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Evidence for the unbinding of the ϕ^4 kink's shape mode

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Abstract: The ϕ^4 double-well theory admits a kink solution, whose rich phenomenology is strongly affected by the existence of a single bound excitation called the shape mode. We find that the leading quantum correction to the energy needed to excite the shape mode is $-0.115567\lambda/M$ in terms of the coupling $\lambda/4$ and the meson mass M evaluated at the minimum of the potential. On the other hand, the correction to the continuum threshold is $-0.433\lambda/M$. A naive extrapolation to finite coupling then suggests that the shape mode melts into the continuum at the modest coupling of $\lambda/4 \sim 0.106M^2$, where the $\mathbb{Z}_2$ symmetry is still broken.

Keywords: Solitons Monopoles and Instantons, Field Theories in Lower Dimensions, Renormalization Regularization and Renormalons

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1 Introduction

The ϕ^4 double well field theory admits a kink solution. This kink appears in a number of physical systems, from polyacetylene [1] to crystals [2] to graphene [3]. Besides the translation mode, it enjoys a single bound excitation, called the shape mode. The shape mode is believed to be responsible for most of the kink's distinctive phenomenology [4] (but see ref. [5]), such as the resonance phenomenon [6, 7] (but see ref. [8]), wobbling kink multiple scattering [9, 10] and spectral walls [11, 12]. While the shape mode deserves and has received significant attention in the literature, quantum corrections to the shape mode have not yet been considered.

The ground state of the quantum ϕ^4 kink has been extensively studied since the pioneering work ref. [13]. However, Euclidean time approaches, which are standard in this field, project out all excited states in the infinite time limit and so it has not so far been possible to study quantum excited states. Recently the Hamiltonian approach of ref. [14] has been extended, from one loop to arbitrary numbers of loops, in a series of papers summarized in ref. [15]. At one loop, it has been known since ref. [13] that excited states are described simply by quantum harmonic oscillators. Therefore the energy difference between a kink with a single excitation and the kink ground state, a quantity which we call the *excitation energy*, is at one loop simply equal to the frequency of the corresponding normal mode as calculated in classical field theory. Using the formalism of ref. [15], the quantum states corresponding to excited kinks were constructed at the next order in ref. [16] and a general formula was provided for the excitation energies. The next order excitation energies depend on the coupling, and so for the first time describe the fate of the excited modes as the dimensionless coupling is taken away from zero.[1]

In ref. [16] a diagrammatic description of the formulas for quantum masses and excitation energies was also introduced. These diagrams are suitable for perturbative calculations

[1] We remind the reader that the coupling must be multiplied by $\hbar$ to become dimensionless. In the classical limit, the dimensionless coupling therefore vanishes. Here we set $\hbar = 1$.

in the one-kink sector. The diagrams are similar to Feynman diagrams, which describe perturbation theory in the vacuum sector, except that momentum space p is replaced by a transform with respect to normal modes labeled by k, which runs over continuous and discrete values. Also vertices are included corresponding to insertions of the momentum operator. Loops which involve a single vertex lead to x-dependent vertex factors. Transforming into the normal mode basis, these lead to k-dependent kink-sector tadpoles which are not removed by ordinary counterterms, which are already fixed to remove tadpoles in the vacuum sector. On the contrary, tadpoles play a rather important role in the kink sector perturbation theory. If an n-loop correction to the ground state energy is of order $O(\lambda^{n-1})$, then it was observed that the order $O(\lambda^{n-1})$ correction to an excitation energy corresponding to a single excitation mode arises from a diagram with only $n-1$ loops. Therefore, in the rest of this note we will refer to $O(\lambda^{n-1})$ corrections to excitation energies as $(n-1)-$loop corrections.

We will report the one-loop quantum correction to the excitation energies of both the kink's shape mode and also the first continuum excitation. We find that the energy needed to excite a shape mode decreases linearly in the coupling, but that the energy needed to excite a continuum mode decreases more rapidly. If this behavior is extrapolated to intermediate couplings, then the shape mode energy will exceed that of a continuum meson and so the shape mode is expected to become unbound, dissolving into the continuum at $\lambda/M^2 \sim 0.422$. This occurs at smaller coupling than other expected features of the model, such as the restoration of the $\phi \to -\phi$ symmetry at $\lambda/M^2 \sim 1.2(1)$ found in ref. [17], or the point at which the meson mass exceeds twice [18, 19] or π times [20] the semiclassical [13, 21] kink mass and one expects mesons to drop out of the spectrum.

2 Review of the ϕ^4 kink

The ϕ^4 double-well quantum field theory is defined by the Hamiltonian

$$H = \int dx \left[\frac{1}{2} : \pi(x)\pi(x) :_a + \frac{1}{2} : \partial_x\phi(x)\partial_x\phi(x) :_a + : \frac{\phi^2}{4} \left(\sqrt{\lambda}\phi(x) - M\sqrt{2} \right)^2 :_a \right] \tag{2.1}$$

where $::_a$ is normal ordering with respect to the operators that create plane wave excitations of the ϕ field about a classical vacuum. It admits a classical kink solution

$$\phi(x,t) = f(x) = \frac{M}{\sqrt{2\lambda}} \left(1 + \tanh\left(\frac{Mx}{2} \right) \right) \tag{2.2}$$

whose mass is classically

$$Q_0 = \frac{M^3}{3\lambda}. \tag{2.3}$$

We remind the reader that, like all 1+1d scalar field theories with canonical kinetic terms and nonderivative interactions, all ultraviolet (UV) divergences in the ϕ^4 theory are caused by loops containing a single vertex and therefore are removed by normal ordering. As a result, no UV counter terms need be included. The two-loop vacuum energy does suffer from an infrared (IR) divergence, which is removed in ref. [15] via the inclusion of an

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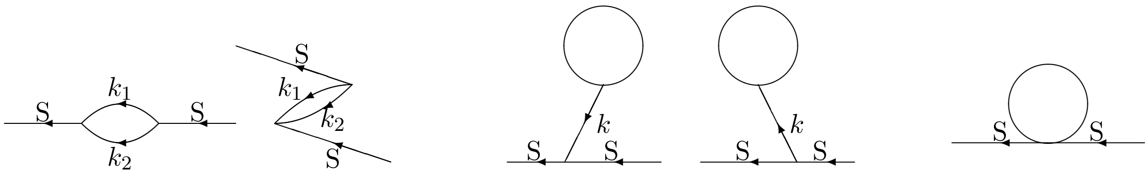

Figure 1. The first two diagrams give $\mu^{(1)}(\mathrm{S})$. The next two are equal and sum to $\mu^{(2)}(\mathrm{S})$. The last diagram is $\mu^{(3)}(\mathrm{S})$. $\mu^{(4)}(\mathrm{S})$ is found by replacing one k in the first two diagrams with a zero mode. A loop factor $\mathcal{I}(x)$ arises for each loop at a single vertex.

IR counterterm. This counterterm is a finite scalar in the Hamiltonian density, and so it contributes equally to both the energies for the excited and the ground state kinks. Therefore it does not contribute to the excitation energy, which is the difference between these two quantities, and it plays no role here.

Small, orthonormal perturbations about the kink solution with frequency ω_k will all be called normal modes. They include a zero-mode proportional to f' as well as continuum modes $g_k(x)$ and a shape mode $g_{\mathrm{S}}(x)$

$$\begin{aligned} g_k(x) &= \frac{e^{-ikx}}{\omega_k\sqrt{M^2+4k^2}}\left[2k^2-M^2+(3/2)M^2\mathrm{sech}^2\left(\frac{Mx}{2}\right)-3iMk\tanh\left(\frac{Mx}{2}\right)\right] \\ g_S(x) &= -i\frac{\sqrt{3M}}{2}\tanh\left(\frac{Mx}{2}\right)\mathrm{sech}\left(\frac{Mx}{2}\right). \end{aligned} \tag{2.4}$$

At tree level the energy required to excite a normal mode is its frequency [13]

$$\omega_k = \sqrt{M^2+k^2}, \qquad \omega_{\mathrm{S}} = \frac{\sqrt{3}}{2}M. \tag{2.5}$$

3 Energy required to excite a shape mode

The one-loop energy required to excite the shape mode is [16]

$$\begin{aligned} \mu(\mathrm{S}) &= \sum_{i=1}^{4}\mu^{(i)}(\mathrm{S}), \qquad \mu^{(1)}(\mathrm{S}) = \int^{+}\frac{d^2k}{(2\pi)^2}\frac{(\omega_{k_1}+\omega_{k_2})\left|V_{k_1k_2\mathrm{S}}\right|^2}{8\omega_{\mathrm{S}}\omega_{k_1}\omega_{k_2}\left(\omega_{\mathrm{S}}^2-(\omega_{k_1}+\omega_{k_2})^2\right)} \\ \mu^{(2)}(\mathrm{S}) &= -\int^{+}\frac{dk}{2\pi}\frac{V_{-\mathrm{S}k\mathrm{S}}V_{\mathcal{I}-k}}{4\omega_{\mathrm{S}}\omega_k^2}, \qquad \mu^{(3)}(\mathrm{S}) = \frac{V_{\mathcal{I}\mathrm{S}-\mathrm{S}}}{4\omega_{\mathrm{S}}} \\ \mu^{(4)}(\mathrm{S}) &= \frac{1}{4Q_0}\int\frac{dk}{2\pi}\left(\frac{\omega_{\mathrm{S}}}{\omega_k}+\frac{\omega_k}{\omega_{\mathrm{S}}}\right)\Delta_{-\mathrm{S}-k}\Delta_{\mathrm{S}k}. \end{aligned} \tag{3.1}$$

The derivation of eq. (3.1) is reviewed in the appendix. The corresponding diagrams, defined in ref. [16], are drawn in figure 1. Here we have defined $\int^{+} dk/(2\pi)$ to be an integral over all real values of k plus the same expression with k replaced by the shape mode. The matrix Δ is defined by

$$\Delta_{IJ} = \int dx g_I(x)g'_J(x) \tag{3.2}$$

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where I and J run over the discrete normal modes B and S as well as the continuum modes k and it is understood that $g_{-S} = g_S^*$.

Let $V^{(n)}(x)$ be the nth functional derivative of H with respect to $\phi(x)$, evaluated at $\phi(x) = f(x)$

$$V^{(3)}(x) = 3\sqrt{2\lambda} M \tanh\left(\frac{Mx}{2}\right), \qquad V^{(4)}(x) = 6\lambda. \tag{3.3}$$

Then we have also defined the symbol

$$V_{k_1\cdots k_n} = \int dx V^{(n)}(x) \prod_{i=1}^{n} g_{k_i}(x), \qquad V_{\mathcal{I}k_1\cdots k_n} = \int dx V^{(2+n)}(x)\mathcal{I}(x) \prod_{i=1}^{n} g_{k_i}(x) \tag{3.4}$$

where the loop function $\mathcal{I}(x)$ is

$$\begin{aligned} \mathcal{I}(x) &= \frac{|g_{\mathrm{S}}(x)|^2}{2\omega_{\mathrm{S}}} + \int \frac{dk}{2\pi} \frac{\left(|g_k(x)|^2 - 1\right)}{2\omega_k} \\ &= \frac{1}{4\sqrt{3}} \mathrm{sech}^2\left(\frac{Mx}{2}\right) \tanh^2\left(\frac{Mx}{2}\right) - \frac{3}{8\pi} \mathrm{sech}^4\left(\frac{Mx}{2}\right). \end{aligned} \tag{3.5}$$

Most of the relevant x integrals were already evaluated analytically in ref. [21]

$$\begin{aligned} \Delta_{Sk} &= i\pi \frac{\sqrt{3}}{16} \frac{(3M^2+4k^2)\sqrt{M^2+4k^2}}{M^{3/2}\omega_k} \mathrm{sech}\left(\frac{\pi k}{M}\right) \\ V_{SSS} &= i\pi \frac{9\sqrt{3\lambda}}{32\sqrt{2}} M^{3/2}, \qquad V_{kSS} = i\pi \frac{3\sqrt{\lambda}}{\sqrt{2}} \frac{k^2\omega_k(M^2-2k^2)}{M^3\sqrt{M^2+4k^2}} \mathrm{csch}\left(\frac{\pi k}{M}\right) \\ V_{k_1k_2S} &= -i\pi \frac{3\sqrt{3\lambda}}{2\sqrt{2}} \frac{\left(\frac{17}{16}M^4 - (k_1^2-k_2^2)^2\right)(M^2+4k_1^2+4k_2^2) + 8M^2k_1^2k_2^2}{M^{3/2}\omega_{k_1}\omega_{k_2}\sqrt{M^2+4k_1^2}\sqrt{M^2+4k_2^2}} \mathrm{sech}\left(\frac{\pi(k_1+k_2)}{M}\right) \end{aligned} \tag{3.6}$$

including some terms with a loop factor $\mathcal{I}(x)$

$$\begin{aligned} V_{\mathcal{I}k} &= i\frac{\sqrt{\lambda}}{\sqrt{6}} \frac{k^2\omega_k}{M^4\sqrt{M^2+4k^2}} \left[\pi(-M^2+2k^2) + 3\sqrt{3}\omega_k^2\right] \mathrm{csch}\left(\frac{\pi k}{M}\right) \\ V_{\mathcal{I}S} &= i\frac{3\sqrt{M\lambda}}{64\sqrt{2}}(3\sqrt{3} - 2\pi). \end{aligned} \tag{3.7}$$

We will also need

$$V_{\mathcal{I}\mathrm{S}-\mathrm{S}} = \left(\frac{3\sqrt{3}\pi - 18}{35\pi}\right)\lambda. \tag{3.8}$$

Substituting these expressions into (3.1) one can immediately evaluate the third and fourth terms

$$\begin{aligned} \mu^{(3)}(\mathrm{S}) &= \left(\frac{3\pi - 6\sqrt{3}}{70\pi}\right)\frac{\lambda}{M} \sim -0.00439962\frac{\lambda}{M} \\ \mu^{(4)}(\mathrm{S}) &= \frac{3\sqrt{3}\pi^2\lambda}{2^{11}M^7} \int \frac{dk}{2\pi} \frac{(7M^2+4k^2)(3M^2+4k^2)^2(M^2+4k^2)}{(M^2+k^2)^{3/2}} \mathrm{sech}^2\left(\frac{\pi k}{M}\right) \\ &\sim 0.27419991\frac{\lambda}{M}. \end{aligned} \tag{3.9}$$

The second term can be decomposed into the contributions $\mu^{(20)}$ and $\mu^{(21)}$ in which the dummy index is the shape mode or a continuum mode respectively

$$\mu^{(20)}(\mathrm{S}) = -\frac{V_{-\mathrm{SSS}}V_{\mathcal{I}-\mathrm{S}}}{4\omega_{\mathrm{S}}^3} = \frac{9\pi}{2^{11}}(3\sqrt{3}-2\pi)\frac{\lambda}{M} \sim -0.01500739\frac{\lambda}{M} \tag{3.10}$$

$$\begin{aligned}\mu^{(21)}(\mathrm{S}) &= -\int\frac{dk}{2\pi}\frac{V_{-\mathrm{S}k\mathrm{S}}V_{\mathcal{I}-k}}{4\omega_{\mathrm{S}}\omega_k^2}\\ &= \frac{\pi\lambda}{4M^8}\int\frac{dk}{2\pi}\frac{k^4(M^2-2k^2)}{M^2+4k^2}\left[(3\sqrt{3}-\pi)M^2+(3\sqrt{3}+2\pi)k^2\right]\operatorname{csch}^2\left(\frac{\pi k}{M}\right)\\ &\sim -0.00339318\frac{\lambda}{M}.\end{aligned}$$

Finally we may decompose the first contribution into terms μ^{1I} with I dummy indices running over continuous modes k

$$\mu^{(10)}(\mathrm{S}) = \frac{(2\omega_{\mathrm{S}})\left|V_{\mathrm{SSS}}\right|^2}{8\omega_{\mathrm{S}}^3\left(\omega_{\mathrm{S}}^2-(2\omega_{\mathrm{S}})^2\right)} = -\frac{\left|V_{\mathrm{SSS}}\right|^2}{12\omega_{\mathrm{S}}^4} = -\frac{9\pi^2}{2^9}\frac{\lambda}{M} \sim -0.17348914\frac{\lambda}{M} \tag{3.11}$$

$$\begin{aligned}\mu^{(11)}(\mathrm{S}) &= 2\int\frac{dk}{2\pi}\frac{(\omega_k+\omega_{\mathrm{S}})\left|V_{k\mathrm{SS}}\right|^2}{8\omega_{\mathrm{S}}^2\omega_k\left(\omega_{\mathrm{S}}^2-(\omega_{\mathrm{S}}+\omega_k)^2\right)}\\ &= \frac{3\pi^2\lambda}{M^8}\int\frac{dk}{2\pi}\frac{\left(2\omega_k+\sqrt{3}M\right)\omega_k k^4}{\left(3M^2-\left(\sqrt{3}M+2\omega_k\right)^2\right)}\frac{(M^2-2k^2)^2}{M^2+4k^2}\operatorname{csch}^2\left(\frac{\pi k}{M}\right)\\ &\sim -0.01112149\frac{\lambda}{M}\end{aligned}$$

$$\begin{aligned}\mu^{(12)}(\mathrm{S}) &= \int\frac{d^2k}{(2\pi)^2}\frac{(\omega_{k_1}+\omega_{k_2})\left|V_{k_1k_2\mathrm{S}}\right|^2}{8\omega_{\mathrm{S}}\omega_{k_1}\omega_{k_2}\left(\omega_{\mathrm{S}}^2-(\omega_{k_1}+\omega_{k_2})^2\right)}\\ &= \frac{9\sqrt{3}\pi^2\lambda}{8M^4}\int\frac{d^2k}{(2\pi)^2}\frac{(\omega_{k_1}+\omega_{k_2})}{\omega_{k_1}^3\omega_{k_2}^3}\operatorname{sech}^2\left(\frac{\pi(k_1+k_2)}{M}\right)\\ &\times\frac{\left[\left(\frac{17}{16}M^4-(k_1^2-k_2^2)^2\right)(M^2+4k_1^2+4k_2^2)+8M^2k_1^2k_2^2\right]^2}{(M^2+4k_1^2)(M^2+4k_2^2)\left(3M^2-4\left(\omega_{k_1}+\omega_{k_2}\right)^2\right)} \sim -0.1823560(2)\frac{\lambda}{M}.\end{aligned}$$

Adding these seven contributions, we find that the one-loop correction to the energy cost of exciting the shape mode is

$$\mu(\mathrm{S}) \sim -0.1155669(2)\frac{\lambda}{M}. \tag{3.12}$$

4 Continuum threshold

We have seen that the cost of exciting the shape mode decreases with increasing coupling. However, this does not guarantee that the shape mode will remain in the spectrum, as the continuum threshold also depends on the coupling. The continuum threshold is defined to be the energy difference between the ground state kink and the lowest energy kink state with a continuum excitation. If the continuum threshold is less than the excitation energy, one expects a shape mode excited state to decay to the kink ground state with a continuum excitation.

In ref. [16] it was shown that at one-loop, the energy required to create a continuum excitation is the sum of the one-loop correction to the meson mass in the vacuum sector plus the nonrelativistic recoil kinetic energy of the kink. The continuum threshold is the mass of the meson with no momentum, and so there is no recoil. This means that the continuum threshold energy is simply equal to the one-loop mass correction in the ϕ^3 theory with mass M and potential $M\sqrt{\lambda/2}\phi^3$, corresponding to a classical vacuum, which can be calculated in old-fashioned perturbation theory or found in a textbook, or can heuristically be read from (3.1) by replacing $f(x)$ with the classical vacuum and the normal modes with plane waves and dividing by a normalization factor of $2\pi\delta(0)$. Any method yields

$$\mu(0) = \frac{9M\lambda}{2}\int \frac{dp}{2\pi}\frac{1}{\omega_p(M^2-4\omega_p^2)} = -\frac{\sqrt{3}}{4}\frac{\lambda}{M} \sim -0.433013\frac{\lambda}{M}. \tag{4.1}$$

Including the leading order and one-loop contributions, the energy required to excite a continuum mode is then

$$m - \frac{\sqrt{3}}{4}\frac{\lambda}{M} \tag{4.2}$$

and that required to excite a shape mode is

$$\frac{\sqrt{3}}{2}m - 0.1155669(2)\frac{\lambda}{M}. \tag{4.3}$$

These are equal when $\lambda/M^2 \sim 0.422$, at which point one expects the shape mode to delocalize into oblivion, drastically affecting the phenomenology. As the coupling constant is $\lambda/4$, this corresponds to a rather small value of the coupling and so it seems reasonable to speculate that the higher order corrections only displace but do not avoid this conclusion.

We have presented evidence that, at reasonably low but finite coupling, the ϕ^4 kink loses its only bound excitation that is not a zero mode. It is not clear from our calculation whether such behavior is generic for kinks, or indeed for solitons. Recently, in ref. [22] (see also [23, 24]), a number of new examples of such bound excitations have been found, and an efficient algorithm was described for generating them. Our method can only be applied to stable kinks, as we require Hamiltonian eigenstates, but using the stability criterion in refs. [25, 26] one may easily filter examples. Our formula (3.1) can then be applied to study the evolution of the bound modes.

A Derivation of the excitation energy

In this appendix we summarize the derivation of the excitation energy (3.1) in ref. [16]. One starts by defining a kink Hamiltonian H' and kink momentum P' in terms of the defining Hamiltonian H and momentum P via

$$H' = \mathcal{D}_f^\dagger H \mathcal{D}_f, \qquad P' = \mathcal{D}_f^\dagger P \mathcal{D}_f \tag{A.1}$$

with the unitary operator $\mathcal{D}_f$ defined by

$$\mathcal{D}_f = \exp\left(-i\int dx f(x)\pi(x)\right). \tag{A.2}$$

This acts by simply shifting $\phi(x)$ by $f(x)$ and is compatible with normal ordering [27]. As (A.1) is a similarity transformation, the kink Hamiltonian H' has the same spectrum as the defining Hamiltonian H, and so it may be used to obtain the energy of any desired state. The kink Hamiltonian is easily evaluated to be

$$H' = Q_0 + H_2 + H_I, \qquad H_I = \int dx \left[\left(-\sqrt{\frac{\lambda}{2}}M + \lambda f(x)\right) :\phi^3(x):_a + \frac{\lambda}{4} :\phi^4(x):_a\right] \tag{A.3}$$
$$H_2 = \int dx \left[\frac{:\pi^2(x):_a}{2} + \frac{:\partial_x\phi(x)\partial_x\phi(x):_a}{2} + \left(\frac{M^2}{2} - 3\left(\frac{M}{2}\right)^2 \operatorname{sech}^2\left(\frac{Mx}{2}\right)\right) :\phi^2(x):_a\right]$$

where Q_0 is the classical energy of the ground state kink. Keeping c equal to unity but not $\hbar$ for a moment, the Hamiltonian has dimensions of $[action][length]^{-1}$ and so ϕ has dimensions of $[action]^{1/2}$, M has dimensions of $[length]^{-1}$ and λ has dimensions of $[action]^{-1}[length]^{-2}$. Therefore $\lambda\hbar/M^2$ is dimensionless and all quantities can be fixed by dimensional analysis up to a function of $\lambda\hbar/M^2$. A power series expansion of this function is equivalently a semiclassical expansion in $\hbar$ and a perturbative expansion in λ/M^2. Setting $\hbar = 1$, we will therefore expand all quantities in λ/M^2. Note that the power series expansion misses some terms that appear in the transseries expansion, for example instanton contributions arising from virtual kink-antikink pairs.[2] We do not yet have a method for calculating such contributions.

The Schrodinger picture field $\phi(x)$ and its conjugate momentum $\pi(x)$ are expanded in two bases: the plane wave basis

$$\phi(x) = \int \frac{dp}{2\pi}\left(A_p^\dagger + \frac{A_{-p}}{2\omega_p}\right) e^{-ipx} \tag{A.4}$$
$$\pi(x) = i\int \frac{dp}{2\pi}\left(\omega_p A_p^\dagger - \frac{A_{-p}}{2}\right) e^{-ipx}$$

and also, following ref. [14], a normal mode basis

$$\phi(x) = \phi_0 g_B(x) + \int^{+} \frac{dk}{2\pi}\left(B_k^\dagger + \frac{B_{-k}}{2\omega_k}\right) g_k(x) \tag{A.5}$$
$$\pi(x) = \pi_0 g_B(x) + i\int^{+} \frac{dk}{2\pi}\left(\omega_k B_k^\dagger - \frac{B_{-k}}{2}\right) g_k(x)$$

where it is implicit that the integral of normal modes k includes a sum over the single shape mode. Corresponding to the two bases of our operator algebra, we define two normal ordering prescriptions. The first, plane wave normal ordering $::_a$, used in the definition of the Hamiltonian, places all A to the right of $A^\dagger$. The second, normal mode normal

[2] In the case of the Sine-Gordon model these correspond to loop corrections in the dual Thirring model.

ordering $::_b$, places all B and π_0 on the right of $B^\dagger$ and ϕ_0. Using the canonical commutation relations one finds

$$[A_p, A_q^\dagger] = 2\pi\delta(p-q) \tag{A.6}$$
$$[\phi_0, \pi_0] = i, \qquad [B_{k_1}, B_{k_2}^\dagger] = 2\pi\delta(k_1 - k_2).$$

These two bases are related by a Bogoliubov transform which can be used to write H_2 in the normal mode basis

$$H_2 = Q_1 + \frac{\pi_0^2}{2} + \int^+ \frac{dk}{2\pi}\omega_k B_k^\dagger B_k \tag{A.7}$$

where Q_1 is the one-loop kink mass and again it is implicit that the integral over k includes a sum over the shape mode. As this is a sum of oscillators, the one-loop ground state $|0\rangle_0$ is the state annihilated by π_0 and all B_k

$$\pi_0|0\rangle_0 = B_k|0\rangle_0 = 0. \tag{A.8}$$

The advantage of this basis is now clear, H_2 is diagonal and so its eigenstates may be used to begin a perturbative expansion of the eigenstates of the full H'.

As usual, a perturbative diagonalization of H' involves the inverse of H_2. As is, this inverse is difficult to define. However P' and H' commute and so we first restrict to states annihilated by P', which are shown to correspond to the translation-invariant eigenstates of H. With this restriction, H_2 is invertible and perturbation theory may proceed as usual.

To perform a perturbative expansion of any given state in the one-kink sector $|\psi\rangle$, we expand it in powers of the coupling $\sqrt{\lambda}/M$

$$|\psi\rangle = \sum_{i=0}^{\infty} |\psi\rangle_i. \tag{A.9}$$

Which states can be treated? We will see shortly that given the first term $|\psi\rangle_0$ one can construct $|\psi\rangle$, and so the question is, which states can be constructed at leading order? These are just the one-loop excitations of the kink, classified long ago in ref. [13]. They consist of the kink ground state plus all excitations of shape modes and any number of continuous excitations $B_k^\dagger$. At higher numbers of loops, multiple shape mode excitations and expected to mix with continuum modes and so, due to the fact that the former are L^2 normalizable and the later only δ-function normalizable, it may be that states with multiple shape mode excitations effectively disappear from the spectrum as a result of this mixing. Physically, multiple shape mode excitations are expected to decay to radiation.

To restrict our Hilbert space to the kernel of P', we further decompose it in terms of normal mode creation operators acting on the one-loop vacuum $|0\rangle_0$

$$|\psi\rangle_i = \sum_{m,n=0}^{\infty} |\psi\rangle_i^{mn}, \qquad |\psi\rangle_i^{mn} = Q_0^{-i/2} \int^+ \frac{d^n k}{(2\pi)^n} \gamma_i^{mn}(k_1 \cdots k_n)\phi_0^m B_{k_1}^\dagger \cdots B_{k_n}^\dagger |0\rangle_0. \tag{A.10}$$

Recall from eq. (2.3) that the classical mass Q_0 is proportional to the meson mass M times $(\lambda/M^2)^{-1}$ and so the $Q_0^{-i/2}$ factor ensures that each $|\psi\rangle_i$ indeed is of order $\left(\sqrt{\lambda}/M\right)^i$ as required by our expansion.

It is easy to show [15] that

$$P' = -\sqrt{Q_0}\pi_0 + P. \tag{A.11}$$

The first term is of order $(\sqrt{\lambda}/M)^{-1/2}$ and the second is of order unity. Therefore the condition $P'|\psi\rangle = 0$ relates adjacent orders

$$\sqrt{Q_0}\pi_0|\psi\rangle_{i+1} = P|\psi\rangle_i. \tag{A.12}$$

The right hand side can be evaluated explicitly using

$$\begin{aligned}P &= -\int dx\pi(x)\partial_x\phi(x) \qquad\qquad (A.13)\\ &= \int\frac{dk}{2\pi}\Delta_{kB}\left[i\phi_0\left(-\omega_k B_k^\dagger + \frac{B_{-k}}{2}\right) + \pi_0\left(B_k^\dagger + \frac{B_{-k}}{2\omega_k}\right)\right]\\ &\quad + i\int\frac{d^2k}{(2\pi)^2}\Delta_{k_1k_2}\left(-\omega_{k_1}B_{k_1}^\dagger B_{k_2}^\dagger + \frac{B_{-k_1}B_{-k_2}}{4\omega_{k_2}} - \frac{1}{2}\left(1+\frac{\omega_{k_1}}{\omega_{k_2}}\right)B_{k_1}^\dagger B_{-k_2}\right).\end{aligned}$$

Matching the coefficients γ in (A.12), one arrives at the recursion relation

$$\begin{aligned}\gamma_{i+1}^{mn}(k_1\cdots k_n) &= \Delta_{k_nB}\left(\gamma_i^{m,n-1}(k_1\cdots k_{n-1}) + \frac{\omega_{k_n}}{m}\gamma_i^{m-2,n-1}(k_1\cdots k_{n-1})\right) \qquad (A.14)\\ &\quad + (n+1)\int^+\frac{dk'}{2\pi}\Delta_{-k'B}\left(\frac{\gamma_i^{m,n+1}(k_1\cdots k_n,k')}{2\omega_{k'}} - \frac{\gamma_i^{m-2,n+1}(k_1\cdots k_n,k')}{2m}\right)\\ &\quad + \frac{\omega_{k_{n-1}}\Delta_{k_{n-1}k_n}}{m}\gamma_i^{m-1,n-2}(k_1\cdots k_{n-2})\\ &\quad + \frac{n}{2m}\int^+\frac{dk'}{2\pi}\Delta_{k_n,-k'}\left(1+\frac{\omega_{k_n}}{\omega_{k'}}\right)\gamma_i^{m-1,n}(k_1\cdots k_{n-1},k')\\ &\quad - \frac{(n+2)(n+1)}{2m}\int^+\frac{d^2k'}{(2\pi)^2}\frac{\Delta_{-k_1',-k_2'}}{2\omega_{k_2'}}\gamma_i^{m-1,n+2}(k_1\cdots k_n,k_1',k_2')\end{aligned}$$

at all $m > 0$, where the arguments k_i are symmetrized at each step. The recursion relation does not fix the terms $m = 0$ because these are not fixed by translation invariance [15]. This is clear from eq. (A.12) as the $m = 0$ terms are in the kernel of π_0 and so do not contribute to the left hand side.

The initial condition of the recursion is the one-loop state which is known exactly, for any given state, as the one-loop Hamiltonian is diagonal. In particular the one-loop kink ground state $|0\rangle_0$ corresponds to the initial condition

$$\gamma_0^{mn}(k_1\cdots k_n) = \delta_{m0}\delta_{n0} \tag{A.15}$$

and the excited state $B_k^\dagger|0\rangle_0$ corresponds to

$$\gamma_0^{mn}(k_1\cdots k_n) = \delta_{m0}\delta_{n1}2\pi\delta(k_1-k). \tag{A.16}$$

In the case of a discrete mode such as the shape mode $B_S^\dagger|0\rangle_0$, 2π times the Dirac delta is replaced by a Kronecker delta selecting the shape mode δ_{k_1S}.

To derive the excitation energy (3.1) one uses the recursion relation to find γ_1^{mn} and γ_2^{mn} at $m > 0$ for both the ground state and also the excited state of interest. For the ground state the result is presented in ref. [15] and for the excited states in ref. [16].

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Next, one needs to derive γ_1^{0n} and γ_2^{0n} for the ground state and excited kinks, which we recall are not fixed by translation invariance. This is done using old fashioned perturbation theory. First the interaction Hamiltonian given in eq. (A.3) is normal mode normal ordered using a Wick's theorem from ref. [28]

$$: \phi^j(x) :_a = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{j!}{2^m m!(j-2m)!} \mathcal{I}^m(x) : \phi^{j-2m}(x) :_b . \tag{A.17}$$

This theorem replaces each contraction of two $\phi(x)$ factors with the loop factor $\mathcal{I}(x)$ defined in eq. (3.5).

The vertices in our diagrammatic approach correspond to the normal mode normal ordered interaction terms. Therefore the $: \phi^3 :_a$ term in H_I leads to two vertices. The first vertex connects three lines and corresponds to the $\phi^3 :_b$ term on the right hand side of eq. (A.17) at $m = 0$. It appears twice in each diagram in the left panel of figure 1. The second vertex involves two lines, one which has one end on the vertex and one which has both ends on the vertex. This vertex corresponds to the $: \phi :_b$ term on the right hand side of (A.17) at $m = 1$. The coefficient of the vertex can be read from (A.17). It depends explicitly on x through the loop factor $\mathcal{I}(x)$. Note that a diagram consisting of only this second vertex and the connected lines is a tadpole diagram and indeed the tadpole diagrams visible in the middle panel of figure 1 contain this vertex. Similarly the last diagram in figure 1 contains the vertex corresponding to the $j = 4$, $m = 1$ term in eq. (A.17).

Once the interactions are normal mode normal ordered, all π_0 and B_k are on the right. Recalling that these annihilate the one-loop kink ground state $|0\rangle_0$, this exercise in perturbation theory is similar to ordinary vacuum-sector perturbation theory and so yields both the states $|0\rangle_i^{0n}$ and also the Hamiltonian eigenvalues corresponding to the energies of the excited and ground state kinks. This standard exercise is performed in detail in refs. [15] and [16] for the two kink states. Subtracting the two energies yields (3.1).

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