



Cosmology of $f(Q)$ gravity in non-flat Universe

Hamid Shabani¹, Avik De², Tee-How Loo³, Emmanuel N. Saridakis^{4,5,6,a} 

¹ Physics Department, Faculty of Sciences, University of Sistan and Baluchestan, Zahedan, Iran

² Department of Mathematical and Actuarial Sciences, Universiti Tunku Abdul Rahman, Jalan Sungai Long, 43000 Cheras, Malaysia

³ Institute of Mathematical Sciences, Faculty of Science, Universiti Malaya, 50603 Kuala Lumpur, Malaysia

⁴ National Observatory of Athens, Lofos Nymfon, 11852 Athens, Greece

⁵ CAS Key Laboratory for Researches in Galaxies and Cosmology, Department of Astronomy, University of Science and Technology of China, Hefei 230026, Anhui, People's Republic of China

⁶ Departamento de Matemáticas, Universidad Católica del Norte, Avda. Angamos 0610, Casilla 1280, Antofagasta, Chile

Received: 7 December 2023 / Accepted: 18 February 2024 / Published online: 19 March 2024
© The Author(s) 2024

Abstract We investigate the cosmological implications of $f(Q)$ gravity, which is a modified theory of gravity based on non-metricity, in non-flat geometry. We perform a detailed dynamical-system analysis keeping the $f(Q)$ function completely arbitrary. As we show, the cosmological scenario admits a dark-matter dominated point, as well as a dark-energy dominated de Sitter solution which can attract the Universe at late times. However, the main result of the present work is that there are additional critical points which exist solely due to curvature. In particular, we find that there are curvature-dominated accelerating points which are unstable and thus can describe the inflationary epoch. Additionally, there is a point in which the dark-matter and dark-energy density parameters are both between zero and one, and thus it can alleviate the coincidence problem. Finally, there is a saddle point which is completely dominated by curvature. In order to provide a specific example, we apply our general analysis to the power-law case, showing that we can obtain the thermal history of the Universe, in which the curvature density parameter may exhibit a peak at intermediate times. These features, alongside possible indications that non-zero curvature could alleviate the cosmological tensions, may serve as advantages for $f(Q)$ gravity in non-flat geometry.

1 Introduction

Modified gravity [1,2] is one of the two main directions that one can follow in order to obtain an improved description of the Universe evolution, both concerning the early (inflation) and late (dark-energy) accelerated phases, as well as concern-

ing the possible observational tensions [3]. In such theories one constructs modifications and extensions of General Relativity which present extra degrees of freedom capable of inducing corrections at the cosmological behavior, both at the background and perturbation level.

There are many ways to construct gravitational modifications. In the simplest ones one starts from the Einstein-Hilbert Lagrangian and adds new terms, resulting to $f(R)$ gravity [4], to $f(G)$ gravity [5], to $f(P)$ gravity [6], to Lovelock gravity [7], to Horndeski/Galileon scalar-tensor theories [8,9] etc. Alternatively, one may start from the torsion-based formulation of gravity and modify it accordingly, resulting to $f(T)$ gravity [10,11], $f(T, T_G)$ gravity [12], $f(T, B)$ gravity [13], scalar-torsion theories [14] etc.

One different class of gravitational modifications arises when one starts from the equivalent formulation of gravity based on non-metricity. Initiated by Nester and Yo [15], based on an affine connection with vanishing curvature and torsion but metric-incompatibility, it was recently extended to $f(Q)$ theory [16]. $f(Q)$ gravity contains general relativity as a particular limit, and has the advantage of possessing second-order field equations. Hence, its cosmological application has attracted the interest of the literature [17–68]. Nevertheless, all of these works focus on spatially-flat Friedmann-Lemaître-Robertson-Walker (FLRW) geometry, in which case the coincident gauge implies that the affine connection field equations can be ignored and thus $f(Q)$ cosmology coincides with $f(T)$ cosmology at the background level [69].

In this work we are interested in investigating $f(Q)$ cosmology in non-flat Universe, in order to reveal possible novel features, having in mind that non-flat geometry [70], apart from being potentially interesting [71–81], might be one way to alleviate cosmological tensions [82]. The manuscript is

^ae-mail: msaridak@phys.uoa.gr (corresponding author)

organized as follows: In Sect. 2 we provide the basic mathematical formalism of symmetric teleparallel and $f(Q)$ gravity. Then, in Sect. 3 we perform a detailed dynamical-system analysis, extracting the general cosmological features, both for a general $f(Q)$ function as well as for a specific power-law example. Finally, in Sect. 4 we summarize our results.

2 Symmetric teleparallel and $f(Q)$ gravity

In this section we briefly review symmetric teleparallel formulation of gravity and its $f(Q)$ extension. In such a formalism one introduces a general affine connection $\Gamma^\alpha_{\beta\gamma}$, defined by $\Gamma^\lambda_{\mu\nu} = \overset{\circ}{\Gamma}^\lambda_{\mu\nu} + L^\lambda_{\mu\nu}$, where $\overset{\circ}{\Gamma}^\lambda_{\mu\nu}$ is the Levi-Civita connection and the disformation tensor is given by

$$L^\lambda_{\mu\nu} = \frac{1}{2}(Q^\lambda_{\mu\nu} - Q_\mu{}^\lambda{}_\nu - Q_\nu{}^\lambda{}_\mu), \tag{1}$$

with the non-metricity tensor given as

$$Q_{\lambda\mu\nu} = \nabla_\lambda g_{\mu\nu}. \tag{2}$$

Additionally, one can define the non-metricity scalar as

$$Q = Q_{\lambda\mu\nu} P^{\lambda\mu\nu} = \frac{1}{4}(-Q_{\lambda\mu\nu} Q^{\lambda\mu\nu} + 2Q_{\lambda\mu\nu} Q^{\mu\lambda\nu} + Q_\lambda Q^\lambda - 2Q_\lambda \tilde{Q}^\lambda), \tag{3}$$

with $Q_\lambda = Q_{\lambda\mu\nu} g^{\mu\nu}$ and $\tilde{Q}_\nu = Q_{\lambda\mu\nu} g^{\lambda\mu}$. Hence, using Q as a Lagrangian gives rise to the same equations with general relativity.

Based on the symmetric teleparallel framework, one can proceed in constructing gravitational modifications, such as $f(Q)$ gravity [16], characterized by the action

$$S = \frac{1}{2\kappa} \int f(Q) \sqrt{-g} d^4x + \int \mathcal{L}_M \sqrt{-g} d^4x, \tag{4}$$

where we have added the matter Lagrangian for completeness, corresponding to an energy-momentum tensor of a perfect fluid $T^m_{\mu\nu} = (p + \rho)u_\mu u_\nu + pg_{\mu\nu}$, with p and ρ the pressure and energy density respectively. Variation of the action with respect to the metric leads to the field equations

$$\frac{2}{\sqrt{-g}} \nabla_\lambda (\sqrt{-g} F P^\lambda_{\mu\nu}) - \frac{1}{2} f g_{\mu\nu} + F (P_{\nu\rho\sigma} Q_\mu{}^{\rho\sigma} - 2P_{\rho\sigma\mu} Q^{\rho\sigma}{}_\nu) = \kappa T^m_{\mu\nu}, \tag{5}$$

where the superpotential $P^\lambda_{\mu\nu}$ is given by

$$P^\lambda_{\mu\nu} = \frac{1}{4} \left(-2L^\lambda_{\mu\nu} + Q^\lambda g_{\mu\nu} - \tilde{Q}^\lambda g_{\mu\nu} - \frac{1}{2} \delta^\lambda_\mu Q_\nu - \frac{1}{2} \delta^\lambda_\nu Q_\mu \right), \tag{6}$$

and with $F(Q) = df(Q)/dQ$. Note that the field equations (5) can be alternatively written as [83]

$$F \overset{\circ}{G}_{\mu\nu} + \frac{1}{2} g_{\mu\nu} (F Q - f) + 2F' P^\lambda_{\mu\nu} \overset{\circ}{\nabla}_\lambda Q = \kappa T^m_{\mu\nu}, \tag{7}$$

where $\overset{\circ}{G}_{\mu\nu} = \overset{\circ}{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{\circ}{R}$, and all the expressions denoted with a $\overset{\circ}{}$ are calculated with respect to the Levi-Civita connection $\overset{\circ}{\Gamma}^\lambda_{\mu\nu}$. Hence, we can re-write them as [22]

$$\overset{\circ}{G}_{\mu\nu} = \frac{\kappa}{F} T^m_{\mu\nu} + \kappa T^{\text{de}}_{\mu\nu}, \tag{8}$$

having defined an effective dark-energy sector of geometrical origin as

$$\kappa T^{\text{de}}_{\mu\nu} = \frac{1}{F} \left[\frac{1}{2} g_{\mu\nu} (f - QF) - 2F' \overset{\circ}{\nabla}_\lambda Q P^\lambda_{\mu\nu} \right], \tag{9}$$

where a prime denotes differentiations with respect to the argument. Lastly, varying the action with respect to the affine connection, and assuming that the matter Lagrangian \mathcal{L}_M does not depend on it, we obtain

$$\nabla_\mu \nabla_\nu (\sqrt{-g} F P^{\nu\mu}{}_\lambda) = 0. \tag{10}$$

Let us apply $f(Q)$ gravity to a cosmological framework. As we mentioned in the Introduction, we consider a non-flat Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime of the form

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \tag{11}$$

where $k = 0, \pm 1$ denotes the spatial curvature. In this case, the non-trivial connection coefficients are given by [43, 84]

$$\begin{aligned} \Gamma^t{}_{tt} &= -\frac{k + \dot{\gamma}}{\gamma}, & \Gamma^t{}_{rr} &= \frac{\gamma}{1 - kr^2}, & \Gamma^t{}_{\theta\theta} &= \gamma r^2, \\ \Gamma^t{}_{\phi\phi} &= \gamma r^2 \sin^2 \theta \\ \Gamma^r{}_{tr} &= -\frac{k}{\gamma}, & \Gamma^r{}_{rr} &= \frac{kr}{1 - kr^2}, & \Gamma^r{}_{\theta\theta} &= -(1 - kr^2)r, \\ \Gamma^r{}_{\phi\phi} &= -(1 - kr^2)r \sin^2 \theta, \\ \Gamma^\theta{}_{t\theta} &= -\frac{k}{\gamma}, & \Gamma^\theta{}_{r\theta} &= \frac{1}{r}, & \Gamma^\theta{}_{\phi\phi} &= -\cos \theta \sin \theta, \\ \Gamma^\phi{}_{t\phi} &= -\frac{k}{\gamma}, & \Gamma^\phi{}_{r\phi} &= \frac{1}{r}, & \Gamma^\phi{}_{\theta\phi} &= \cot \theta, \end{aligned} \tag{12}$$

where $\gamma(t)$ is a non-zero function of time. The corresponding non-metricity scalar Q can be calculated from (12) as [43, 84–88]

$$Q(t) = -3 \left[2H^2 + \left(\frac{3k}{\gamma} - \frac{\gamma}{a^2} \right) H - \frac{2k}{a^2} - k \frac{\dot{\gamma}}{\gamma^2} - \frac{\dot{\gamma}}{a^2} \right]. \tag{13}$$

Therefore, inserting into the field equations (7) we obtain the modified Friedmann equations

$$-\frac{1}{2}f - \left(3H^2 + 3\frac{k}{a^2} - \frac{1}{2}Q\right)F + \frac{3}{2}\dot{Q}\left(\frac{k}{\gamma} + \frac{\gamma}{a^2}\right)F' + \kappa\rho^m = 0 \tag{14}$$

$$\frac{1}{2}f + \left(3H^2 + 2\dot{H} + \frac{k}{a^2} - \frac{1}{2}Q\right)F + \dot{Q}\left(2H + \frac{3k}{2\gamma} - \frac{1}{2}Q\right)F' + \kappa p^m = 0. \tag{15}$$

3 Cosmological behavior

In this section we investigate in detail the cosmological evolution of a Universe governed by $f(Q)$ gravity in non-flat geometry. In order to achieve that we perform a dynamical-system analysis [89,90], which allows one to extract the global features of a cosmological scenario independently of the specific initial conditions [91–97]. Note that the dynamical-system analysis for $f(Q)$ gravity has been performed in the literature [98–102], however it remains in the flat FLRW case, while as we will see in the following the inclusion of spatial curvature leads to novel qualitative features.

We start by defining dimensionless variables in order to rewrite Eqs. (14)–(15) as an autonomous system. For simplicity we will focus on dust matter, namely we consider $p^m = 0$, while concerning the $\gamma(t)$ form we assume the simple case $\gamma(t) = \epsilon a(t)$ (analysis of the general case is straightforward, with the inclusion of an extra variable [102]). In particular, we have

$$\begin{aligned} x_1 &= -\frac{f}{6H^2F}, & x_2 &= \frac{Q}{6H^2}, & x_3 &= \frac{\dot{F}}{HF}, & x_4 &= \frac{1}{2Ha}, \\ \Omega^m &= \frac{\kappa\rho^m}{3H^2F}, & \Omega^k &= -\frac{k}{H^2a^2}, \\ r &= -\frac{QF}{f} = \frac{x_2}{x_1}, & m &= \frac{QF'}{F}, \end{aligned} \tag{16}$$

where Ω^m and Ω^k denote the contributions to the dark matter and the spatial curvature energy densities, and thus according to (9) the first Friedmann equation (14) is written as $1 = \Omega^m + \Omega^k + \Omega^{de}$. The two parameters m and r parametrize the $f(Q)$ form as a function $m(r)$, while the variable x_4 has been introduced in order to break the degeneracy between positive and negative H (since it is H^2 that appears in the other variables).

Using the above dimensionless variables we result to the four-dimensional autonomous system

$$\frac{dx_1}{dN} = -\frac{x_2x_3}{m} - x_1(x_3 + 3\mathcal{A}), \tag{17}$$

$$\frac{dx_2}{dN} = \frac{x_2x_3}{m} - 3x_2\mathcal{A}, \tag{18}$$

$$\frac{dx_3}{dN} = -x_3\left(3 + x_3 + \frac{3}{2}\mathcal{A}\right), \tag{19}$$

$$\frac{\Omega^k}{dN} = -\Omega^k(2 + 3\mathcal{A}), \tag{20}$$

with $\mathcal{A} \equiv \frac{2\dot{H}}{3H^2} = -1 + x_1 + x_2 + \frac{x_3}{3}(\zeta x_4 - 2) + \frac{\Omega^k}{3}$ and $\zeta = -3\frac{k}{\epsilon} + \epsilon$, and where $N = \ln a$. Hence, the total equation-of-state parameter is just $w^{eff} = -1 - 2\dot{H}/3H^2 = -1 - \mathcal{A}$. Finally, we mention here that since $r = x_2/x_1$, the condition $dr/dN = rx_3(1 + \frac{1+r}{m}) = 0$ implies that the critical points of the system (17)–(20) must satisfy either $r = 0$ (or equivalently $x_2 = 0$), or $x_3 = 0$, or $m(r) = -(1 + r)$, while when the conditions $x_1 = 0, x_2 = 0$ and $x_3 \neq 0$ simultaneously hold the relation $m(r) = -(1 + r)$ must be considered.

3.1 General $f(Q)$ form

We start by performing the analysis for a general $f(Q)$ form, namely for a general $m(r)$ function. As we will see, in this case the intersections of the curve $m(r)$ with the line $m = -r - 1$ can play an important role in the way that the critical point corresponding to dark-matter dominated era connects to those exhibiting dark-energy domination.

In the general case the critical points of the system (17)–(20) are presented in Table 1. As can be seen, for a general $f(Q)$ function there exist seven critical points, or curves of critical points, with different physical features. Note that all m and m' values must be calculated at probable intersections of $m(r)$ with $m = -r - 1$, which happen at the roots $r_i, i = 1, 2, \dots$

The physical properties of these critical points are the following:

- Point P^m : It corresponds to dark-matter ($\Omega^m = 1$) dominated era with total equation-of-state parameter $w^{eff} = 0$. Its eigenvalues imply that it is a saddle point and thus it can be the intermediate state of the Universe.
- Point P^k : It corresponds to a curvature-dominated era and it is a saddle point.
- Curve of points P^{ds} : It corresponds to a dark-energy dominated Universe (since $\Omega^m = \Omega^k = 0$ we have $\Omega^{de} = 1$), with $w^{eff} = -1$, namely to the de Sitter solution. Although it has a zero eigenvalue, application of the center manifold theorem [89,90] shows that this point is stable and thus it can attract the Universe at late times.
- Point P^1 : This point exists only for non-flat geometry. It corresponds to a curvature-dominated solution if $k = \pm\epsilon^2$, and it is unstable for every values of m and m' .
- Point P^2 : This point is physical (i.e. having $0 \leq \Omega^m \leq 1$) only for $k = 1$ and for $\epsilon^2 \leq 1$. In this case Ω^m and Ω^{de} are both between 0 and 1 and thus this point can alleviate the

Table 1 The critical points in the general $f(Q)$ case, namely with general $m(r)$. We have defined $j_1 = \frac{6[k(m-1)+(m+1)\epsilon^2]}{m(3k-\epsilon^2)}$, $j_2 = \frac{6[(m+1)(k-\epsilon^2)m'+2km]}{m(3k-\epsilon^2)}$, $l_1 = -\frac{3(-\sqrt{33k^2-30k\epsilon^2+\epsilon^4+k+\epsilon^2})}{4(k-\epsilon^2)}$ and $l_2 = \frac{3(\sqrt{33k^2-30k\epsilon^2+\epsilon^4+k+\epsilon^2})}{4(\epsilon^2-k)}$

Fixed point	Coordinates $(x_1, x_2, x_3, \Omega^k)$	Eigenvalues	Ω^m	Ω^k	w^{eff}
P^m	$(0, 0, 0, 0)$	$[3, 3, -\frac{3}{2}, 1]$	1	0	0
P^k	$(0, 0, 0, 1)$	$[-2, -1, 2, 2]$	0	1	$-\frac{1}{3}$
P^{ds}	$(x_1, 1 - x_1, 0, 0)$	$[-3, -2, 0, -3]$	0	0	-1
[0.75 ex] P^1	$(0, 0, -2, -\frac{\epsilon^2}{k})$ with $k \neq 0$	$\left\{ \begin{aligned} &[1, 2, 2 - \frac{2}{m}, 2(\frac{1}{m} + 1)m' + 4], \\ &k = -\epsilon^2, \\ &\{2, \pm\infty, 2 - \frac{2}{m}, 2(\frac{1}{m} + 1)m' + 4\}, \\ &k = +1, \epsilon \rightarrow 1^\pm \end{aligned} \right.$	0	$-\frac{\epsilon^2}{k}$	$-\frac{1}{3}$
P^2	$(0, 0, -6[\frac{2k}{\epsilon^2-3k} + 1], 0)$	$[\frac{3}{2}, -\frac{8\epsilon^2}{\epsilon^2-3k} + 1, j_1, j_2]$	$\frac{4\epsilon^2}{\epsilon^2-3k}$	0	$-\frac{8k}{\epsilon^2-3k} - 3$
[0.75 ex] P^3	$(\frac{2k}{k-\epsilon^2}, 0, -6, 0)$	$\left\{ \begin{aligned} &[-8, -\frac{6(m+1)}{m}, l_1, l_2], \\ &(l_1 = -3, l_2 = 3), k = -\epsilon^2 \end{aligned} \right.$	$\frac{8k}{\epsilon^2-k} + 4$	0	-3
P^4	$(\frac{k(m-1)+(m+1)\epsilon^2}{m[k(m+2)+m\epsilon^2]}, \frac{(m+1)[k(m-1)+(m+1)\epsilon^2]}{m[k(m+2)+m\epsilon^2]}, -\frac{6m}{2m+1}, 0)$	$\left\{ \begin{aligned} &[\frac{6}{2m+1} - 2, 3, \frac{3}{2m+1}, \frac{6(m+1)(m'+1)}{2m+1}], \\ &k = -\epsilon^2 \end{aligned} \right.$	$\frac{2(m-1)(k+\epsilon^2)}{k(m+2)+m\epsilon^2}$	0	$\frac{2}{2m+1} - 1$

coincidence problem. Additionally, it has $-1 \leq w^{eff} \leq -1/3$ and thus it corresponds to accelerated solution. The fact that it is unstable makes this point a good candidate for the description of inflation with a successful exit.

- Point P^3 : This point is physical only for $k = -1$ and for $1 \leq \epsilon^2 \leq 5/3$, in which case Ω^m and Ω^{de} are both between 0 and 1. It corresponds to super-acceleration and it is unstable.
- Curve of points P^4 : The properties of this curve cannot be inferred without specifying $m(r)$, namely the $f(Q)$ form.

In summary, $f(Q)$ cosmology in non-flat Universe exhibits the desired features of saddle matter-dominated era and stable late-time dark-energy era. However, apart from these, we obtain interesting features that arise solely from non-zero curvature, such as a point which can alleviate the coincidence problem, or a point that corresponds to a curvature-driven inflation which is unstable and thus it can easily acquire a successful inflation exit. Nevertheless, since some features cannot be extracted for the general $f(Q)$ form, in the following subsection we examine a specific $f(Q)$ case.

3.2 Application for $f(Q) = \eta Q^n$

Let us apply the above general analysis in the case $f(Q) = \eta Q^n$. Such a choice, according to (16) corresponds to $m = n - 1 = const.$ and $r = -n = const.$, and thus $x_2 = r x_1$, which implies that variable x_2 is not needed. We first examine the flat case and then we continue to $k = \pm 1$.

3.2.1 $k = 0$

In this case Ω^k is absent and we acquire a two-dimensional system, namely (17) and (19) with zero Ω^k terms. The corresponding physical critical points are shown in Table 2. As we can see, we obtain an unstable dark-matter dominated point, namely p^m , as well as a stable dark-energy dominated de Sitter solution p^{de} . However, for $1 \leq m \leq 2$ we obtain point p^b , in which Ω^m and Ω^{de} are both between 0 and 1 and thus this point can alleviate the coincidence problem, while it has $-3/5 \leq w^{eff} \leq -1/3$. Note that this point for $1 \leq m$ is unstable. In Fig. 1 we depict the critical points, using for convenience the new variables $x \equiv X/\sqrt{1 - X^2 - Y^2}$ and $y \equiv Y/\sqrt{1 - X^2 - Y^2}$ in order to compactify them.

3.2.2 $k = -1$

In the case of $k \neq 0$ the system of dynamical equations contains (17), (19) and (20). The critical points are presented in Table 3. As can be seen there is the unstable dark-matter dominated point Q^m , and the unstable curvature-dominated point Q^k . Additionally, there exist a stable dark-energy dominated de Sitter solution Q^{ds} . Moreover, similarly to points P^4 in Table 1 and p^b in Table 2 above, there is a point Q^3 in which Ω^m and Ω^{de} are both between 0 and 1 and thus this point can alleviate the coincidence problem.

In the presence of spatial curvature the two points Q^1 and Q^2 appear, too. Q^1 corresponds to P^2 of Table 1, and it is physical only for $k = 1$ and for $\epsilon^2 \leq 1$. It has both Ω^m and Ω^{de} between 0 and 1 and thus this point can alleviate the coincidence problem. Additionally, it has $-1 \leq w^{eff} \leq -1/3$ and therefore it corresponds to accelerated solution. The fact

Table 2 The critical points for the case $f(Q) = \eta Q^n$, with $k = 0$

Fixed point	Coordinates (x_1, x_3)	Eigenvalues	Ω^m	w^{eff}
p^m	(0, 0)	$[3, -\frac{3}{2}]$	1	0
p^{ds}	$(-\frac{1}{m}, 0)$	$[-3, -3]$	0	-1
p^b	$(\frac{m+1}{m^2}, -\frac{6m}{2m+1})$	$[-\frac{3(-2m^2 + \sqrt{1-4m^2(m+1)(7m+5)+1})}{4m(2m+1)}, \frac{3(2m^2 + \sqrt{1-4m^2(m+1)(7m+5)-1})}{4m(2m+1)}]$	$2 - \frac{2}{m}$	$\frac{2}{2m+1} - 1$

Table 3 The critical points for the case $f(Q) = \eta Q^n$, with $k = \pm 1$

Fixed point	Coordinates (x_1, x_3, Ω^k)	Eigenvalues	Ω^m	Ω^k	w^{eff}
Q^m	(0, 0, 0)	$[3, -\frac{3}{2}, 1]$	1	0	0
Q^k	(0, 0, 1)	$[-2, -1, 2]$	0	1	$-\frac{1}{3}$
Q^{ds}	$(-\frac{1}{m}, 0, 0)$	$[-3, -3, -2]$	0	0	-1
[0.75 ex] Q^1	$(0, -6[\frac{2k}{\epsilon^2-3k} + 1], 0)$	$[\frac{3}{2}, -\frac{8\epsilon^2}{\epsilon^2-3k}, \frac{6[k(m-1)+(m+1)\epsilon^2]}{m(3k-\epsilon^2)}]$	$\frac{4\epsilon^2}{\epsilon^2-3k}$	0	$-\frac{8k}{\epsilon^2-3k} - 3$
[0.75 ex] Q^2	$(0, -2, -\frac{\epsilon^2}{k})$	$[2 - \frac{2}{m}, 1, 2]$	0	$-\frac{\epsilon^2}{k}$	$-\frac{1}{3}$
Q^3	$(\frac{k(m-1)+(m+1)\epsilon^2}{m[k(m+2)+m\epsilon^2]}, -\frac{6m}{2m+1}, 0)$	$\left\{ \begin{aligned} &[\frac{6}{2m+1} - 2, 3, \frac{3}{2m+1}], \\ &k = -\epsilon^2 \end{aligned} \right.$	$\frac{2(m-1)(k+\epsilon^2)}{k(m+2)+m\epsilon^2}$	0	$\frac{2}{2m+1} - 1$

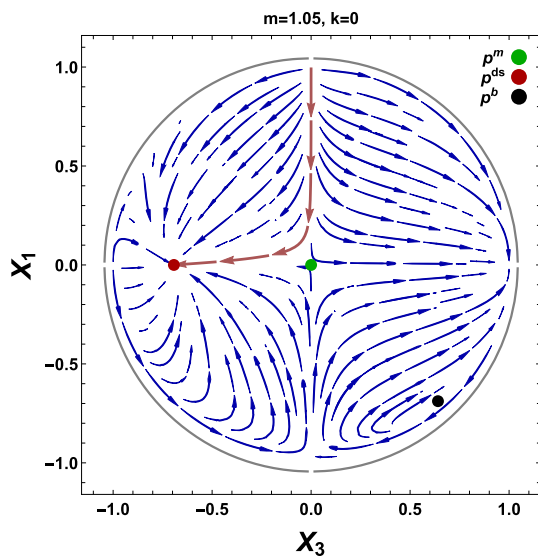


Fig. 1 The phase-space behavior in the specific case of $f(Q) = \eta Q^n$ gravity, for $k = 0$. The Universe passes through the saddle matter-dominated point p^m at intermediate times, before it results to the dark-energy dominated de Sitter solution p^{ds}

that it is unstable makes this point a good candidate for the description of inflation. Furthermore, point Q^2 corresponds to P^1 , namely it describes a curvature-dominated solution if $k = \pm\epsilon^2$, and it is unstable.

In order to present the above features in a more transparent way, we proceed to numerical investigation and in Fig. 2 we depict the corresponding phase-space behavior in the $X_1 - X_3$ plane. In the left panel the system starts from the inflationary

point Q^1 , then it passes close to the matter-dominated point Q^m and finally it results to the dark-energy dominated de Sitter solution Q^{ds} . As we can see, the role of spatial curvature is crucial in obtaining such a thermal history of the Universe. Moreover, for completeness, in the right panel of Fig. 2 we present a parameter-case in which the trajectories starting from Q^3 do not approach Q^m efficiently.

In the left panel of Fig. 3 we provide the redshift-evolution of the density parameters, while in the right panel we depict the evolution of the deceleration and total equation-of-state parameters (note that $\ln(1+z) = -\ln a = -N$). Interestingly enough, we observe a transition from the initial accelerated expansion stage, to the intermediate matter-dominated non-accelerating era, and then to the final accelerated expansion phase. Additionally, note that the spatial curvature density parameter grows when the domination of the matter and the dark energy phases is reversed.

Finally, in order to illustrate the behavior of the phase-space trajectories near the curvature-dominated points Q^k and Q^2 , we focus on the $x_1 = 0$ plane. In upper panel of Fig. 4 we display Q^m , Q^k and Q^2 in the $x_1 = 0$ plane for $k = -1$. Note that in the $x_1 = 0$ plane one cannot indicate the point Q^{ds} for which one acquires $x_1 = -1/m$. As we can see, for particular initial values the dark-matter dominated phase falls between two different epochs with considerable values of Ω^k . In particular, the Universe evolves from an epoch with curvature domination to the dark-matter dominated era and then to another curvature-dominated epoch. The lower panels of Fig. 4 show the time evolution of the density parameters, and the deceleration and total equation-of-state parameters.

Fig. 2 The phase-space behavior in the specific case of $f(Q) = \eta Q^n$ gravity, for $k = -1$. Left panel: for the choice $\epsilon = 0.1$ the system starts from the inflationary point Q^1 , then it passes close to the matter-dominated point Q^m and finally it results to the dark-energy dominated de Sitter solution Q^{ds} . Right panel: for the choice $\epsilon = 1$ the system starts from the scaling point Q^3 and it results to the dark-energy dominated de Sitter solution Q^{ds} without passing sufficiently close to the matter-dominated point Q^m

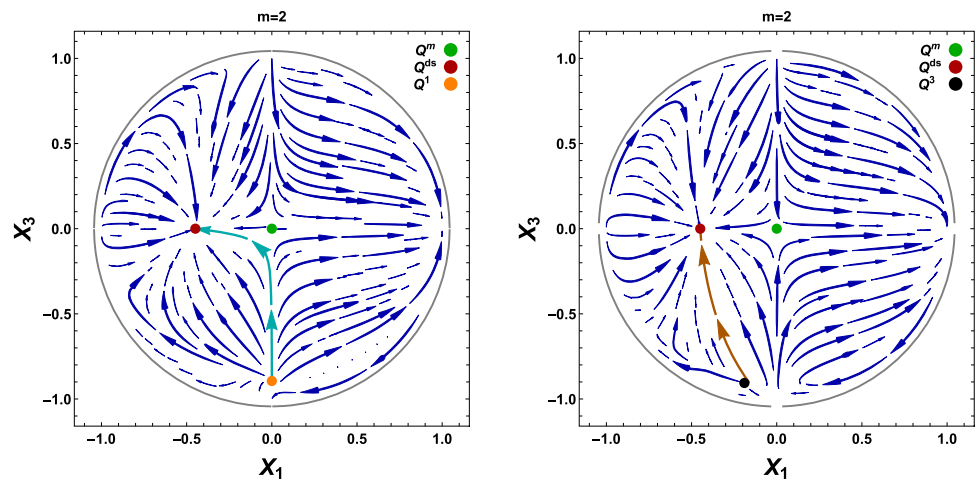
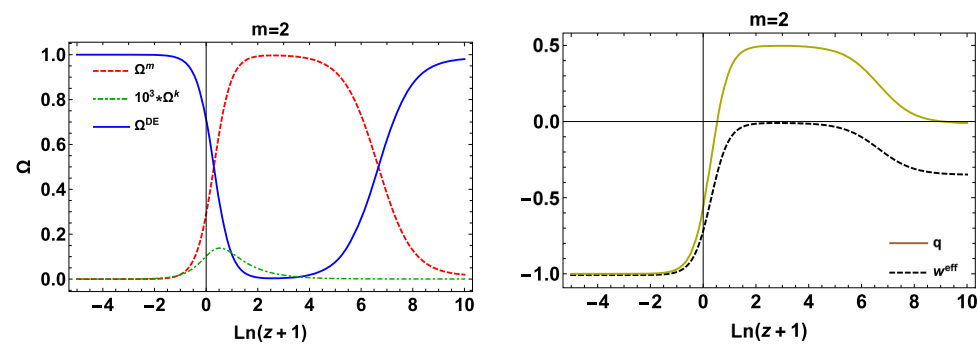


Fig. 3 The redshift evolution of the density parameters (left panel) and of the deceleration and equation-of-state parameters (right panel) in the specific case of $f(Q) = \eta Q^n$ gravity, for $k = -1$ and $\epsilon = 0.1$. We have set the initial conditions $x_{1i} = -1.05 \times 10^{-10}$, $x_{3i} = -2.00013$ and $\Omega_i^k = 5 \times 10^{-7}$



As we can see, the transition $Q^2 - Q^m - Q^k - Q^{ds}$ is also possible for a particular set of the initial values.

3.2.3 $k = +1$

In the case of $k = +1$ the system of dynamical equations (17), (19) and (20) exhibits the critical points presented in Table 3. In particular, one has points Q^m and Q^{ds} , however in this case Q^k and Q^1 are not physical. Point Q^2 corresponds to a closed spatial curvature dominated era. Finally, Q^3 exists, in which Ω^m and Ω^{de} are between 0 and 1 and thus it can alleviate the coincidence problem. In Fig. 5 we plot the phase-space trajectories in both $\Omega^k = 0$ and $x_1 = 0$ planes. As we can see, we cannot obtain any transition between Q^m and Q^2/Q^3 . Hence, we conclude that under positive spatial curvature we can only obtain the usual transition from matter to dark-energy dominated phases.

4 Concluding remarks

In this manuscript we investigated the cosmological implications of $f(Q)$ gravity, which is a modified theory of gravity based on non-metricity, in non-flat FLRW geometry. After presenting the relevant cosmological equations, we performed a detailed dynamical-system analysis in order to

reveal the global features of the evolution, independently of the initial conditions.

Firstly, we performed the analysis keeping the $f(Q)$ function completely arbitrary. As we showed, the cosmological scenario admits a dark-matter dominated point, which is saddle and thus it can be the intermediate state of the Universe, as well as dark-energy dominated de Sitter solution which is stable and thus it can attract the Universe at late times. However, the main result of the present work is that there are additional critical points and curves of critical points which exist solely due to curvature

In particular, we found that there are points which are curvature-dominated and correspond to accelerating expansion, and the fact that they are unstable makes them good candidates for the description of inflation. Additionally, there is a point in which the dark-matter and dark-energy density parameters are both between zero and one, and thus it can alleviate the coincidence problem. Finally, there is a saddle point which is completely dominated by curvature.

In order to provide a specific example, we applied our general analysis to the power-law case $f(Q) = \eta Q^n$. In this specific model, the Universe exhibits the general features presented above, namely a saddle matter-dominated point and a late-time dark-energy dominated attractor. Furthermore, it has points that exist only in the non-flat case, which can alleviate the coincidence problem, as well as curvature-

Fig. 4 Upper panel: the phase-space behavior in the specific case of $f(Q) = \eta Q^n$ gravity, in the x_1 -plane, for $k = -1$ and $\epsilon = 1$. The orange curve represents the transition from the curvature-dominated point Q^2 to the matter-dominated point Q^m and then to the curvature-dominated solution Q^k . Lower panels: the corresponding redshift evolution of the density parameters and of the deceleration and equation-of-state parameters. We have set the initial conditions $x_{1i} = -1.5 \times 10^{-7}$, $x_{3i} = -2.1$ and $\Omega_i^k = 0.9$, related to the black dot near Q^2 in the upper panel

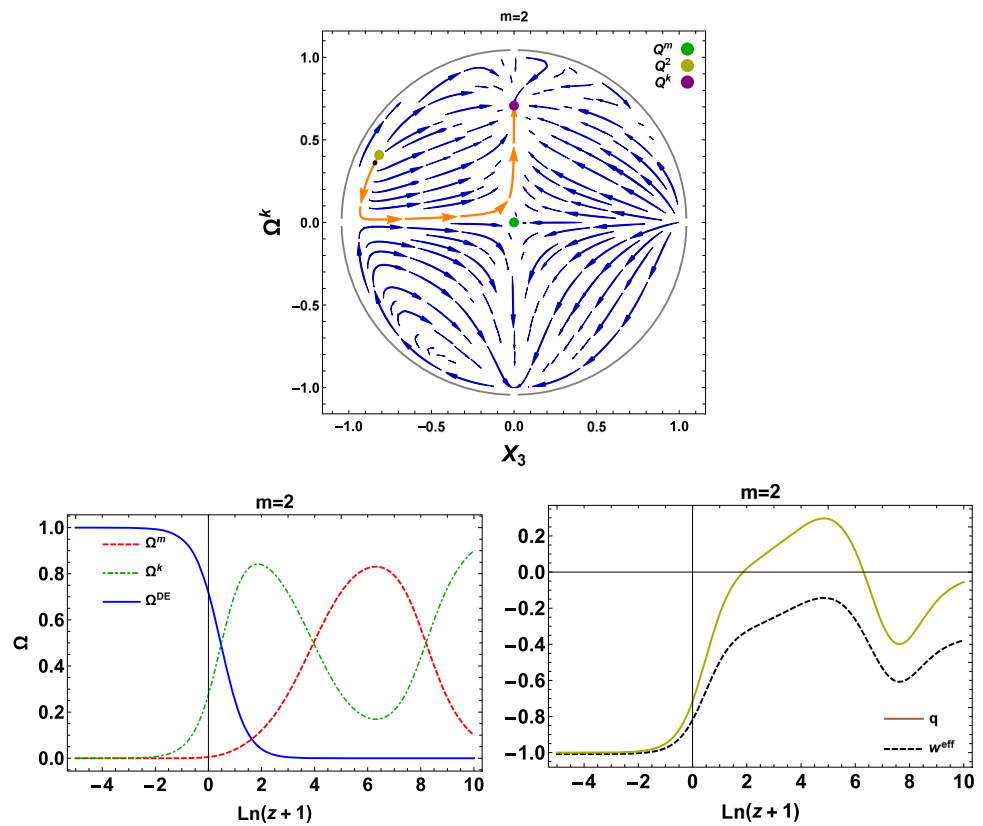
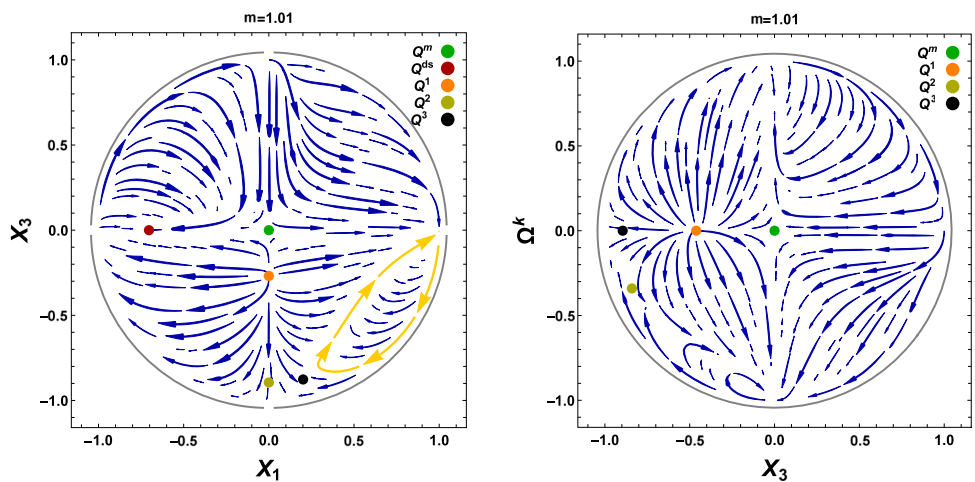


Fig. 5 The phase-space behavior in the specific case of $f(Q) = \eta Q^n$ gravity, for $k = +1$, in the $X_1 - X_3$ plane (left panel) and in the $Q^k - X_3$ plane (right panel). The unstable Q^1 point stands in between Q^m and Q^2/Q^3 and thus it blocks transitions between them, and hence we can only obtain the usual transition from matter to dark-energy dominated phases



dominated accelerating unstable points that can describe the early-time inflationary epoch. In this case we performed a numerical investigation showing that the system in the open geometry case exhibits a transition from the initial accelerated expansion stage, to the intermediate matter-dominated non-accelerating era, and then to the final accelerated expansion phase, while the curvature density parameter exhibits a peak at intermediate times.

In summary, $f(Q)$ cosmology in non-flat Universe exhibits the desired behavior known from the flat case, however it additionally exhibits qualitatively novel features that arise

solely from non-zero curvature. This fact, alongside possible indications that non-zero curvature could alleviate the cosmological tensions, makes it both interesting and necessary to further investigate modified gravity, and in particular $f(Q)$ gravity, in non-flat geometry.

Acknowledgements This research was partially supported by the UTAR Research Fund Scheme. ENS would like to acknowledge the contribution of the COST Action CA21136 ‘‘Addressing observational tensions in cosmology with systematics and fundamental physics (CosmoVerse)’’.

Data availability This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and no experimental data were used.]

Code availability My manuscript has no associated code/software. [Authors' comment: Code/Software sharing not applicable to this article as no code/software was generated or analysed during the current study.]

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>. Funded by SCOAP³.

References

- E.N. Saridakis et al. [CANTATA], (Springer, 2021). [arXiv:2105.12582](https://arxiv.org/abs/2105.12582) [gr-qc]
- S. Capozziello, M. De Laurentis, Phys. Rep. **509**, 167 (2011)
- E. Abdalla, G. Franco Abellán, A. Aboubrahim, A. Agnello, O. Akarsu, Y. Akrami, G. Alestas, D. Aloni, L. Amendola, L. A. Anchordoqui, et al. JHEAp **34**, 49–211 (2022)
- A.A. Starobinsky, Phys. Lett. B **91**, 99 (1980)
- S. Nojiri, S.D. Odintsov, Phys. Lett. B **631**, 1 (2005)
- C. Erices, E. Papantonopoulos, E.N. Saridakis, Phys. Rev. D **99**(12), 123527 (2019)
- D. Lovelock, J. Math. Phys. **12**, 498 (1971)
- G.W. Horndeski, Int. J. Theor. Phys. **10**, 363–384 (1974)
- C. Deffayet, G. Esposito-Farese, A. Vikman, Phys. Rev. D **79**, 084003 (2009)
- G.R. Bengochea, R. Ferraro, Phys. Rev. D **79**, 124019 (2009)
- Y.F. Cai, S. Capozziello, M. De Laurentis, E.N. Saridakis, Rep. Prog. Phys. **79**(10), 106901 (2016)
- G. Kofinas, E.N. Saridakis, Phys. Rev. D **90**, 084044 (2014)
- S. Bahamonde, C.G. Böhmmer, M. Wright, Phys. Rev. D **92**(10), 104042 (2015)
- C.Q. Geng, C.C. Lee, E.N. Saridakis, Y.P. Wu, Phys. Lett. B **704**, 384–387 (2011)
- J.M. Nester, H.J. Yo, Chin. J. Phys. **37**, 113 (1999)
- J. Beltrán Jiménez, L. Heisenberg, T. Koivisto, Phys. Rev. D **98**(4), 044048 (2018)
- F.K. Anagnostopoulos, S. Basilakos, E.N. Saridakis, Phys. Lett. B **822**, 136634 (2021)
- W. Khylllep, A. Paliathanasis, J. Dutta, Phys. Rev. D **103**(10), 103521 (2021)
- S. Mandal, D. Wang, P.K. Sahoo, Phys. Rev. D **102**, 124029 (2020)
- B.J. Barros, T. Barreiro, T. Koivisto, N.J. Nunes, Phys. Dark Univ. **30**, 100616 (2020)
- J. Lu, X. Zhao, G. Chee, Eur. Phys. J. C **79**(6), 530 (2019)
- A. De, L.T. How, Phys. Rev. D **106**(4), 048501 (2022)
- A. De, S. Mandal, J.T. Beh, T.H. Loo, P.K. Sahoo, Eur. Phys. J. C **82**(1), 72 (2022)
- R. Solanki, A. De, S. Mandal, P.K. Sahoo, Phys. Dark Univ. **36**, 101053 (2022)
- R. Solanki, A. De, P.K. Sahoo, Phys. Dark Univ. **36**, 100996 (2022)
- P. Sarmah et al., Phys. Dark Univ. **40**, 101209 (2023)
- A. De, D. Saha, G. Subramaniam, A.K. Sanyal, [arXiv:2209.12120](https://arxiv.org/abs/2209.12120) [gr-qc]
- J.T. Beh, T.H. Loo, A. De, Chin. J. Phys. **77**, 1551–1560 (2022)
- J. Beltrán Jiménez, L. Heisenberg, T. S. Koivisto, S. Pekar, Phys. Rev. D **101**(10), 103507 (2020)
- N. Frusciante, Phys. Rev. D **103**(4), 044021 (2021)
- L. Atayde, N. Frusciante, Phys. Rev. D **104**(6), 064052 (2021)
- J. Ferreira, T. Barreiro, J. Mimoso, N.J. Nunes, Phys. Rev. D **105**(12), 123531 (2022)
- S. Capozziello, R. D'Agostino, Phys. Lett. B **832**, 137229 (2022)
- G.N. Gadbail, S. Mandal, P.K. Sahoo, Phys. Lett. B **835**, 137509 (2022)
- R. Lazkoz, F.S.N. Lobo, M. Ortiz-Baños, V. Salzano, Phys. Rev. D **100**(10), 104027 (2019)
- I. Ayuso, R. Lazkoz, V. Salzano, Phys. Rev. D **103**(6), 063505 (2021)
- F. Bajardi, D. Vernieri, S. Capozziello, Eur. Phys. J. Plus **135**(11), 912 (2020)
- A. Lymperis, JCAP **11**, 018 (2022)
- P. Sahoo, A. De, T.H. Loo, P.K. Sahoo, Commun. Theor. Phys. **74**(12), 125402 (2022)
- F. D'Ambrosio, S.D.B. Fell, L. Heisenberg, S. Kuhn, Phys. Rev. D **105**(2), 024042 (2022)
- M. Li, D. Zhao, Phys. Lett. B **827**, 136968 (2022)
- N. Dimakis, A. Paliathanasis, T. Christodoulakis, Class. Quantum Gravity **38**(22), 225003 (2021)
- M. Hohmann, Phys. Rev. D **104**(12), 124077 (2021)
- A. Kar, S. Sadhukhan, U. Debnath, Mod. Phys. Lett. A **37**(28), 2250183 (2022)
- W. Wang, H. Chen, T. Katsuragawa, Phys. Rev. D **105**(2), 024060 (2022)
- I. Quiros, Phys. Rev. D **105**(10), 104060 (2022)
- S. Mandal, P.K. Sahoo, Phys. Lett. B **823**, 136786 (2021)
- I.S. Albuquerque, N. Frusciante, Phys. Dark Univ. **35**, 100980 (2022)
- G. Papagiannopoulos, S. Basilakos, E.N. Saridakis, Phys. Rev. D **106**(10), 103512 (2022)
- F.K. Anagnostopoulos, V. Gakis, E.N. Saridakis, S. Basilakos, Eur. Phys. J. C **83**(1), 58 (2023)
- S. Arora, P.K. Sahoo, Ann. Phys. **534**(8), 2200233 (2022)
- L. Pati, S.A. Narawade, S.K. Tripathy, B. Mishra, Eur. Phys. J. C **83**(5), 445 (2023)
- S.K. Maurya, K. Newton Singh, S.V. Lohakare, B. Mishra, Fortsch. Phys. **70**(11), 2200061 (2022)
- S. Capozziello, M. Shokri, Phys. Dark Univ. **37**, 101113 (2022)
- N. Dimakis, M. Roumeliotis, A. Paliathanasis, P.S. Apostolopoulos, T. Christodoulakis, Phys. Rev. D **106**(12), 123516 (2022)
- R. D'Agostino, R.C. Nunes, Phys. Rev. D **106**(12), 124053 (2022)
- S.A. Narawade, B. Mishra, Ann. Phys. **535**(5), 2200626 (2023)
- E.D. Emtsova, A.N. Petrov, A.V. Toporensky, Eur. Phys. J. C **83**(5), 366 (2023)
- S. Bahamonde, G. Trenkler, L.G. Trombetta, M. Yamaguchi, Phys. Rev. D **107**(10), 104024 (2023)
- S.A. Narawade, S.H. Shekh, B. Mishra, W. Khylllep, J. Dutta, [arXiv:2303.01985](https://arxiv.org/abs/2303.01985) [gr-qc]
- J. Ferreira, [arXiv:2303.12674](https://arxiv.org/abs/2303.12674) [astro-ph.CO]
- O. Sokoliuk, S. Arora, S. Prharaj, A. Baransky, P.K. Sahoo, Mon. Not. R. Astron. Soc. **522**(1), 252–267 (2023)
- A.Y. Shaikh, Eur. Phys. J. Plus **138**(3), 301 (2023)
- M. Jan, A. Ashraf, A. Basit, A. Caliskan, E. Güdekli, Symmetry **15**(4), 859 (2023)
- N. Dimakis, M. Roumeliotis, A. Paliathanasis, T. Christodoulakis, [arXiv:2304.04419](https://arxiv.org/abs/2304.04419) [gr-qc]

66. M. Koussour, A. De, Eur. Phys. J. C **83**(5), 400 (2023)
67. J.A. Nájera, C.A. Alvarado, C. Escamilla-Rivera, [arXiv:2304.12601](https://arxiv.org/abs/2304.12601) [gr-qc]
68. L. Atayde, N. Frusciante, [arXiv:2306.03015](https://arxiv.org/abs/2306.03015) [astro-ph.CO]
69. L. Järv, M. Rünkla, M. Saal, O. Vilson, Phys. Rev. D **97**(12), 124025 (2018)
70. P. Coles, G.F.R. Ellis, *Is the Universe Open or Closed?* (Cambridge University Press, Cambridge, 1997)
71. W. Yang, W. Giarè, S. Pan, E. Di Valentino, A. Melchiorri, J. Silk, Phys. Rev. D **107**(6), 063509 (2023)
72. S. Chatzidakis, A. Giacomini, P.G.L. Leach, G. Leon, A. Paliathanasis, S. Pan, JHEAp **36**, 141–151 (2022)
73. G. Subramaniam, A. De, T.H. Loo, Y.K. Goh, Phys. Dark Univ. **41**, 101243 (2023)
74. M. Cruz, S. Lepe, Class. Quantum Gravity **35**(15), 155013 (2018)
75. E. Di Valentino, A. Melchiorri, J. Silk, Nat. Astron. **4**(2), 196–203 (2019)
76. S. Vagnozzi, E. Di Valentino, S. Gariazzo, A. Melchiorri, O. Mena, J. Silk, Phys. Dark Univ. **33**, 100851 (2021)
77. S. Vagnozzi, A. Loeb, M. Moresco, Astrophys. J. **908**(1), 84 (2021)
78. S. Dhawan, J. Alsing, S. Vagnozzi, Mon. Not. R. Astron. Soc. **506**(1), L1–L5 (2021)
79. A.A. Asgari, A.H. Abbassi, J. Khodagholizadeh, Eur. Phys. J. C **74**, 2917 (2014)
80. A. Glanville, C. Howlett, T.M. Davis, Mon. Not. R. Astron. Soc. **517**(2), 3087–3100 (2022)
81. S. Bahamonde, K.F. Dialektopoulos, M. Hohmann, J. Levi Said, C. Pfeifer, E.N. Saridakis, Eur. Phys. J. C **83**(3), 193 (2023)
82. E. Di Valentino, A. Melchiorri, J. Silk, Nat. Astron. **4**(2), 196–203 (2019)
83. D. Zhao, Eur. Phys. J. C **82**, 303 (2022)
84. F. D'Ambrosio, L. Heisenberg, S. Kuhn, Class. Quantum Gravity **39**(2), 025013 (2022)
85. N. Dimakis, A. Paliathanasis, M. Roumeliotis, T. Christodoulakis, Phys. Rev. D **106**, 043509 (2022)
86. N. Dimakis, M. Roumeliotis, A. Paliathanasis, P.S. Apostolopoulos, T. Christodoulakis, Phys. Rev. D **106**(12), 123516 (2022)
87. A. De, T.H. Loo, Class. Quantum Gravity **40**(11), 115007 (2023)
88. A. Paliathanasis, Phys. Dark Univ. **41**, 101255 (2023)
89. A.A. Coley, *Dynamical Systems and Cosmology* (Kluwer, 2003)
90. S. Bahamonde, C.G. Böhm, S. Carloni, E.J. Copeland, W. Fang, N. Tamanini, Phys. Rep. **775–777**, 1–122 (2018)
91. E.J. Copeland, A.R. Liddle, D. Wands, Phys. Rev. D **57**, 4686–4690 (1998)
92. M.R. Setare, E.N. Saridakis, Phys. Rev. D **79**, 043005 (2009)
93. T. Matos, J.R. Luevano, I. Quiros, L.A. Urena-Lopez, J.A. Vazquez, Phys. Rev. D **80**, 123521 (2009)
94. E.J. Copeland, S. Mizuno, M. Shaeri, Phys. Rev. D **79**, 103515 (2009)
95. G. Leon, J. Saavedra, E.N. Saridakis, Class. Quantum Gravity **30**, 135001 (2013)
96. M.A. Skugoreva, E.N. Saridakis, A.V. Toporensky, Phys. Rev. D **91**, 044023 (2015)
97. A. Hernández-Almada, G. Leon, J. Magaña, M.A. García-Aspeitia, V. Motta, E.N. Saridakis, K. Yesmakhanova, A.D. Millano, Mon. Not. R. Astron. Soc. **512**(4), 5122–5134 (2022)
98. S.A. Narawade, L. Pati, B. Mishra, S.K. Tripathy, Phys. Dark Univ. **36**, 101020 (2022)
99. W. Khylllep, J. Dutta, E.N. Saridakis, K. Yesmakhanova, Phys. Rev. D **107**(4), 044022 (2023)
100. C.G. Boehmer, E. Jensko, R. Lazkoz, Universe **9**(4), 166 (2023)
101. S.A. Narawade, S.P. Singh, B. Mishra, [arXiv:2303.06427](https://arxiv.org/abs/2303.06427) [gr-qc]
102. H. Shabani, A. De, T.H. Loo, Eur. Phys. J. C **83**, 535 (2023)