

Boundary correlators in WZW model on AdS_2

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ABSTRACT: Boundary correlators of elementary fields in some 2d conformal field theories defined on AdS_2 have a particularly simple structure. For example, the correlators of the Liouville scalar happen to be the same as the correlators of the chiral component of the stress tensor on a plane restricted to the real line. Here we show that an analogous relation is true also in the WZW model: boundary correlators of the WZW scalars have the same structure as the correlators of chiral Kac-Moody currents. This is checked at the level of the tree and one-loop Witten diagrams in AdS_2 . We also compute some tree-level correlators in a generic σ -model defined on AdS_2 and show that they simplify only in the WZW case where an extra Kac-Moody symmetry appears. In particular, the terms in 4-point correlators having logarithmic dependence on 1d cross-ratio cancel only at the WZW point. One motivation behind this work is to learn how to compute AdS_2 loop corrections in 2d models with derivative interactions related to the study of correlators of operators on Wilson loops in string theory in AdS.

KEYWORDS: AdS-CFT Correspondence, Conformal Field Models in String Theory

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1 Introduction

Study of σ -models in AdS_2 is of interest for several reasons (see, e.g., [1–3]). Here we will consider correlators of elementary σ -model fields in Euclidean AdS_2 with Poincare metric $ds^2 = \frac{1}{z^2}(dt^2 + dz^2)$. While in flat space the scattering amplitudes of massless scalar fields in perturbative vacuum are ambiguous due to IR divergences (see, e.g., [4]) the coordinate-space boundary correlators in AdS_2 are well-defined and are constrained by 1d conformal invariance. One interesting question is how the structure of these correlators is further restricted by hidden symmetries of the σ -model and how to compute AdS_2 loop corrections in a way consistent with these underlying symmetries.

Since a classical σ -model in curved 2d space is Weyl-invariant (with the scalar field not transforming), defined on AdS_2 it is formally the same as on a half-plane $ds^2 = dt^2 + dz^2$, $z > 0$. This is true also for models that are conformally invariant at quantum level, where Weyl anomaly is decoupled from σ -model correlators. Compared to a generic boundary CFT set-up here we are interested in (i) the standard AdS_2 (or Dirichlet) boundary conditions $\varphi(\mathbf{t}, z)|_{z \rightarrow 0} = z^\Delta \Phi(\mathbf{t}) + \dots$ for an elementary field with mass $m^2 = \Delta(\Delta - 1)$; (ii) correlators of elementary fields φ rather than composite operators with good 2d conformal transformation properties. The 1d boundary operators dual to the massless σ -model fields with Dirichlet b.c. will thus have $\Delta = 1$. In contrast to the Liouville theory case discussed in [5–7] it turns out that the classical 2d conformal invariance of the bulk σ -model theory does not sufficiently constrain the structure of the tree-level boundary correlators. For example, the tree-level boundary four-point functions are still non-trivial log functions of 1d cross-ratio. An important difference is that while the σ -model field is a scalar on which conformal symmetry acts trivially, the Liouville field transforms non-trivially under the conformal transformations.¹ Similar tree-level correlators (containing logs) were found also for the fields of the Nambu action in AdS_2 [1, 2], but they appear already in the case of 2-derivative σ -model vertices.

To study the role of additional σ -model symmetries here we will consider the example of the WZW model [13, 14] which has an infinite-dimensional Kac-Moody (KM) symmetry $g' = u(w)g v(\bar{w})$, $w = \mathbf{t} + iz$. It appears for the special value of the ratio of the coefficients of the principal chiral model (PCM) and WZ terms in the action when the classical equations of motion admit a chiral decomposition (the resulting model is then conformal and KM invariant also at the quantum level). Like the Virasoro symmetry in the Liouville case here the KM symmetry will impose rigid constraints on the AdS_2 boundary correlators of the elementary fields φ_a parametrizing g . In particular, the KM symmetry rules out the presence of log terms in the four-point correlators, both at the tree and the quantum level.

As we will argue below, the AdS_2 boundary correlators of the massless fields φ_a defined in the standard way as

$$\langle \Phi_{a_1}(\mathbf{t}_1) \cdots \Phi_{a_n}(\mathbf{t}_n) \rangle \equiv \lim_{z_i \rightarrow 0} \prod_{i=1}^n z_i^{-\Delta} \langle \varphi_{a_1}(\mathbf{t}_1, z_1) \cdots \varphi_{a_n}(\mathbf{t}_n, z_n) \rangle_{\text{AdS}_2}, \quad \Delta = 1, \quad (1.1)$$

¹The theories in flat space and in AdS_2 correspond to different vacua [5, 8, 9]. The Liouville field in AdS_2 has a constant vacuum with the fluctuation field with $m^2 = 2$ and thus $\Delta = 2$. Its boundary correlators are constrained by 1d Virasoro symmetry and thus are exactly the same as the 2d stress-tensor correlators restricted to the boundary [6, 7]. This generalizes also to the Toda theory (see also [10–12]).

are constrained by the underlying KM symmetry so that they are equal, up to a universal prefactor, to the correlators of the chiral component of the WZW current $J^a(w) \sim \text{tr}(t^a \partial_w g g^{-1})$, $w = \mathbf{t} + iz$, restricted to the boundary. This is formally equivalent to the “identification” of the boundary operator associated to φ_a with the chiral component of the current $J_a(w \rightarrow \mathbf{t})$

$$\Phi_a(\mathbf{t}) \rightarrow \kappa J_a(w)|_{z \rightarrow 0}, \quad \kappa = \sqrt{\frac{2}{k}}. \tag{1.2}$$

Here k is the WZW level. For comparison, in the Liouville theory case the role of the $\Delta = 1$ current $J \equiv J_w$ (the generator of KM symmetry) is played by the $\Delta = 2$ chiral stress tensor $T \equiv T_{ww}$ (the generator of the Virasoro symmetry)² and the proportionality coefficient was $\kappa = -4\sqrt{\frac{c-1}{6c^2}}$ where $c = 1 + 6(b^{-1} + b)^2$ is the Liouville central charge [7].³ In the WZW case the KM symmetry implies the Virasoro symmetry but is much stronger: as already mentioned above, the boundary correlators in conformal σ -models that do not have an extra KM symmetry have much more complicated structure.⁴

Explicitly, the standard OPE relation for the chiral components of the KM current (see, e.g., [19])

$$J^a(w)J^b(w') \sim \frac{k\delta^{ab}}{(w-w')^2} + \frac{f^{abc}J^c(w)}{w-w'} + \dots, \tag{1.3}$$

determines all higher current correlators to be given by

$$\langle J^{a_1}(w_1) J^{a_2}(w_2) \rangle = \frac{k \delta^{a_1 a_2}}{(w_1 - w_2)^2}, \tag{1.4}$$

$$\langle J^{a_1}(w_1) J^{a_2}(w_2) J^{a_3}(w_3) \rangle = \frac{k f^{a_1 a_2 a_3}}{w_{12} w_{13} w_{23}}, \tag{1.5}$$

$$\langle J^{a_1}(w_1) J^{a_2}(w_2) J^{a_3}(w_3) J^{a_4}(w_4) \rangle = \frac{k^2 \delta^{a_1 a_2} \delta^{a_3 a_4}}{w_{12}^2 w_{34}^2} + \frac{k f^{a_1 a_2 b} f^{a_3 a_4 b}}{w_{12} w_{34} w_{23} w_{24}} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4). \tag{1.6}$$

Below we will explicitly reproduce (1.4), (1.5), (1.6) with $w_i \rightarrow \mathbf{t}_i$ as the expressions for the boundary correlators of the WZW fields (1.1) computed in the $1/k$ perturbation theory in AdS_2 with the identification (1.2).

A semiclassical argument of why the boundary correlators of φ_a are related to the restriction of the current correlators to the boundary of half-plane can be given as follows

²In the Liouville (or Toda) case the Virasoro symmetry becomes realized as a reparametrizations of the boundary and thus completely fixes the structure of the correlators modulo overall powers of the coordinate-independent factor κ .

³This close analogy may not be accidental given that the Liouville theory may be obtained by a Hamiltonian reduction from the $\text{SL}(2)$ WZW model [15, 16].

⁴The key point is that the elementary σ -model field transforms non-trivially under the KM symmetry (like the Liouville field was transforming under the conformal symmetry). Note also that the simplification of the form of boundary correlators in the case of KM symmetry is analogous to what happens in the AdS/CFT examples when the bulk theory has higher symmetry thus constraining also the correlators of the dual boundary CFT. An example is provided by the vectorial AdS/CFT where the symmetry in question is a higher spin symmetry [17, 18].

(for a similar though more involved argument in the Liouville theory case see [7]). Starting with the expression $J^a \sim \text{tr}(t^a \partial_w g g^{-1}) \rightarrow \partial_w \varphi^a + \mathcal{O}(\varphi^2)$ (where $\partial_w = \frac{1}{2}(\partial_z - i\partial_t)$) and using the boundary condition $\varphi_a(\mathbf{z}, \mathbf{t})|_{\mathbf{z} \rightarrow 0} \rightarrow \mathbf{z} \Phi_a(\mathbf{t}) + \dots$ we find that (up to an overall normalization constant) $J^a|_{\mathbf{z} \rightarrow 0} \rightarrow \Phi^a$.

To demonstrate the correspondence (1.2) we shall start in section 2 with the example of the $SL(2, \mathbb{R})$ WZW model on AdS_2 and compute boundary correlators of its fields in the leading tree-level approximation. We shall also consider the corresponding PCM_q theory (i.e. the PCM with a WZ term with coefficient $\propto q$), and show that the four-point correlators simplify (with logs of coordinates cancelling out) and thus can be matched with the correlators of the chiral currents only at the WZW point ($q^2 = 1$) when the model has an extra KM symmetry. In section 3 we shall repeat the computation of the tree boundary correlators for a generic σ -model including the case of PCM_q for an arbitrary group G .

In section 4 we shall test the relation between the boundary correlators of the WZW fields and the chiral currents (1.2) beyond the classical (large k) limit by computing the one-loop corrections to the two-point and three-point boundary correlators. Like in similar computations in the Liouville and Toda theories in AdS_2 [7, 11, 12] this requires an explicit evaluation of loop integrals in AdS_2 which is subtle in the present case of the σ -model theory with two derivatives in the vertices. We shall argue that there exists a particular computational scheme in which the one-loop terms in the WZW field boundary correlators vanish, implying that the proportionality coefficient κ in (1.2) does not receive $1/k$ correction and thus its expression in (1.2) is expected to be exact.

It is interesting to note that while in flat space the scattering amplitudes for the massless WZW fields vanish [4, 20] their coordinate-space boundary correlators in AdS_2 are non-vanishing. Their structure, however, is simple being dictated by the KM symmetry. One may wonder if with some natural definition of the S-matrix in AdS they may actually correspond to trivial scattering in AdS_2 or on half-plane. We will address this question in section 5. There is a close analogy with what happens in the Liouville theory [5] where the full quantum S-matrix was argued to be trivial [21, 22]. We shall discuss the idea of defining AdS_2 scattering amplitudes by Fourier transform of boundary correlators or using the prescription of [5] (cf. [23]) and argue that this leads to trivial three-point scattering amplitudes also in the present WZW case.

Some concluding remarks will be made in section 6. Appendix A will list our notation and conventions. In appendix B we shall discuss a constraint imposed by global symmetry on boundary two-point functions in the $SL(2, \mathbb{R})$ WZW model. In appendix C we shall revisit the computation of the one-loop corrections to the two-point boundary correlators in $SL(2, \mathbb{R})$ WZW model using an alternative form of the action and emphasizing some subtle scheme-dependence issues.

2 Boundary correlators in $SL(2, \mathbb{R})$ WZW model on AdS_2

To demonstrate the correspondence (1.2) we shall first consider the example of the $SL(2, \mathbb{R})$ WZW model and compute its boundary correlators on AdS_2 in the leading-order (tree) approximation. It is useful to view this WZW model as a special case of the PCM_q , i.e.

the principal chiral model with an additional WZ term. This allows one to investigate the consequences of the Kac-Moody symmetry appearing at the WZW point for the structure of the boundary correlators.

2.1 Action

The action for the PCM_q may be written as

$$S = \frac{1}{2\pi\lambda^2} \left[-\frac{1}{2} \int_{\Sigma} d^2x \text{Tr}(g^{-1}\partial_{\mu}g g^{-1}\partial^{\mu}g) + \frac{i}{3}q \int_{\mathbf{B}^3} \text{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg) \right], \quad q = \frac{1}{2}k\lambda^2, \quad (2.1)$$

where k is the coefficient of the WZ term, Σ is a Riemann surface and \mathbf{B}^3 is the 3d extension of Σ such that $\partial\mathbf{B}^3 = \Sigma$. When

$$\lambda = \sqrt{\frac{2}{|k|}}, \quad \text{i.e.} \quad q = \text{sign } k = \pm 1, \quad (2.2)$$

the action (2.1) reduces to the WZW model action.

Assuming $k > 0$, a generic $\text{SL}(2, \mathbb{R})$ group element may be represented in the Gauss decomposition form (see, e.g., [24])

$$g(x) = \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{\lambda}{\sqrt{2}}\phi} & 0 \\ 0 & e^{\frac{\lambda}{\sqrt{2}}\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\lambda^2}{2}\tilde{\psi} & 1 \end{pmatrix}. \quad (2.3)$$

Then the action (2.1) (written on generic curved 2-space with metric \mathbf{g}) becomes

$$S = \frac{1}{4\pi} \int d^2x \sqrt{\mathbf{g}} \left[\mathbf{g}^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\phi + e^{b\phi} (\mathbf{g}^{\mu\nu} + iq\epsilon^{\mu\nu}) \partial_{\mu}\psi \partial_{\nu}\tilde{\psi} \right], \quad b \equiv \sqrt{2}\lambda. \quad (2.4)$$

Here $\epsilon^{\mu\nu} = \frac{\epsilon^{\mu\nu}}{\sqrt{\mathbf{g}}}$ is the standard antisymmetric tensor. This action may be interpreted as that of a σ -model with AdS_3 target space and particular B -field coupling.

Specializing to the $q = 1$ WZW point and the Euclidean AdS_2 background (see appendix A for our notation and conventions)

$$ds^2 = \frac{dt^2 + dz^2}{z^2} = -4 \frac{dw d\bar{w}}{(w - \bar{w})^2}, \quad w = \mathbf{t} + iz, \quad \bar{w} = \mathbf{t} - iz, \quad z > 0, \quad (2.5)$$

we get the following expression for the corresponding $\text{SL}(2, \mathbb{R})$ WZW action

$$S = \int d^2w (\partial\phi\bar{\partial}\phi + e^{b\phi}\partial\psi\bar{\partial}\tilde{\psi}) = \int d^2w \left(\partial\phi\bar{\partial}\phi + \partial\psi\bar{\partial}\tilde{\psi} + b\phi\partial\psi\bar{\partial}\tilde{\psi} + \frac{1}{4}b^2\phi^2\partial\psi\bar{\partial}\tilde{\psi} + \dots \right), \quad b = \frac{2}{\sqrt{k}}. \quad (2.6)$$

As the conformal factor of the metric decouples, this is formally the same as the WZW action on a flat half-plane $z > 0$. However, it will be useful to phrase the computation of the boundary correlators in the AdS_2 language.

We shall assume that the massless fields $\phi, \psi, \tilde{\psi}$ are subject to the standard (Dirichlet) boundary conditions and thus they should be dual to the boundary operators with dimension $\Delta = 1$, i.e. the asymptotic expansion of these fields near $z = 0$ is

$$z \rightarrow 0 : \quad \phi(\mathbf{t}, z) = z\Phi(\mathbf{t}) + \dots, \quad \psi(\mathbf{t}, z) = z\Psi(\mathbf{t}) + \dots, \quad \tilde{\psi}(\mathbf{t}, z) = z\tilde{\Psi}(\mathbf{t}) + \dots. \quad (2.7)$$

Our aim will be to compute the tree level boundary correlation functions (1.1) for the fields in (2.6) and then match them with the correlators of KM currents.

2.2 Propagators

The bulk-to-bulk propagator of a massless scalar in AdS₂ with a standard normalization $\frac{1}{2} \int d^2w \partial^\mu \phi \partial_\mu \phi$ is

$$G_{\Delta=1}(\eta) = -\frac{1}{4\pi} \log \eta, \quad (2.8)$$

where the geodesic distance is defined as

$$\eta = \frac{u}{u+2}, \quad u = \frac{(\mathbf{t} - \mathbf{t}')^2 + (\mathbf{z} - \mathbf{z}')^2}{2\mathbf{z}\mathbf{z}'}. \quad (2.9)$$

Hence, for the field ϕ in (2.6) we have (cf. (A.7))⁵

$$g_{\phi\phi}(w, w') = \langle \phi(w) \phi(w') \rangle = \begin{array}{c} w \quad \quad w' \\ \bullet \text{---} \bullet \end{array} = 2\pi G_{\Delta=1}(\eta) \equiv g(\eta) = -\frac{1}{2} \log \eta(w, w'). \quad (2.10)$$

The bulk-to-bulk propagator of the pair of fields $\psi, \tilde{\psi}$ is similarly

$$g_{\psi\tilde{\psi}}(w, w') = \begin{array}{c} w \quad \quad w' \\ \bullet \text{---} \bullet \end{array} = \langle \psi(w) \tilde{\psi}(w') \rangle = 2g(w, w') = -\log \eta(w, w'). \quad (2.11)$$

Given the structure of the perturbative (small b or large k) expansion in (2.6), it is useful also to quote the propagators for the differentiated fields

$$\begin{aligned} g_{\partial\psi\bar{\partial}\tilde{\psi}}(w, w') &= \langle \partial\psi(w) \bar{\partial}\tilde{\psi}(w') \rangle = \begin{array}{c} \partial\psi(w) \quad \quad \bar{\partial}\tilde{\psi}(w') \\ \bullet \text{---} \bullet \end{array} \\ &= \partial_w \bar{\partial}_{w'} g_{\psi\tilde{\psi}}(w, w') = \partial_w \left(\frac{1}{\bar{w} - \bar{w}'} - \frac{1}{w - \bar{w}'} \right) = \frac{1}{(w - \bar{w}')^2} + \pi \delta^{(2)}(w - w'), \end{aligned} \quad (2.12)$$

where we used the relations (A.6). The δ -function piece here will be important to account for below.

To compute the boundary correlators, we will also need the bulk-to-boundary propagators

$$\begin{aligned} g_{\phi\phi}^\partial(\mathbf{t}; w') &= \lim_{z \rightarrow 0} \frac{1}{z} g_{\phi\phi}(\mathbf{t}, \mathbf{z}; \mathbf{t}', \mathbf{z}') = \frac{2z'}{(\mathbf{t}' - \mathbf{t})^2 + z'^2} = \frac{-i}{\mathbf{t} - w'} + \frac{i}{\mathbf{t} - \bar{w}'} \equiv g^\partial(\mathbf{t}; w'), \\ g_{\psi\bar{\partial}\tilde{\psi}}^\partial(\mathbf{t}; w') &= \lim_{z \rightarrow 0} \frac{1}{z} \langle \psi(w) \bar{\partial}\tilde{\psi}(w') \rangle = \frac{2i}{(\mathbf{t} - \bar{w}')^2} = \bar{\partial}_{w'} \frac{2i}{(\mathbf{t} - \bar{w}')} = 2 \bar{\partial}' g^\partial(\mathbf{t}; w'), \\ g_{\tilde{\psi}\partial\psi}^\partial(\mathbf{t}; w') &= \lim_{z \rightarrow 0} \frac{1}{z} \langle \tilde{\psi}(w) \partial\psi(w') \rangle = \frac{-2i}{(\mathbf{t} - w')^2} = \partial_{w'} \frac{-2i}{(\mathbf{t} - w')} = 2 \partial' g^\partial(\mathbf{t}; w'). \end{aligned} \quad (2.13)$$

2.3 Tree-level boundary correlation functions

2.3.1 Two- and three- point functions

Considering the boundary-to-boundary case of the propagators (2.10), we get the following two-point functions (using the notation in (1.1))

$$\langle \Phi(\mathbf{t}_1) \Phi(\mathbf{t}_2) \rangle = \frac{2}{\mathbf{t}_{12}^2}, \quad \langle \Psi(\mathbf{t}_1) \tilde{\Psi}(\mathbf{t}_2) \rangle = \frac{4}{\mathbf{t}_{12}^2}, \quad \mathbf{t}_{ij} \equiv \mathbf{t}_i - \mathbf{t}_j. \quad (2.14)$$

These have the same form as the boundary restriction of (1.4).

⁵Note that w, w' in the propagators are labels of the points on half-plane: the propagators may also depend on the anti-holomorphic coordinates \bar{w}, \bar{w}' but we do not indicate this to simplify the notation.

The only non-zero three-point function is $\langle \Phi \Psi \tilde{\Psi} \rangle$, which, at the tree level (leading order in $1/k$), is computed by the Witten diagram⁶

$$(2.15)$$

We have

$$A_3(t_1, t_2, t_3) \equiv \langle \Phi(t_1) \Psi(t_2) \tilde{\Psi}(t_3) \rangle = -b \int d^2w g_{\phi\phi}^\partial(t_1, w) g_{\psi\tilde{\psi}}^\partial(t_2, w) g_{\tilde{\psi}\psi}^\partial(t_3, w). \quad (2.16)$$

Using the propagators in (2.13), we get

$$A_3(t_1, t_2, t_3) = -8b \int_0^\infty dz \int_{-\infty}^\infty dt \frac{z}{\pi [(t-t_1)^2 + z^2] (-t+t_2+iz)^2 (t-t_3+iz)^2}. \quad (2.17)$$

This integral can be done by first computing the residues in the t integration variable. Integrating then over z one finds

$$\langle \Phi(t_1) \Psi(t_2) \tilde{\Psi}(t_3) \rangle = \frac{4ib}{t_{12}t_{23}t_{31}}. \quad (2.18)$$

This has again the same structure as the real-line limit of (1.5).

2.3.2 Four-point functions

We now turn to the four-point functions the computation of which is little more involved. The only non-vanishing cases are the correlators $\langle \Psi^2 \tilde{\Psi}^2 \rangle$ and $\langle \Phi^2 \Psi \tilde{\Psi} \rangle$.

$\langle \Psi^2 \tilde{\Psi}^2 \rangle$. At tree level this correlator is given by the following Witten diagrams

$$(2.19)$$

We can represent the result as

$$\langle \Psi(t_1) \tilde{\Psi}(t_2) \Psi(t_3) \tilde{\Psi}(t_4) \rangle = A_4(t_1, t_2, t_3, t_4) + A_4(t_1, t_4, t_3, t_2), \quad (2.20)$$

⁶Here the dashed circle represents the boundary of AdS_2 and solid lines represent the propagators of the corresponding fields in the bulk.

where

$$A_4(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = (-b)^2 \int d^2w d^2w' g_{\psi\bar{\psi}}^\partial(\mathbf{t}_1, w) g_{\psi\bar{\psi}}^\partial(\mathbf{t}_2; w) g_{\phi\phi}(w, w') g_{\psi\bar{\psi}}^\partial(\mathbf{t}_3, w') g_{\psi\bar{\psi}}^\partial(\mathbf{t}_4; w')$$

$$= 2^4 b^2 \tilde{H}(\mathbf{t}_2, \mathbf{t}_1, \mathbf{t}_4, \mathbf{t}_3), \quad (2.21)$$

$$\tilde{H}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) \equiv \int d^2w d^2w' \partial g^\partial(\mathbf{t}_1, w) \bar{\partial} g^\partial(\mathbf{t}_2; w) g(w, w') \partial' g^\partial(\mathbf{t}_3, w') \bar{\partial}' g^\partial(\mathbf{t}_4; w'). \quad (2.22)$$

To compute this integral we may first integrate by parts at the vertex w ,

$$\tilde{H}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = - \int d^2w d^2w' \frac{-i}{\mathbf{t}_1 - w} \bar{\partial} g^\partial(\mathbf{t}_2; w) \partial g(w, w') \partial' g^\partial(\mathbf{t}_3, w') \bar{\partial}' g^\partial(\mathbf{t}_4; w') \quad (2.23)$$

$$= \int d^2w d^2w' \frac{-iz'}{\pi^2(-\bar{w} + \mathbf{t}_2)^2(w - \mathbf{t}_1)(\mathbf{t}_4 - \bar{w}')^2(-\mathbf{t}_3 + w')^2(w - w')(w - \bar{w}')} ,$$

where we ignored 2-derivative terms assuming

$$\partial' \bar{\partial}' g^\partial(\mathbf{t}; w') = 0. \quad (2.24)$$

Indeed, possible terms with $\delta^{(2)}(\mathbf{t} - w')$ and its derivative may be neglected here as they localize the bulk point to the boundary, and hence give zero contributions after performing the bulk integral.

The integral in (2.23) can be evaluated by applying the residue theorem

$$\tilde{H}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = \frac{\log\left(\frac{\mathbf{t}_{12}\mathbf{t}_{34}}{\mathbf{t}_{14}\mathbf{t}_{23}}\right)^2 + i\pi(\text{sgn } \mathbf{t}_{12} + \text{sgn } \mathbf{t}_{23} + \text{sgn } \mathbf{t}_{34} + \text{sgn } \mathbf{t}_{41})}{4\mathbf{t}_{13}^2\mathbf{t}_{24}^2} - \frac{1}{2\mathbf{t}_{12}\mathbf{t}_{13}\mathbf{t}_{24}\mathbf{t}_{34}}. \quad (2.25)$$

Then from (2.20) we finally obtain

$$\langle \Psi(\mathbf{t}_1) \tilde{\Psi}(\mathbf{t}_2) \Psi(\mathbf{t}_3) \tilde{\Psi}(\mathbf{t}_4) \rangle = 2^4 b^2 \tilde{H}(\mathbf{t}_2, \mathbf{t}_1, \mathbf{t}_4, \mathbf{t}_3) + 2^4 b^2 \tilde{H}(\mathbf{t}_4, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) = \frac{8b^2}{\mathbf{t}_{12}\mathbf{t}_{23}\mathbf{t}_{34}\mathbf{t}_{41}}. \quad (2.26)$$

Remarkably, all logarithmic (and sign function) terms present in (2.25) cancel out in the sum of the two exchange Witten diagrams. This cancellation is crucial in order to be able to match (2.26) with the correlators of the KM currents that are rational functions of the differences of points (cf. (1.6)).

$\langle \Phi^2 \Psi \tilde{\Psi} \rangle$. This correlator is given by the sum of the following three diagrams

$$(2.27)$$

It can be written as

$$\langle \Phi(\mathbf{t}_1) \Phi(\mathbf{t}_2) \Psi(\mathbf{t}_3) \tilde{\Psi}(\mathbf{t}_4) \rangle = B_4(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) + B_4(\mathbf{t}_2, \mathbf{t}_1, \mathbf{t}_3, \mathbf{t}_4) + C_4(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4), \quad (2.28)$$

where the explicit form of B_4 and C_4 is

$$\begin{aligned} B_4(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) &= b^2 \int d^2w d^2w' g_{\phi\phi}^\partial(\mathbf{t}_1, w) g_{\phi\phi}^\partial(\mathbf{t}_2; w') g_{\partial\psi\bar{\partial}\tilde{\psi}}(w', w) g_{\psi\bar{\partial}\tilde{\psi}}^\partial(\mathbf{t}_3, w') g_{\tilde{\psi}\partial\psi}^\partial(\mathbf{t}_4; w) \\ &= B_4^{\text{reg}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) - C_4(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4), \end{aligned} \quad (2.29)$$

$$C_4(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = -b^2 \int d^2w g_{\phi\phi}^\partial(\mathbf{t}_1, w) g_{\phi\phi}^\partial(\mathbf{t}_2; w) g_{\psi\bar{\partial}\tilde{\psi}}^\partial(\mathbf{t}_3, w) g_{\tilde{\psi}\partial\psi}^\partial(\mathbf{t}_4; w). \quad (2.30)$$

Here B_4^{reg} and C_4 are the contributions from the regular and singular parts of the internal propagators in (2.12), respectively (note that in (2.29) the singular δ -function part in the propagator (2.12) turns the exchange diagram into a contact diagram). Using the explicit form of the propagators we get

$$\begin{aligned} B_4^{\text{reg}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) &= \frac{16b^2}{\pi^2} \int d^2w d^2w' \frac{zz'}{(t_4 - t - iz)^2 (t_3 - t' + iz')^2 [(t - t_1)^2 + z^2] [(t' - t_2)^2 + z'^2]} \\ &\quad \times [-t + t' + i(z + z')]^{-2}, \end{aligned} \quad (2.31)$$

$$C_4(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = \frac{-16b^2}{\pi} \int d^2w \frac{z^2}{(t_3 - t + iz)^2 (t_4 - t - iz)^2 [(t_1 - t)^2 + z^2] [(t_2 - t)^2 + z^2]}. \quad (2.32)$$

Using the same method as for the previous four-point function, we obtain

$$B_4^{\text{reg}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = -\frac{2ib^2}{t_{23}^2 t_{14}^2} \left[i \log \left(\frac{t_{12} t_{34}}{t_{13} t_{24}} \right)^2 + \pi (-\text{sgn } t_{12} + \text{sgn } t_{13} - \text{sgn } t_{24} + \text{sgn } t_{34}) \right], \quad (2.33)$$

$$\begin{aligned} C_4(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) &= 2b^2 \left[-i \frac{i \log \left(\frac{t_{12} t_{34}}{t_{14} t_{23}} \right)^2 + \pi \text{sgn}(t_{12}) - \pi \text{sgn}(t_{14}) + \pi \text{sgn}(t_{23}) + \pi \text{sgn}(t_{34})}{t_{13}^2 t_{24}^2} \right. \\ &\quad + i \frac{-i \log \left(\frac{t_{12} t_{34}}{t_{13} t_{24}} \right)^2 + \pi \text{sgn}(t_{12}) - \pi \text{sgn}(t_{13}) + \pi \text{sgn}(t_{24}) - \pi \text{sgn}(t_{34})}{t_{14}^2 t_{23}^2} \\ &\quad \left. + \frac{2}{t_{13} t_{14} t_{23} t_{24}} \right]. \end{aligned} \quad (2.34)$$

Inserting these results into (2.28) gives⁷

$$\langle \Phi(\mathbf{t}_1) \Phi(\mathbf{t}_2) \Psi(\mathbf{t}_3) \tilde{\Psi}(\mathbf{t}_4) \rangle = B_4^{\text{reg}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) + B_4^{\text{reg}}(\mathbf{t}_2, \mathbf{t}_1, \mathbf{t}_3, \mathbf{t}_4) - C_4(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = -\frac{4b^2}{t_{13} t_{14} t_{23} t_{24}}. \quad (2.35)$$

⁷It is easy to verify that B_4 is related to A_4 in (2.20) as $B_4(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = A_4(\mathbf{t}_1, \mathbf{t}_4, \mathbf{t}_3, \mathbf{t}_2)$. This relation can be easily understood using integration by parts. Indeed, integrating by parts the cubic vertex, one can transfer the derivatives acting on the internal leg to the external legs; then the first diagram in (2.27) reduces to the second diagram in (2.19).

As in the case of the correlator $\langle \Psi(t_1) \tilde{\Psi}(t_2) \Psi(t_3) \tilde{\Psi}(t_4) \rangle$ in (2.26), both the logarithms and the sign functions again cancel out.

2.4 Matching AdS₂ boundary correlators with correlators of chiral currents

Let us now compare the above boundary correlators with the correlators of the chiral WZW currents on the plane restricted to the real line. The correlation functions of the currents are a direct consequence of the KM algebra (1.3). Adapted to the SL(2, ℝ) case the OPEs of the three currents (H, J^+, J^-) read (see, e.g., [24])

$$H(w)H(0) \sim \frac{k}{w^2}, \quad H(w)J^\pm(0) \sim \mp \frac{i\sqrt{2}J^\pm(0)}{w}, \quad J^+(w)J^-(0) \sim \frac{2k}{w^2} - \frac{2i\sqrt{2}H(0)}{w}. \tag{2.36}$$

From (2.36) we conclude that: (i) the two-point functions are

$$\langle H(w_1)H(w_2) \rangle = \frac{k}{w_{12}^2}, \quad \langle J^+(w_1)J^-(w_2) \rangle = \frac{2k}{w_{12}^2}, \tag{2.37}$$

(ii) the only non-vanishing three-point function is

$$\langle H(w_1)J^+(w_2)J^-(w_3) \rangle = -2\sqrt{2} \frac{ik}{w_{12}w_{13}w_{23}}, \tag{2.38}$$

and (iii) the non-trivial four-point functions are

$$\begin{aligned} \langle J^+(w_1)J^-(w_2)J^+(w_3)J^-(w_4) \rangle &= 4k^2 \left(\frac{1}{w_{23}^2w_{14}^2} + \frac{1}{w_{12}^2w_{34}^2} \right) - \frac{8k}{w_{12}w_{23}w_{14}w_{34}}, \\ \langle J^+(w_1)J^-(w_2)H(w_3)H(w_4) \rangle &= 2k^2 \frac{1}{w_{12}^2w_{34}^2} - \frac{4k}{w_{13}w_{14}w_{23}w_{24}}. \end{aligned} \tag{2.39}$$

These four-point functions are non-trivial in the sense that in addition to the k^2 contribution they also have a term linear in k (cf. (1.6)).⁸ Comparing to boundary correlators discussed above, the k^2 term is a counterpart with disconnected AdS₂ Witten diagram contribution, while the order k term corresponds to connected exchange and contact contributions due to non-trivial bulk interactions. In fact, it is possible to establish the precise matching between the tree-level 2-point and 3-point boundary correlators in (2.14), (2.18) and the current correlators (2.37), (2.38) restricted to the real line using the following identification (cf. (1.2))

$$\Phi = \kappa H, \quad \Psi = \kappa J^+, \quad \tilde{\Psi} = \kappa J^-, \quad \kappa = \sqrt{\frac{2}{k}} = \frac{b}{\sqrt{2}}. \tag{2.40}$$

A non-trivial consistency check is that the connected 4-point boundary correlators (2.26), (2.36) then also match with the non-trivial order k parts of the 4-current correlators in (2.39). The fact that there is just a single universal proportionality coefficient κ follows from the global group symmetry of the WZW model (this is true not only at the tree level but also to all orders in $1/k$).

⁸For instance, a four-point function which is non-vanishing but trivial in the above sense is

$$\langle H(w_1)H(w_2)H(w_3)H(w_4) \rangle = \frac{k^2}{4} \left(\frac{1}{w_{12}^2w_{34}^2} + \frac{1}{w_{13}^2w_{24}^2} + \frac{1}{w_{14}^2w_{23}^2} \right).$$

2.5 Boundary correlators in PCM_q on AdS_2

Let us now go back to the $\text{SL}(2, \mathbb{R})$ principal chiral model with a general coefficient q of the WZ term in (2.4) to emphasize that its boundary correlators have a complicated structure already at the tree level (containing, in particular, logarithmic terms found also in a similar σ -model context in [1, 2]). In contrast to the Liouville and Toda theories discussed in [6, 7, 10] here the classical conformal symmetry of PCM_q (2.4) is not enough to sufficiently constrain the boundary correlators.⁹ The correlators simplify precisely at the WZW point $q^2 = 1$ and this may be attributed to the emerging KM symmetry (that implies chiral decomposition in flat space).

The action (2.4) in AdS_2 expanded in powers of b reads (cf. (2.6))¹⁰

$$\begin{aligned}
 S &= \int d^2w \left(\partial\phi\bar{\partial}\phi + \frac{1+q}{2} e^{b\phi} \partial\psi\bar{\partial}\tilde{\psi} + \frac{1-q}{2} e^{b\phi} \bar{\partial}\psi\partial\tilde{\psi} \right) \\
 &= \int d^2w \left(\partial\phi\bar{\partial}\phi + \partial\psi\bar{\partial}\tilde{\psi} + \frac{b(1+q)}{2} \phi\partial\psi\bar{\partial}\tilde{\psi} + \frac{b(1-q)}{2} \phi\bar{\partial}\psi\partial\tilde{\psi} \right. \\
 &\quad \left. + \frac{b^2(1+q)}{8} \phi^2\partial\psi\bar{\partial}\tilde{\psi} + \frac{b^2(1-q)}{8} \phi^2\bar{\partial}\psi\partial\tilde{\psi} + \dots \right). \tag{2.41}
 \end{aligned}$$

Repeating the calculation of the tree-level four-point $\langle \Psi^2 \tilde{\Psi}^2 \rangle$ correlator we find (cf. (2.20))

$$\langle \Psi(t_1) \tilde{\Psi}(t_2) \Psi(t_3) \tilde{\Psi}(t_4) \rangle = \mathcal{A}_4(t_1, t_2, t_3, t_4) + \mathcal{A}_4(t_1, t_4, t_3, t_2), \tag{2.42}$$

$$\begin{aligned}
 \mathcal{A}_4(t_1, t_2, t_3, t_4) &= \left(\frac{1+q}{2} \right)^2 A_4(t_1, t_2, t_3, t_4) + \left(\frac{1-q}{2} \right)^2 A_4(t_2, t_1, t_4, t_3) \\
 &\quad + \frac{1-q^2}{4} \left[A_4(t_2, t_1, t_3, t_4) + A_4(t_1, t_2, t_4, t_3) \right], \tag{2.43}
 \end{aligned}$$

where A_4 is given by (2.21), (2.22), (2.25). Similarly, for the $\langle \Phi^2 \Psi \tilde{\Psi} \rangle$ correlator we get

$$\langle \Phi(t_1) \Phi(t_2) \Psi(t_3) \tilde{\Psi}(t_4) \rangle = \mathcal{B}_4(t_1, t_2, t_3, t_4) + \mathcal{B}_4(t_2, t_1, t_3, t_4) + \mathcal{C}_4(t_1, t_2, t_3, t_4), \tag{2.44}$$

$$\mathcal{B}_4(t_1, t_2, t_3, t_4) = \left(\frac{1+q}{2} \right)^2 B_4(t_1, t_2, t_3, t_4) + \left(\frac{1-q}{2} \right)^2 B_4(t_1, t_2, t_4, t_3), \tag{2.45}$$

$$\mathcal{C}_4(t_1, t_2, t_3, t_4) = \frac{1+q}{2} C_4(t_1, t_2, t_3, t_4) + \frac{1-q}{2} C_4(t_1, t_2, t_4, t_3), \tag{2.46}$$

where B_4 and C_4 are given by (2.29), (2.30), (2.33), (2.34). We conclude that these four-point functions contain logarithmic terms. These cancel only at the WZW point $q^2 = 1$ allowing one to relate these boundary correlators to the “connected” part of the correlators of the chiral WZW currents as explained above.

⁹As we remarked in the Introduction, this may be related to the fact that the σ -model fields transform as scalars (i.e. trivially) under the conformal group.

¹⁰While the AdS_2 conformal factor decouples at the tree level, this will no longer be so at the quantum level as this model will have UV divergences and thus conformal anomaly (assuming reparametrization-covariant regularization) unless $q^2 = 1$.

3 Tree-level boundary correlators in generic σ -model on AdS₂

Let us now consider the AdS₂ boundary correlators in a general σ -model expanded near a constant background. This includes, in particular, the case of a WZW model for a general group G . We shall again demonstrate the cancellation of the logarithmic terms in the four-point correlators at the WZW point and match them with the connected part of the correlators of the chiral currents (1.6) restricted to the real line.

3.1 Action

Let us start with a general bosonic σ -model with coupling functions (G_{ab}, B_{ab}) and expand it in normal coordinates near the origin using $G_{ab}(X) = \delta_{ab} - \frac{1}{3}R_{abcd}(0)X^cX^d + \mathcal{O}(X^3)$ and $B_{ab}(X) = B_{ab}(0) + \frac{1}{3}H_{abc}(0)X^c + \mathcal{O}(X^2)$. Then its Euclidean action may be written as¹¹

$$\begin{aligned}
 S &= \frac{1}{4\pi} \int d^2x \sqrt{g} \left[g^{\mu\nu} G_{ab}(X) + i\epsilon^{\mu\nu} B_{ab}(X) \right] \partial_\mu X^a \partial_\nu X^b \\
 &= \frac{1}{4\pi} \int d^2x \sqrt{g} \left[g^{\mu\nu} \left(\delta_{ab} - \frac{1}{3}R_{abcd}(0)X^cX^d + \dots \right) + \frac{i}{3}\epsilon^{\mu\nu} (H_{abc}(0)X^c + \dots) \right] \partial_\mu X^a \partial_\nu X^b.
 \end{aligned}
 \tag{3.1}$$

In what follows we will consider the leading terms in this action parametrized as

$$S = \frac{1}{4\pi} \int d^2x \sqrt{g} \left(\partial_\mu X^a \partial^\mu X^a - P_{abcd} X^a X^c \partial_\mu X^b \partial^\mu X^d + iQ_{abc} \epsilon^{\mu\nu} X^a \partial_\mu X^b \partial_\nu X^c + \dots \right),
 \tag{3.2}$$

where the constant real coupling functions P and Q are given by

$$P_{abcd} \equiv \frac{1}{3}R_{abcd}(0), \quad Q_{abc} \equiv \frac{1}{3}H_{abc}(0).
 \tag{3.3}$$

Thus Q is totally antisymmetric and P has algebraic symmetries of the curvature

$$P_{abcd} = -P_{bacd}, \quad P_{abcd} = -P_{abdc}, \quad P_{abcd} = P_{cdab},
 \tag{3.4}$$

$$P_{abcd} + P_{acdb} + P_{adbc} = 0.
 \tag{3.5}$$

To account for the manifest symmetry of the 4-vertex in (3.2) in (a, c) it is useful to introduce also the corresponding symmetrization of P_{abcd}

$$\tilde{P}_{abcd} \equiv \frac{1}{2}(P_{abcd} + P_{cbad}) = -\frac{1}{2}(P_{bacd} + P_{bcad}) = \frac{1}{2}(P_{badc} + P_{dabc}),
 \tag{3.6}$$

$$\tilde{P}_{abcd} = \tilde{P}_{badc}, \quad \tilde{P}_{abcd} = \tilde{P}_{cbad}, \quad \tilde{P}_{abcd} = \tilde{P}_{adcb}.
 \tag{3.7}$$

Then specifying to the AdS₂ background the action (3.2) may be written as (cf. (2.6))

$$S = \int d^2w \left(\partial X^a \bar{\partial} X^a - \tilde{P}_{abcd} X^a X^c \partial X^b \bar{\partial} X^d + Q_{abc} X^a \partial X^b \bar{\partial} X^c + \dots \right).
 \tag{3.8}$$

¹¹We ignore the overall coupling factor or $\frac{1}{\alpha'}$ that can be absorbed into a rescaling of X^a and then appears in R and H .

The action (3.2) represents as a particular case the expansion of the PCM_q (2.1) for an arbitrary group G . Let us normalize the generators $\{t_a\}$ and the invariant bilinear form of the Lie algebra of G as¹²

$$[t_a, t_b] = f_{ab}{}^c t_c, \quad \text{tr}(t_a t_b) = \delta_{ab}. \quad (3.9)$$

Then choosing the parametrization of the group field as¹³ $g = e^{-i\lambda t_a X^a}$ we find that in the PCM_q case

$$\text{PCM}_q: \quad P_{abcd} = \mathfrak{p} f_{abe} f_{cde}, \quad Q_{abc} = \mathfrak{q} f_{abc}, \quad \mathfrak{p} = -\frac{1}{12} \lambda^2, \quad \mathfrak{q} = -\frac{1}{3} i \lambda q. \quad (3.10)$$

The WZW theory corresponds to the choice (2.2), i.e. $\mathfrak{q} = -\text{sign}(k) \frac{1}{3} i \lambda$, $\lambda = \sqrt{\frac{2}{|k|}}$. In what follows we shall assume that $k > 0$.

3.2 Tree-level AdS_2 boundary correlation functions

The fields X^a in (3.8) are massless and thus, assuming the Dirichlet boundary conditions, we have (cf. (2.7))

$$X^a(\mathbf{t}, \mathbf{z})|_{\mathbf{z} \rightarrow 0} = \mathbf{z} X^a(\mathbf{t}) + \dots \quad (3.11)$$

They should correspond to the boundary operators with dimension $\Delta = 1$. As in (2.10), their bulk-to-bulk AdS_2 propagator is given by

$$g_{ab}(w, w') = \langle X_a(w) X_b(w') \rangle = \delta_{ab} g(\eta) = -\frac{1}{2} \delta_{ab} \log \eta(w, w'), \quad (3.12)$$

while the bulk-to-boundary propagator is

$$g_{ab}^\partial(\mathbf{t}; w') = \lim_{\mathbf{z} \rightarrow 0} \frac{1}{\mathbf{z}} g_{ab}(\mathbf{t}, \mathbf{z}; \mathbf{t}', \mathbf{z}') = \delta_{ab} \frac{2\mathbf{z}'}{(\mathbf{t}' - \mathbf{t})^2 + \mathbf{z}'^2} = \delta_{ab} g^\partial(\mathbf{t}, w'). \quad (3.13)$$

Then the boundary two-point function is (cf. (2.14))

$$\langle X_a(\mathbf{t}_1) X_b(\mathbf{t}_2) \rangle = \frac{2\delta_{ab}}{\mathbf{t}_{12}^2}. \quad (3.14)$$

Starting with (3.8) it is straightforward also to compute the three-point function

$$\langle X_a(\mathbf{t}_1) X_b(\mathbf{t}_2) X_c(\mathbf{t}_3) \rangle = \begin{array}{c} X_a(\mathbf{t}_1) \\ \bullet \\ | \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ X_b(\mathbf{t}_2) \quad X_c(\mathbf{t}_3) \end{array} \quad = \frac{6iQ_{abc}}{\mathbf{t}_{12}\mathbf{t}_{23}\mathbf{t}_{13}}. \quad (3.15)$$

The connected tree-level four-point function receives contributions from both exchange diagrams and contact diagrams

$$\langle X_a(\mathbf{t}_1) X_b(\mathbf{t}_2) X_c(\mathbf{t}_3) X_d(\mathbf{t}_4) \rangle = G_{abcd}^{\text{exch}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) + G_{abcd}^{\text{cont}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4). \quad (3.16)$$

¹²We assume that t_a are Hermitian and thus the structure constants $f_{ab}{}^c$ are purely imaginary. The group indices are raised or lowered by δ_{ab} , implying that $f_{abc} = f_{ab}{}^c$ is fully anti-symmetric. Repeated group indices are summed over, regardless of their positions.

¹³Notice that this parametrization is different from (2.3) used in the $\text{SL}(2, \mathbb{R})$ case.

Exchange diagrams. The exchange part contains contributions of the three different channels

$$G_{abcd}^{\text{exch}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = G_{abcd}^{\text{exch}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) + G_{acbd}^{\text{exch}}(\mathbf{t}_1, \mathbf{t}_3, \mathbf{t}_2, \mathbf{t}_4) + G_{adcb}^{\text{exch}}(\mathbf{t}_1, \mathbf{t}_4, \mathbf{t}_3, \mathbf{t}_2). \quad (3.17)$$

As these are related by permutations (crossing), we only need to compute one of them

$$G_{abcd}^{\text{exch}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = Q_{abe}Q_{cde} \cdot \text{Diagram} \quad (3.18)$$

Note that using integration by parts, one can always arrange so that the derivatives in the cubic vertex in (3.8) act only on the two external legs.¹⁴ Then the 6 terms in the cubic vertex can be written as

$$K(\mathbf{t}_1, \mathbf{t}_2, w) = 3\partial g^\partial(\mathbf{t}_1, w)\bar{\partial}g^\partial(\mathbf{t}_2, w) - 3\bar{\partial}g^\partial(\mathbf{t}_1, w)\partial g^\partial(\mathbf{t}_2, w), \quad (3.19)$$

where the two terms arise from the two ways of acting by derivative on the external legs and the factor of 3 comes from rearranging other similar cubic terms. Using (3.19), we find for the exchange diagram

$$G_{abcd}^{\text{exch}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = Q_{abe}Q_{cde} \int d^2w d^2w' K(\mathbf{t}_1, \mathbf{t}_2, w) g(w, w') K(\mathbf{t}_3, \mathbf{t}_4, w) \\ = 9Q_{abe}Q_{cde}H(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4), \quad (3.20)$$

$$H(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = \tilde{H}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) - \tilde{H}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_4, \mathbf{t}_3) - \tilde{H}(\mathbf{t}_2, \mathbf{t}_1, \mathbf{t}_3, \mathbf{t}_4) + \tilde{H}(\mathbf{t}_2, \mathbf{t}_1, \mathbf{t}_4, \mathbf{t}_3). \quad (3.21)$$

Here \tilde{H} was defined in (2.22) and computed in (2.25).

Contact diagrams. Since the quartic vertex in (3.8) contains derivatives, the contact contribution may also be represented as a sum of the three contributions

$$G_{abcd}^{\text{cont}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = G_{abcd}^{\text{cont}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) + G_{acbd}^{\text{cont}}(\mathbf{t}_1, \mathbf{t}_3, \mathbf{t}_2, \mathbf{t}_4) + G_{abd c}^{\text{cont}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_4, \mathbf{t}_3), \quad (3.22)$$

$$G_{abcd}^{\text{cont}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = 2\tilde{P}_{badc} \cdot \text{Diagram 1} + 2\tilde{P}_{bcda} \cdot \text{Diagram 2} \\ + 2\tilde{P}_{adcb} \cdot \text{Diagram 3} + 2\tilde{P}_{abcd} \cdot \text{Diagram 4} \quad (3.23)$$

¹⁴Note that $X^a(\partial X^b \bar{\partial} X^c - \partial X^c \bar{\partial} X^b)$ is fully anti-symmetric in a, b, c up to a total derivative.

Here we have indicated explicitly the coupling tensors appearing from each diagram (the factor of 2 arises from two ways of contracting the two legs without derivative). Explicitly, we get

$$\begin{aligned}
 G_{abcd}^{\text{cont}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) &= 2\tilde{P}_{abcd} \left[\int d^2w \partial g(\mathbf{t}_1, w) g(\mathbf{t}_2, w) \bar{\partial} g(\mathbf{t}_3, w) g(\mathbf{t}_4, w) + (\mathbf{t}_1 \leftrightarrow \mathbf{t}_3) \right] \\
 &\quad + 2\tilde{P}_{abcd} \left[\int d^2w g(\mathbf{t}_1, w) \partial g(\mathbf{t}_2, w) g(\mathbf{t}_3, w) \bar{\partial} g(\mathbf{t}_4, w) + (\mathbf{t}_2 \leftrightarrow \mathbf{t}_4) \right],
 \end{aligned}
 \tag{3.24}$$

where we used (3.7). As a result,

$$G_{abcd}^{\text{cont}}(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = 4\tilde{P}_{abcd} l(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4),
 \tag{3.25}$$

where

$$\begin{aligned}
 l(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) &\equiv \int d^2w \partial g(\mathbf{t}_1, w) g(\mathbf{t}_2, w) \bar{\partial} g(\mathbf{t}_3, w) g(\mathbf{t}_4, w) + (\mathbf{t}_1 \leftrightarrow \mathbf{t}_3) \\
 &= \int d^2w g(\mathbf{t}_1, w) \partial g(\mathbf{t}_2, w) g(\mathbf{t}_3, w) \bar{\partial} g(\mathbf{t}_4, w) + (\mathbf{t}_2 \leftrightarrow \mathbf{t}_4) \\
 &= \frac{1}{\mathbf{t}_{14}^2 \mathbf{t}_{23}^2} \log \left(\frac{\mathbf{t}_{12} \mathbf{t}_{34}}{\mathbf{t}_{13} \mathbf{t}_{24}} \right)^2 + \frac{1}{\mathbf{t}_{12}^2 \mathbf{t}_{34}^2} \log \left(\frac{\mathbf{t}_{14} \mathbf{t}_{23}}{\mathbf{t}_{13} \mathbf{t}_{24}} \right)^2 + \frac{2}{\mathbf{t}_{12} \mathbf{t}_{14} \mathbf{t}_{23} \mathbf{t}_{34}}.
 \end{aligned}
 \tag{3.26}$$

Collecting the contributions in (3.17), (3.20) and (3.25), (3.26), the four-point function in (3.16) may be written as

$$\begin{aligned}
 \langle \mathcal{X}_a(\mathbf{t}_1) \mathcal{X}_b(\mathbf{t}_2) \mathcal{X}_c(\mathbf{t}_3) \mathcal{X}_d(\mathbf{t}_4) \rangle &= 4\alpha_s l(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) + 4\alpha_t l(\mathbf{t}_1, \mathbf{t}_3, \mathbf{t}_2, \mathbf{t}_4) + 4\alpha_u l(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_4, \mathbf{t}_3) \\
 &\quad + 9\beta_s H(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) + 9\beta_t H(\mathbf{t}_1, \mathbf{t}_3, \mathbf{t}_2, \mathbf{t}_4) + 9\beta_u H(\mathbf{t}_1, \mathbf{t}_4, \mathbf{t}_3, \mathbf{t}_2),
 \end{aligned}
 \tag{3.27}$$

where in the r.h.s. we suppressed the indices (a, b, c, d) introducing the symbolic notation (s, t, u) stand for different channels)¹⁵

$$\alpha_s = \tilde{P}_{abcd}, \quad \alpha_t = \tilde{P}_{acbd}, \quad \alpha_u = \tilde{P}_{abdc},
 \tag{3.28}$$

$$\beta_s = Q_{abe} Q_{cde}, \quad \beta_t = Q_{ace} Q_{bde}, \quad \beta_u = Q_{ade} Q_{cbe}.
 \tag{3.29}$$

Using the expressions for the integrals H in (3.21), (2.25) and l in (3.26) one can compute the four-point function (3.27) explicitly.

We find that the logarithmic terms in (3.27) cancel if the following relations are satisfied

$$\beta_s = \frac{8}{9}(\alpha_s - \alpha_u), \quad \beta_t = \frac{8}{9}(\alpha_t - \alpha_u), \quad \beta_u = \frac{8}{9}(\alpha_s - \alpha_t).
 \tag{3.30}$$

This implies that

$$\beta_s - \beta_t - \beta_u = Q_{abe} Q_{cde} - Q_{ace} Q_{bde} - Q_{ade} Q_{cbe} = Q_{abe} Q_{cde} + Q_{cae} Q_{bde} + Q_{ade} Q_{bce} = 0,
 \tag{3.31}$$

¹⁵Note that the permutations of legs on the first and second lines of (3.27) are different.

and also that

$$\frac{9}{4}Q_{abe}Q_{cde} = \tilde{P}_{abcd} - \tilde{P}_{abdc} = P_{abcd} + P_{cbad} - P_{abdc} - P_{dbac} = 3P_{abcd}, \quad \text{i.e. } P_{abcd} = \frac{3}{4}Q_{abe}Q_{cde}, \quad (3.32)$$

where we used (3.6) and symmetry properties of the curvature tensor in (3.4) and (3.5). Written in terms of R and H in (3.3) this reads

$$R_{abcd} = \frac{1}{4}H_{abe}H_{cde}. \quad (3.33)$$

Interestingly, the trace of this relation, i.e. $R_{ac} = \frac{1}{4}H_{abe}H_{cbe}$, is the same as the vanishing of the one-loop beta-function [25] of the σ -model in (3.1).

In the group space case (3.10) the condition (3.31) is automatically satisfied due to the Jacobi identity for the structure constants. The condition (3.32) or (3.33) reduces to

$$q^2 = \frac{4}{3}p, \quad \text{i.e. } q^2 = 1, \quad (3.34)$$

i.e. is valid only in the WZW model case (cf. (2.2), (3.10)).

3.3 WZW model case: matching with correlators of chiral currents

Thus the cancellation of the logarithmic terms in the four-point boundary correlators of a generic σ -model in AdS_2 happens only in the WZW model. This generalizes the observation made in section 2 in the $\text{SL}(2, \mathbb{R})$ WZW case. Then the resulting expression for the connected four-point correlator (3.27) may be written as (using (3.10) with $q = 1$, i.e. $\lambda^2 = \frac{2}{k}$)

$$\langle X_a(t_1)X_b(t_2)X_c(t_3)X_d(t_4) \rangle = \frac{4}{k} \left(\frac{f_{abe}f_{cde}}{t_{12}t_{13}t_{23}t_{34}} + \frac{f_{ace}f_{dbe}}{t_{12}t_{13}t_{23}t_{24}} + \frac{f_{ade}f_{bce}}{t_{12}t_{13}t_{14}t_{23}} \right). \quad (3.35)$$

As in the $\text{SL}(2, \mathbb{R})$ case (cf. (2.26), (2.36)), we can now explicitly check the correspondence between AdS_2 boundary correlators and holomorphic correlation functions of Kac-Moody currents.

The basic OPE relation for the WZW chiral currents on the plane (1.3) gives the two-point function (1.4). Higher point correlators can be obtained by repeatedly using the OPE (1.3).¹⁶ In particular, one finds

$$\langle J_a(w_1)J_b(w_2)J_c(w_3) \rangle = \frac{kf_{abc}}{w_{12}w_{13}w_{23}}, \quad (3.36)$$

$$\langle J_a(w_1)J_b(w_2)J_c(w_3)J_d(w_4) \rangle = \frac{k^2\delta_{ab}\delta_{cd}}{w_{12}^2w_{34}^2} + \frac{kf_{abe}f_{cde}}{w_{12}w_{34}w_{23}w_{24}} + \left(\begin{matrix} w_2 \leftrightarrow w_3 \\ b \leftrightarrow c \end{matrix} \right) + \left(\begin{matrix} w_2 \leftrightarrow w_4 \\ b \leftrightarrow d \end{matrix} \right). \quad (3.37)$$

¹⁶The current-current OPE translates into the (recursion) relation [26]

$$\langle J_{a_1}(w_1) \dots J_{a_n}(w_n) \rangle = \sum_{j=2}^n \langle J_{a_2}(w_2) \dots J_{a_{j-1}}(w_{j-1}) \left[\frac{k\delta_{a_1 a_j}}{(w_1 - w_j)^2} + \frac{f_{a_1 a_j c} J_c(w_j)}{w_1 - w_j} \right] J_{a_{j+1}}(w_{j+1}) \dots J_{a_n}(w_n) \rangle.$$

The mutual locality of the KM currents implies a trivial (meromorphic) singularity structure and the solution of the above relation is simply obtained by isolating poles as in (3.36)–(3.38).

The “connected” part of (3.37) may be written as

$$\begin{aligned} \langle J_a(w_1)J_b(w_2)J_c(w_3)J_d(w_4) \rangle_{\text{conn}} &= k \left(\frac{f_{abe}f_{cde}}{w_{12}w_{34}w_{23}w_{24}} + \frac{f_{ace}f_{dbe}}{w_{13}w_{24}w_{23}w_{34}} + \frac{f_{ade}f_{bce}}{w_{14}w_{23}w_{43}w_{24}} \right) \\ &= k \left(\frac{f_{abe}f_{cde}}{w_{12}w_{13}w_{23}w_{34}} + \frac{f_{ace}f_{dbe}}{w_{12}w_{13}w_{23}w_{24}} + \frac{f_{ade}f_{bce}}{w_{12}w_{13}w_{14}w_{23}} \right). \end{aligned} \tag{3.38}$$

Here in the second line we wrote an equivalent expression (expressing crossing symmetry of the four-point function) that is a consequence of the Jacobi identity for the structure constants.

Restricting the points to the real line ($w_i \rightarrow t_i$) we can identify the two-point (1.4) and three-point (1.5) correlators of the currents with the corresponding boundary correlators in (3.14) and (3.15) up to an overall universal factor κ^n where $n = 2, 3, \dots$ is the number of legs. Explicitly, this amounts to the formal identification (assuming $k > 0$)

$$\mathcal{X}^a \rightarrow \kappa J^a, \quad \kappa = \sqrt{\frac{2}{k}}. \tag{3.39}$$

Indeed, the two-point functions match if $\kappa^2 = \frac{2}{k}$, while the three-point functions match for Q_{abc} in (3.15) related to f_{abc} in (3.36) as in (3.10) and $\kappa^3 = \frac{2}{k}\lambda$. Furthermore, the four-point correlator (3.35) is also in precise agreement with the boundary restriction of the connected part of the correlator of four currents in (3.38).

As already mentioned in the Introduction, one can give a simple semiclassical argument supporting the relation (3.39), i.e. the expression for κ that, we remark, is same as in (2.40). Starting with the expression for the w -component of the chiral current consistent with $k > 0$ and (1.3), (3.36) (see, e.g., [19]) $J^a = -k \text{tr}(t^a \partial g g^{-1})$ and using the parametrization $g = e^{-i\lambda t_a X^a}$ we get in the $z \rightarrow 0$ limit (for the boundary asymptotics in (3.11))

$$z \rightarrow 0: \quad J^a = ik\lambda \partial X^a + \dots = ik\lambda \frac{1}{2}(\partial_t - i\partial_z)(zX^a + \dots) = \frac{1}{2}k\lambda X^a + \dots. \tag{3.40}$$

This suggests the identification $J^a = \frac{1}{2}k\lambda X^a$ as in (3.39) where λ is given by (2.2), i.e. $\kappa = \sqrt{\frac{2}{k}}$.

4 Quantum corrections to boundary correlators in $SL(2, \mathbb{R})$ WZW model

The above discussion was restricted to consideration of tree-level terms in the boundary correlators in AdS_2 . Let us now try to test the relation between the boundary correlators of WZW fields and chiral currents (1.2) beyond the classical (large k) limit. This requires determining loop corrections to AdS_2 boundary correlators. Similar computations were done in the Liouville and Toda theories in [7, 11, 12] and it was found that the analogs of the coefficient κ in (1.2) that relate boundary correlators of elementary fields in AdS_2 to correlators of CFT currents (stress tensor and \mathcal{W} -symmetry currents) restricted to real line receive quantum corrections.

In the present WZW model case, the simplicity of the semiclassical argument in (3.40) suggests instead that the relation (1.2) or (2.40), (3.39) may be exact.¹⁷

To provide support to this conjecture below we shall consider the computation of one-loop corrections to the two-point and three-point boundary correlators (1.1) on the example of the $SL(2, \mathbb{R})$ WZW model. A central issue will be the choice of a UV regularization and subtraction scheme consistent with underlying $SL(2, \mathbb{R})$ symmetry of the model. It turns out to be possible to relate the scheme ambiguity to the definition of the propagator at the coinciding points, i.e. to the choice of the renormalized value of the self-contraction contributions.

4.1 One-loop corrections to the two-point correlators

Let us start with computing the one-loop corrections to the tree-level two-point functions (2.14) for the fields in the action (2.6), i.e. to the boundary correlators $\langle \Psi \tilde{\Psi} \rangle$ and $\langle \Phi \Phi \rangle$.

4.1.1 $\langle \Psi \tilde{\Psi} \rangle$

One-loop corrections to the $\psi, \tilde{\psi}$ propagator in AdS_2 come from the following diagrams:

$$\text{regular} + \left[\text{\delta-function} + \text{circle} \right] + \text{tadpole} . \quad (4.1)$$

Here we have separated the contributions of the regular and δ -function terms in (2.12) combining the latter with the self-contraction diagram corresponding to the vertex $\phi^2 \psi \tilde{\psi}$ in (2.6) as both are proportional to the free scalar propagator at the coinciding points, i.e. $g(w, w)$ (cf. (2.10), (2.11)). The last tadpole diagram with a ψ loop is linearly divergent and may be removed by imposing the normalization condition $\langle \phi \rangle = 0$.

The first contribution in (4.1) involving the regular part of the second derivative of the propagator in (2.12) (with legs taken to the boundary) is given by

$$D_{\Psi \tilde{\Psi}}(\mathbf{t}_{12}) = \text{regular} = 2^2 \frac{b^2}{\pi^2} \hat{E}(\mathbf{t}_{12}), \quad (4.2)$$

$$\begin{aligned} \hat{E}(\mathbf{t}_{12}) &\equiv \int d^2 w d^2 w' \frac{1}{(\mathbf{t} - \mathbf{t}_1 - i\mathbf{z})^2} \frac{1}{(\mathbf{t}' - \mathbf{t}_2 + i\mathbf{z}')^2} \frac{1}{[\mathbf{t} - \mathbf{t}' + i(\mathbf{z} + \mathbf{z}')]^2} g(w, w') \\ &= \int d^2 w d^2 w' \frac{1}{(\bar{w} - \mathbf{t}_1)^2} \frac{1}{(w' - \mathbf{t}_2)^2} \frac{1}{(w - \bar{w}')^2} g(w, w') = \partial_{\mathbf{t}_1} \partial_{\mathbf{t}_2} J(\mathbf{t}_{12}), \end{aligned} \quad (4.3)$$

$$J(\mathbf{t}_{12}) \equiv \int d^2 w d^2 w' \frac{1}{\bar{w} - \mathbf{t}_1} \frac{1}{w' - \mathbf{t}_2} \frac{1}{(w - \bar{w}')^2} g(w, w'). \quad (4.4)$$

¹⁷One could wonder if the level k in (1.2) may get a familiar quantum shift by the dual Coxeter number of G (i.e. $k \rightarrow k + c_G$) which is known to appear from the quantum jacobian transformation from the group fields g to currents and in the Sugawara construction of the stress tensor and related computation of the central charge. As we shall argue below, there exists a natural computation scheme in which this does not apparently happen in the present case of κ in (1.2).

By formal shifting and rescaling w, w' one may try to argue that the integral J should be independent of t_1, t_2 . However, it is IR divergent and thus requires a regularization. A regularization will then be expected to give $J \sim \log(\Lambda^{-1}|t_1 - t_2|)$ and thus a finite $\sim \frac{1}{t_{12}^2}$ contribution to $\widehat{E}(t_{12})$. Indeed, integrating by parts the formal expression in (4.4) we get (using (2.11))¹⁸

$$J = \int d^2w d^2w' \frac{1}{\bar{w}-t_1} \frac{1}{w'-t_2} \frac{1}{w-\bar{w}'} \partial_w g(w, w') = \int_0^1 dx \int_0^\infty dy \frac{\pi^2(1+6iy+2ixy-8y^2)}{(-i+2y)(-i+2y+2xy)^2}. \quad (4.5)$$

This is divergent due to the contribution from the $y \rightarrow +\infty$ region where the integrand scales as $\sim 1/(x^2y)$. A cutoff on the z, z' integrals near zero in (4.4) translates into the modified integration range $0 < y < \frac{\Lambda}{t_{12}}$, $\Lambda \rightarrow \infty$. Then we find for the regularized integral¹⁹

$$J(t_{12}; \Lambda) = \int_0^1 dx \int_0^{\Lambda/t_{12}} dy \frac{\pi^2(1+6iy-2ixy+8y^2)}{(-i+2y)(-i+2y+2xy)^2} \stackrel{\Lambda \rightarrow \infty}{=} -\frac{\pi^2}{2} \log \frac{\Lambda}{t_{12}} + \text{finite}, \quad (4.6)$$

and thus

$$\widehat{E}(t_{12}) = \partial_{t_1} \partial_{t_2} \left[-\frac{\pi^2}{2} \log \frac{\Lambda}{t_{12}} + \dots \right] = \frac{\pi^2}{2t_{12}^2}. \quad (4.7)$$

Including also the contribution of the square bracket terms in (4.1) which depend on regularized value of $g(w, w)$ we finish with the following one-loop (i.e. order $b^2 \sim \frac{1}{k}$) correction to the tree-level boundary correlator (2.14)

$$\langle \Psi(t_1) \widetilde{\Psi}(t_2) \rangle_{1\text{-loop}} = \frac{2b^2}{t_{12}^2} (1 - g_0), \quad g_0 \equiv g(w, w). \quad (4.8)$$

Thus a particular scheme choice where $g_0 = 1$ would lead to the vanishing of the one-loop correction.

To put this in a more general context, while the WZW is UV finite in the sense that there is no coupling renormalization, there may still be a wave function renormalization (i.e. UV divergent Z -factor in the off-shell 2-point function). This should be accounted for in the definition of the S-matrix: the scattering amplitudes defined in terms of correlators with extra powers of Z will be automatically finite (see, e.g., a discussion in [27] and refs. there). Similar considerations should apply to the analog of S-matrix in AdS (see section 5) and thus to the boundary correlators. Here we will effectively by-pass this subtlety by simply assuming a particular subtraction under which the wave-function renormalization factor is trivial.²⁰ It remains an open question how this scheme extends in a consistent way to higher loop orders.

4.1.2 $\langle \Phi \Phi \rangle$

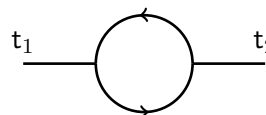
The one-loop correction to the boundary two-point function $\langle \Phi(t_1) \Phi(t_2) \rangle$ is given by the sum of two diagrams: a bubble and a self-contraction diagram.

¹⁸The integral over z, z' here may be split and turned into a double integral over z, Z with $0 < z < Z$. Then setting $Z = yt_{12}$, $z = xyt_{12}$ one is to integrate over $0 < x < 1$ and $0 < y < \infty$.

¹⁹We first integrate over x and then add and subtract the leading term of the $y \rightarrow \infty$ expansion.

²⁰For some recent discussions of one-loop self-energy corrections in AdS see [28–30].

Bubble. The bubble contribution is



$$= 4 \frac{b^2}{\pi^2} \left[\widehat{D}(\mathbf{t}_{12}) + \pi \widehat{D}_+(\mathbf{t}_{12}) + \pi \widehat{D}_-(\mathbf{t}_{12}) + \pi^2 \widehat{D}_{\text{cont}}(\mathbf{t}_{12}) \right]. \quad (4.9)$$

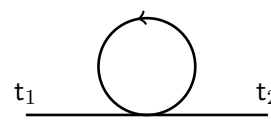
Here we decomposed the derivatives of both propagators (2.12) in the loop into the regular and δ -function parts getting four terms: with no δ -function factors (\widehat{D}), with one (\widehat{D}_\pm) and with two ($\widehat{D}_{\text{cont}}$). Explicitly,

$$\begin{aligned} \widehat{D}(\mathbf{t}_{12}) &= \int d^2w d^2w' \frac{z}{(\mathbf{t} - \mathbf{t}_1)^2 + z^2} \frac{z'}{(\mathbf{t}' - \mathbf{t}_2)^2 + z'^2} \frac{1}{[(\mathbf{t} - \mathbf{t}')^2 + (\mathbf{z} + \mathbf{z}')^2]^2}, \\ \widehat{D}_\pm(\mathbf{t}_{12}) &= \int d^2w d^2w' \frac{z}{(\mathbf{t} - \mathbf{t}_1)^2 + z^2} \frac{z'}{(\mathbf{t}' - \mathbf{t}_2)^2 + z'^2} \frac{1}{[\mathbf{t} - \mathbf{t}' \pm i(\mathbf{z} + \mathbf{z}')]^2} \delta(w - w') \\ &= -\frac{1}{4} \int d^2w \frac{1}{(\mathbf{t} - \mathbf{t}_1)^2 + z^2} \frac{1}{(\mathbf{t} - \mathbf{t}_2)^2 + z^2}. \end{aligned} \quad (4.10)$$

Integrating over $\mathbf{t}, \mathbf{t}', \mathbf{z}'$ gives

$$\widehat{D}(\mathbf{t}_{12}) = \frac{\pi^2}{2} \int_0^\infty dz \frac{1}{z(\mathbf{t}_{12}^2 + 4z^2)}, \quad \widehat{D}_\pm(\mathbf{t}_{12}) = -\frac{\pi}{2} \int_0^\infty dz \frac{1}{z(\mathbf{t}_{12}^2 + 4z^2)} = -\frac{1}{\pi} \widehat{D}(\mathbf{t}_{12}). \quad (4.11)$$

Self-contraction. With the same decomposition of the two derivatives of the propagator in the loop (2.12) we get



$$= -4 \frac{b^2}{\pi} \left[\widehat{D}_\pm(\mathbf{t}_{12}) + \pi \widehat{D}_{\text{cont}}(\mathbf{t}_{12}) \right]. \quad (4.12)$$

As a result, $\widehat{D}_{\text{cont}}$ here exactly cancels against the double δ -function part in the bubble diagram (4.9). The total expression for the one-loop correction is then

$$\langle \Phi \Phi \rangle_{1\text{-loop}} = \frac{4b^2}{\pi^2} (\widehat{D} + 2\pi \widehat{D}_\pm) - \frac{4b^2}{\pi} \widehat{D}_\pm = \frac{4b^2}{\pi^2} (\widehat{D} + \pi \widehat{D}_\pm) = 0, \quad (4.13)$$

where we used (4.11), i.e. $\widehat{D}_\pm = -\frac{1}{\pi} \widehat{D}$.

A more rigorous derivation of (4.13) requires introducing a regularization factor z^ε in each (formally divergent) AdS integral. Then

$$\begin{aligned} \widehat{D}_\varepsilon(\mathbf{t}_{12}) &= \int d^2w d^2w' z^\varepsilon z'^\varepsilon \frac{z}{(\mathbf{t} - \mathbf{t}_1)^2 + z^2} \frac{z'}{(\mathbf{t}' - \mathbf{t}_2)^2 + z'^2} \frac{1}{[(\mathbf{t} - \mathbf{t}')^2 + (\mathbf{z} + \mathbf{z}')^2]^2} \\ &= \frac{1}{\mathbf{t}_{12}^{2-2\varepsilon}} \frac{\pi^3 4^{-\varepsilon} (2\varepsilon + 1) \cot(\pi\varepsilon) \Gamma(-2\varepsilon - 2) \Gamma(\varepsilon + 2)}{\Gamma(-\varepsilon)}, \\ \widehat{D}_{\pm, \varepsilon}(\mathbf{t}_{12}) &= -\frac{1}{4} \int d^2w z^{2\varepsilon} \frac{1}{(\mathbf{t} - \mathbf{t}_1)^2 + z^2} \frac{1}{(\mathbf{t} - \mathbf{t}_2)^2 + z^2} = -\frac{1}{\mathbf{t}_{12}^{2-2\varepsilon}} 2^{-2-2\varepsilon} \pi^2 \csc(\pi\varepsilon). \end{aligned} \quad (4.14)$$

Then expanding for small ε gives

$$\widehat{D}_\varepsilon(\mathbf{t}_{12}) = \frac{1}{\mathbf{t}_{12}^2} \left[\frac{\pi^2}{4\varepsilon} + \frac{\pi^2}{4} \log \frac{\mathbf{t}_{12}^2}{4} + \dots \right], \quad \widehat{D}_{\pm,\varepsilon}(\mathbf{t}_{12}) = \frac{1}{\mathbf{t}_{12}^2} \left[-\frac{\pi}{4\varepsilon} - \frac{\pi}{4} \log \frac{\mathbf{t}_{12}^2}{4} + \dots \right], \quad (4.15)$$

leading again to (4.13).

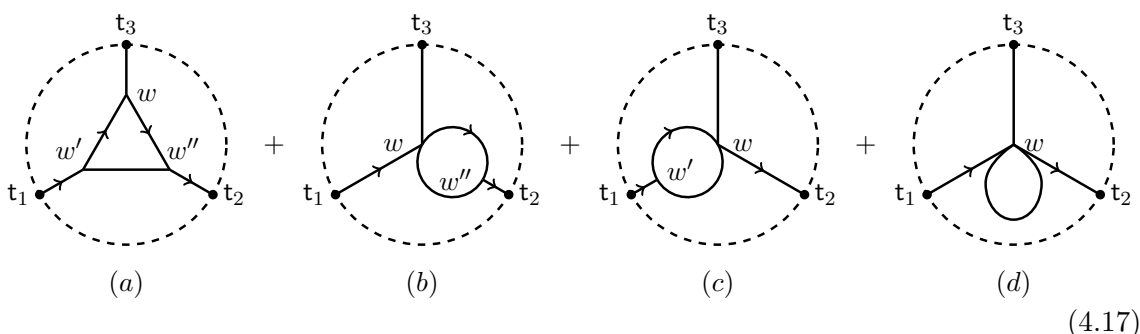
Compared to (4.8) the vanishing result in (4.13) suggests that for consistency with global symmetry (see (B.6)) the value of $g_0 = g(w, w)$ to be used in (4.8) should be indeed

$$g_0 = 1. \quad (4.16)$$

4.2 One-loop correction to the three-point correlator

Let us now compute the one-loop correction to the three-point function $\langle \Psi(\mathbf{t}_1) \widetilde{\Psi}(\mathbf{t}_2) \Phi(\mathbf{t}_3) \rangle$ with the tree-level value given by in (2.18).²¹ There are two types of contributions: from the triangle diagram and its “limits”, and from the “self-energy” corrections to the propagators in the tree-level diagram (2.15).

Triangle. The first is given by the following set of diagrams



i.e.

$$\langle \Psi(\mathbf{t}_1) \widetilde{\Psi}(\mathbf{t}_2) \Phi(\mathbf{t}_3) \rangle_{1\text{-loop}}^{\text{triangle}} = V_a(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) + V_b(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) + V_c(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) + V_d(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3). \quad (4.18)$$

We will again separate the contributions coming from regular and δ -function parts of derivatives of internal propagators in (2.12). Then the “regular” part of V_a is given by²²

$$\begin{aligned} V_a^{\text{reg}} &= \left(\frac{-2b}{\pi} \right)^3 \int d^2w d^2w' d^2w'' \frac{1}{(\bar{w}' - \mathbf{t}_1)^2} \frac{1}{(w' - \bar{w})^2} \frac{1}{(w - \bar{w}'')^2} \frac{1}{(w'' - \mathbf{t}_2)^2} \frac{z}{(\mathbf{t} - \mathbf{t}_3)^2 + z^2} g(w', w'') \\ &= \left(\frac{-2b}{\pi} \right)^3 \int d^2w d^2w' d^2w'' \frac{1}{(\bar{w}' - \mathbf{t}_1)^2} \frac{\partial_{w'} g(w', w'')}{w' - \bar{w}} \frac{1}{(w - \bar{w}'')^2} \frac{1}{(w'' - \mathbf{t}_2)^2} \frac{z}{(\mathbf{t} - \mathbf{t}_3)^2 + z^2}, \end{aligned} \quad (4.19)$$

²¹Similar loop corrected three-point functions in AdS_2 have been considered in the Liouville theory [7], the abelian or non-abelian Toda theory [11], and the $\mathcal{N} = 1$ supersymmetric Liouville theory [12]. In all those cases, the analysis has been semi-analytic because some contribution required a numerical evaluation. In the present WZW model all calculations will be fully analytical due to the simpler structure of virtual exchanges.

²²In the integration by parts we may ignore the δ -function from $\partial_{w'} \frac{1}{\bar{w}' - \mathbf{t}_1}$ and its derivatives as they localize the bulk integral to the boundary.

This is the triple integral over a half-plane of a rational integrand. Applying the residue theorem gives

$$V_a^{\text{reg}}(t_1, t_2, t_3) = \left(\frac{-2b}{\pi}\right)^3 \frac{\pi^3}{4} \frac{i}{t_{12}t_{13}t_{23}} = -\frac{2ib^3}{t_{12}t_{13}t_{23}}. \quad (4.20)$$

The contribution V_a^δ with only one δ -function from (2.12) turns out to precisely cancel the regular parts of the contributions of the two diagrams (b) and (c), i.e. $V_b^{\text{reg}} + V_c^{\text{reg}}$.

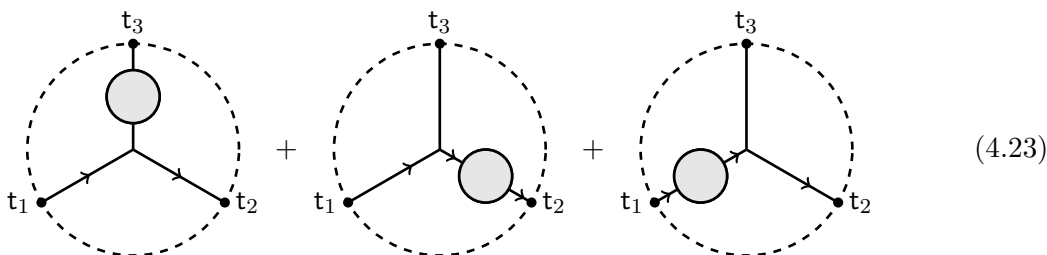
The contributions of the diagrams with the δ -function parts of the derivatives of all of the internal ($\psi, \tilde{\psi}$) propagators reduce to that of the diagram (d) (with different overall factors). Explicitly, the contribution $V_a^{2\delta}$ with the two δ -functions $\delta^{(2)}(w-w')$, $\delta^{(2)}(w-w'')$ in the diagram (a) is given by $V_a^{2\delta} = -2V_d$ (accounting for the symmetry factor of the ϕ loop) and $V_b^{1\delta} = V_c^{1\delta} = -V_a^{2\delta}$, i.e.

$$V^\delta(t_1, t_2, t_3) = V_a^{2\delta} + V_b^{1\delta} + V_c^{1\delta} + V_d = \frac{1}{2}V_a^{2\delta} = \frac{2ib^3g(w,w)}{t_{12}t_{23}t_{13}}. \quad (4.21)$$

Thus finally (using the notation g_0 in (4.8))

$$\langle \Psi(t_1)\tilde{\Psi}(t_2)\Phi(t_3) \rangle_{1\text{-loop}}^{\text{triangle}} = -\frac{2ib^3}{t_{12}t_{23}t_{13}}(1-g_0). \quad (4.22)$$

Self-energy corrections. The contribution of the corresponding diagrams (here gray circles stand for sums of relevant one-loop diagrams as in (4.1) and (4.9), (4.12))



may be represented as

$$\langle \Psi(t_1)\tilde{\Psi}(t_2)\Phi(t_3) \rangle_{1\text{-loop}}^{\text{self-energy}} = \langle \Psi(t_1)\tilde{\Psi}(t_2)\Phi(t_3) \rangle_{\text{tree}} \times \left(\frac{\langle \Phi\Phi \rangle_{1\text{-loop}}}{\langle \Phi\Phi \rangle_{\text{tree}}} + 2\frac{\langle \Psi\tilde{\Psi} \rangle_{1\text{-loop}}}{\langle \Psi\tilde{\Psi} \rangle_{\text{tree}}} \right), \quad (4.24)$$

with the full 1-loop correction to three-point function thus given by

$$\langle \Psi(t_1)\tilde{\Psi}(t_2)\Phi(t_3) \rangle_{1\text{-loop}} = \langle \Psi(t_1)\tilde{\Psi}(t_2)\Phi(t_3) \rangle_{1\text{-loop}}^{\text{triangle}} + \langle \Psi(t_1)\tilde{\Psi}(t_2)\Phi(t_3) \rangle_{1\text{-loop}}^{\text{self-energy}}. \quad (4.25)$$

In view of (4.22) to (4.8), (4.13) we conclude that for the special scheme choice (4.16) under which the two-point functions do not receive one-loop corrections the same is true also for the three-point function (4.25). Then comparing to the correlators of chiral currents in (2.37), (2.38) this suggests that the coefficient κ in (2.40) does not receive quantum corrections.²³

²³We are assuming that the quantum theory is defined by the path integral with the WZW action (2.1), (2.3), (2.6) where the overall coefficient k or b in (2.6) has its classical value (an action with a shifted k would correspond to a different scheme choice). It is not clear if the quantum effective action [31, 32] given by the WZW action with $k \rightarrow k + c_G$ (that reproduces correlators of currents computed in perturbation theory on a plane) is a possible starting point in computing boundary correlators of elementary fields of the WZW theory in AdS₂.

In appendix C we will further elaborate on the issue of the scheme dependence of the one-loop corrections to the boundary correlators starting with a classically equivalent action in terms of redefined fields.

5 Boundary correlators and scattering amplitudes on AdS₂

While the scattering amplitudes for the massless WZW fields in flat space is known to vanish [4, 20], we have seen that the coordinate-space boundary correlators for WZW fields AdS₂ are non zero. Their structure, however, is simple being dictated by the KM symmetry. One may wonder if with some natural definition of the AdS S-matrix they may actually correspond to trivial scattering in AdS₂ or on half-plane. Below we will attempt to clarify this issue.²⁴

It is useful first to recall what happened in the Liouville theory — how triviality of scattering in AdS₂ emerges in that case. The flat space scattering in this theory was argued to be trivial in [21], based on previous results about the energy-momentum eigenstates in finite volume [33–36].²⁵ The scattering in a non-trivial Liouville vacuum or effectively in AdS₂ space was discussed in [5].²⁶ Ref. [5] have shown that at the tree level there exists a perturbative expansion which is infrared safe and leads to trivial S-matrix. This conclusion was generalized and proved in more formal way in [22].

One may attempt to define S-matrix in AdS space by specifying suitable “in” and “out” states and computing amputated bulk correlators (as in flat space LSZ formula). In addition to the question of which asymptotic states to use (cf. [23]) a major technical problem is how to explicitly construct the Lorentzian AdS scattering amplitudes starting directly from the Euclidean coordinate-space boundary correlators.

Below we shall first outline the general relation between the AdS scattering amplitudes and the Lorentzian boundary correlators. Then we shall discuss the Euclidean \rightarrow Lorentzian correlator reconstruction problem in the case of the Liouville theory relating it to the approach of [5]. Finally, we shall comment on the simplest scattering amplitude in the WZW theory in AdS₂ using an analogous method.

²⁴Starting with scattering amplitudes in AdS, one can in principle define a flat space limit of the associated S-matrix, but this requires to scale the masses/dimensions of external particles. In the massless 2d case, it is unclear how to do this and whether it is possible to directly related the AdS S-matrix to the flat space one.

²⁵To avoid infrared problems, the theory may considered on a circle, where the Liouville field φ can be expressed in terms of a free field $\varphi^{(0)}$ by means of a quantum Bäcklund transformation. All energy-momentum eigenstates on the circle can be obtained by acting on the vacuum with the modes of the stress tensor $T_{mn}^{(0)}$ of the Bäcklund field. In [21], it was argued that the dynamical properties of the infinite volume multi-particle states are equivalent to the large radius limit of the (free) $T_{mn}^{(0)}$ eigenstates. This implies that the S-matrix is trivial.

²⁶As a normalizable translation-invariant ground state does not exist in Liouville theory in flat space, ref. [5] considered, following [8], the theory in a non-invariant domain-wall background that spontaneously breaks translation invariance and “semi-compactifies” space to a half-line. The resulting model can be identified with the Liouville theory in AdS₂ geometry.

5.1 Massive scalar S-matrix on AdS₂

Let us start a scalar field theory in AdS₂ with mass parameter $m^2 = \Delta(\Delta - 1)$. Let us consider a Witten diagram with one propagator connected to a bulk point (t, z) (here t is real Minkowski time, and $z \geq 0$ is the radial AdS₂ Poincare coordinate). Ignoring dependence on other external points, it may be symbolically represented as²⁷

$$\mathcal{G}(t, z) = (t, z) \bullet \underset{\mathcal{D}}{\circlearrowleft} \overset{(t', z')}{\circlearrowright} \Gamma = \int dt' dz' \mathcal{D}(t, z; t', z') \Gamma(t', z'), \quad (5.1)$$

where $\mathcal{D} \equiv G_\Delta$ is the Lorentzian massive scalar propagator with Dirichlet boundary conditions²⁸

$$\mathcal{D}(t, z; t', z') = \frac{\mathcal{C}_\Delta}{(2u)^\Delta} {}_2F_1\left(\Delta, \Delta, 2\Delta, -\frac{2}{u}\right), \quad \mathcal{C}_\Delta = \frac{\Gamma(\Delta)}{2\sqrt{\pi}\Gamma(\Delta+1/2)}, \quad u(x, x') = \frac{(z-z')^2 - (t-t')^2}{2zz'}, \quad (5.2)$$

and Γ stands for the rest of the diagram (i.e. with one line amputated). The propagator \mathcal{D} may be written as

$$\mathcal{D}(t, z; t', z') = \frac{1}{2} \int_0^\infty \frac{d\omega}{\omega} e^{-i\omega|t-t'|} f_\omega(z) f_\omega(z'), \quad (5.3)$$

where the functions $\{f_\omega(z)\}_{\omega>0}$ are eigenmodes of the kinetic operator for a scalar field in AdS₂

$$\left(\partial_z^2 + \omega^2 - \frac{m^2}{z^2}\right) f_\omega(z) = 0. \quad (5.4)$$

They form a basis in $z \in [0, \infty)$ with normalization

$$\int_0^\infty d\omega f_\omega(z) f_\omega(z') = \delta(z - z'), \quad \int_0^\infty dz f_\omega(z) f_{\omega'}(z) = \delta(\omega - \omega'). \quad (5.5)$$

$f_\omega(z)$ can be identified with the wave function of the asymptotic state with energy ω created by the scalar field. Its explicit form for the Dirichlet boundary condition is

$$f_\omega(z) = a(\omega) \sqrt{z} J_{\Delta-\frac{1}{2}}(\omega z), \quad (5.6)$$

where the normalization $a(\omega)$ is determined by (5.5). The corresponding scattering amplitude $\mathcal{A}(\omega_1, \dots, \omega_N)$ may be formally defined as

$$\mathcal{A}(\omega_1, \dots, \omega_N) = \int \left(\prod_{i=1}^N dt_i dz_i e^{i\omega_i t_i} f_{\omega_i}^{(\Delta_i)}(z_i) \right) \Gamma(t_1, z_1; \dots; t_N, z_N), \quad (5.7)$$

where in Γ we included the external leg labels and the subscript in $f_\omega^{(\Delta)}(z)$ is (5.6) indicates the corresponding value of Δ (in the case of multi-scalar scattering with different masses).

²⁷Here t is Minkowski time related to Euclidean AdS₂ time t used above by $t = it$.

²⁸As in (2.8), this is for the standard normalization of the action, i.e. $S = \frac{1}{2} \int d^2x \sqrt{g} [(\partial\phi)^2 + m^2 \phi^2 + \dots]$.

5.1.1 Comments on relation to boundary correlators

It is possible to formally “derive” a relation between (5.7) and a Fourier transform of the coordinate-space boundary correlators. Let us consider one leg in (5.1) taken to the boundary, i.e. define the boundary correlator

$$A(t) = \lim_{z \rightarrow 0} z^{-\Delta} \mathcal{G}(t, z) = t \left(\begin{array}{c} (t', z') \\ \mathcal{D} \end{array} \right) \Gamma, \quad (5.8)$$

where the circular line on the left denotes AdS₂ boundary. Substituting (5.3) into (5.1) and computing (5.8) using that $f_\omega(\omega z) \sim z^\Delta$ for $z \rightarrow 0$, we find that the Fourier transform of $A(t)$ is actually the same as the scattering amplitude in (5.7). Indeed, (here c_Δ is a coefficient dependent only on Δ)²⁹

$$\begin{aligned} \mathcal{A}(\omega) &\equiv \int dt e^{i\omega t} A(t) = c_\Delta \int dt e^{i\omega t} \int dt' dz' \int_0^\infty \frac{d\omega'}{\omega'} e^{-i\omega'|t-t'|} a(\omega') \omega'^\Delta f_{\omega'}(z') \Gamma(t', z') \\ &= c_\Delta \int dt' dz' e^{i\omega t'} \int_0^\infty \frac{d\omega'}{\omega'} \frac{i a(\omega') \omega'^\Delta}{\omega^2 - \omega'^2} f_{\omega'}(z') \Gamma(t', z'). \end{aligned} \quad (5.9)$$

Evaluating the integral over ω' by picking (one-half of) the contribution from the pole at $\omega' = \omega$, we obtain

$$\mathcal{A}(\omega) = \mathcal{N}(\omega) \int dt dz e^{i\omega t} f_\omega(z) \Gamma(t, z), \quad (5.10)$$

in agreement with (5.7). The same result is found by directly considering the boundary limit of (5.1). This amounts to replacing the bulk propagator \mathcal{D} by the bulk-to-boundary expression

$$A(t) = \lim_{z \rightarrow 0} z^{-\Delta} \mathcal{G}(t, z) = c_\Delta \int dt' dz' \left[\frac{z'}{-(t-t')^2 + z'^2} \right]^\Delta \Gamma(t', z'), \quad (5.11)$$

and taking the Fourier transform of (5.11) in Cauchy principal value sense (i.e. summing half of the two residues at $t = t' \pm z'$).³⁰ For $\omega > 0$, it reads

$$c_\Delta \int dt e^{i\omega t} \left[\frac{z'}{-(t-t')^2 + z'^2} \right]^\Delta = \frac{2^{-\frac{1}{2}-\Delta} \pi}{\Gamma(\Delta + 1/2)} \omega^{\Delta+\frac{1}{2}} \sqrt{z} J_{\Delta-1/2}(\omega z) e^{i\omega t'}. \quad (5.12)$$

and thus implies again (5.10).

To summarize, we have shown that under a certain prescription, one can start with the N -leg boundary correlator for fields with dual conformal dimensions $\Delta_1, \dots, \Delta_N$

$$A(t_1, \dots, t_N) = \lim_{z_i \rightarrow 0} z_1^{-\Delta_1} \dots z_N^{-\Delta_N} \int \left(\prod_{i=1}^N dt'_i dz'_i \mathcal{D}(t_i, z_i; t'_i, z'_i) \right) \Gamma(t'_1, z'_1; \dots; t'_N, z'_N), \quad (5.13)$$

²⁹Here and in the following, integrals z are restricted to the AdS₂ region, i.e. $z \geq 0$.

³⁰At this stage this is just a formal prescription. More precisely, one should shift the integration contour by adding causal $i\epsilon$ shifts, see below.

take its Fourier transform in each leg and as result find an alternative representation for the scattering amplitude $\mathcal{A}(\omega_1, \dots, \omega_N)$ in (5.7), i.e.

$$\mathcal{A}(\omega_1, \dots, \omega_N) = \int \left(\prod_{i=1}^N dt_i e^{i\omega_i t_i} \right) A(t_1, \dots, t_N). \quad (5.14)$$

Let us note that the amputated Green's function Γ in (5.7), as well as the boundary correlator in (5.14), are the Lorentzian ones. In general, the explicit analytical continuation of the boundary correlators from the Euclidean to the Lorentz signature should be done according to the general prescriptions based on reconstruction theorems [37] as discussed more recently in [38–40]. In particular, to compute the fully time-ordered Wightman function from the Euclidean correlators, one replaces $t_i \rightarrow t_i - i\varepsilon_i$ with $\varepsilon_i > \varepsilon_j$ when $t_i > t_j$ and then takes $\varepsilon_i \rightarrow 0$.³¹ The Fourier transform of the resulting expression is expected to give the scattering amplitude and to match (5.7).

5.1.2 Tree level scattering in Liouville theory on AdS₂

To illustrate the relation between (5.7) and (5.14) let us consider again the Liouville theory following [5]. The basic $1 \rightarrow 2$ particle production process $\varphi \rightarrow \varphi + \varphi$ here is particularly simple: at tree level it involves the amputated 3-point function that is just a constant. Let us begin by (5.7). The off-shell wave functions (5.6) are

$$f_{\alpha, \omega}(\mathbf{t}, \mathbf{z}) = e^{i\alpha t} \sqrt{\omega \mathbf{z}} J_{3/2}(\omega \mathbf{z}), \quad \omega > 0. \quad (5.15)$$

The on-shell condition is $\alpha^2 = \omega^2$, namely $\alpha = \pm\omega$. Besides, $f_{-\omega, \omega}(\mathbf{t}, \mathbf{z}) = f_{-\omega, -\omega}(\mathbf{t}, \mathbf{z})$, and we can simultaneously treat both signs of ω , i.e. “in” or “out” states. Up to irrelevant constants, the scattering amplitude for a 3-particle process may be written as

$$\mathcal{A}_3(\alpha_1, \omega_1; \alpha_2, \omega_2; \alpha_3, \omega_3) \sim \delta(\alpha_1 + \alpha_2 + \alpha_3) \bar{\mathcal{A}}_3(\omega_1, \omega_2, \omega_3), \quad (5.16)$$

where

$$\bar{\mathcal{A}}_3(\omega_1, \omega_2, \omega_3) = \int_0^\infty \frac{dz}{z^2} f_{\omega_1}(z) f_{\omega_2}(z) f_{\omega_3}(z) = \sqrt{\omega_1 \omega_2 \omega_3} \int_0^\infty \frac{dz}{\sqrt{z}} J_{3/2}(\omega_1 z) J_{3/2}(\omega_2 z) J_{3/2}(\omega_3 z). \quad (5.17)$$

We may now use the known value of the following definite integral³²

$$\int_0^\infty \frac{dz}{z^{\nu-1}} J_\nu(\omega_1 z) J_\nu(\omega_2 z) J_\nu(\omega_3 z) = \frac{2^{\nu-1} \mathcal{S}^{2\nu-1}}{\sqrt{\pi} (\omega_1 \omega_2 \omega_3)^\nu \Gamma(\nu + \frac{1}{2})}, \quad (5.18)$$

where

$$\mathcal{S} = \frac{1}{4} \sqrt{(\omega_1 + \omega_2 + \omega_3)(-\omega_1 + \omega_2 + \omega_3)(\omega_1 - \omega_2 + \omega_3)(\omega_1 + \omega_2 - \omega_3)}, \quad (5.19)$$

³¹As first discussed in [41], the analytical continuation can be done at the level of Mellin amplitudes, see [42–44].

³²Useful integrals involving three Bessel functions are discussed in [45, 46].

is the area of a triangle with sides $\omega_1, \omega_2, \omega_3$ (if $\omega_1, \omega_2, \omega_3$ do not form a triangle, the integral is zero). From (5.18) ref. [5] found the following expression for (5.17)

$$\bar{\mathcal{A}}_3 = \sqrt{\frac{2}{\pi}} \frac{S^2}{\omega_1 \omega_2 \omega_3}. \quad (5.20)$$

As the kinematically allowed 3-particle processes are associated with a degenerate triangle with vanishing area $S = 0$ one finds that $\mathcal{A}_3 = 0$. This calculation has been extended in [5] to the 4-particle scattering processes that were also found to vanish.

To try to recover (5.20) as a Fourier transform (5.14) of the boundary correlator we need first to analytically continue the Euclidean boundary 3-point function $\sim \frac{1}{t_{12}^2 t_{13}^2 t_{23}^2}$ to the Lorentzian signature (to get the Lorentzian time-ordered 3-point function). Evaluating the associated Fourier transform seems far from trivial because the d^3t integration region has to be split according to the time ordering and suitable $\pm i\varepsilon$ shifts have to be introduced.³³ In principle, another approach is to look for an analytic continuation of the triple- K integral representation of the Euclidean 3-point function [47]. Such analytic continuations have been recently discussed in [48].

Let us note that continuation to Lorentzian signature and time-like momenta requires an analytic continuation of expressions involving the Appel function and this is known to be related to triple- J integrals for special arguments, see eq. (7.1) of [49] and also [45, 46]. This procedure is yet to be investigated in detail, but let us note that

$$\int_0^\infty dz z^{1-\nu+\varepsilon} K_\nu(\omega_1 z) K_\nu(\omega_2 z) K_\nu(\omega_3 z) = \frac{\pi^2}{2\varepsilon} \frac{2^{\nu-1} S^{2\nu-1}}{\sqrt{\pi} (\omega_1 \omega_2 \omega_3)^\nu \Gamma(\nu + \frac{1}{2})} + \mathcal{O}(1). \quad (5.21)$$

This relation shows that with a simple (although ad hoc) regularization of the triple- K integral, the triple- J integral (relevant for the scattering amplitude) shows up as the residue at the singular pole. The fact that leading singularities of divergent triple K integrals may contain physical objects has been discussed in the Euclidean context in [50]. It would be interesting to understand the relation between their analysis and relations like (5.21).

5.2 Massless scattering case

In view of the subtleties involved in extracting the scattering amplitudes from the Euclidean boundary correlators, here we shall consider massless scattering following the approach of [5] based on (5.7). Let us start with the simplest $1 \rightarrow 2$ process and emphasize the difference between models with derivative-independent scalar ϕ^3 vertex and with $\phi(\partial\phi)^2$ σ -model type (classically) conformally invariant vertices.

For a massless scalar we have $\Delta = 1$ or $\Delta - \frac{1}{2} = \frac{1}{2}$ in (5.6) and for a ϕ^3 interaction vertex the analog of the integral in (5.17) representing the tree level $1 \rightarrow 2$ particle production

³³The Fourier representation of the Wightman Lorentzian 3-point function with fixed time ordering $\langle \mathcal{O}(t_1) \mathcal{O}(t_2) \mathcal{O}(t_3) \rangle$, $t_1 > t_2 > t_3$ is discussed in [39].

amplitude is

$$\int_0^\infty \frac{dz}{\sqrt{z}} J_{1/2}(\omega_1 z) J_{1/2}(\omega_2 z) J_{1/2}(\omega_3 z) = \frac{1}{2\sqrt{2} \pi^{3/2} \sqrt{\omega_1 \omega_2 \omega_3}} \left[\Omega \log(\Omega^2) - \omega_{12} \log(\omega_{12}^2) - \omega_{13} \log(\omega_{13}^2) - \omega_{23} \log(\omega_{23}^2) \right], \quad (5.22)$$

where we defined $\omega_{12} = \omega_1 + \omega_2 - \omega_3$, etc., and $\Omega = \omega_1 + \omega_2 + \omega_3$. The integral (5.22) does not vanish on-shell. For instance, if $\omega_3 \rightarrow \omega_1 + \omega_2$ it has a finite non-zero limit.³⁴

In the σ -model case in flat space the 3-point amplitude vanishes due to on-shell kinematics. This is less automatic in the AdS₂ case. Let us consider the case of a general σ -model in the parametrization used in (3.2), (3.8) where the cubic vertex in the WZW case is $\sim f_{abc} \partial X^a \bar{\partial} X^b X^c$. Because of antisymmetry of f_{abc} the vertex is effectively $\sim \frac{1}{2} (\partial X^a \bar{\partial} X^b - \bar{\partial} X^a \partial X^b) X^c$. Let us first consider the contribution of the first term and then antisymmetrize in momenta. We will need the wave functions

$$\begin{aligned} f_{\pm\omega,\omega}(t, z) &= e^{\pm i\omega t} \sin(z\omega), \\ \partial f_{\pm\omega,\omega}(t, z) &\propto (\partial_t + \partial_z) e^{\pm i\omega t} \sin(z\omega) = \omega e^{\pm i\omega(t+z)}, \\ \bar{\partial} f_{\pm\omega,\omega}(t, z) &\propto (\partial_t - \partial_z) e^{\pm i\omega t} \sin(z\omega) = -\omega e^{\pm i\omega(t-z)}. \end{aligned} \quad (5.23)$$

Starting from (5.7), suppressing the group indices and defining $\alpha_i = \pm\omega_i$ we find (cf. (5.17), (5.22))

$$\begin{aligned} \mathcal{A}'_3 &= \int dt dz \partial f_{\alpha_1, \omega_1}(t, z) \bar{\partial} f_{\alpha_2, \omega_2}(t, z) f_{\alpha_3, \omega_3}(t, z) \propto \int dt dz \omega_1 \omega_2 e^{i\alpha_1(t+z)} e^{i\alpha_2(t-z)} e^{i\alpha_3 t} \sin(z\omega_3) \\ &\propto \delta(\alpha_1 + \alpha_2 + \alpha_3) \omega_1 \omega_2 \int_0^\infty dz \left[e^{i(\alpha_1 - \alpha_2 + \alpha_3)z} - e^{i(\alpha_1 - \alpha_2 - \alpha_3)z} \right] \\ &= \delta(\alpha_1 + \alpha_2 + \alpha_3) \omega_1 \omega_2 \left[\pi \delta(\alpha_1 - \alpha_2 + \alpha_3) + \frac{i}{\alpha_1 - \alpha_2 + \alpha_3} - \pi \delta(\alpha_1 - \alpha_2 - \alpha_3) - \frac{i}{\alpha_1 - \alpha_2 - \alpha_3} \right] \\ &\propto \delta(\alpha_1 + \alpha_2 + \alpha_3) \omega_1 \omega_2 \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2}. \end{aligned} \quad (5.24)$$

Here we used that $\int_0^\infty dz e^{i\omega z} = \pi \delta(\omega) + i\omega^{-1}$. We are still to antisymmetrize in $\omega_1 \leftrightarrow \omega_2$, but since the expression in (5.24) is symmetric, the final result is thus zero. Thus the 3-point scattering amplitude vanishes also in AdS₂.³⁵

As for the 4-particle scattering amplitude, in the Liouville theory in AdS₂ it was found to vanish in a non-trivial way, due to a cancellation of different contributions [5]. It would be interesting to see if it also vanishes in the WZW theory in AdS₂. A possible reason of why this may happen is the absence of non-trivial structures in the corresponding Euclidean boundary correlators, i.e. the cancellation of logarithmic terms that happens in

³⁴Notice that we can put an arbitrary scale μ in the logarithms in (5.22) since $\Omega - \omega_{12} - \omega_{13} - \omega_{23} = 0$.

³⁵Let us note that dealing with massless 2d fields requires extra care. The wave function f in (5.23) is not vanishing for $z \rightarrow \infty$. Thus, integration by parts is not a priori allowed in (5.24). That means that the starting form of the action may be important as the contribution of boundary terms (produced by integrations by parts) may be non-trivial.

the Liouville theory [6, 7] and that we also observed above for the WZW limit of a general σ -model. To establish this link it remains to derive the AdS_2 scattering amplitudes from Euclidean boundary correlators in a systematic way.

6 Concluding remarks

In this paper we considered boundary correlators of elementary fields of 2d σ -models in AdS_2 . Similar problem appears in the study of correlators of operators on a Wilson line in the strong-coupling description in terms of the $\text{AdS}_5 \times S^5$ Nambu string action in the static gauge [1, 2]. One motivation is to learn how to compute loop Witten diagrams in AdS_2 in models with derivative interactions. We have observed, in particular, that the structure of four-point correlators simplifies (with logarithmic terms of the 1d cross-ratio cancelling out) only in the WZW case when the σ -model has an extra KM symmetry. In that case the boundary correlation functions of the WZW fields are found to be the same as the correlators of the chiral WZW currents on the plane restricted to the real line.

Another possible motivation is related to the search for new integrable 2d σ -models using S-matrix based criteria as in the massive case. If one expands near a trivial σ -model vacuum in flat 2d space one gets massless scattering amplitudes which, in general, suffer from IR ambiguities [4, 51]. If instead one considers the σ -model on AdS_2 then its coordinate-space boundary correlators are better defined and one may try to find the analogs of the standard integrability constraints (S-matrix factorization and no particle creation) directly in terms of them. As any 2d σ -model is classically Weyl invariant, the tree-level problem in AdS_2 is equivalent to the same problem on flat half-plane with particular (Dirichlet) boundary conditions. Hidden conserved charges that exist in a classically integrable σ -model on a plane should lead to constraints on the corresponding Euclidean boundary correlators and the associated S-matrix on half-plane. This should also extend to the quantum level if the σ -model is quantum scale invariant (like the WZW model).

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A Notation and conventions

The AdS_2 metric is

$$ds^2 = \frac{dt^2 + dz^2}{z^2} = -4 \frac{dw d\bar{w}}{(w - \bar{w})^2}, \quad w = t + iz, \quad z > 0, \quad (\text{A.1})$$

and we use the conventions

$$\partial \equiv \partial_w = \frac{1}{2}(\partial_t - i\partial_z), \quad \bar{\partial} \equiv \partial_{\bar{w}} = \frac{1}{2}(\partial_t + i\partial_z), \quad (\text{A.2})$$

$$\epsilon^{tz} = -\epsilon^{zt} = z^2, \quad \epsilon^{w\bar{w}} = -\epsilon^{\bar{w}w} = -2iz^2, \quad g^{zz} = g^{tt} = z^2, \quad g^{w\bar{w}} = g^{\bar{w}w} = 2z^2. \quad (\text{A.3})$$

We also define the integration measure as follows

$$d^2w = dz dt, \quad d^2w = \frac{d^2w}{\pi}. \quad (\text{A.4})$$

Our convention for the δ -function is

$$\delta^{(2)}(w) = \delta(t)\delta(z), \quad \int d^2w \delta^{(2)}(w) f(w) = f(0), \quad d^2w = dt dz, \quad w = t + iz, \quad (\text{A.5})$$

so that one has

$$\partial \frac{1}{\bar{w}} = \pi \delta^{(2)}(w), \quad \bar{\partial} \frac{1}{w} = \pi \delta^{(2)}(w). \quad (\text{A.6})$$

The bulk propagator of a massless field in AdS_2 with the action normalized as $S = \int_{\text{AdS}_2} d^2w \partial\phi\bar{\partial}\phi$ is given by

$$g(\eta) = -\frac{1}{2} \log \eta(w, w'), \quad (\text{A.7})$$

where the geodesic distance η is defined in (2.9). The associated bulk-to-boundary propagator is

$$g^\partial(t; w') = \lim_{z \rightarrow 0} \frac{1}{z} g(t, z; t', z') = \frac{2z'}{(t' - t)^2 + z'^2} = \frac{-i}{t - w'} + \frac{i}{t - \bar{w}'}. \quad (\text{A.8})$$

B Global symmetry constraints in $\text{SL}(2, \mathbb{R})$ WZW model

Let us consider the consequences of the global invariance of the WZW action (2.6) under $g \rightarrow Ug$ where U is a $\text{SL}(2, \mathbb{R})$ matrix that may be chosen as

$$U = \begin{pmatrix} 1 + \gamma & \rho \\ \varepsilon & 1 - \gamma \end{pmatrix}, \quad (\text{B.1})$$

where $(\gamma, \rho, \varepsilon)$ are constant parameters. The infinitesimal transformation of the fields in (2.3) reads

$$\delta\phi = \sqrt{k}(-\gamma + \psi\varepsilon), \quad \delta\psi = 2\psi\gamma + \rho - \psi^2\varepsilon, \quad \delta\tilde{\psi} = k e^{-\frac{2}{\sqrt{k}}\phi} \varepsilon. \quad (\text{B.2})$$

In particular, taking $\gamma = \rho = 0$ and rescaling $\varepsilon \rightarrow b\varepsilon$ where $b = \frac{2}{\sqrt{k}}$ gives

$$\delta\phi = 2\psi\varepsilon, \quad \delta\psi = -b\psi^2\varepsilon, \quad \delta\tilde{\psi} = 4b^{-1} e^{-b\phi} \varepsilon. \quad (\text{B.3})$$

The action (2.6) is readily checked to be invariant under (B.3) (using integration by parts). Using the boundary asymptotics (2.7) we get from the $z \rightarrow 0$ limit of (B.3) the following transformation of the corresponding boundary fields

$$\delta\Phi(t) = 2\Psi(t)\varepsilon + \mathcal{O}(\varepsilon^2, z), \quad \delta\Psi(t) = \mathcal{O}(z), \quad \delta\tilde{\Psi}(t) = 4[b^{-1}z^{-1} - \Phi(t)]\varepsilon + \mathcal{O}(\varepsilon^2, z). \quad (\text{B.4})$$

Assuming the computational scheme preserves the global $SL(2, \mathbb{R})$ symmetry, it then imposes constraints on the boundary correlators. In view of the symmetry rotating ψ into $\tilde{\psi}$ one should have $\langle \Phi(\mathbf{t}_1) \tilde{\Psi}(\mathbf{t}_2) \rangle = 0$. Applying the variation (B.4) to this relation gives

$$0 = 2\langle \Psi(\mathbf{t}_1) \tilde{\Psi}(\mathbf{t}_2) \rangle + 4\langle \Phi(\mathbf{t}_1) 4[b^{-1}z^{-1} - \Phi(\mathbf{t}_2)] \rangle + \mathcal{O}(\varepsilon, z). \quad (\text{B.5})$$

The $SL(2, \mathbb{R})$ symmetry implies that the tadpole $\langle \Phi \rangle$ should vanish (ϕ is shifted by the parameter θ_1 in (B.1)).³⁶ We thus find the following relation

$$\langle \Psi(\mathbf{t}_1) \tilde{\Psi}(\mathbf{t}_2) \rangle = 2 \langle \Phi(\mathbf{t}_1) \Phi(\mathbf{t}_2) \rangle. \quad (\text{B.6})$$

This relation is expected to hold at the quantum level assuming the above $SL(2, \mathbb{R})$ symmetry is preserved by the computational scheme. This is a necessary condition for matching the correlation functions of chiral currents on which $SL(2, \mathbb{R})$ acts linearly.

C Alternative computation of one-loop boundary correlators in $SL(2, \mathbb{R})$ WZW model

Here we shall revisit the computation of the one-loop corrections to the two-point boundary correlators in $SL(2, \mathbb{R})$ WZW model discussed in section 4.1. We shall use an alternative form of the action in terms of redefined field variables. Local field redefinitions are, in general, expected to leave the physical (boundary) correlators invariant provided they are properly defined (taking into account wave-function renormalization factors, etc.).³⁷ Here we shall first follow a naive approach ignoring this subtlety.

Let us start with the action (2.6) and represent it in terms of the redefined fields $(\chi, \tilde{\chi})$ defined by

$$\psi = e^{-b\phi/2} \chi, \quad \tilde{\psi} = e^{-b\phi/2} \tilde{\chi}. \quad (\text{C.1})$$

Then up to the quartic terms (2.6) is given by³⁸

$$S = \int d^2w \left[\partial\phi\bar{\partial}\phi + \partial\chi\bar{\partial}\tilde{\chi} - \frac{b}{2}(\chi\bar{\partial}\tilde{\chi}\partial\phi + \tilde{\chi}\partial\chi\bar{\partial}\phi) + \frac{b^2}{4}\chi\tilde{\chi}\partial\phi\bar{\partial}\phi + \dots \right]. \quad (\text{C.2})$$

Let us now compute the one-loop correction to the boundary two-point function for ϕ , i.e. $\langle \Phi\Phi \rangle$. It receives contributions from several bubble diagrams (with the cubic vertices from (C.2)) and a self-contraction diagram (with the quartic vertex from (C.2)).

³⁶Note that the one-loop contribution to $\langle \Phi \rangle$ given by the tadpole with $(\psi, \tilde{\psi})$ propagator computed with a cutoff $z > \varepsilon$ is linearly divergent

$$\mathbf{t}_1 \text{---} \bigcirc \sim \int d^2w \frac{z}{(t - \mathbf{t}_1)^2 + z^2} \frac{1}{(2iz)^2} \rightarrow -\frac{1}{4} \int_{-\infty}^{\infty} dt \int_{\varepsilon}^{\infty} dz \frac{1}{z(t^2 + z^2)} = -\frac{\pi}{4\varepsilon}.$$

This divergence is to be subtracted in a $SL(2, \mathbb{R})$ preserving scheme (see also discussion below (4.1)).

³⁷This is easy to see at the tree level: redefinitions like $\varphi \rightarrow \varphi + \varphi^2 + \dots$ with φ subject to the boundary conditions like (2.7) produce terms of higher order in $z \rightarrow 0$ in the correlators.

³⁸Note that the cubic term can be rewritten as $\chi\bar{\partial}\tilde{\chi}\partial\phi + \tilde{\chi}\partial\chi\bar{\partial}\phi = \chi(\bar{\partial}\tilde{\chi}\partial\phi - \partial\tilde{\chi}\bar{\partial}\phi) - \chi\tilde{\chi}\bar{\partial}\partial\phi$. The first two terms here represent the standard WZ term, while the last term can be removed by a redefinition of ϕ under which an extra quartic term will be generated.

There are two bubble diagrams where both cubic vertices are of the same type:

$$\begin{array}{c}
 \begin{array}{c} \leftarrow \\ \text{t}_1 \end{array} \begin{array}{c} w \\ \bar{\partial}\phi \end{array} \begin{array}{c} \bar{\partial}\tilde{\chi} \\ \chi \end{array} \begin{array}{c} \chi \\ \bar{\partial}\tilde{\chi} \end{array} \begin{array}{c} w' \\ \partial\phi \end{array} \begin{array}{c} \text{t}_2 \\ \leftarrow \end{array} \\
 + \\
 \begin{array}{c} \leftarrow \\ \bar{\partial}\phi \end{array} \begin{array}{c} \partial\chi \\ \tilde{\chi} \end{array} \begin{array}{c} \tilde{\chi} \\ \partial\chi \end{array} \begin{array}{c} \bar{\partial}\phi \end{array} \begin{array}{c} \leftarrow \end{array} .
 \end{array} \tag{C.3}$$

As these two diagrams are complex conjugate of each other, it is enough to focus on the contribution of the first one:

$$\begin{aligned}
 I(\mathbf{t}_1, \mathbf{t}_2) &= \int d^2w d^2w' \partial_w g_{\phi\phi}^b(\mathbf{t}_1, w) \partial_{w'} g_{\phi\phi}^b(\mathbf{t}_2, w') \bar{\partial}_{w'} g_{\chi\tilde{\chi}}(w, w') \bar{\partial}_w g_{\chi\tilde{\chi}}(w', w) \\
 &= - \int d^2w d^2w' \frac{1}{(\mathbf{t}_1 - w)^2 (\mathbf{t}_2 - w')^2} \bar{\partial}_{w'} g_{\chi\tilde{\chi}}(w, w') \bar{\partial}_w g_{\chi\tilde{\chi}}(w', w) .
 \end{aligned} \tag{C.4}$$

Here we used the notation in (A.4), (2.10)–(2.13) (the free propagators of $\chi, \tilde{\chi}$ fields are the same as of $\psi, \tilde{\psi}$). As the integrand is a rational function one may apply the residue theorem to do the \mathbf{t}, \mathbf{t}' integral. It turns out that no pole survives,³⁹ implying that the \mathbf{t}, \mathbf{t}' integral gives zero. Thus (C.3) gives a vanishing contribution.

The remaining bubble diagram with two different cubic vertices and the self-contraction diagram are represented by

$$\begin{array}{c}
 \begin{array}{c} \leftarrow \\ \text{t}_1 \end{array} \begin{array}{c} w \\ \bar{\partial}\phi \end{array} \begin{array}{c} \tilde{\chi} \\ \partial\chi \end{array} \begin{array}{c} \chi \\ \bar{\partial}\tilde{\chi} \end{array} \begin{array}{c} w' \\ \partial\phi \end{array} \begin{array}{c} \text{t}_2 \\ \leftarrow \end{array} \\
 + \\
 \begin{array}{c} \leftarrow \\ \bar{\partial}\phi \end{array} \begin{array}{c} \tilde{\chi} \\ \partial\phi \end{array} \begin{array}{c} \chi \\ \partial\phi \end{array} \begin{array}{c} \leftarrow \\ \text{t}_2 \end{array} \\
 \equiv \left(\frac{b}{2}\right)^2 E(\mathbf{t}_1, \mathbf{t}_2) ,
 \end{array} \tag{C.5}$$

plus complex conjugate diagrams. Explicitly,

$$\begin{aligned}
 E(\mathbf{t}_1, \mathbf{t}_2) &= \int d^2w d^2w' \bar{\partial}_w g_{\phi\phi}^b(\mathbf{t}_1, w) \partial_{w'} g_{\phi\phi}^b(\mathbf{t}_2, w') \partial_w \bar{\partial}_{w'} g_{\chi\tilde{\chi}}(w, w') g_{\chi\tilde{\chi}}(w', w) \\
 &\quad - \int d^2w \bar{\partial}_w g_{\phi\phi}^b(\mathbf{t}_1, w) \partial_w g_{\phi\phi}^b(\mathbf{t}_2, w') g_{\chi\tilde{\chi}}(w, w) \\
 &= \int d^2w d^2w' \frac{1}{(\mathbf{t}_1 - \bar{w})^2 (\mathbf{t}_2 - w')^2} \frac{1}{(w - \bar{w}')^2} g_{\chi\tilde{\chi}}(w', w) ,
 \end{aligned} \tag{C.6}$$

where the contribution of the self-contraction diagram is exactly cancelled by the part of the bubble diagram associated with the δ -function piece in the derivatives of the propagator (cf. (2.12)). It is easy to see that, (4.3), (4.7) and (C.6) happen to differ by an overall factor only, although coming from different two-point functions, i.e.

$$E(\mathbf{t}_1, \mathbf{t}_2) = \frac{2}{\pi^2} \widehat{E}(\mathbf{t}_{12}) = \frac{1}{\mathbf{t}_{12}^2} . \tag{C.7}$$

³⁹This is true for $\mathbf{z} > \mathbf{z}'$ and $\mathbf{z} < \mathbf{z}'$, respectively (recall that $w = \mathbf{t} + iz, w' = \mathbf{t}' + iz'$). In the case of $\mathbf{z} = \mathbf{z}'$, one would encounter a factor of $1/(\mathbf{t} - \mathbf{t}')^2$ in the integrand, which leads to a divergence when performing the \mathbf{t}, \mathbf{t}' integral. A more careful treatment with an explicit regularization may lead to a non-vanishing contribution, but we will not explore this here. Note that a similar subtlety happens also when $\mathbf{z} = 0$ or $\mathbf{z}' = 0$.

Thus the final expression for one-loop correction is given by (taking into account the contribution of the complex conjugate to (C.5))

$$\langle \Phi(\mathbf{t}_1)\Phi(\mathbf{t}_2) \rangle_{1\text{-loop}} = 2 \times \frac{b^2}{4} E(\mathbf{t}_1, \mathbf{t}_2) = \frac{b^2}{2\mathbf{t}_{12}^2}. \tag{C.8}$$

Curiously, this is different from the vanishing result in (4.13).

Assuming the symmetry relation (B.6), the result in (C.8) corresponds to

$$\langle \Psi(\mathbf{t}_1)\tilde{\Psi}(\mathbf{t}_2) \rangle_{1\text{-loop}} = \frac{b^2}{\mathbf{t}_{12}^2}. \tag{C.9}$$

This matches the expression in (4.8) provided one chooses $g_0 = g(w, w) = \frac{1}{2}$ (instead of $g_0 = 1$ in (4.16)).

One may also compute the one-loop correction to the three-point function (4.25). Using (4.24) and (C.8), (C.9) we get

$$\langle \Psi(\mathbf{t}_1)\tilde{\Psi}(\mathbf{t}_2)\Phi(\mathbf{t}_3) \rangle_{1\text{-loop}}^{\text{self-energy}} = -\frac{3i b^3}{\mathbf{t}_{12}\mathbf{t}_{23}\mathbf{t}_{13}}, \quad \langle \Psi(\mathbf{t}_1)\tilde{\Psi}(\mathbf{t}_2)\Phi(\mathbf{t}_3) \rangle_{1\text{-loop}}^{\text{triangle}} = \frac{2i b^3}{\mathbf{t}_{12}\mathbf{t}_{23}\mathbf{t}_{13}}(g_0 - 1). \tag{C.10}$$

Then instead of the vanishing result in (4.25) for $g_0 = 1$ found in section 4 here we get

$$\langle \Phi(\mathbf{t}_1)\Psi(\mathbf{t}_2)\tilde{\Psi}(\mathbf{t}_3) \rangle_{1\text{-loop}} = \frac{ib^3}{\mathbf{t}_{12}\mathbf{t}_{23}\mathbf{t}_{13}}(2g_0 - 5). \tag{C.11}$$

The resulting one-loop corrected expressions for the boundary correlators could be, in principle, reconciled with the corresponding correlators of the currents provided the relations between κ and k in (2.40) and between b and k in the action (2.4) are modified from their tree-level form.

A more consistent approach should be to define the boundary correlator with the “wave-function” renormalization factors included and that should ensure the invariance of the result under field redefinitions. Then the expressions in this appendix found starting with the redefined action (C.2) could be reconciled with the approach used in section 4.⁴⁰ This remains to be clarified further.

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⁴⁰One may need also to carefully take into account contributions of boundary terms from integration by parts. Note also that the use of the symmetry relation (B.6) probably requires a particular choice of the scheme, i.e. the value of the propagator at coinciding points $g(w, w)$ and its derivatives. For example, using the action (C.2) to compute explicitly the 1-loop correction of the two-point function $\langle \Psi\tilde{\Psi} \rangle$, one would encounter the self-contraction diagram like the third diagram in (4.1). The quartic vertex in (C.2) requires us to deal with $\partial_w\partial_{\bar{w}}g(w, w)$ due to ϕ running in the loop. The regularization of such derivative term $\partial_w\partial_{\bar{w}}g(w, w)$ was discussed in [2].

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