Excited ρ mesons in $B_c \to \psi^{(\prime)} K K_S$ decays

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In this paper, exclusive decays $B_c \to J/\psi K K_S$ and $B_c \to \psi(2S) K K_S$ are analyzed. It is shown that contributions of the excited ρ mesons should be taken into account to describe these decays. It is also shown that, unlike the corresponding τ lepton decays, peaks in m_{KK_S} distributions caused by these resonances are clearly seen and can be easily separated. Theoretical predictions for the branching fractions of the reactions and $m_{\psi K}$ distributions are also presented.

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I. INTRODUCTION

The lightest vector hadron, i.e., the $\rho(770)$ meson, has been studied in detail. One cannot say the same, however, about its excited partners, $\rho(1450)$, $\rho(1570)$, and $\rho(1700)$. For these mesons, only neutral states were observed, mainly in the ee and $\pi\pi$ channels. Their decays into the KK pair are hard to detect.

One of the reactions that can be used to observe KK decay of the charged excited ρ meson is the τ lepton decay $\tau \to \nu_\tau K K_S$. This process was first studied experimentally by the CLEO Collaboration in 1996 [1]. Recently a more detailed result, obtained by the *BABAR* Collaboration, appeared in [2,3]. According to Ref. [4], CEO data can be explained theoretically using Flatte formalism [5] and taking into account contributions of three ρ mesons. It should be interesting to check this approach on new *BABAR* data.

There is, however, a fundamental problem with using τ lepton decays to analyze contributions of the excited ρ mesons. It is evident that in this reaction the available energy is limited by the mass of τ lepton, $m_\tau = 1.77$ GeV, and, for example, $\rho(1570)$ can hardly be observed. It is clear, on the other hand, that a larger energy range is available in the decays of the heavier particles, e.g., the B_c meson. In a series of papers (see, for example, [6–11]), it was shown how the QCD factorization theorem can be used to connect differential branching fraction of light mesons' production in exclusive τ lepton and B_c meson decays. Predictions presented in this article are in good agreement with experimental results [12–16]. It could be interesting to try such an approach for $B_c \to J/\psi K K_S$ and $B_c \to \psi(2S) K K_S$ decays.

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The rest of the paper is organized as follows. In the next section, we use data on $\tau \to \nu_\tau KK$ decay obtained by the CLEO Collaboration to determine the coupling constants of the excited ρ mesons decays into KK_S pair. In Sec. III, these results are used to make theoretical predictions for the branching fractions of $B_c \to \psi^{(\prime)} K K_S$ decays and distributions over different kinematical variables. A short discussion is presented in the last section. In the Appendix, a brief description of the spectral function formalism is given.

II. $\tau \to \nu_\tau K K_S$ DECAYS

Let us first consider KK_S pair production in τ lepton decay. The Feynman diagram describing this process is shown in Fig. 1, and the corresponding amplitude can be written in the form

$$\mathcal{M}_\tau = \frac{G_F}{\sqrt{2}} \bar{u}_\nu(k)\gamma^\mu(1+\gamma_5)u_\tau(P)F(q^2)(p_1-p_2)_\mu, \quad (1)$$

where P, k, $p_{1,2}$ are the momenta of the initial lepton, τ neutrino, and final K mesons, respectively (in the following we will neglect the difference in K and K_S masses), $q = p_1 + p_2$ is the momentum of the virtual W boson, and $F(q^2)$ is the form factor of $W \to KK_S$ transition. It is clear

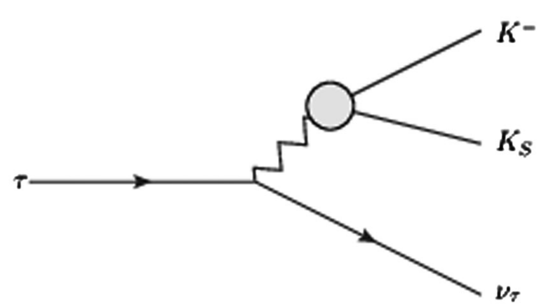

FIG. 1. Feynman diagram for $\tau \to \nu_\tau K K_S$ decay.

that the quantum numbers of the final KK_S pair should be equal to $I^G(J^P) = 1^+(1^-)$, so this transition should be saturated by contributions of the charged ρ meson and its excitations. It is convenient to use the Flatte parametrization of the form factor [5] and write it in the form

$$F(s) = \sum_i c_i^K BW_i(s), \tag{2}$$

where the summation is performed over the intermediate ρ mesons: $\rho(770)$, $\rho' = \rho(1450)$, and $\rho'' = \rho(1700)$, c_i^K are the coupling constants,

$$BW_i(s) = \frac{m_i^2}{m_i^2 - s - i\sqrt{s}\Gamma_i(s)}, \tag{3}$$

m_i is the mass of the corresponding particle, and $\Gamma_i(\rho)$ is the energy-dependent width of $\rho \to 2\pi$ decay. Since final π mesons in these decays are in P wave state, the latter width can be calculated as

$$\Gamma_i(s) = \frac{m_i^2}{s}\left(\frac{1 - 4m_\pi^2/s}{1 - 4m_\pi^2/m_i^2}\right)^{3/2}\Gamma_i, \tag{4}$$

where $\Gamma_i = \Gamma_i(m_i^2)$ is the decay widths of the corresponding meson on its mass shell.

The model parameters m_i, Γ_i, and c_i^K can be determined from analysis of the experimental data, especially q-distributions. If we are such distributions only, we can use the spectral functions formalism, described briefly in the Appendix. In this framework, the differential width of $\tau \to \nu_\tau KK_S$ decay can be written as

$$\frac{d\Gamma(\tau \to \nu_\tau KK_S)}{d\sqrt{q^2}} = 2\sqrt{q^2}\frac{G_F^2}{16\pi m_\tau}\frac{(m_\tau^2 - q^2)^2}{m_\tau^3}(m_\tau^2 + 2q^2)\rho_T(q^2), \tag{5}$$

where

$$\rho_T(q^2) = \left(1 - \frac{4m_K^2}{q^2}\right)^{3/2}\frac{|F(q^2)|^2}{48\pi^2} \tag{6}$$

is the transversal spectral function of $W \to KK_S$ transition. Experimental analysis of the considered decay was performed, for example, by the CLEO [1] and *BABAR* [2,3] Collaborations. In paper [4] obtained by the CLEO Collaboration, results were used to determine the values of the model parameters m_i, Γ_i, and c_i^K. According to this paper, the following values of the parameters should be used in order to describe CLEO results

$$m_\rho = 775\,\text{MeV}, \quad \Gamma_\rho = 150\,\text{MeV}, \quad c_\rho^K = 1.195 \pm 0.009, \tag{7}$$

$$m_{\rho'} = 1465\,\text{MeV}, \quad \Gamma_{\rho'} = 400\,\text{MeV}, \quad c_{\rho'}^K = -0.112 \pm 0.010, \tag{8}$$

$$m_{\rho''} = 1720\,\text{MeV}, \quad \Gamma_{\rho''} = 250\,\text{MeV}, \quad c_{\rho''}^K = -0.083 \pm 0.019. \tag{9}$$

In the left panel of Fig. 2, we show the resulting q dependence of the differential width in comparison with experimental data obtained by the CLEO and *BABAR* Collaborations. It is clear that the agreement with these results is pretty good. The contributions of the exited ρ mesons (especially ρ'' one), however, can hardly be seen since these mesons lie almost on the upper limit of the allowed phase space. Indeed, the relation (5) is universal and only the spectral function depends on the final hadronic state, so this relation can be rewritten in the form

$$\frac{d\Gamma(\tau \to \nu_\tau KK_S)}{d\sqrt{q^2}} = \frac{d\Gamma(\tau \to \nu_\tau \mu\nu_\mu)}{d\sqrt{q^2}}\frac{\rho_T(q^2)}{\rho_T^{\mu\nu}(q^2)}, \tag{10}$$

where the transverse spectral function of the leptonic pair is $\rho_T^{\mu\nu}(q^2) = 1/(6\pi^2)$. Transferred momentum distribution of the semileptonic τ decay is shown in Fig. 2, and it is clearly seen that in the region of excited ρ mesons is strongly suppressed. That is why it could be interesting to study

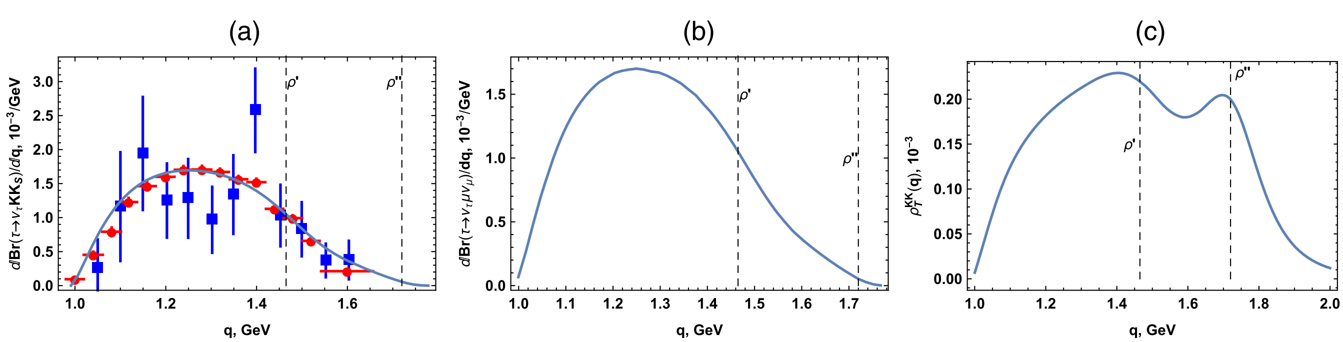

FIG. 2. Transferred momentum distribution of $\tau \to \nu_\tau KK_S$ decay (a), $\tau \to \nu_\tau \mu\bar{\nu}_\mu$ decay (b), and $\rho_T^{KK_S}$ spectral function (c).

production of KK_S pair in some other experiments. In the next section, we will perform the calculation of $B_c \to \psi^{(\prime)} KK_S$ decays and show that in this case the contributions of the excited states are much more clear.

III. $B_c \to \psi^{(\prime)} KK_S$ DECAYS

The decay $B_c \to \psi^{(\prime)} KK_S$ is described as shown in the Fig. 3 Feynman diagram. The corresponding matrix element can be written as

$$\mathcal{M}(B_c \to \psi^{(\prime)} KK_S) = \frac{G_F V_{cb}}{\sqrt{2}} a_1 \langle V - A\rangle_\mu F(q^2)(p_1 - p_2)^\mu, \tag{11}$$

where a_1 is the Wilson coefficient describes the effect of soft gluon interaction [17], and the matrix element of $B_c \to \psi^{(\prime)} W$ can be parametrized as

$$\langle V - A\rangle_\mu = \left[2M_\psi A_0(q^2)\frac{q^\mu q^\nu}{q^2} + (M_{B_c} - M_\psi)A_1(q^2)\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) - A_2(q^2)q^\nu\left(P^\mu + k^\mu - \frac{M_{B_c}^2 - M_\psi^2}{q^2}q^\mu\right) - 2i\frac{V(q^2)}{M_{B_c} + M_\psi}e^{\mu\nu\alpha\beta}P_\alpha k_\beta\right]\epsilon_\nu. \tag{12}$$

In this expression, P, k, $p_{1,2}$ are the momenta of the B_c meson, final vector charmonium, and K mesons, respectively, ϵ_μ is the polarization vector of $\psi^{(\prime)}$, $q = P - k$ is the momentum of virtual weak boson, M_{B_c} and M_ψ are the masses of the corresponding particles, and $V(q^2)$, $A_{0,1,2}(q^2)$ are dimensionless form factors, whose numerical values will be discussed later. It should be noted, however, that since we neglect the mass difference between K and K_S mesons, we have $q^\mu(p_1 - p_2)_\mu = 0$ and the form factor $A_0(q^2)$ does not contribute.

If we are interested in q distribution only, we can use the formalism of the spectral functions and the corresponding decay width is equal to

$$\frac{d\Gamma(B_c \to \psi^{(\prime)} KK_S)}{dq^2} = \frac{G_F^2 V_{cb}^2 a_1^2 \rho_T(q^2)}{128\pi M_{B_c} M_\psi^2 (M_{B_c} + M_\psi)^2}\sqrt{1 - \frac{(M_\psi + q)^2}{M_{B_c}^2}}\sqrt{1 - \frac{(M_\psi - q)^2}{M_{B_c}^2}} \times [\Delta_1^4(\Delta_1^4 A_2^2(q^2) + 8M_\psi^2 q^2 V^2(q^2)) + \Delta_2^4(M_{B_c} + M_\psi)^4 A_1^2(q^2) - 2\Delta_3^6(M_{B_c} + M_\psi)^2 A_1(q^2)A_2(q^2)], \tag{13}$$

where

$$\Delta_1^4 = M_{B_c}^4 - 2M_{B_c}^2(M_\psi^2 + q^2) + (M_\psi^2 - q^2)^2, \tag{14}$$

$$\Delta_2^4 = M_{B_c}^4 - 2M_{B_c}^2(M_\psi^2 + q^2) + M_\psi^4 + 10M_\psi^2 q^2 + q^4, \tag{15}$$

$$\Delta_3^6 = M_{B_c}^6 - 3M_{B_c}^4(M_\psi^2 + q^2) + M_{B_c}^2(3M_\psi^4 + 2M_\psi^2 q^2 + 3q^4) - (M_\psi^2 - q^2)^2(M_\psi^2 + q^2). \tag{16}$$

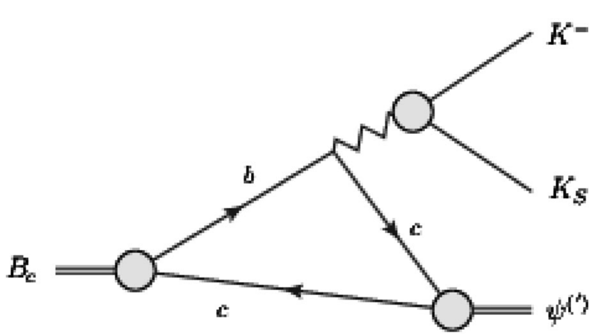

FIG. 3. Feynman diagram for $B_c \to KK_S$ decay.

Let us discuss the parametrizations of the $B_c \to J/\psi KK_S$ decay first. It is clear that the corresponding form factors are essentially nonperturbative, so some other methods such as QCD sum rules of Potential Models should be used for their calculation. This topic is widely discussed in the literature. In the following, we will use the results presented in works [18] (QCD sum rules were used in this work, in the following we will refer to it as SR) and [19] (in this case the author use potential model, PM in the following). It is clear that $A_0(q^2)$ form factor does not give contributions to the process under consideration. Transferred momentum dependence of all other form factors for models used in our work is shown in Fig. 4. Using these values it is easy to see that the branching fractions of the decay in different form factors models are equal to

$$\mathrm{Br}_{\mathrm{SR}}(B_c \to J/\psi KK_S) = (6.9 \pm 0.1) \times 10^{-5}, \tag{17}$$

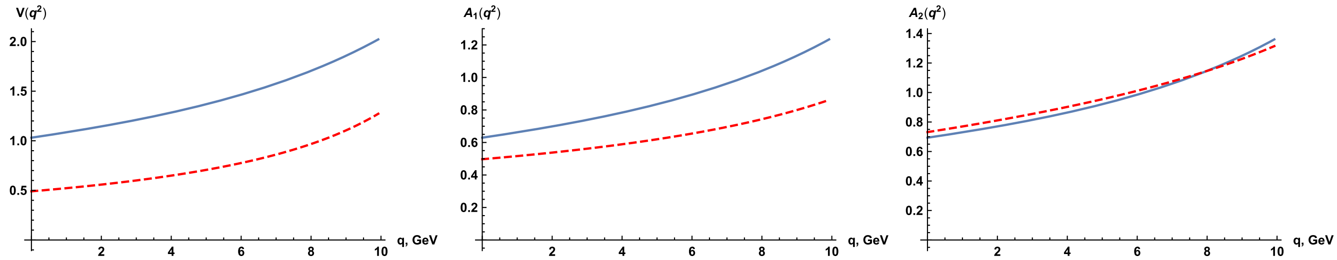

FIG. 4. $B_c \to J/\psi W$ form factors. Solid blue and dashed red lined correspond to SR and PM form factor sets, respectively.

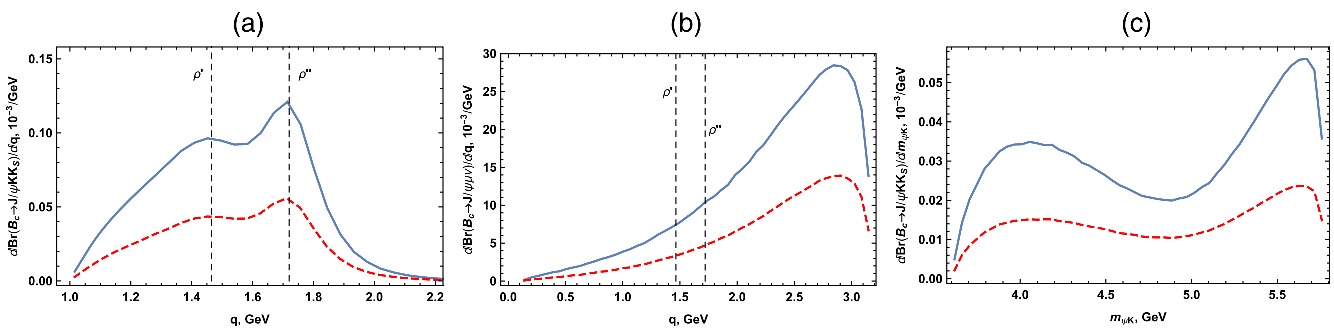

FIG. 5. $B_c \to J/\psi\mu\nu$ and $B_c \to J/\psi KK_S$ distributions. Solid blue and dashed red lines correspond to SR and PM form factor sets, respectively. Vertical dashed lines show the position of excited ρ resonances.

$$\mathrm{Br}_{\mathrm{PM}}(B_c \to J/\psi KK_S) = (3.1 \pm 0.05) \times 10^{-5}, \qquad (18)$$

where the uncertainty is caused by the experimental error in $\tau \to \nu_\tau KK_S$ branching fractions [2,3]. The corresponding $\sqrt{q^2}$ distributions are shown in Fig. 5(a). One can see that, unlike $\tau \to \nu_\tau KK_S$ decay, the contributions of the excited ρ mesons are clearly seen and can be easily separated. This is because in the case of B_c meson decay the branching fraction of the semileptonic reaction $B_c \to J/\psi\mu\nu$ is not suppressed in $q \sim m_{\rho'}$ region [see Fig. 5(b)]. It is also interesting to note that form of the distributions produced by different form factor sets is almost the same with the only difference in overall normalization. The reason is that, as can be seen from the left panel of Fig. 4, in the energy region that is significant for our task, SR and PM form factors are almost proportional to each other.

The distribution of the considered branching fraction over the invariant mass of the $J/\psi K$ pair can be also observed experimentally. It is clear that this distribution cannot be obtained using spectral function formalism, so we need to calculate the corresponding squared matrix element. As a result, we have

$$\begin{aligned}\frac{d^2\Gamma(B_c \to \psi^{(\prime)} KK_S)}{dq^2 dm_{\psi 1}^2} = & \frac{G_F^2 V_{cb}^2 a_1^2 |F(q^2)|^2}{2048\pi^3 M_{B_c}^3 M_\psi^2 (M_{B_c} + M_\psi)^2}\{-2(M_{B_c} + M_\psi)^2(m_{\psi 1}^2 - m_{\psi 2}^2)^2 A_1 A_2 (M_{B_c}^2 - M_\psi^2 - q^2) \\ & - 4M_\psi^2 V^2 (M_{B_c}^4 (4m_K^2 - q^2) - 2M_{B_c}^2 (4m_K^2 - q^2)(M_\psi^2 + q^2) \\ & + 4m_K^2 (M_\psi^2 - q^2)^2 + q^2 (m_{\psi 1}^4 - 2m_{\psi 1}^2 m_{\psi 2}^2 + m_{\psi 2}^4 - (M_\psi^2 - q^2)^2)) \\ & + (m_{\psi 1}^2 - m_{\psi 2}^2)^2 A_2^2 (M_{B_c}^4 - 2M_{B_c}^2 (M_\psi^2 + q^2) + (M_\psi^2 - q^2)^2) \\ & + (M_{B_c} + M_\psi)^4 A_1^2 (-(16 m_K^2 M_\psi^2 - (m_{\psi 1}^2 - m_{\psi 2}^2)^2 - 4M_\psi^2 q^2))\}, \end{aligned} \qquad (19)$$

where $m_{\psi 1,2}^2 = (k + p_{1,2})^2$ are the corresponding Dalitz variables (according to momentum conservation $q^2 + m_{\psi 1}^2 + m_{\psi 2}^2 = M_{B_c}^2 + M_\psi^2 + 2m_K^2$). The corresponding distribution is shown in Fig. 5(c). It should be noted that two peaks in these distributions do not correspond to any resonances, but come from the form of $B_c \to \psi^{(\prime)}\mu\nu$ matrix element.

The form factors of the $B_c \to \psi(2S)W$ transition were also studied, for example, in papers [18,19] and we show them in Fig. 6. Using these form factors, it is easy to

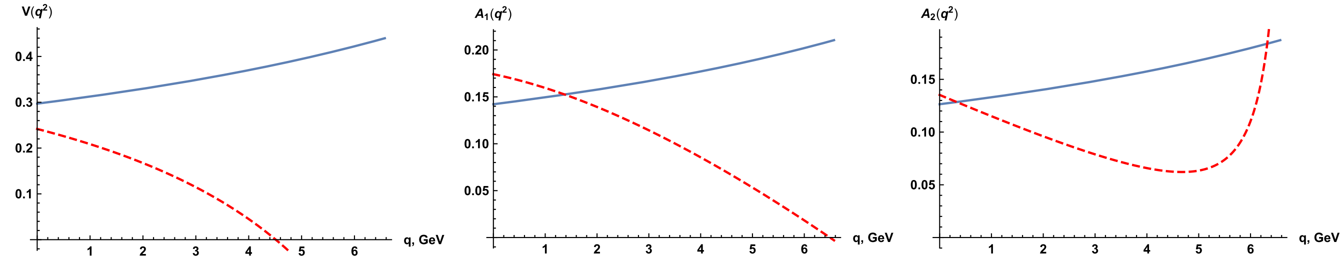

FIG. 6. $B_c \to \psi(2S)W$ form factors. Notations are the same as in Fig. 4.

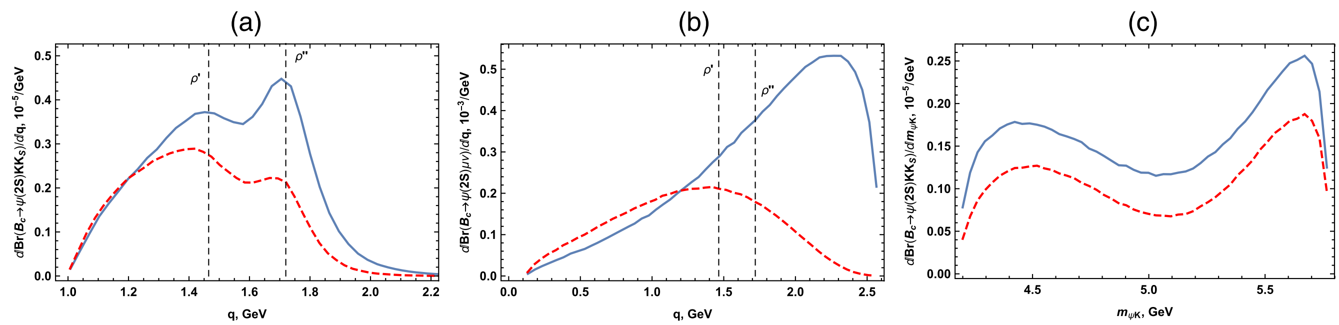

FIG. 7. $B_c \to \psi(2S)\mu\nu$ and $B_c \to \psi(2S)KK_S$ distributions. Notations are the same as in Fig. 5.

calculate the branching fractions of $B_c \to \psi(2S)KK_S$ decay in different models:

$$\mathrm{Br}_{\mathrm{SR}}(B_c \to J/\psi KK_S) = (2.6 \pm 0.04) \times 10^{-6}, \qquad (20)$$

$$\mathrm{Br}_{\mathrm{PM}}(B_c \to \psi(2S)KK_S) = (1.7 \pm 0.03) \times 10^{-6}. \qquad (21)$$

The distributions over KK_S and $\psi(2S)K$ invariant masses are shown in Fig. 7. Note that in this case the forms of q distributions for different form factor sets are quite different from each other.

IV. CONCLUSION

In this article, production of the KK_S pair in exclusive τ and B_c decays is discussed. It is clear that this final state can be produced only from the decay of the virtual vector charged particle, i.e., the ρ meson and its excitations. As a result, experimental investigation of the decays can give us additional information about masses and widths of these particles and the coupling constants of $\rho^{(\prime)} \to KK_S$ decays.

The decay $\tau \to KK_S\nu_\tau$ was studied experimentally, for example, in the recent *BABAR* papers [2,3]. According to analysis presented in [4], these results can be explained by taking into account contributions of the $\rho(770)$ meson and its two excitations, $\rho(1450)$ and $\rho(1700)$. It is clear, however, that the τ lepton's mass is not very large, so the peak caused by the last resonance peak cannot be seen in m_{KK_S} distribution. For this reason, it could be interesting to study KK_S pair production in decays of a heavier particle, e.g., the B_c meson.

In our paper, we perform such an analysis and give theoretical description of $B_c \to J/\psi KK_S$ and $B_c \to \psi(2S)KK_S$ decays. It is clear that the form factors of $B_c \to \psi^{(\prime)}$ transitions are required for calculations, and in our work we used two different sets of these form factors, obtained using QCD sum rules and Potential models. According to our results, peaks caused both by $\rho(1450)$ and $\rho(1700)$ resonances are clearly seen in m_{KK_S} distributions and can be easily separated. The branching fractions of the considered decays are also calculated.

Since the final K_S meson will be detected in $K_S \to \pi\pi$ decay, the observed state of the considered here decays will be $\psi^{(\prime)}K\pi\pi$. According to [11] the same final state can be produced also in the decay chain $B_c \to \psi^{(\prime)}K_1 \to \psi^{(\prime)}K\rho \to \psi^{(\prime)}K\pi\pi$ and the branching fractions of these reactions are significantly larger than the branching fractions of the decays considered in our article. It should be noted, however, that the same can also be said about the corresponding τ lepton decays, but both decays modes were observed.

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APPENDIX: SPECTRAL FUNCTION FORMALISM

Let us describe briefly the formalism of the spectral functions [20] used in our paper. The amplitudes of the decay $A \to BW(q) \to BR$ can be written in the factorized form,

$$\mathcal{M}(A \to BR) = \mathcal{H}_\mu (A \to BW) \epsilon_\mu^R, \tag{A1}$$

where R stands for KK_S or $\mu\nu$ pairs, while A, B are τ, ν_τ in the case of the semileptonic decay and B_c, $\psi^{(\prime)}$ for $B_c \to \psi^{(\prime)} R$ process. In the above expression, $\mathcal{H}_\mu$ is the vertex of $A \to BW$ transition and ϵ_μ^R is the effective polarization vector of virtual W boson. The corresponding decay width equals

$$d\Gamma = \frac{1}{2J_A + 1} \frac{1}{2M_A} d\Phi_{n+1}(P_A \to P_B k_1 \ldots k_n) \sum_{\text{pol}} |\mathcal{M}|^2, \tag{A2}$$

where P_A, P_B are the momenta of the corresponding particles, n is the number of particles in the system R, and k_i are the momenta of these particles. Used in this expression, the Lorentz-invariant phase space $d\Phi_n$ is defined as

$$d\Phi_{n+1}(P_A \to P_B k_1 \ldots k_n) = (2\pi)^4 \delta^4 \left(P_A - P_B - \sum_i k_i \right) \frac{d^3 P_B}{(2\pi)^3 2E_B} \prod_i \frac{d^3 k_i}{(2\pi)^3 2E_i} \tag{A3}$$

and satisfies the following recurrence relation:

$$\begin{aligned} d\Phi_{n+1}(P_A \to P_B k_1 \ldots k_n) &= \frac{dq^2}{2\pi} d\Phi_2(P_A \to P_B q) d\Phi_n(q \to k_1 \ldots k_n) \\ &= \frac{dq^2}{2\pi} \frac{\lambda(q^2)}{8\pi} d\Phi_n(q \to k_1 \ldots k_n), \end{aligned} \tag{A4}$$

where $q = P_A - P_B$ and

$$\lambda(q^2) = \sqrt{1 - \frac{(M_B + \sqrt{q^2})^2}{M_A^2}} \sqrt{1 - \frac{(M_B - \sqrt{q^2})^2}{M_A^2}} \tag{A5}$$

is the velocity of the particle B. Using these relations, one can rewrite the expression (A2) in the form

$$\frac{d\Gamma}{dq^2} = \frac{1}{2M_A(2J_A + 1)} \frac{\lambda(q^2)}{8\pi} [\mathcal{H}_T^2(q^2) \rho_T^R(q^2 + \mathcal{H}_L^2(q^2 \rho_L^R(q^2))], \tag{A6}$$

where

$$\mathcal{H}_L^2(q^2) = q^\mu q^\nu \mathcal{H}_\mu \mathcal{H}_\nu^*, \qquad \mathcal{H}_T^2 = (q^\mu q^\nu - q^2 g^{\mu\nu}) \mathcal{H}_\mu \mathcal{H}_\nu^* \tag{A7}$$

and spectral functions $\rho_{L,T}^R(q^2)$ are defined as

$$\begin{aligned} &\int \frac{d\Phi_n(q \to k_1 \ldots k_n)}{2\pi} \epsilon_\mu^R \epsilon_\nu^{*R} \\ &\quad = q_\mu q_\nu \rho_L^R(q^2) + (q_\mu q_\nu - q^2 g_{\mu\nu}) \rho_T^R(q^2). \end{aligned} \tag{A8}$$

It is easy to check that both for $R = KK_S$ and $\mu\nu$ the effective polarization vector is transversal, i.e., $\epsilon_\mu^R q^\mu = 0$. As a result, the longitudinal spectral function vanishes and for the transverse one we have the relation

$$\rho_T^R(q^2) = -\frac{g^{\mu\nu}}{3q^2} \int \frac{d\Phi_n}{2\pi} \epsilon_\mu^R \epsilon_\nu^{*R}. \tag{A9}$$

It is important to note that in the relation (A6) only the spectral functions $\rho_{L,T}^R$ depend on the final system R. Since only transverse spectral functions contribute, all other factors in (A6) cancel, and we have the relation

$$\frac{d\Gamma(A \to BR_1)/dq^2}{d\Gamma(A \to BR_1)/dq^2} = \frac{\rho_T^{R_1}(q^2)}{\rho_T^{R_2}(q_2)} \tag{A10}$$

that is Eq. (10).

With the help of the above expressions, it is easy to calculate the spectral functions of KK_S and $\mu\nu$ pairs. In the first case, for example, we have

$$\begin{aligned} \epsilon_\mu^{KK_S} &= F(q^2)(p_1 - p_2)_\mu, \\ \rho_T(q^2) &= \left(1 - \frac{4m_K^2}{q^2}\right)^{3/2} \frac{|F(q^2)|^2}{48\pi^2}, \end{aligned} \tag{A11}$$

that is Eq. (6). In the case of semileptonic decay, on the other hand,

$$\epsilon_\alpha^{\mu\nu} = \bar{u}(k_\mu) \gamma_\alpha v(k_\nu), \qquad \rho_T^{\mu\nu}(q^2) = \frac{1}{6\pi^2}. \tag{A12}$$

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