{-(x-y)^2}}{\left(\alpha(1-\alpha)(-(x-y)^2)\right)^{1-\eta/2}} \\
&\quad = -i\, m^\eta \frac{2\pi^{1+\eta/2}}{\Gamma(1+\eta/2)} \, \frac{\pi\eta}{4\sin(\pi\eta/2)} \, \frac{\Gamma(\eta/2)}{\Gamma(\eta)} \, \frac{\sqrt{-(x-y)^2 + i\eta}}{\left((-(x-y)^2 + i\eta\right)^{1-\eta/2}}
\end{aligned} \tag{4.17}$$

Expanding the result in powers of η and putting back the original $i\epsilon$ for Minkowski space gives

$$\int d^2z \, \frac{1}{(-(x-z)^2 + i\epsilon)(-(y-z)^2 + i\epsilon)} = -4i\pi \, \frac{\log\left(m\sqrt{-(x-y)^2 + i\epsilon}\right)}{(-(x-y)^2 + i\epsilon)} \tag{4.18}$$

so (4.13) becomes

$$\pm\, \mu^2 \, \frac{2\xi}{\pi^2 m} \, \frac{\log\left(m\sqrt{-(x-y)^2 + i\epsilon}\right)}{\sqrt{-(x-y)^2 + i\epsilon}} \tag{4.19}$$

Dimensional regularization works simply enough in this case, but it will be useful to understand the integral in different ways. The long-distance behavior should be independent of the details of our short distance regularization. In particular, we can cut off the short distance behavior in (4.15) at $1/m$ by adding a $1/m^2$ term to get

$$\begin{aligned}
&-\, i \int_0^1 d\alpha \int d^2z \, \frac{\sqrt{-(x-y)^2}}{\left(z^2 + \alpha(1-\alpha)(-(x-y)^2) + 1/m^2\right)^2} \\
&\qquad\qquad = -i\pi \int_0^1 d\alpha \, \frac{1}{\left(\alpha(1-\alpha)(-(x-y)^2) + 1/m^2\right)}
\end{aligned} \tag{4.20}$$

In the long-distance limit, $-(x-y)^2 \gg 1/m^2$, this again gives (4.18).

[10] This is simply related to the subtractive regularization of the 2-point function of the fermion-bilinears.

The key things in (4.19) are the appearance of the logarithm of m and the absence of any logarithm of μ. This again shows that the matching is happening at the scale m and there are no IR divergences. The log of m does not indicate that conformal invariance is broken. Rather, it is exactly what we would expect if the conformal symmetry of the 2-point functions at long distances is not broken to this order in μ^2 but the parity eigenstate operators, $O_{\pm+}$ have scaling dimensions $d_\pm$ that change in opposite directions when the mass term is turned on. Adding (4.19) to the zeroth-order contribution gives

$$\frac{\xi m}{2\pi^2}\,\frac{1}{\sqrt{-(x-y)^2+i\epsilon}} \pm \mu^2\,\frac{2\xi}{\pi^2 m}\,\frac{\log\left(m\sqrt{-(x-y)^2+i\epsilon}\right)}{\sqrt{-(x-y)^2+i\epsilon}} \tag{4.21}$$

which is the expansion to leading nontrivial order in μ of

$$\langle 0|T\,O_{\pm+}(x)\,O_{\pm+}(y)|0\rangle_\mu = \xi\,\frac{m^{2-2d_\pm}}{2\pi^2}\,\frac{1}{\left(-(x-y)^2+i\epsilon\right)^{d_\pm}} \tag{4.22}$$

where

$$d_\pm = \frac{1\mp 4\mu^2/m^2}{2} \tag{4.23}$$

5 4-point functions

The non-zero 4-point functions in the massless theory at long distances can be calculated simply from (3.16) and (4.5). They are[11]

$$\begin{aligned}
&\langle 0|T\,O_{\pm+}(x_1)\,O_{\pm+}(x_2)\,O_{\pm+}(x_3)\,O_{\pm+}(x_4)|0\rangle_0 \\
&\quad = 2\left(\frac{\xi m}{2\pi}\right)^4 \frac{1}{(\xi m)^2} \sum_{\substack{3\,\neq\,\text{perms}\\ \{\text{jklm}\}=\{1234\}}} \\
&\qquad \times\sqrt{\frac{\left(-(x_j-x_k)^2+i\epsilon\right)\left(-(x_l-x_m)^2+i\epsilon\right)}{\left(-(x_j-x_l)^2+i\epsilon\right)\left(-(x_j-x_m)^2+i\epsilon\right)\left(-(x_k-x_l)^2+i\epsilon\right)\left(-(x_k-x_m)^2+i\epsilon\right)}}
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
&\langle 0|T\,O_{++}(x_1)\,O_{-+}(x_2)\,O_{++}(x_3)\,O_{-+}(x_4)|0\rangle_0 \\
&\quad = -\,2\left(\frac{\xi m}{2\pi}\right)^4 \frac{1}{(\xi m)^2} \sum_{\substack{3\,\neq\,\text{perms}\\ \{\text{jklm}\}=\{1234\}}} (-1)^{j+k} \\
&\qquad \times\sqrt{\frac{\left(-(x_j-x_k)^2+i\epsilon\right)\left(-(x_l-x_m)^2+i\epsilon\right)}{\left(-(x_j-x_l)^2+i\epsilon\right)\left(-(x_j-x_m)^2+i\epsilon\right)\left(-(x_k-x_l)^2+i\epsilon\right)\left(-(x_k-x_m)^2+i\epsilon\right)}}
\end{aligned} \tag{5.2}$$

[11]Note the particular pattern of O_{++}s and O_{-+}s in (5.2). This is just a convenience to make the result easy to write down compactly.

We begin by discussing (5.1). The order μ^2 correction to (5.1) is

$$
\begin{aligned}
i\left(\frac{\xi m}{2\pi}\right)^4 & \left(\frac{\xi^2\mu^2}{\pi}\right)\int d^2z \\
& \times \langle 0|\,T\,\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-(x_1)} \pm e^{-i\sqrt{2\pi}\,\mathcal{D}_-(x_1)}\right)\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-(x_2)} \pm e^{-i\sqrt{2\pi}\,\mathcal{D}_-(x_2)}\right) \\
& \times \left(e^{i\sqrt{2\pi}\,\mathcal{D}_-(x_3)} \pm e^{-i\sqrt{2\pi}\,\mathcal{D}_-(x_3)}\right)\left(e^{i\sqrt{2\pi}\,\mathcal{D}_-(x_4)} \pm e^{-i\sqrt{2\pi}\,\mathcal{D}_-(x_4)}\right) \\
& \times \left(e^{2i\sqrt{2\pi}\,\mathcal{D}_-(z)} + e^{-2i\sqrt{2\pi}\,\mathcal{D}_-(z)} - 2\right)|0\rangle_0
\end{aligned}
\tag{5.3}
$$

As for the 2-point function, the third term in the last line gives an irrelevant vacuum energy contribution while the first and second terms give effects of the following form:

$$
\pm\, i\mu^2\,\frac{\xi^6 m^4}{8\pi^5}\int d^2z\;\langle 0|T\,e^{\pm i\sqrt{2\pi}\,\mathcal{D}(x_j)}\,e^{\pm i\sqrt{2\pi}\,\mathcal{D}(x_k)}\,e^{\pm i\sqrt{2\pi}\,\mathcal{D}(x_l)}\,e^{\mp i\sqrt{2\pi}\,\mathcal{D}(x_m)}e^{\mp 2i\sqrt{2\pi}\,\mathcal{D}(z)}|0\rangle_0
\tag{5.4}
$$

(the j, k and l indices have the same sign in the exponent) which gives

$$
\pm\, i\mu^2\,\frac{\xi^2}{8\pi^5}\sum_m\int d^2z\,\frac{\left(-(z-x_m)^2+i\epsilon\right)\prod_{\substack{j<k\\ j,k\neq m}}\sqrt{-(x_j-x_k)^2+i\epsilon}}{\prod_{\substack{n=\\ j,k,l}}\left(-(z-x_n)^2+i\epsilon\right)\sqrt{-(x_n-x_m)^2+i\epsilon}}
\tag{5.5}
$$

So we need the integral[12]

$$
\begin{aligned}
&\int d^2z\,\frac{\left(-(z-x_m)^2\right)}{\prod_{\substack{n=\\ j,k,l}}\left(-(z-x_n)^2+i\epsilon\right)} \qquad (5.6)\\
&= 2\int[d\alpha]\,d^2z\,\delta\left(1-\sum_{\substack{n=\\ j,k,l}}\alpha_n\right)\frac{\left(-(z-x_m)^2\right)}{\left(\sum_{\substack{n=\\ j,k,l}}\alpha_n\left(-(z-x_n)^2+i\epsilon\right)\right)^3}\\
&= 2\int[d\alpha]\,d^2z\,\delta\left(1-\sum_{\substack{n=\\ j,k,l}}\alpha_n\right)\frac{\left(-(z-x_m)^2\right)}{\left(-z^2+\sum_{\substack{n=\\ j,k,l}}\left(2\alpha_n(zx_n)-\alpha_n x_n^2\right)+i\epsilon\right)^3}\\
&= 2\int[d\alpha]\,d^2\tilde z\,\delta\left(1-\sum_{\substack{n=\\ j,k,l}}\alpha_n\right)\frac{\left(-\left(\tilde z-\left(\sum_{\substack{n=\\ j,k,l}}\alpha_n x_n\right)-x_m\right)^2\right)}{\left(-\tilde z^2-\left(\sum_{\substack{n=\\ j,k,l}}\alpha_n x_n^2\right)+\left(\sum_{\substack{n=\\ j,k,l}}\alpha_n x_n\right)^2+i\epsilon\right)^3}\\
&= 2\int[d\alpha]\,d^2z\,\delta\left(1-\sum_{\substack{n=\\ j,k,l}}\alpha_n\right)\frac{\left(-z^2-\left(x_m+\left(\sum_{\substack{n=\\ j,k,l}}\alpha_n x_n\right)\right)^2\right)}{\left(-z^2-\left(\sum_{\substack{n=\\ j,k,l}}\left(\alpha_n x_n^2\right)\right)+\left(\sum_{\substack{n=\\ j,k,l}}\alpha_n x_n\right)^2+i\epsilon\right)^3}
\end{aligned}
$$

Now we can Wick rotate and do the z integration

$$
= -2i\int[d\alpha]\,d^2z\,\delta\left(1-\sum_{\substack{n=\\ j,k,l}}\alpha_n\right)\frac{(z^2+a)}{(z^2+b)^3} = -i\pi\int[d\alpha]\,\delta\left(1-\sum_{\substack{n=\\ j,k,l}}\alpha_n\right)\frac{b+a}{b^2}
\tag{5.7}
$$

[12]The $i\epsilon$ is not necessary in the numerator.

where

$$a = -\left(x_m - \left(\sum_{\substack{n=\\ j,k,l}} \alpha_n x_n\right)\right)^2 \qquad\qquad b = \left(\sum_{j=1}^{3}\left(-\alpha_j x_j^2\right)\right) + \left(\sum_{\substack{n=\\ j,k,l}} \alpha_n x_n\right)^2 \tag{5.8}$$

and because $\sum_{n=j,k,l} \alpha_n = 1$ in the integral, we can write

$$b = -\sum_{j<k} \alpha_j \alpha_k (x_j - x_k)^2 \qquad\qquad b + a = -\sum_{\substack{n=\\ j,k,l}} \alpha_n (x_m - x_n)^2 \tag{5.9}$$

If we cut off the integral at short distance as in (4.20), (5.7) becomes

$$= -i\pi \int [d\alpha]\, \delta\left(1 - \sum_{\substack{n=\\ j,k,l}} \alpha_n\right) \frac{\left(-\sum_{\substack{n=\\ j,k,l}} \alpha_n (x_m - x_n)^2\right)}{\left(-\sum_{j<k} \alpha_j \alpha_k (x_j - x_k)^2 + 1/m^2\right)} \tag{5.10}$$

In the long-distance limit, $-(x_j - x_k)^2 \gg 1/m^2$, this gives (suppressing the $i\epsilon$s in the result)

$$\begin{aligned} &\int d^2 z\, \frac{(-(z - x_m)^2)}{\prod_{\substack{n=\\ j,k,l}} (-(z - x_n)^2 + i\epsilon)} \\ &\quad \to -2i\pi \sum_{j\neq m} \frac{-(x_m - x_j)^2}{(x_k - x_j)^2 (x_l - x_j)^2} \log\left(m\sqrt{-(x_k - x_j)^2}\sqrt{-(x_l - x_j)^2}/\sqrt{-(x_k - x_l)^2}\right) \end{aligned} \tag{5.11}$$

Putting all this together gives

$$\begin{aligned} &\pm\mu^2 \frac{\xi^2}{4\pi^4} \sum_m \frac{\prod_{\substack{j<k\\ j,k\neq m}} \sqrt{-(x_j - x_k)^2}}{\prod_{\substack{n=\\ j,k,l}} \sqrt{-(x_n - x_m)^2}} \\ &\quad \times \sum_{j\neq m} \frac{-(x_m - x_j)^2}{(x_k - x_j)^2 (x_l - x_j)^2} \log\left(m\sqrt{-(x_k - x_j)^2}\sqrt{-(x_l - x_j)^2}/\sqrt{-(x_k - x_l)^2}\right) \\ &= \pm\mu^2 \frac{\xi^2}{2\pi^4} \sum_{\substack{\text{pairs } \{j,k\}=\\ \{1,2\},\{1,3\},\{1,4\}}} \frac{\sqrt{-(x_j - x_k)^2}\sqrt{-(x_l - x_m)^2}}{\sqrt{-(x_j - x_l)^2}\sqrt{-(x_j - x_m)^2}\sqrt{-(x_k - x_l)^2}\sqrt{-(x_k - x_m)^2}} \\ &\quad \times \log\left(\frac{m^4\sqrt{-(x_j - x_l)^2}\sqrt{-(x_j - x_m)^2}\sqrt{-(x_k - x_l)^2}\sqrt{-(x_k - x_m)^2}}{\sqrt{-(x_j - x_k)^2}\sqrt{-(x_l - x_m)^2}}\right) \end{aligned} \tag{5.12}$$

Adding the zeroth order term gives

$$\begin{aligned} &2\left(\frac{\xi m}{2\pi}\right)^4 \left(\frac{1}{\xi m}\right)^2 \sum_{\substack{\text{pairs } \{j,k\}=\\ \{1,2\},\{1,3\},\{1,4\}}} \frac{\sqrt{-(x_j - x_k)^2}\sqrt{-(x_l - x_m)^2}}{\sqrt{-(x_j - x_l)^2}\sqrt{-(x_j - x_m)^2}\sqrt{-(x_k - x_l)^2}\sqrt{-(x_k - x_m)^2}} \\ &\quad \left(1 \pm \frac{4\mu^2}{m^2} \log\left(\frac{m^4\sqrt{-(x_j - x_l)^2}\sqrt{-(x_j - x_m)^2}\sqrt{-(x_k - x_l)^2}\sqrt{-(x_k - x_m)^2}}{\sqrt{-(x_j - x_k)^2}\sqrt{-(x_l - x_m)^2}}\right)\right) \end{aligned} \tag{5.13}$$

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This is the expansion to order μ^2 of

$$\frac{\xi^2 m^{4-4d_\pm}}{8\pi^4} \sum_{\substack{\text{pairs } \{j,k\}= \\ \{1,2\},\{1,3\},\{1,4\}}} \frac{\left(-(x_j-x_k)^2\right)^{d_\pm}\left(-(x_l-x_m)^2\right)^{d_\pm}}{\left(-(x_j-x_l)^2\right)^{d_\pm}\left(-(x_j-x_m)^2\right)^{d_\pm}\left(-(x_k-x_l)^2\right)^{d_\pm}\left(-(x_k-x_m)^2\right)^{d_\pm}} \tag{5.14}$$

This is consistent with the result (4.22) for the 2-point function and is the simplest possible result consistent with conformal invariance for the $O_{\pm+}$ fields with scaling dimensions $d_\pm$.

In the mixed correlator, (5.2), the pieces of the calculation are very similar but they get put together very differently. The first order correction to (5.2) is

$$\begin{aligned}
&-i\mu^2\,\frac{\xi^6 m^4}{8\pi^5}\int d^2z\,\Big(\langle 0|T\,e^{-i\sqrt{2\pi}\,\mathcal{D}(x_1)}\,e^{i\sqrt{2\pi}\,\mathcal{D}(x_2)}\,e^{i\sqrt{2\pi}\,\mathcal{D}(x_3)}\,e^{i\sqrt{2\pi}\,\mathcal{D}(x_4)}e^{-2i\sqrt{2\pi}\,\mathcal{D}(z)}|0\rangle_0 \\
&\qquad -\langle 0|T\,e^{i\sqrt{2\pi}\,\mathcal{D}(x_1)}\,e^{-i\sqrt{2\pi}\,\mathcal{D}(x_2)}\,e^{i\sqrt{2\pi}\,\mathcal{D}(x_3)}\,e^{i\sqrt{2\pi}\,\mathcal{D}(x_4)}e^{-2i\sqrt{2\pi}\,\mathcal{D}(z)}|0\rangle_0 \\
&\qquad +\langle 0|T\,e^{i\sqrt{2\pi}\,\mathcal{D}(x_1)}\,e^{i\sqrt{2\pi}\,\mathcal{D}(x_2)}\,e^{-i\sqrt{2\pi}\,\mathcal{D}(x_3)}\,e^{i\sqrt{2\pi}\,\mathcal{D}(x_4)}e^{-2i\sqrt{2\pi}\,\mathcal{D}(z)}|0\rangle_0 \\
&\qquad -\langle 0|T\,e^{i\sqrt{2\pi}\,\mathcal{D}(x_1)}\,e^{i\sqrt{2\pi}\,\mathcal{D}(x_2)}\,e^{i\sqrt{2\pi}\,\mathcal{D}(x_3)}\,e^{-i\sqrt{2\pi}\,\mathcal{D}(x_4)}e^{-2i\sqrt{2\pi}\,\mathcal{D}(z)}|0\rangle_0\Big) \\
&=i\mu^2\,\frac{\xi^2}{8\pi^5}\sum_m(-1)^m\int d^2z\,\frac{\left(-(z-x_m)^2\right)\prod_{\substack{j<k\\ j,k\neq m}}\sqrt{-(x_j-x_k)^2+i\epsilon}}{\prod_{\substack{n=\\ j,k,l}}\left(-(z-x_n)^2+i\epsilon\right)\quad\sqrt{-(x_n-x_m)^2+i\epsilon}}
\end{aligned}\tag{5.15}$$

Except for a factor of $(-1)^m$ in the sum, this is proportional to (5.5). So following the same steps and putting everything together now gives

$$\begin{aligned}
&\mu^2\,\frac{\xi^2}{4\pi^4}\sum_m(-1)^m\frac{\prod_{\substack{j<k\\ j,k\neq m}}\sqrt{-(x_j-x_k)^2}}{\prod_{\substack{n=\\ j,k,l}}\sqrt{-(x_n-x_m)^2}}\sum_{j\neq m}\frac{-(x_m-x_j)^2}{(x_k-x_j)^2(x_l-x_j)^2} \\
&\qquad\times\log\left(m\sqrt{-(x_k-x_j)^2}\sqrt{-(x_l-x_j)^2}/\sqrt{-(x_k-x_l)^2}\right) \\
&=\mu^2\,\frac{\xi^2}{4\pi^4}\left(\prod_{\alpha\neq\beta}\frac{1}{\sqrt{-(x_\alpha-x_\beta)^2}}\right)\left(\left(-(x_1-x_3)^2\right)\left(-(x_2-x_4)^2\right)\log\left(\frac{-(x_2-x_4)^2}{-(x_1-x_3)^2}\right)\right. \\
&\quad-\left(-(x_1-x_2)^2\right)\left(-(x_3-x_4)^2\right)\log\left(\frac{-(x_2-x_4)^2}{-(x_1-x_3)^2}\right) \\
&\quad\left.-\left(-(x_1-x_4)^2\right)\left(-(x_2-x_3)^2\right)\log\left(\frac{-(x_2-x_4)^2}{-(x_1-x_3)^2}\right)\right) \\
&=\frac{2\mu^2}{m^2}\log\left(\frac{-(x_2-x_4)^2}{-(x_1-x_3)^2}\right)\langle 0|T\,O_{++}(x_1)\,O_{-+}(x_2)\,O_{++}(x_3)\,O_{-+}(x_4)|0\rangle_0
\end{aligned}\tag{5.16}$$

Adding the zeroth order term gives

$$\left(1+\frac{2\mu^2}{m^2}\log\left(\frac{-(x_2-x_4)^2}{-(x_1-x_3)^2}\right)\right)\langle 0|T\,O_{++}(x_1)\,O_{-+}(x_2)\,O_{++}(x_3)\,O_{-+}(x_4)|0\rangle_0 \tag{5.17}$$

which is the expansion of

$$\frac{\left(-(x_2-x_4)^2\right)^{2\mu^2/m^2}}{\left(-(x_1-x_3)^2\right)^{2\mu^2/m^2}}\langle 0|T\,O_{++}(x_1)\,O_{-+}(x_2)\,O_{++}(x_3)\,O_{-+}(x_4)|0\rangle_0 \tag{5.18}$$

So again conformal invariance is satisfied.

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6 Higher $2n$-point functions

The calculation of higher $2n$-point correlators is straightforward but quickly gets complicated. The leading correction to the $2n$-point function involves integrals of the form

$$\int d^2z \, \frac{\prod_{j=n+2}^{2n}\left(-(z-x_j)^2\right)}{\prod_{j=1}^{n+1}\left(-(z-x_j)^2+i\epsilon\right)} \tag{6.1}$$

It would be a difficult task to calculate this using the direct methods of section 5, but we can use (4.18) and (5.11) and linear algebra to write down the result without further integration.

Here is how this works for the the 6-point function with the integral

$$\int d^2z \, \frac{\left(-(z-x_5)^2\right)\left(-(z-x_6)^2\right)}{\prod_{j=1}^{n+1}\left(-(z-x_1)^2+i\epsilon\right)\left(-(z-x_2)^2+i\epsilon\right)\left(-(z-x_3)^2+i\epsilon\right)\left(-(z-x_4)^2+i\epsilon\right)} \tag{6.2}$$

The point is that the product

$$\left(-(z-x_5)^2\right)\left(-(z-x_6)^2\right) \tag{6.3}$$

can be written as a linear combination of $\left(-(z-x_j)^2\right)\left(-(z-x_k)^2\right)$, $\left(-(z-x_5)^2\right)\left(-(z-x_k)^2\right)$, and $\left(-(z-x_j)^2\right)\left(-(z-x_6)^2\right)$ where j and k are less than 5. In fact, this can be done in many different ways because the z dependence can be written in terms of the nine combinations

$$\{z_+^2,\, z_+,\, 1\} \times \{z_-^2,\, z_-,\, 1\} \tag{6.4}$$

where

$$z_\pm = z^0 \pm z^1 \tag{6.5}$$

So we can write

$$\left(-(z-x_5)^2\right)\left(-(z-x_6)^2\right) = \sum_{\{j,k\}\neq\{5,6\}}^{9\text{ terms}} \beta(j,k)\left(-(z-x_j)^2\right)\left(-(z-x_k)^2\right) \tag{6.6}$$

Then setting (6.3) equal to (6.6) gives 9 linear equations for the 9 βs depending on the $x_{j\pm}$. Then in each of the terms in the sum in (6.6), one or two of the factors cancel with factors in the denominator and the integral reduces to (4.18) or (5.11).

A similar procedure can be used to calculate the leading corrections to the $2n$-point function in terms of n^2 βs.

While this is simple to describe, I have not found a choice of βs that leads to any simple or intuitive result.

7 More questions

The analysis above gives some nontrivial checks of the conjecture that the 2-flavor Schwinger model has an unbroken conformal sector even when small equal and opposite fermion masses are turned on and describes the calculation of the leading corrections to

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all flavor-diagonal correlators in a systematic expansion in the fermion mass parameters. But the simple calculational scheme used here leaves some questions unanswered. Can the matrix analysis of 6-point and higher correlators in sections 6 be simplified. What happens in higher orders in μ^2? Do the scaling dimensions of the fermion-bilinears, (4.23), give a clue to the nature of the phase transition that must occur between $\mu^2 \ll m^2$ and $\mu^2 \gg m^2$? [3] Are there observable effects at high energies of the non-trivial dimensions in the conformal sector? Can the analysis be extended to include other fermion bilinears and the non-abelian chiral symmetry. [15] And most importantly, does this solvable model give any clue to how an *unbroken* conformal sector might show up in the particle physics of our 3+1 dimensional world?[13] I believe that it is worth studying this model further.

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