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Finslerian extension of an anisotropic strange star in the domain of modified gravity

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Abstract In this article, we apply the Finsler spacetime to develop the Einstein field equations in the extension of modified geometry. Following Finsler geometry, which is focused on the tangent bundle with a scalar function, a scalar equation should be the field equation that defines this structure. This spacetime maintains the required causality properties on the generalized Lorentzian metric manifold. The matter field is coupled with the Finsler geometry to produce the complete action. The developed Einstein field equations are employed on the strange stellar system to improve the study. The interior of the system is composed of a strange quark matter, maintained by the MIT bag equation of state. In addition, the modified Tolman–Oppenheimer–Volkov (TOV) equation is formulated. In particular, the anisotropic stress attains the maximum at the surface. The mass-central density variation confirms the stability of the system.

1 Introduction

General relativity (GR) is based on a spacetime manifold furnished with a metric tensor consisting of the Lorentzian signature. The Einstein equations are determined from the metric. The geodesic equation of the system helps to determine the motion of the particles. Different characteristics and behaviour of spacetime in four dimensions together with higher dimensions, have been studied by several theoretical physicists [1–5]. Most investigations are concentrated on spacetime, and specifically the Einstein field equations. There are a variety of reasons for investigating gravitational theories. Many of them utilize theoretical predictions, and have pointed out that GR should be superseded in a more general aspect, and a few have tested results. In addition to astrophysical relevance, there are vast applications of spacetime to de Sitter gauge theory, induced gravity, string theory, and anti-de Sitter/conformal field theory (ADS/CFT) correspondence [6–8].

Various observational, simulation, and experimental studies are being pursued to understand the formation and evolution of galaxies, the dynamics and morphology of galaxies, early-stage formation of the universe, reionization of the universe, rotational curves of galaxies, formation of stars, different stages of stars, and the merger of binary compact objects [9–16]. The parametrized post-Newtonian (PPN) formalism is a mathematical mechanism to determine the deviation of GR and experimental results [17–19]. However, the PPN formalism is confined to metric theories of gravitation. Observations and measurements of the magnitude and redshift of supernovas are also innovative studies [20–22]. The outcomes of the data analysis of supernovae reveal that at present, the decelerating parameter (q) lies in the domain of $-1.0 \leq q \leq -0.5$ [23,24]. The accelerating phase of the universe is confirmed by the negative values of the decelerating parameter. The net outcomes of these results are the contrary behaviour described by GR.

The observation of the motion of point particles provides an adept explanation of the physical properties of spacetime. The geodesic of the geometry of spacetime can be considered as the observed trajectory. The appropriateness of the geometry can be verified from the matching of a predicted geodesic with the observed curve. Finsler geometry is where the manifold is accounted for with a Finsler function. It has a 1-homogeneous function on the tangent bundle of space-

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time, and the length measure for curves is associated with the Lorentzian metric. The generalized expression for the length of a curve on a manifold generalized metric geometry [25–27]

$$S[\gamma] = \int \mathrm{d}\tau \mathscr{F}(x(\tau), \dot{x}(\tau)).$$

The Lorentzian metric g_{ab} determines the function $\mathscr{F}$ as $\mathscr{F}(x, \dot{x}) = |g_{ab}(x)\dot{x}^a \dot{x}^b|^{1/2}$. Here, x and $\dot{x}$ stand for position vector and tangent vector, respectively. According to the definition, the geometric fields and the objects depend not only on the points of the manifold but also on its tangent directions.

It is impossible to describe the dynamic physical processes without a clock. Due to the versatility of the Finsler clock postulate, Finsler geometry has emerged in different contexts to describe physics. The importance of Finsler geometry has been realized and explored from the viewpoint of fundamental approaches of theoretical physics: from various aspects of different gravity theories; from hyperbolic polynomials defined in the generalized backgrounds [28]; for defining the modified dispersion relations of the Planck scale in the effective classical geometry and the corresponding breaking of local Lorentz invariance [29–31]; and in both linear and nonlinear optical media of covariant formulations for electrodynamical systems [32,33]. The Finslerian extension also provides an acceptable explanation of the anomaly that GR does not fulfil for the phenomenological level, astronomical and cosmological data, dark matter, and dark energy [34–36].

Buchdahl [37] proposed a modification of GR by introducing the Ricci scalar R as an arbitrary function $f(R)$ in the Einstein–Hilbert action, which was further investigated in similar research [38,39]. Different modifications were subsequently achieved by modifying the geometric term of the action: $f(G)$ gravity [40,41], $f(\mathbb{T})$ gravity [42,43], $f(R, G)$ gravity [44], and so on (where G and $\mathbb{T}$ are the Gauss–Bonnet scalar and torsion scalar, respectively). Regarding the validation of $f(R)$ gravity, it fails to uphold the solar system tests [45,46]. Such gravity theory is also unaccounted for in the stable stellar configuration [47,48]. In addition, scalar-tensor gravity and $f(R)$ gravity are classically equivalent to each other [49,50]. Several studies have investigated strange stellar objects in a non-modified frame [51–55].

Harko et al. [1] defined $f(R, \mathscr{T})$ gravity by including the matter Lagrangian with any arbitrary form of the Ricci scalar (R) as well as the trace of the energy–momentum tensor ($\mathscr{T}$). Theoretical divisions of cosmology and astrophysics have successfully studied $f(R, \mathscr{T})$ gravity [56–62]. For the self-gravitating, spherically symmetric system, the effects of stability for the locally isotropic system was explored by Sharif et al. [63]. Noureen et al. introduced perturbation effects in the system. Several scientists have also studied different characteristics of dynamical instability of spherically symmetric anisotropic collapsing stars [64–67]. The Palatini approach of $f(R, \mathscr{T})$ gravity, independent of a metric, is presented in the literature [68,69]. The hydrostatic equation for a stellar system was reviewed for isotropic and anisotropic systems by Moraes et al. [70] and Deb et al. [71]. Coupling of matter with a curvature reveals that the energy–momentum tensor is non-conserved ($\nabla_\mu T^{\mu\nu} \neq 0$) [1,72], i.e., the presence of the additional force is due to the coupling. Hence, it can be concluded that gravity violates the equivalence principle of GR [73].

Interestingly, Chakraborty [74], considering the coupling of matter and geometry, applied the restriction to the case where the test particle moves along a geodesic. As a consequence, it was shown that the matter originating from two non-interacting fluids within the stellar system conserves the effective energy–momentum tensor. From the modification of gravitation, the Lagrangian introduces a supplementary force in $f(R, \mathscr{T})$ gravity, which is used to stabilize the stellar system in addition to the hydrodynamic force, anisotropic force, and gravitational force [71]. Shabani and Farhaudi [59] provide the consequences of a cosmological and solar system in $f(R, \mathscr{T})$ gravity, which were consistent with the observational data. Confirmation of the dark matter galactic effects and gravitational lensing also support the validity of the modified theory [75]. The application of modified gravity in Finsler spacetime has also been studied in the literature [76,77].

In this article, we define the Finsler structure based on the following characteristics:

(i) The Finsler function is the fundamental variable of the geometry that has a homogeneous scalar equation on the tangent bundle.
(ii) The geometric structure is constructed from the Finsler function and is of simplified form.
(iii) The fundamental dynamical variable is no longer the metric in Finsler geometry as it is in semi-Riemannian geometry.
(iv) By the variation of the action integral, the field equation is obtained.
(v) For pseudo-Riemannian geometry, the Finslerian spacetime geometry becomes similar to the dynamics determined from the Einstein field equations.
(vi) The modified gravity maintains the system.
(vii) The interior of the stellar system is composed of up (u), down (d), and strange (s) quarks, and the matter distribution is maintained by the phenomenological MIT bag equation of state (EOS).

From the physics point of view, the geometry is a non-metric spacetime geometry, but the main motivation for considering it is that it introduces an intrinsic local anisotropy.

This anisotropy contributes to the structure of astrophysical objects through the so-called *Finslerian parameter*.

The manuscript is organized as follows: A concise definition of the Finsler spacetime is given in Sect. 2.1. In Sect. 2.2 we introduce the action principle and complete the gravity equation including matter and as the stage where we develop the Einstein field equation for modified gravity with a Finslerian background. The quadratic form of the Finsler structure is a semi-definite Riemannian structure, and we show its consistency with the Einstein field equations. We review the formation of the stellar system and hydrostatic equation with the MIT bag model equation of state (EOS) in Sect. 3. Section 4 is devoted to a discussion of the stellar system and the generalized mass-radius limit for strange stellar configurations. In Appendix 1, we explain the constant flag in two-dimensional Finsler space.

2 Finsler spacetime

The generalization of modified gravity presented in this article is based on the description of spacetime. In Sect. 2.1, we follow up the basic notion of geometry on the Finsler spacetime [78]. The generation of the Lorentzian metric spacetime is introduced here. In Sect. 2.2, we develop the Einstein field equations for the modified theory of gravity, i.e., the Lagrangian density is any arbitrary function of the Ricci scalar and trace of the energy–momentum tensor in the Finslerian extension. The field equation is developed from the action principle, where the total action is a combination of matter and geometry.

2.1 The definition

The definition of Finsler spacetime has been generalized in the literature [4,79–81] from the original definition by Beem [82]. It has been formulated and employed to describe a variety of indefinite Finsler lengths.

A Finsler spacetime $(\mathbb{M}, L)$ is a four-dimensional smooth manifold. Here $L : \mathrm{TM} \rightarrow \mathbb{R}$ is a continuous function on the tangent bundle, known as the Finsler–Lagrange function, which satisfies the following criteria:

(i) L is positively homogeneous of degree 2 with respect to the fibre coordinates of TM.
(ii) L is reversible in the sense $|L(x, -y)| = |L(x, y)|$.
(iii) The Euler–Lagrange equation $\frac{\mathrm{d}}{\mathrm{d}\mathscr{T}}\dot{\partial}_i L - \partial_i L = 0$. For every initial condition $(x, \dot{x}) \in \mathscr{T} \cup \mathscr{N}$, there exists a unique solution, with $\mathscr{N}$ the kernel of L.
(iv) L is smooth, and in respect to the fibre coordinate, the Hessian g^L_{ab} of L so that $g^L_{ab} = \frac{1}{2}\partial_a\partial_b L$.
(v) For the preimage $L^{-1}(0, \infty) \subset \mathrm{TM}$, there is a connected component $\mathscr{T}$, such that on $\mathscr{T}$, the smooth g^L exists with a Lorentzian signature $(+, -, -, -)$.

The essence of four conical sub-bundles of $\mathrm{TM}\backslash\{0\}$ originated from the difficulty in defining Finsler spacetime. This characterizes the properties of the indefinite Finsler geometry as follows:

(a) $\mathscr{N}$ is the sub-bundle where $L = 0$ and the fibre $\mathscr{N}_x = \mathscr{N} \cap T_x M$.
(b) $\mathscr{A}$ is the sub-bundle with a smooth L and non-degenerate g^L where the fibre is $\mathscr{A}_x = \mathscr{A} \cap T_x M$ and is known as the set of admissible vectors.
(c) $\mathscr{A}_0 = \mathscr{A}\backslash\mathscr{N}$ is the sub-bundle where L is used for normalization with the fibre $\mathscr{A}_{0x} = \mathscr{A}_0 \cap T_x M$.
(d) $\mathscr{T}$ is the conic sub-bundle where $L > 0$ and the fibre $\mathscr{T}_x = \mathscr{T} \cap T_x M$. The signature of the L metric is the Lorentzian signature $(+, -, -, -)$.

The extensive section is the assurance of the existence of the convex cone $\mathscr{T}_x$ in each tangent space $T_x M$ from the definition of $\mathscr{T}$. The convexity of the $\mathscr{T}_x$ is elaborately studied in the literature [4]. The interrelations, such as $\mathscr{A}_0 \subset \mathscr{A}$ and $\mathscr{T} \subset \mathscr{A}_0$, differentiate the earlier definitions of Finsler spacetime. There is no correlation between $\mathscr{N}$ and $\mathscr{A}$; thus we can consider that the L is not differentiable along the direction $L(x, \dot{x}) \neq 0$ and $L(x, \dot{x}) = 0$ [81].

2.2 Basic formalism

The Einstein field equations can be derived from the action principle. The action is the integral of the Lagrangian density over spacetime. Total action can be defined as a combination of the matter and the Einstein–Hilbert action, which couples gravity to matter as follows:

$$S = kS_{EH} + S_M.$$

Let us now consider a Finsler space $(\mathbb{M}, \mathscr{F})$. In Finslerian language, the Einstein–Hilbert action can be considered over the sphere bundle Σ given by

$$S_{EH} = \int_{\Sigma} \mathrm{d}^4\hat{x}\mathrm{d}^3\theta\sqrt{g}\sqrt{h}(f_{ab}y^a y^b)|_{\Sigma}. \tag{1}$$

No restriction is required of $\sqrt{g}$ and $\sqrt{h}$ to Σ. During the calculation, we omit the subscript $|_{\Sigma}$ for the restriction of the functions to Σ, and all functions are meant to be evaluated there.

All quantities in the action are a function of $\mathscr{F}$ in respect of g. The action is varied with respect to $\mathscr{F}$. The dynamics of $\mathscr{F}$, which describes the equation of motion, are equivalent to the Einstein equations:

$$\delta S_{EH} = \int d^4\hat{x}d^3\theta\sqrt{g}\sqrt{h}\Big(\frac{1}{2}f_{ab}g^{ab}\delta g_{ab} + f_R\delta R_{ab} + \frac{1}{2}f_{ab}h^{ab}\delta h_{ab} - 2f_R R_{ab}\frac{\delta\mathscr{F}}{\mathscr{F}} + 3f_\tau(T_{ab} - g_{ab}L_m))\delta g_{ab}\Big)y^a y^b, \tag{2}$$

where $f_R = \partial f/\partial Ric$, $f_\tau = \partial f/\partial\tau$, and the function $f = f_{ab}y^a y^b$ over the sphere bundle [26,83]. Here, $\sqrt{g}$ and R_{ab} are independent of the variation of θ.

The variation of $h^{ab}\delta h_{ab}$ can be defined as

$$h^{ab}\delta h_{ab} = (g^{ab} - y^a y^b)\delta g_{ab} - 6\frac{\delta\mathscr{F}}{\mathscr{F}}. \tag{3}$$

The correlation of δg_{ab} and $\frac{\delta\mathscr{F}}{\mathscr{F}}$ can be written as $\delta g_{ab}(\hat{x}) = 2g_{ab}\frac{\delta\mathscr{F}}{\mathscr{F}}$. On substituting Eq. (3) and the correlations in the variational Eq. (2), we have

$$\delta S_{EH} = \int_\Sigma d^4\hat{x}d^3\theta\sqrt{g}\sqrt{h}\Big(2fg_{ab} - 6f_{ab} + 6f_\tau(T_{ab} - g_{ab}L_m)\Big)y^a y^b\frac{\delta\mathscr{F}}{\mathscr{F}}. \tag{4}$$

The matter action of the Finsler space is based only on the Lagrangian density (L) of the system, which is a scalar on the space. Therefore, we can consider that it depends only on the manifold geometry. In the Finslerian setting with Finsler function ($\mathscr{F}$) considered as a fundamental variable that determines spacetime, the matter action for matter fields ψ_i looks like

$$S_M = \int_\Sigma d^4\hat{x}d^3\theta\sqrt{g}\sqrt{h}L(g, \psi_i).$$

Due to the independence of L and g from θ over the fibre coordinates, we can integrate the system on the manifold $\mathbb{M}$, which leads to the standard matter action if we divide out the volume of the 3-sphere.

The energy–momentum tensor of the matter under consideration can be defined as the calculus of variation of the matter action with respect to the metric. In the Finsler setting, the variation with respect to the Finsler function leads to an expression that involves the energy–momentum tensor of p-form fields on Lorentzian metric spacetime as T^{ab} and its trace $T = T^{ab}g_{ab} = 4L + 2g_{ab}\frac{\partial L}{\partial g_{ab}}$, following Pfeifer and Wohlfarth [4]. The variation with respect to the Finsler function is as follows:

$$\delta S_M = \int_\Sigma d^4\hat{x}d^3\theta\sqrt{g}\sqrt{h}(12T_{ab} - 2Tg_{ab})y^a y^b\frac{\delta\mathscr{F}}{\mathscr{F}}. \tag{5}$$

Combining the Einstein–Hilbert action with the matter leads to the total action that couples gravity to matter as follows:

$$S[F, \psi_i] = kS_{EH} + S_M.$$

After performing the variation with respect to $\mathscr{F}$,

$$\delta S[\mathscr{F}, \psi_i] = k\delta S_EH + \delta S_M = \int_\Sigma d^4\hat{x}d^3\theta\sqrt{g}\sqrt{h}\Big(k(2fg_{ab} - 6f_{ab} + 6f_\tau(T_{ab} - g_{ab}L_m)) + (12T_{ab} - 2Tg_{ab})\Big) y^a y^b\frac{\delta\mathscr{F}}{\mathscr{F}}. \tag{6}$$

The following equation allows us to determine the structure of spacetime:

$$\Big((3f + Ric)g_{ab} - 6f_{ab} + 6f_\tau(T_{ab} - g_{ab}L_m)\Big)y^a y^b = -\frac{12T_{ab}}{k}y^a y^b. \tag{7}$$

The tensors in the bracket are y-independent due to consideration of the space with a vanishing Cartan tensor.

We consider $f = Ric + 2\eta\tau$, a linear combination form of Ric and τ, with a constant η as adopted by Harko et al. [1]. In this study, we assume $L_m = -\mathscr{P}$, with $\mathscr{P} = \frac{1}{3}(p_r + 2p_t)$, and set the coupling constant $k = \frac{c^4}{4\pi_F G}$.

The second derivative of Eq. (7) with respect to fibre coordinates results in the Einstein equations as follows:

$$R_{ab} - \frac{1}{2}Rg_{ab} = \frac{8\pi_F G}{c^4}T_{ab} + \eta(\tau g_{ab} + 2g_{ab}\mathscr{P}). \tag{8}$$

The effective energy–momentum tensor of the system can be defined as

$$T^{eff}_{ab} = T_{ab} + \frac{\eta c^4}{8\pi_F G}(\tau g_{ab} + 2g_{ab}\mathscr{P}). \tag{9}$$

Hereafter, we shall consider the geometrized unit, i.e., $G = c = 1$.

Now, the covariant divergence of the stress–energy tensor is

$$\nabla^a T_{ab} = -\frac{\eta}{8\pi}\Big\{g_{ab}\nabla^a\tau + 2\nabla^a(g_{ab}\mathscr{P})\Big\}. \tag{10}$$

Following the above, we can write

$$T_{eff}{}^a_{b;a} = 0.$$

3 Basic equation for the stellar system

To define the stellar structure, we assume the Finsler structure is of the form

$$\mathscr{F}^2 = -e^{\lambda(r)}y^t y^t + e^{\nu(r)}y^r y^r + r^2\overline{\mathscr{F}}^2(\theta, \phi, y^\theta, y^\phi). \tag{11}$$

The metric structure coefficient can be written as

$$g_{\mu\nu} = \frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu}\left(\frac{1}{2}\mathscr{F}^2\right),$$

where $(g^{\mu\nu}) = (g_{\mu\nu})^{-1}$, and we also note that each $g_{\mu\nu}$ is homogeneous of degree zero in y.

For a non-zero vector $y = y^\mu(\frac{\partial}{\partial x^\mu})\mid_p \in T_p\mathbb{M}$, $\mathscr{F}$ induces an inner product on $T_p\mathbb{M}$ which is given by

$$g_y(u, v) = g_{\mu\nu}(x, y)u^\mu v^\nu,$$

where $u = u^\mu(\frac{\partial}{\partial x^\mu})\mid_p,\ v = v^\mu(\frac{\partial}{\partial x^\mu})\mid_p \in T_p\mathbb{M}\backslash\{0\}$.

Hence, the metric potential of the system can be defined as

$$g_{\mu\nu} = diag(-e^{\lambda(r)}, e^{\nu(r)}, r^2\overline{g_{ij}}),$$

where the term $\overline{g_{ij}}$ arises from $\overline{\mathscr{F}}^2$.

The energy–momentum tensor of the anisotropic system can be considered in the following form:

$$T^\mu_\nu = -(\rho + p_t)u^\mu u_\nu + p_t\delta^\mu_\nu + (p_t - p_r)v^\mu v_\nu, \tag{12}$$

with u_ν and v_ν the four-velocity and radial four-vector, respectively. The energy density, the radial and tangential pressures of the anisotropic fluid are respectively represented by ρ, p_r, and p_t.

The Einstein field equations for an anisotropic stellar system are in the form

$$\frac{\nu' e^{-\nu}}{r} - \frac{e^{-\nu}}{r^2} + \frac{\overline{Ric}}{r^2} = 8\pi_F\left(\rho + \frac{\eta}{24\pi_F}(3\rho - p_r - 2p_t)\right) = 8\pi_F\rho^{eff}, \tag{13}$$

$$\frac{\lambda' e^{-\nu}}{r} + \frac{e^{-\nu}}{r^2} - \frac{\overline{Ric}}{r^2} = 8\pi_F\left(p_r - \frac{\eta}{24\pi_F}(3\rho - p_r - 2p_t)\right) = 8\pi_F p_r^{eff}, \tag{14}$$

$$e^{-\nu}\left[\frac{\lambda''}{2} + \frac{\lambda'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\lambda' - \nu'}{2r}\right] = 8\pi_F\left(p_t - \frac{\eta}{24\pi_F}(3\rho - p_r - 2p_t)\right) = 8\pi_F p_t^{eff}, \tag{15}$$

where $\overline{Ric}$ represents the Ricci scalar, derived from $\overline{\mathscr{F}}^2$.

To define the strange stellar system, we consider monotonically decreasing non-singular matter density within the spherically symmetric system, as considered by Mak and Harko [84], in the following form:

$$\rho(r) = \rho_c\left[1 - \left(1 - \frac{\rho_0}{\rho_c}\right)\frac{r^2}{R^2}\right], \tag{16}$$

where ρ_c and ρ_0 are the central and surface densities, respectively.

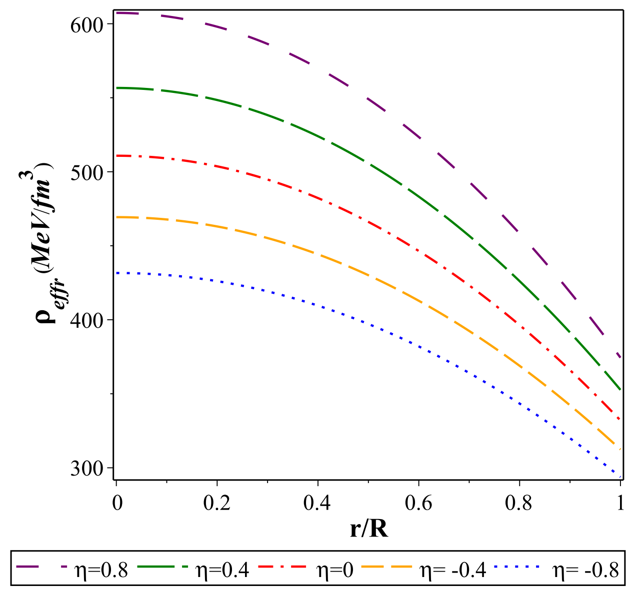

Fig. 1 Variation of the density (ρ) as a function of the fractional radial coordinate r/R, with bag constant (B_g) = 83 MeV/fm^3 and Finsler parameter $(\overline{Ric})$ = 1.2 for the $LMC\ X-4$

Figure 1 shows the variation of density with the fractional radial function for different coupling constants.

We presume that the internal matter distribution of the strange stellar system is defined by the phenomenological MIT bag model EOS followed by Chodos et al. [85]. The three flavoured quarks considered as the basic foundation of the bag are regarded as massless and non-interacting. Following this, the total quark pressure can be assumed as

$$p_r = \sum_f p^f - B_g,$$

where p^f represents the pressure of the up (u), down (d), and strange (s) quarks, respectively, and the vacuum energy density (also known as bag constant) of the system is B. Here, the pressure of individual quarks is related to the energy density ρ^f of the individual quarks as $p^f = \frac{1}{3}\rho^f$.

The energy density of each de-confined quark is as follows:

$$\sum_f \rho^f = \rho + B_g.$$

Hence, the correlation of the energy and pressure inside the strange stellar system can be interpreted as

$$p_r = \frac{1}{3}(\rho - 4B_g). \tag{17}$$

All the corrections required for the energy and pressure functions of strange quark matter have been maintained by introducing the ad hoc bag function.

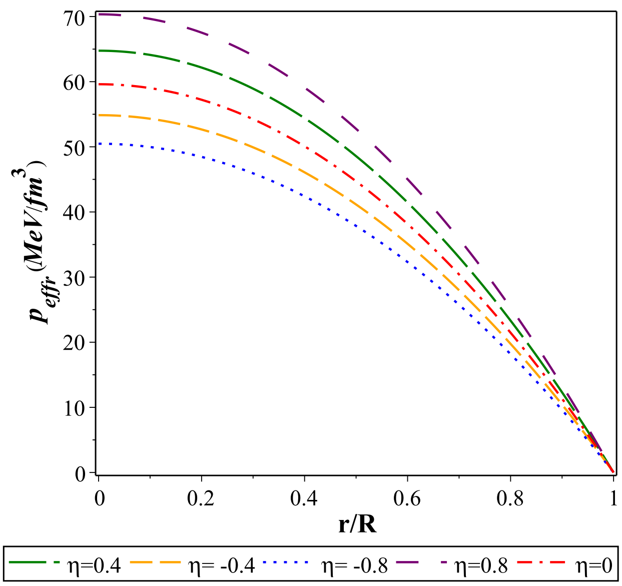

Fig. 2 Variation of the radial pressure (p_r) as a function of the fractional radial coordinate r/R, with bag constant $(B_g) = 83$ MeV/fm^3 and Finsler parameter $(\overline{Ric}) = 1.2$ for the $LMC\ X-4$

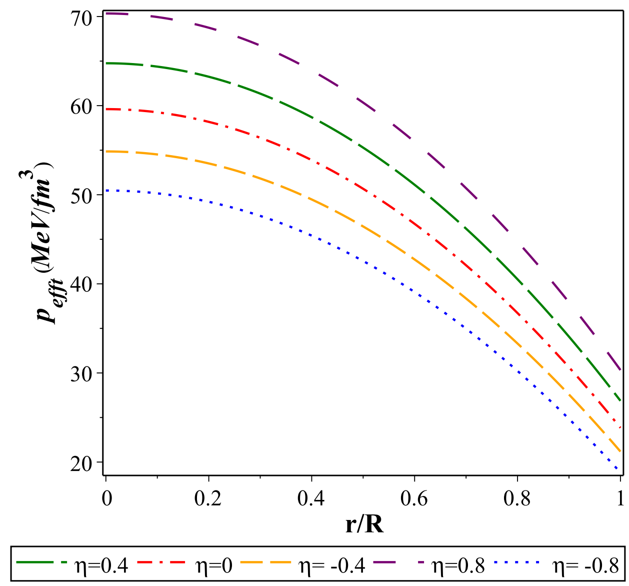

Fig. 3 Variation of the tangential pressure (p_t) as a function of the fractional radial coordinate r/R, with bag constant $(B_g) = 83$ MeV/fm^3 and Finsler parameter $(\overline{Ric}) = 1.2$ for the $LMC\ X-4$

The radial pressure must be on the surface of a stellar system; therefore, from Eq. (17) we can conclude

$$\rho_0 = 4B_g,$$

where ρ_0 is the surface density (i.e., at $r = R$).

Hence, the modified form is as follows:

$$p_r = \frac{1}{3}(\rho - \rho_0). \tag{18}$$

Following Moraes et al. [86], we consider that the tangential component of pressure inside the system is related to the matter density in the form

$$p_t = \rho c_1 + c_2. \tag{19}$$

The variations of the physical quantities, like the radial and tangential pressure shown in Figs. 2 and 3, respectively, are determined in reference of the fractional radial coordinate for different coupling constants.

From the conservation equation of the stress–energy tensor, we obtain the hydrostatic equation of the strange stellar system in the following form:

$$-p_r' - \frac{\lambda'}{2}(\rho + p_r) + \frac{2}{r}(p_t - p_r) + \frac{\eta}{24\pi_F}(3\rho' - p_r' - 2p_t') = 0.$$

Now, following Eq. (14), the simplified form of the above equation can be written as

$$p_r' = -\Bigg[p_r\Bigg\{\Bigg(4\pi_F r^3 + \frac{\eta r^3}{6}(\rho + p_r) + 2r\overline{Ric} - 3m\Bigg)\Bigg\}$$
$$-2p_t(r\overline{Ric} - m) - \frac{\eta r^3}{6}(3\rho - 2p_t) + m\rho$$
$$-\frac{\eta r}{24\pi_F}(3\rho' - 2p_t')(r\overline{Ric} - m)\Bigg]\Big/(r\overline{Ric} - m)$$
$$\times\left(r - \frac{\eta r}{24\pi_F}\right). \tag{20}$$

The expected result for the hydrostatic equilibrium condition of the non-modified gravity in the Finslerian background for the strange stellar system can be obtained from $\eta = 0$.

4 Discussion and conclusion

In order to enhance the analysis of a feasible Finslerian generalization of the Einstein equations, we have developed an action-based modified Einstein field equation (Eq. 7), evaluating the Finsler function of the Finsler spacetime. Our Finsler gravity theory incorporates the definition of the matter fields coupled with the Finsler spacetime by the principle which produces the necessary action from the Lagrangian norm on Lorentzian spacetime. We have obtained the Einstein field equations through variance with regard to the basic function of geometry. It suggests that in the metric geometry limit, it becomes comparable to the Einstein field equations for coupling variable $(\eta) = 0$ [52]. To develop a physically stable system, we choose $\overline{Ric} \geq 1$.

As a further formal development, we present a model of the strange stellar system in the modified gravity background in the extension of the Finslerian structure. The modified

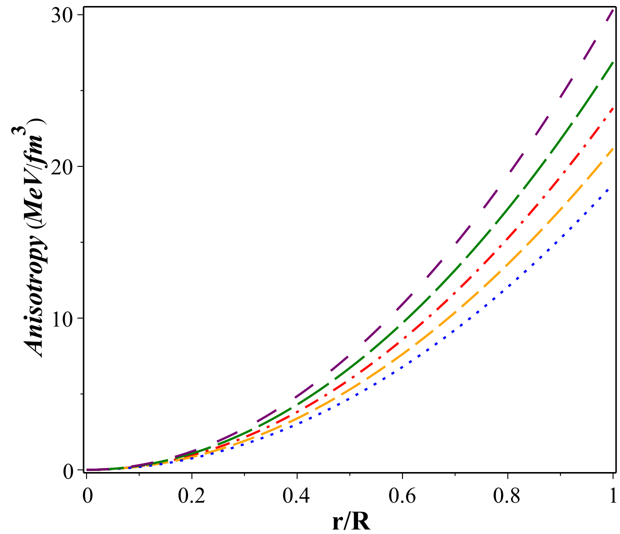

Fig. 4 Variation of the anisotropic stress (Δ) as a function of the fractional radial coordinate r/R, with bag constant (B_g) = 83 MeV/fm^3 and Finsler parameter ($\overline{Ric}$) = 1.2 for the $LMC\ X-4$

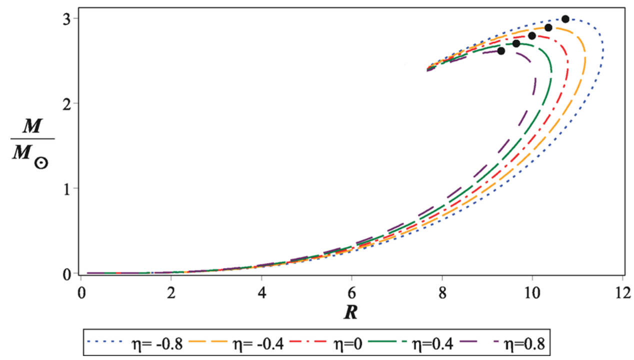

Fig. 5 Variation of the mass of a strange star as a function of radius. Solid circles represent the maximum mass and radius of the respective curves. Here, curves are drawn for B_g = 83 MeV/fm^3 and Finsler parameter ($\overline{Ric}$) = 1.2

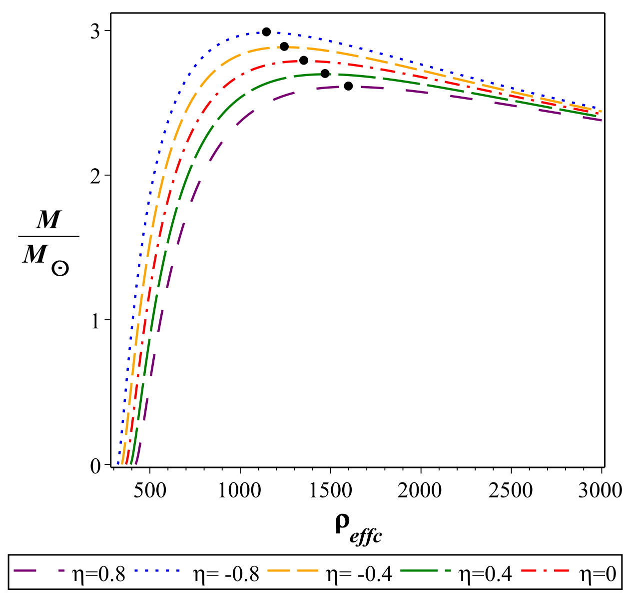

Fig. 6 Variation of the mass of a strange star as a function of the central density (ρ_c). Solid circles represent the maximum mass and radius of the respective curves. Curves are drawn for B_g = 83 MeV/fm^3 and Finsler parameter ($\overline{Ric}$) = 1.2

density and pressure are developed from the coupled matter action, which depends on the behaviour of the coupling constant.

The overall force operating is the anisotropic flow in addition to the differential pressure provided by the gravity operating on the shell by the substance (mass) within it. This determines the fluid element's hydrostatic equilibrium, which is at rest within the structure, and the overlying matter, which decreases with the radial coordinate. The difference in the stress of the tangential and radial component of pressures (anisotropic stress) is displayed in Fig. 4. In particular, the anisotropic flow is shown to be well defined over the system and reaches the maximum at the surface of the stellar model.

The variation of the total mass M in terms of normalized $M_\odot$ with respect to the radius for a chosen value of $\overline{Ric}$ = 1.2 and $B_g = 83\,\text{MeV/fm}^3$ is presented in Fig. 5. The maximum mass point corresponds to the specific η, and the radius is marked by a solid circle. An increase in the η increases the maximum mass and respective radius. We obtain the maximum mass for $\eta = 0$ at 2.788 $M_\odot$, with a radius of 10.002 km. It is interesting to note that the mass decreases by 6.33% and the corresponding radius decreases by 6.88% for $\eta = 0.8$. Further, we found that for $\eta = -0.8$, the total mass increases by 7.06% and the radius increases by 7.43%. Therefore, we can conclude that the higher coupling parameter compacted the stellar system. All variations for the mass–radius relation are suitable for the singularity condition.

The essential condition for a stellar system to be stable is $\frac{dM}{d\rho_c} > 0$. The variation of the stellar mass in $M_\odot$ with the central density ρ_c is shown in Fig. 6. The variation indicates the central density attains a value of 1355.445 MeV/fm^3 for the maximum mass 2.788 $M_\odot$, corresponding to $\overline{Ric}$ =1.2 and $\eta = 0$, whereas the highest values of the central density are 1602.481 MeV/fm^3 and 1149.021 MeV/fm^3 for η = 0.8 and -0.8, respectively. Complete circles over the curves show where the highest mass amounts are found with central density.

The lower mass gap of 2.5–5 $M_\odot$, which lies between the heaviest known neutron star and the lightest defined black hole, has always astonished scientists. Recent observations by Advanced LIGO and Virgo have indicated two events, GW200210 and GW190814 [87–89], where the second companion masses of $2.83^{+0.47}_{-0.42}$ $M_\odot$ and $2.6^{+0.1}_{-0.1}$ $M_\odot$, respectively, lie within this mass gap. It is possible that the

Table 1 Numerical values of the physical parameters for different coupling constant η for the strange star $LMC\ X-4$ of mass $1.29M_{\odot}$ ($1\ M_{\odot} = 1.475$ km) with $\overline{Ric} = 1.2$ and $B_g = 83$ Mev/fm^3

Value of η	$\eta=-0.8$	$\eta=-0.4$	$\eta=0.0$	$\eta=0.4$	$\eta=0.8$
Predicted radius (km)	9.946	9.708	9.475	9.246	9.021
ρ_{effc} (g/cm^3)	7.694×10^{14}	8.366×10^{14}	9.106×10^{14}	9.923×10^{14}	10.830×10^{14}
ρ_{effo} (g/cm^3)	5.237×10^{14}	5.569×10^{14}	5.918×10^{14}	6.287×10^{14}	6.674×10^{14}
P_{effc} (dyne/cm^2)	8.087×10^{34}	8.789×10^{34}	9.550×10^{34}	10.370×10^{34}	11.270×10^{34}
$\frac{2M}{R}$	0.38	0.39	0.40	0.41	0.42
Red shift (Z_s)	0.27	0.28	0.29	0.30	0.31

Table 2 Numerical values of the physical parameters for different $\overline{Ric}$ for the strange star $LMC\ X-4$ of mass $1.29M_{\odot}$ ($1\ M_{\odot} = 1.475$ km) with coupling parameter $\eta = 0.4$ and $B_g = 83$ Mev/fm^3

Value of $\overline{Ric}$	$\overline{Ric}=1$	$\overline{Ric}=1.1$	$\overline{Ric}=1.2$
Predicted radius (km)	9.456	9.349	9.246
ρ_{effc} (g/cm^3)	8.653×10^{14}	9.284×10^{14}	9.923×10^{14}
ρ_{effo} (g/cm^3)	6.731×10^{14}	6.290×10^{14}	6.287×10^{14}
P_{effc} (dyne/cm^2)	6.731×10^{34}	8.541×10^{34}	10.370×10^{34}
$\frac{2M}{R}$	0.402	0.407	0.412
Red shift (Z_s)	0.293	0.299	0.304

mass gap does not actually exist but rather is a result of other restrictions. The secondary companion is typically thought to be a very light black hole due to the maximum mass restrictions of known nuclear EOS and the GR relativistic Tolman–Oppenheimer–Volkov (TOV) equation. Depending on the coupling parameter and Finsler parameter, the modified gravity equation for a compact object combined with a known EOS can result in a more massive stellar structure than GR. In further work, we are investigating the gravitational echoes and tidal deformity of the structure.

We have developed a concise study in tabular format (Table 1) for the observed mass of $LMC-X4$ of different physical parameters. The study is developed for $B_g = 83$ Mev/fm^3, and $\overline{Ric} = 1.2$ under the chosen values of η as -0.8, -0.4, 0.0 , 0.4, and 0.8. On the other hand, Table 2 shows the diversity of the physical parameters with the variation of the Finsler parameter $\overline{Ric}$.

According to the present study, it is clear that with the increase in the coupling parameter and the Finsler parameter, the radii of the system decrease and the central density increases significantly, which indicates that the Finslerian background is a strong candidate for describing the compact system.

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Appendix: Two-dimensional Finsler space with a constant flag

The appearance of the Finsler spacetime with a Lorentzian signature is the point of interest. On the Finsler manifold, every point of the Finsler structure $\mathscr{F}$ is not positive-definite. $\mathscr{F}$=0 for the massless stipulation. Finsler space can be classified into two types: (i) Riemannian space, where $\mathscr{F}^2$ is quadratics in y, and (ii) Randers space [90], where

$$\mathscr{F}(x, y) \equiv a(x, y) + b(x, y), \tag{.1}$$

with $a = \sqrt{\tilde{a}_{\mu\nu}(x)y^{\mu}y^{\nu}}$, where $\tilde{a}_{ij}$ is the Riemannian metric, and $b = \tilde{b}_{\mu}(x)y^{\mu}$, where $\tilde{b}_{\mu}$ is a 1 form.

Let us consider the isometric transformation of x under the infinitesimal coordinate transformation as follows:

$$\tilde{x}^{\mu} = x^{\mu} + \epsilon \tilde{V}^{\mu}.$$

The corresponding y transforms as

$$\tilde{y}^{\mu} = y^{\mu} + \epsilon \frac{\partial \tilde{V}}{\partial x^{\nu}} y^{\nu},$$

with $|\epsilon| << 1$.

The Finsler structure can be expanded by incorporating the transformations of x and y. The expansion is considered up to the first order in $|\epsilon|$ as follows:

$$\tilde{\mathscr{F}}(\tilde{x}, \tilde{y}) = \tilde{\mathscr{F}}(x, y) + \epsilon \tilde{V}^{\mu} \frac{\partial \mathscr{F}}{\partial x^{\mu}} + \epsilon y^{\nu} \frac{\partial \tilde{V}^{\mu}}{\partial x^{\mu}} \frac{\partial \mathscr{F}}{\partial y^{\mu}}. \quad (.2)$$

Using the expansion of the structure, the Killing equation of the space can be expressed as

$$K_V(\mathscr{F}) \equiv \tilde{V}^{\mu} \frac{\partial \mathscr{F}}{\partial x^{\mu}} + y^{\nu} \frac{\partial \tilde{V}^{\mu}}{\partial x^{\mu}} \frac{\partial \mathscr{F}}{\partial y^{\mu}} = 0. \quad (.3)$$

By introducing the Randers length element defined in Eq. (.1), and since the rational and irrational terms of the Killing equation are independent of each other,

$$\tilde{V}_{\mu|\nu} + \tilde{V}_{\nu|\mu} = 0,$$
$$\tilde{V}^{\mu} \frac{\partial \tilde{b}_{\nu}}{\partial x^{\mu}} + \tilde{b}_{\mu} \frac{\partial \tilde{V}^{\mu}}{\partial x^{\nu}} = 0. \quad (.4)$$

In Riemannian background, for a fixed radial coordinate r, if the metric has the form $\mathscr{F}_{RS} = \sqrt{(y^{\theta})^2 + \sin\eta (y^{\phi})^2}$, then the system can be considered spherical symmetric with constant curvature. The "Finslerian sphere" is equivalent to the spherical symmetry of Riemannian space. Most celestial objects should possess spherical symmetry. The "Finslerian sphere" also preserves the maximum possible symmetry. This is the topological equivalent of a sphere from the mathematical definition. The flag curvature in Finsler geometry is generally the sectional curvature of the Riemannian frame. The constant Ricci scalar and the constant flag curvature are equivalent to each other. A two-dimensional Finsler space has only one independent Killing vector [91]. Bao et al. [26] provide the two-dimensional Randers–Finsler space with constant positive flag curvature λ = 1 as follows:

$$\mathscr{F}_{FS} = \frac{\sqrt{(1-\epsilon^2 \sin^2\theta) y^{\theta} y^{\theta} + \sin^2\theta y^{\phi} y^{\phi}}}{1-\epsilon^2 \sin^2\theta} - \frac{\epsilon^2 \sin^2\theta \; y^{\phi}}{1-\epsilon^2 \sin^2\theta}.$$

with $0 \leq \epsilon \leq 1$. For $\epsilon = 0$, the metric returns to the Riemann sphere.

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