Eur. Phys. J. C (2022) 82:624
https://doi.org/10.1140/epjc/s10052-022-10578-5

THE EUROPEAN
PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

On quasinormal modes in 4D black hole solutions in the model with anisotropic fluid

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Received: 1 February 2022 / Accepted: 3 July 2022 / Published online: 19 July 2022

Abstract We consider a family of four-dimensional black hole solutions from Dehnen et al. (Grav Cosmol 9:153 arXiv:gr-qc/0211049, 2003) governed by natural number $q = 1, 2, 3, \dots$, which appear in the model with anisotropic fluid and the equations of state: $p_r = -\rho(2q - 1)^{-1}$, $p_t = -p_r$, where p_r and p_t are pressures in radial and transverse directions, respectively, and $\rho > 0$ is the density. These equations of state obey weak, strong and dominant energy conditions. For $q = 1$ the metric of the solution coincides with that of the Reissner–Nordström one. The global structure of solutions is outlined, giving rise to Carter–Penrose diagram of Reissner–Nordström or Schwarzschild types for odd $q = 2k + 1$ or even $q = 2k$, respectively. Certain physical parameters corresponding to BH solutions (gravitational mass, PPN parameters, Hawking temperature and entropy) are calculated. We obtain and analyse the quasinormal modes for a test massless scalar field in the eikonal approximation. For limiting case $q = +\infty$, they coincide with the well-known results for the Schwarzschild solution. We show that the Hod conjecture which connect the Hawking temperature and the damping rate is obeyed for all $q \geq 2$ and all (allowed) values of parameters.

1 Introduction

The decaying oscillations such as quasinormal modes (QNMs) [2–12] are at presence a very interesting and popular topic of investigations. A possible application of QNMs may be related to gravitational waves [13–15] emitted during the ringdown (final) stage of binary black hole (BH) mergers. It is belived that the frequencies of gravitational waves may be calculated by using certain superpositions of QNMs. The importance of these experiments is following: the analysis of experimental data may clarify the nature of gravity in the regime of strong fields.

In solving the quasinormal mode (QNM) problem for certain physical tasks (e.g. related to asymptotically flat black hole solutions) one should seek the solutions to a wave equation of the form $\Phi(t, x) = e^{-i\omega t}\Phi_*(x)$, where $\Phi_* = \Phi_*(x)$ obeys a Schrödinger-type equation

$$\left(-\epsilon^2 \frac{d^2}{dx^2} + V(x)\right)\Phi_* = \omega^2\Phi_*, \quad (1.1)$$

with $x \in (-\infty, +\infty)$ usually appearing as tortoise coordinate and $\epsilon > 0$, while typically $\epsilon = 1$ [8–11].

For a certain class of spherically symmetric solutions (which contain Schwarzschild, Reissner–Nordström ones and the solutions considered in the body of the paper) the potential is a smooth function obeying $V(x) > 0$, which tends to 0 either when $x \to -\infty$ (in approaching to horizon) or $x \to +\infty$ (in approaching to spatial infinity). By choosing (typically) the QNM frequencies ω as complex numbers obeying $\mathrm{Re}\,\omega > 0$ and $\mathrm{Im}\,\omega < 0$, one get the wave functions $\Phi(t, x) = e^{-i\omega t}\Phi_*(x)$ to be damped in time as $t \to +\infty$, while $|\Phi_*(x)|$ has an exponential growth (in $|x|$) as $|x| \to \infty$. The QNMs [10,11] are usually calculated by a analytical continuation method [5–7]. According to Ref. [12] this method reads as follows: one should start with the Schrödinger equation for a wave function $\Psi = \Psi(x)$

$$\left(-\hbar^2 \frac{d^2}{dx^2} - V(x)\right)\Psi = E\Psi. \quad (1.2)$$

It describes a (non-relativistic) quantum particle of mass 1/2 "moving" in the potential $-V(x)$. Let us suppose that the Schrödinger operator corresponding to (1.2) has non-empty

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discrete spectrum $E_n = E(\hbar, n| - V)$, where $n = 0, 1, \ldots$. (The corresponding eigen functions $\Psi = \Psi_n(x)$ should be exponentially decaying as $x \to \pm\infty$). Due to Ref. [12] one should put for QNM frequencies

$$\omega^2 = -E(\hbar = i\epsilon, n| - V). \tag{1.3}$$

Here $n = 0, 1, \ldots$ is called as overtone number.

In this article we deal with 4D black hole solutions from Ref. [1]. These solutions take place in the model with anisotropic fluid with the following equations of state:

$$p_r = -\rho/(2q - 1), \quad p_t = \rho/(2q - 1), \tag{1.4}$$

where p_r and p_t are pressures in radial and transverse directions, respectively, $\rho > 0$ is the density and $q = 1, 2, 3, \ldots$ is the natural number. (In (1.4) we put $c = 1$.) It may be readily verified that these equations of state obey weak ($\rho \geq 0$, $\rho + p_i \geq 0$), strong ($\rho + \sum_j p_j \geq 0$, $\rho + p_i \geq 0$) and dominant ($\rho \geq |p_i|$) energy conditions (here $(p_i) = (p_r, p_t, p_t)$).

Here we obtain and analyse the QNMs for a test massless scalar field in the eikonal approximation which is the main subject of the paper. By product we present the global structure of BH solutions under consideration and calculate certain physical parameters corresponding to them (gravitational mass, PPN parameters, Hawking temperature and entropy).

The paper is organised as follows. In Sect. 2 we present the black hole solutions from Ref. [1]. In Sect. 3 we analyse the global structure of the solutions. In Sect. 4 we calculate certain physical parameters which correspond to the solutions under consideration. In Sect. 5 we find the frequences of QNMs in the eikonal approximation which correspond to massless test scalar field in the background metric of our BH solutions with anisotropic fluid for $q = 1, 2, 3, \ldots$. Section 6 is devoted to special (integrable) cases $q = 1, 2, 3$ and the limiting case $q = +\infty$. In Sect. 7 we verify the validity of the Hod conjecture [16] for the solutions under consideration with $q > 1$.

2 The black hole solution

Here we consider the solutions to Einstein equations

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = \kappa T^\mu_\nu, \tag{2.1}$$

where $\kappa = 8\pi G/c^4$, G is Newton gravitational constant and c is speed of light.

The solutions under consideration are defined on the four-dimensional manifold with topology

$$M = \mathbb{R}_{(\text{radial})} \times \mathbb{S}^2 \times \mathbb{R}_{(\text{time})}. \tag{2.2}$$

Here the spherical coordinate system is used: $x^\mu = (r, \theta, \phi, t)$ with signature $(+++, -)$. The energy-momentum tensor of anisotropic fluid is taken as

$$(T^\mu_\nu) = \text{diag}\left(p_r, \ p_t, p_t, \ -\rho c^2\right), \tag{2.3}$$

and the equations of state read

$$p_r = -\rho c^2(2q - 1)^{-1}, \quad p_t = -p_r. \tag{2.4}$$

Here ρ is the mass density, p_r and p_t are pressures in radial and orthogonal (to radial) directions, respectively.

The parameter q describes relations between the pressures and the mass density; $q > 0$, $q \neq 1/2$. In the present paper, the parameter q is taken to be a natural number to avoid the non-analytical behaviour of the metric at the (would be) horizon.

The solution has the following form [1]:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (H(r))^{2/q}\left[\frac{dr^2}{1 - \frac{2\mu}{r}} + r^2 d\Omega^2 \right.$$
$$\left. -(H(r))^{-4/q}\left(1 - \frac{2\mu}{r}\right)c^2dt^2\right], \tag{2.5}$$

$$\kappa\rho c^2 = \frac{(2q - 1)P(P + 2\mu)(1 - 2\mu r^{-1})^{q-1}}{H(r)^{2+\frac{2}{q}}\ r^4}, \tag{2.6}$$

where the function $H(r)$ reads as follows:

$$H(r) = 1 + \frac{P}{2\mu}\left[1 - \left(1 - \frac{2\mu}{r}\right)^q\right]. \tag{2.7}$$

The metric on the sphere $\mathbb{S}^2$ is denoted by $d\Omega^2$; parameters $P, \mu > 0$ are arbitrary. Originally we put $r > 2\mu = r_h$ but the domain of definition of the metric will be extended below.

The equations of motion (2.1) imply the following relation for the scalar curvature

$$R[g] = -\kappa T^\mu_\mu = \frac{2(q - 1)}{2q - 1}\kappa\rho c^2, \tag{2.8}$$

which will be used below for identifying the singularities of solutions for $q = 2, 3, 4, \ldots$.

3 The global structure of the solution

In what follows we will use the following relation for the metric

$$ds^2 = -A(r)c^2dt^2 + (A(r))^{-1}dr^2 + C(r)d\Omega^2, \tag{3.1}$$

where

$$A = A(r) = (H^2(r))^{-1/q}\left(1 - \frac{2\mu}{r}\right), \tag{3.2}$$

$$C = C(r) = (H^2(r))^{1/q}r^2. \tag{3.3}$$

Here $A = A(r)$ is so-called "red shift function", $C(r) > 0$ is "area function".

The global structure of the solutions above may be studied by analysing the behaviour of the "redshift function" ($A(r)$)

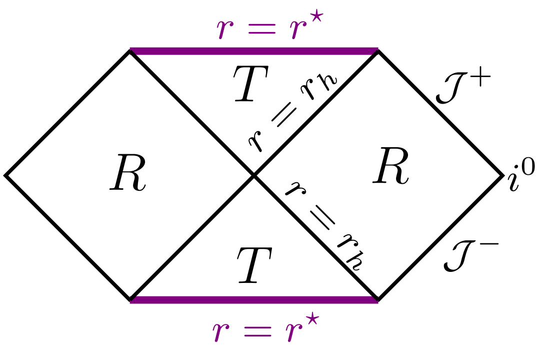

Fig. 1 Carter–Penrose diagram for case $q = 2, 4, 6, \ldots$

and the "area function" $C(r)$ (the factor at $d\Omega^2$) at critical points corresponding to horizons or singularities. The Carter–Penrose diagrams can be constructed for various values of the parameters using the standard algorithm [17]. For our BH solutions it was done in Ref. [18].

In what follows we denote by $r = r^\star$ the maximal root of the equation $H(r) = 0$. We have $r^\star < 0$ for odd $q = 2k + 1$ and $r^\star > 0$ for even $q = 2k$.

There are three classes of important critical points of the radial coordinate r for the metric (3.1):

(1) $r = r_h \equiv 2\mu$. This point corresponds to a regular external horizon.
(2) $r = r^\star$. This point corresponds to the singularity.
(3) $r = 0$ for odd $q = 2k + 1$. This point corresponds to internal horizon.

We introduce the following notations. Let **Sch**$[r_1, r_2]$ ($r_1 < r_2$) be a Carter–Penrose diagram of Schwarzschild type with a singularity at a point r_1 and a regular horizon at r_2 (Fig. 1). Similarly, we denote by **RN**$[r_1, r_2, r_3]$ ($r_1 < r_2 < r_3$) the diagram of Reissner–Nordström type with singularity at r_1, an internal horizon at r_2, and an external horizon at r_3 (Fig. 2).

As a result of analysis, we conclude that the structure of diagrams depends mostly on the parity of the parameter q:

- For $q = 2m$, $m \in \mathbb{N}$, we have a diagram of type **Sch**$[r^\star, r_h]$.
- For $q = 2m + 1$ the diagram is of type **RN**$[r^\star, 0, r_h]$.

Extremal case. Let us consider an extremal case of the solution under consideration when $\mu \to +0$. By using relations (2.5)–(2.7) we get in the limit $\mu \to +0$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (H_e(r))^{2/q} \left[dr^2 + r^2 d\Omega^2 - (H_e(r))^{-4/q} c^2 dt^2 \right], \quad (3.4)$$

$$\kappa \rho c^2 = \frac{(2q-1)P^2}{(H_e(r))^{2+\frac{2}{q}} r^4}, \quad (3.5)$$

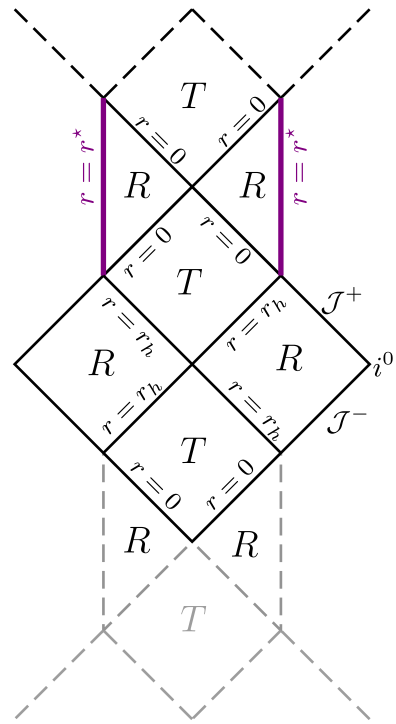

Fig. 2 Carter–Penrose diagram for case $q = 1, 3, 5, \ldots$

where

$$H_e(r) = 1 + \frac{Pq}{r}, \quad (3.6)$$

with $P > 0$. For $q > 1$ the metric (3.4) describes a naked singularity corresponding to $r \to +0$. Indeed, using relations (2.8) and (3.5) we obtain for the scalar curvature

$$R[g] = \frac{2(q-1)P^2}{(r + Pq)^{2+\frac{2}{q}} r^{2-\frac{2}{q}}}. \quad (3.7)$$

For $q = 2, 3, 4, \ldots$ we are led to relation: $R[g] \to +\infty$ as $r \to +0$, which tells us about the singularity corresponding to $r = +0$. For $q = 1$ the metric (3.4) is coinciding with the metric of extremal Reissner–Nordström solution with "double" horizon corresponding to $r = +0$ and singularity (center) at $r = -P + 0$.

4 Physical parameters

In this section we deal with some physical parameters of the solutions. Here we put for simplicity $c = \hbar = k_B = 1$.

4.1 Gravitational mass and PPN parameters

Let us consider the four-dimensional space-time with the metric (2.5) for $r > 2\mu$. Introducing a new radial variable $\bar{R}$

by the relation:

$$r = \bar{R}\left(1+\frac{\mu}{2\bar{R}}\right)^2, \tag{4.1}$$

we rewrite the metric in the following form:

$$ds^2 = H^{2/q}\left[-H^{-4/q}\left(\frac{1-\frac{\mu}{2\bar{R}}}{1+\frac{\mu}{2\bar{R}}}\right)^2 dt^2 + \left(1+\frac{\mu}{2\bar{R}}\right)^4 \delta_{ij}dx^i dx^j\right] \tag{4.2}$$

$i, j = 1, 2, 3$. Here $\bar{R}^2 = \delta_{ij}x^i x^j$.

The parametrized post-Newtonian (Eddington) parameters are defined by the well-known relations

$$g_{00} = -(1 - 2V + 2\beta V^2) + O(V^3), \tag{4.3}$$

$$g_{ij} = \delta_{ij}(1 + 2\gamma V) + O(V^2), \tag{4.4}$$

$i, j = 1, 2, 3$. Here

$$V = \frac{GM}{\bar{R}} \tag{4.5}$$

is the Newtonian potential, M is the gravitational mass and G is the gravitational constant.

From (4.2)–(4.4) we obtain:

$$GM = \mu + \frac{P}{q} \tag{4.6}$$

and

$$\beta - 1 = \frac{qA_f}{2(GM)^2}, \tag{4.7}$$

$$\gamma = 1, \tag{4.8}$$

where

$$A_f = P(P + 2\mu), \tag{4.9}$$

or, equivalently, $P = -\mu + \sqrt{\mu^2 + A_f} > 0$.

The parameter $\beta - 1$ is proportional to the ratio of two physical parameters: the anisotropic fluid density parameter A_f and the gravitational radius squared $(GM)^2$.

4.2 Hawking temperature and entropy

The Hawking temperature of the black hole may be calculated using the well-known relation [19]

$$T_H = \frac{1}{4\pi\sqrt{-g_{00}g_{rr}}}\frac{d(-g_{00})}{dr}\bigg|_{horizon}, \tag{4.10}$$

where here $g_{rr} = (A(r))^{-1}$, see (3.1).

We get

$$T_H = \frac{1}{8\pi\mu}\left(1+\frac{P}{2\mu}\right)^{-2/q}. \tag{4.11}$$

Here $q = 1, 2, \ldots$.

The Bekenstein–Hawking (area) entropy $S = \mathcal{A}/(4G)$, corresponding to the horizon at $r = 2\mu$, where $\mathcal{A}$ is the horizon area, reads

$$S_{BH} = \frac{4\pi\mu^2}{G}\left(1+\frac{P}{2\mu}\right)^{2/q}. \tag{4.12}$$

5 Quasinormal modes

In this section we derive quasinormal modes (in eikonal approximation) for our static and spherically symmetric solution (for given q) with the metric given (initially) in the following general form

$$ds^2 = -A(u)dt^2 + B(u)du^2 + C(u)d\Omega^2, \tag{5.1}$$

where $A(u)$, $B(u)$, $C(u) > 0$ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Note that in this section and below we use the Planck units, i.e. we put $\hbar = G = c = 1$.

We consider a test massless scalar field defined in the background given by the metric (4.2). The equation of motion in general is written in the form of the covariant Klein–Fock–Gordon equation

$$\Delta\Psi = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\Psi) = 0. \tag{5.2}$$

where $\mu, \nu = 0, 1, 2, 3$. In order to solve this equation we separate variables in function Ψ as follows

$$\Psi = e^{-i\omega t}e^{-\gamma}\Psi_*(u)Y_{lm}, \tag{5.3}$$

where Y_{lm} are the spherical harmonics, l is the multipole quantum number, $l = 0, 1, \ldots$ and $m = -l, \ldots, 0, \ldots, l$.

Equation (5.2), after using (5.3) yields the equation describing the radial function $\Psi_*(u)$ and having a Schrödinger-like form

$$\frac{d^2\Psi_*(u)}{du^2} + \left\{\frac{B}{A}\omega^2 - \frac{B}{C}l(l+1) - \gamma'' - (\gamma')^2\right\}\Psi_*(u) = 0 \tag{5.4}$$

where

$$\gamma = \frac{1}{2}\ln(B^{-1}C\sqrt{AB}) \tag{5.5}$$

and $\gamma' = d\gamma/du$, $\gamma'' = \frac{d^2\gamma}{du^2}$.

Taking into account above expressions one can examine our black hole solution which has the following form

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + H^a r^2 d\Omega^2, \tag{5.6}$$

where $A(r)$ and $C(r)$ according to Eq. (2.5) can be written as

$$A(r) = A = H^{-a}\left(1 - \frac{2\mu}{r}\right), \tag{5.7}$$

$$C(r) = C = H^a r^2 = \exp(2\gamma), \quad a = 2/q, \tag{5.8}$$

where

$$H(r) = 1 + \frac{P}{2\mu}\left[1 - \left(1 - \frac{2\mu}{r}\right)^q\right] = 1 + p(1 - z^q) \tag{5.9}$$

is the moduli function, $\mu > 0$, $P > 0$, $p = P/(2\mu)$, $q = 1, 2, \ldots$ and

$$z = 1 - \frac{2\mu}{r}, \quad r = \frac{2\mu}{1 - z}. \tag{5.10}$$

We note that $0 < z < 1$ for $r > 2\mu$.

After using the "tortoise" coordinate transformation

$$dr_* = \frac{dr}{A(r)} \tag{5.11}$$

the metric takes the following form

$$ds^2 = -Adt^2 + Adr_*^2 + Cd\Omega^2\,. \tag{5.12}$$

For the choice of the tortoise coordinate as a radial one ($u = r_*$) we have $A = B$ and

$$\gamma = \frac{1}{2}\ln C = \frac{1}{2}\ln(H^a r^2), \tag{5.13}$$

$a = 2/q$.

Thus, the Klein–Fock–Gordon equation becomes

$$\frac{d^2\Psi_*}{dr_*^2} + \left\{\omega^2 - V\right\}\Psi_* = 0, \tag{5.14}$$

where ω is the (cyclic) frequency of the quasinormal mode and $V = V(r) = V(r(r_*))$ is the effective potential

$$V = \mathcal{V} + \delta\mathcal{V}, \tag{5.15}$$

$$\mathcal{V} = \frac{l(l+1)A}{H^a r^2} = \frac{l(l+1)z(1-z)^2(1+p(1-z^q))^{-2a}}{(2\mu)^2}, \tag{5.16}$$

$$\delta\mathcal{V} = \gamma'' + (\gamma')^2 = (\sqrt{C})''/\sqrt{C}, \tag{5.17}$$

so that $\mathcal{V}$ is the eikonal part of the effective potential. Here and below we denote $F' = \frac{dF}{dr_*} = A\frac{dF}{dr}$.

In what follows we consider the so-called eikonal approximation when $l \gg 1$.

The maximum of the eikonal part of the effective potential is found from the extremum condition

$$\mathcal{V}' = A\frac{d\mathcal{V}}{dr} = 0 \tag{5.18}$$

or, equivalently,

$$\nu = \frac{d\mathcal{V}}{\mathcal{V}dz} = \frac{1}{z} - \frac{2}{1-z} + \frac{4pz^{q-1}}{1 + p(1 - z^q)} = 0, \tag{5.19}$$

or

$$pz^{q+1} - 3pz^q + (1+p)(3z - 1) = 0. \tag{5.20}$$

Proposition 1 *For any $P > 0$, $\mu > 0$ and $q \in \mathbf{N}$, the extremality relation (5.18) has only one solution for $r > 2\mu$, which is the point of maximum for $\mathcal{V}(r)$.*

The proposition is proved in Appendix. We denote this point of extremum by r_0. In terms of variable z we get that the point $z_0 = 1 - 2\mu/r_0$ is a unique solution to Eq. (5.20) for $z \in (0, 1)$.

The maximum of the eikonal part of the effective potential thus becomes

$$\mathcal{V}_0 = \mathcal{V}(r_0) = \frac{l(l+1)}{H^{2a}(r_0)r_0^2}\left(1 - \frac{2\mu}{r_0}\right). \tag{5.21}$$

In Fig. 3 we plot the reduced eikonal part of the effective potential $\mathcal{V}/(l(l+1))$ ($l \neq 0$) as a function of the radial coordinate r (left panel) and the tortoise coordinate r_* (right panel).

As can be seen from examples presented in figure for special fixed values of P and μ, the maximum of the effective potential is largest for $q = +\infty$ case and smallest for $q = 1$ case. The case with $q = 2$ is in the middle. At large distances the effective potential tends to zero, as expected.

The second derivative with respect to the tortoise coordinate in the point of extremum is given by

$$\mathcal{V}_0'' = \left.\frac{d^2\mathcal{V}}{dr_*^2}\right|_{r_*=r_*(r_0)} = A_0^2\left.\frac{d^2\mathcal{V}}{dr^2}\right|_{r=r_0} = A_0^2\left(\frac{dz}{dr}\right)^2\Bigg|_{r=r_0}\left.\frac{d^2\mathcal{V}}{dz^2}\right|_{z=z_0} = A_0^2\left(\frac{2\mu}{r_0^2}\right)^2\left.\frac{d^2\mathcal{V}}{dz^2}\right|_{z=z_0} \tag{5.22}$$

where $A_0 = A(r_0)$, see (5.7). The calculation of second derivative gives us

$$\left.\frac{d^2\mathcal{V}}{dz^2}\right|_{z=z_0} = \left.\frac{d}{dz}(\nu\mathcal{V})\right|_{z=z_0} = \mathcal{V}_0\left.\frac{d\nu}{dz}\right|_{z=z_0}, \tag{5.23}$$

where $\nu = \nu(z)$ is defined in (5.19). We get

$$\left.\frac{d\nu}{dz}\right|_{z=z_0} = -\frac{1}{z_0^2} - \frac{2}{(1-z_0)^2} + \frac{4p^2qz_0^{2q-2}}{(1+p(1-z_0^q))^2} + \frac{4p(q-1)z_0^{q-2}}{1+p(1-z_0^q)}. \tag{5.24}$$

The last two terms in this relation may be simplified by using the relation for the third term in (5.19). We obtain

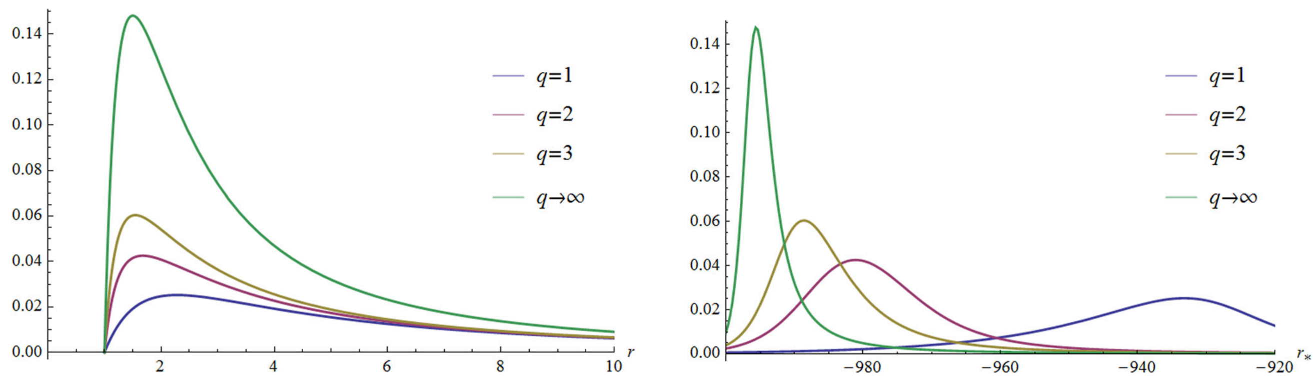

Fig. 3 The graphical representation of the reduced potential $\mathcal{V}/(l(l+1))$ as a function of the radial coordinate r (left panel) and the tortoise coordinate r_* (right panel) for $P = 2\mu = 1$, $q = 1, 2, 3$ and the limiting case $q \to +\infty$

$$\frac{dv}{dz}\bigg|_{z=z_0} = -\frac{1}{z_0^2} - \frac{2}{(1-z_0)^2} + \frac{q}{4}\left(\frac{1}{z_0} - \frac{2}{1-z_0}\right)^2 + \frac{q-1}{z_0}\left(-\frac{1}{z_0} + \frac{2}{1-z_0}\right) = -\frac{3q}{4z_0^2} + \frac{q-2}{z_0(1-z_0)^2}. \quad (5.25)$$

Thus, by using (5.22), (5.23) and (5.25) we find

$$\mathcal{V}_0'' = -\frac{1}{2}A_0^2\left(\frac{2\mu}{r_0^2}\right)^2 \mathcal{V}_0\mathcal{B}(z_0), \quad (5.26)$$

where

$$\mathcal{B}(z) = \frac{3}{2}q - \frac{2(q-2)z}{(1-z)^2}. \quad (5.27)$$

The square of the cyclic frequency in the eikonal approximation reads as following [10,11]

$$\omega^2 = \mathcal{V}_0 - i\left(n + \frac{1}{2}\right)\sqrt{-2\mathcal{V}_0''} + O(1), \quad (5.28)$$

where $l \gg 1$ and $l \gg n$. Here $n = 0, 1, \ldots$ is the overtone number. By choosing an appropriate sign for ω we get the asymptotic relations (as $l \to +\infty$) on real and imaginary parts of complex ω in the eikonal approximation

$$\mathrm{Re}(\omega) = \left(l + \frac{1}{2}\right) H_0^{-a} r_0^{-1} z_0^{1/2} + O\left(\frac{1}{l+\frac{1}{2}}\right), \quad (5.29)$$

$$\mathrm{Im}(\omega) = -\left(n + \frac{1}{2}\right) H_0^{-a} \mu r_0^{-2} \mathcal{B}_0^{1/2} + O\left(\frac{1}{l+\frac{1}{2}}\right), \quad (5.30)$$

where $H_0 = H(r_0)$ (see (5.9)), $r_0 = 2\mu/(1-z_0)$, and $z_0 \in (0, 1)$ is solution to master equation (5.20) and $\mathcal{B}_0 = \mathcal{B}(z_0)$, where $\mathcal{B}(z)$ is defined in (5.27).

We note that the parameters of the unstable circular null geodesics around stationary spherically symmetric and asymptotically flat black holes are in correspondence with the eikonal part of quasinormal modes of these black holes. See [20–22] and references therein. Due to Ref. [23] this correspondence is valid if certain restrictions on perturbations are imposed.

6 Special cases $q = 1, 2, 3$ and the limiting case $q = +\infty$

In this section we consider eikonal QNM for three cases $q = 1, 2, 3$ when the master equation (5.20) may be solved in radicals for all values of $p > 0$ and also in the limiting case $q = +\infty$.

6.1 The case $q = 1$

Let us consider the case $q = 1$ ($a = 2$). In this case the master equation (5.20) is just a quadratic one with two roots:

$$z_{\pm} = \frac{-3 \pm \sqrt{4p(p+1)+9}}{2p}, \quad (6.1)$$

Here

$$z_+ = z_0 = z_0(1, p), \quad (6.2)$$

is belonging to interval $(0, 1)$, while $z_- < 0$ is irrelevant for our consideration. We have

$$\frac{\partial z_0(1,p)}{\partial p} = \frac{3\sqrt{4p(p+1)+9} - 2p - 9}{2p^2\sqrt{4p(p+1)+9}} > 0. \quad (6.3)$$

We get that the fuction $z_0(1, p)$ is monotonically increasing and have the following limits: $z_0(1, p) \to 1/3$ as $p \to +0$ and $z_0(1, p) \to 1$ as $p \to +\infty$. For all values $p > 0$ we have

$$1/3 < z_0(1, p) < 1. \quad (6.4)$$

In this case the eikonal QNM (see (5.29) and (5.30)) read

$$\mathrm{Re}(\omega) = \left(l + \frac{1}{2}\right) H_0^{-2} r_0^{-1} z_0^{1/2} + O\left(\frac{1}{l + \frac{1}{2}}\right), \quad (6.5)$$

$$\mathrm{Im}(\omega) = -\left(n + \frac{1}{2}\right) H_0^{-2} \mu r_0^{-2} \sqrt{\frac{3}{2} + \frac{2z_0}{(1 - z_0)^2}} + O\left(\frac{1}{l + \frac{1}{2}}\right), \quad (6.6)$$

where $H_0 = 1 + \frac{P}{r_0}$, $r_0 = 2\mu/(1 - z_0)$ and $z_0 = z_0(1, p)$ is defined in (6.2).

It may be verified that relations (6.5), (6.6) may be rewritten as follows

$$\mathrm{Re}(\omega) = \left(l + \frac{1}{2}\right) \sqrt{\frac{\bar{M}}{\bar{r}_0^3} - \frac{Q^2}{2\bar{r}_0^4}} + O\left(\frac{1}{l + \frac{1}{2}}\right), \quad (6.7)$$

$$\mathrm{Im}(\omega) = -\left(n + \frac{1}{2}\right) \sqrt{\frac{\bar{M}}{\bar{r}_0^3} - \frac{Q^2}{2\bar{r}_0^4}} \sqrt{\frac{3\bar{M}}{\bar{r}_0} - \frac{2Q^2}{\bar{r}_0^2}} + O\left(\frac{1}{l + \frac{1}{2}}\right), \quad (6.8)$$

where $\bar{r}_0 = r_0 + P$, $\bar{M} = \mu + P = GM$ and

$$A_f = P(P + 2\mu) = \frac{1}{2} Q^2. \quad (6.9)$$

Here $\bar{r}_0$ corresponds to the position of the unstable, circular photon orbit in the Reissner–Nordström spacetime with the metric

$$ds^2 = -\bar{f}(\bar{r})dt^2 + (\bar{f}(\bar{r}))^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega_2^2, \quad (6.10)$$

where $\bar{f}(\bar{r}) = 1 - \frac{2GM}{\bar{r}} + \frac{Q^2}{2\bar{r}^2}$, with Q^2 given by (6.9). Our AF (anisotropic fluid) metric (2.5) for $q = 1$ is coinciding with the Reissner–Nordström one (6.10) when the following relation for radial coordinates $\bar{r} = r + P$ is imposed.

Relations (6.7), (6.8) for Reissner–Nordström spacetime were obtained in Ref. [24] for $n = 0$.

6.2 The case $q = 2$

Now we put $q = 2$ ($a = 1$). The master equation (5.20) in this case is just cubic one. It has a unique (real) solution $z_0 = z_0(2, p)$ for any $p > 0$ belonging to interval $(1/3, 1)$

$$z_0 = z_0(2, p) = Z^{1/3} - p^{-1} Z^{-1/3} + 1, \quad (6.11)$$

where

$$Z = Z(p) = \frac{1}{p}\left(\sqrt{1 + \frac{1}{p}} - 1\right). \quad (6.12)$$

The function $Z(p)$ is monotonically decreasing from $+\infty$ to $+0$ and has the asymptotics:

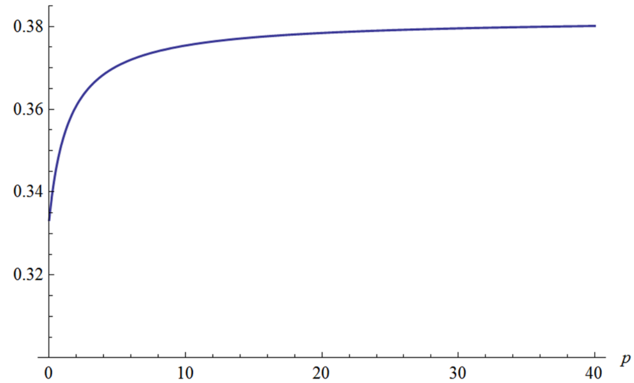

Fig. 4 The graphical representation of the function $z_0 = z_0(3, p)$

$Z(p) \sim p^{-3/2}(1 - \sqrt{p} + O(p))$ as $p \to +0$ and $Z(p) \sim 2^{-1} p^{-2}$ as $p \to +\infty$ which imply $z_0(2, p) \to 1/3$ as $p \to +0$ and $z_0(2, p) \to 1$ as $p \to +\infty$. It may be verified that the function $z_0(2, p)$ is monotonically inreasing from $1/3$ to 1.

The eikonal QNM for $q = 2$ read

$$\mathrm{Re}(\omega) = \left(l + \frac{1}{2}\right) H_0^{-1} r_0^{-1} z_0^{1/2} + O\left(\frac{1}{l + \frac{1}{2}}\right), \quad (6.13)$$

$$\mathrm{Im}(\omega) = -\left(n + \frac{1}{2}\right) H_0^{-1} \mu r_0^{-2} \sqrt{3} + O\left(\frac{1}{l + \frac{1}{2}}\right), \quad (6.14)$$

where $H_0 = 1 + \frac{P}{2\mu}\left[1 - \left(1 - \frac{2\mu}{r_0}\right)^2\right]$, $r_0 = 2\mu/(1 - z_0)$, and $z_0 = z_0(2, p)$ is defined in (6.11).

6.3 The case $q = 3$

Let us consider the last case $q = 3$, when the master equation (5.20) of fourth power has a solution in radicals (which was obtained by Mathematica):

$$z_0 = z_0(3, p) = \frac{1}{2}\sqrt{X} - \frac{\sqrt{Y}}{4\sqrt{3}} + \frac{3}{4}, \quad (6.15)$$

$$Y = 12 Z^{1/3} + 27 + 20\left(1 + \frac{1}{p}\right) Z^{-1/3}, \quad (6.16)$$

$$Z = \frac{p + 1}{2p^2}(9 + 3^{-3/2}\sqrt{2187 - 500p(p + 1)}), \quad (6.17)$$

$$X = -\frac{3\sqrt{3}}{2}\left(1 - \frac{8}{p}\right) Y^{-1/2} - Z^{1/3} - \frac{5(p + 1)}{3p} Z^{-1/3} + \frac{9}{2}. \quad (6.18)$$

It may be verified that $z_0 = z_0(3, p)$, given by relations (6.15)–(6.18), is real for all $p > 0$ and obey $1/3 < z_0 < \frac{3 - \sqrt{5}}{2} \approx 0,382$. This property is graphically illustrated on Fig. 4.

Relations (6.5), (6.6) in this case reads as follows

$$\mathrm{Re}(\omega) = \left(l + \frac{1}{2}\right) H_0^{-2/3} r_0^{-1} z_0^{1/2} + O\left(\frac{1}{l + \frac{1}{2}}\right), \tag{6.19}$$

$$\mathrm{Im}(\omega) = -\left(n + \frac{1}{2}\right) H_0^{-2/3} \mu r_0^{-2} \left(\frac{9}{2} - \frac{2z_0}{(1-z_0)^2}\right)^{1/2} + O\left(\frac{1}{l + \frac{1}{2}}\right), \tag{6.20}$$

where $H_0 = 1 + \frac{P}{2\mu}\left[1 - \left(1 - \frac{2\mu}{r_0}\right)^3\right]$, $r_0 = 2\mu/(1 - z_0)$, and $z_0 = z_0(3, p)$ is given by (6.15).

6.4 The case $q = +\infty$

In this case the relations (5.29) and (5.30) for QNM in eikonal approximation read as follows

$$\mathrm{Re}(\omega) = \left(l + \frac{1}{2}\right) \sqrt{\frac{\mu}{r_0^3}} + O\left(\frac{1}{l + \frac{1}{2}}\right), \tag{6.21}$$

$$\mathrm{Im}(\omega) = -\left(n + \frac{1}{2}\right) \sqrt{\frac{\mu}{r_0^3}} + O\left(\frac{1}{l + \frac{1}{2}}\right), \tag{6.22}$$

where $r_0 = 3\mu = 3GM$ corresponds the position where the black hole effective potential attains its maximum. We note that $r_0 = 3\mu$ is the radius of the photon sphere for the Schwarzschild black hole with the metric

$$ds^2 = -\left(1 - \frac{2\mu}{r}\right) dt^2 + \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{6.23}$$

which is coinciding with limiting case of our AF metric (2.5) when $q = +\infty$.

We note that relations (6.21), (6.22) for Schwarzschild spacetime were obtained in Ref. [5].

Remark Here we restrict our choice of a test field by a massless (spin-zero, non-charged) scalar field which is the simplest "perturbation" to study. It may be shown that the consideration of a test Maxwell field on our black hole background will lead us to two equations on functions: $\Psi_{*,a} = a_{lm}(r_*)$ and $\Psi_{*,b} = b_{lm}(r_*)$, which are certain combinations of coefficients (and their derivatives) coming from decomposing of vector potential in (vector) spherical harmonics. These equations (one of them is just an integrability condition) look like Eq. (5.14) but with another potential $V = \mathcal{V}$, instead of (5.15) ($\delta V = 0$ in this case). Thus, we will obtain the same spectrum of QNM in eikonal approximation for a test Maxwell field as for a massless scalar field considered here.

7 Hod conjecture

Here we verify the conjecture by Hod [16] on the existence of quasi-normal modes obeying the inequality

$$|\mathrm{Im}(\omega)| \leq \pi T_H, \tag{7.1}$$

where T_H is Hawking temperature.

We note the Hod conjecture has been tested in theories with higher curvature corrections such as the Einstein–Dilaton–Gauss–Bonnet and Einstein-Weyl for the Dirac field (with positive result) [25]. (For negative result see Ref. [26].) Recently, we have also verified the Hod conjecture (with positive result) for a solution with dyon-like dilatonic black hole [27] for certain values of dimensionless parameter $a \in [0, 1]$.

Here we verify this conjecture by using the obtained eikonal relations (5.30) for $\mathrm{Im}(\omega)$ and the relation for the Hawking temperature (4.11). For our purpose it is sufficient to check the validity of the inequality

$$y = y(p, q) \equiv \frac{|\mathrm{Im}(\omega_{\mathrm{eik}})(n = 0)|}{\pi T_H} = \left[\frac{1 + p}{1 + p(1 - z_0^q)}\right]^{2/q} \times (1 - z_0)^2 \sqrt{\frac{3}{2}q - \frac{2(q-2)z_0}{(1-z_0)^2}} < 1, \tag{7.2}$$

for all $p = P/\mu > 0$, $q = 2, 3, \ldots$, where $z_0 = z_0(p, q)$ is unique solution to master equation (5.20), which obeys $0 < z_0 < 1$, see Lemma in Appendix.

In (7.2) we use the limiting "eikonal value" given by the first term in (5.30) for the lowest overtone number $n = 0$.

Proposition 2 *The dimensionless parameter $y = y(p, a)$ from (7.2) obeys the inequality: $y < 1$ for all $p > 0$ and $q \in \{2, 3, 4, \ldots\}$.*

Proof First we consider the case $q > 2$. In what follows we use the relation

$$\frac{1}{3} < z_0 = z_0(p, q) < 0.4, \tag{7.3}$$

for all $p > 0$ and $q > 2$. Indeed, it follows from relations (A.16)–(A.18) given at Appendix that

$$\frac{1}{3} < z_0 = z_0(p, q) < z_*(q) \leq z_*(3) = \frac{3 - \sqrt{5}}{2} \approx 0,382, \tag{7.4}$$

for all $p > 0$ and $q \geq 3$. Thus, relation (7.3) is correct.

In what follows we use the following splitting

$$y = y_1 y_2 y_3, \tag{7.5}$$

$$y_1 = \left[\frac{1 + p}{1 + p(1 - z_0^q)}\right]^{2/q}, \quad y_2 = (1 - z_0)^2, \quad y_3 = \sqrt{\mathcal{B}(z_0)}, \tag{7.6}$$

where $\mathcal{B}(z) = \frac{3}{2}q - \frac{2(q-2)z}{(1-z)^2}$.

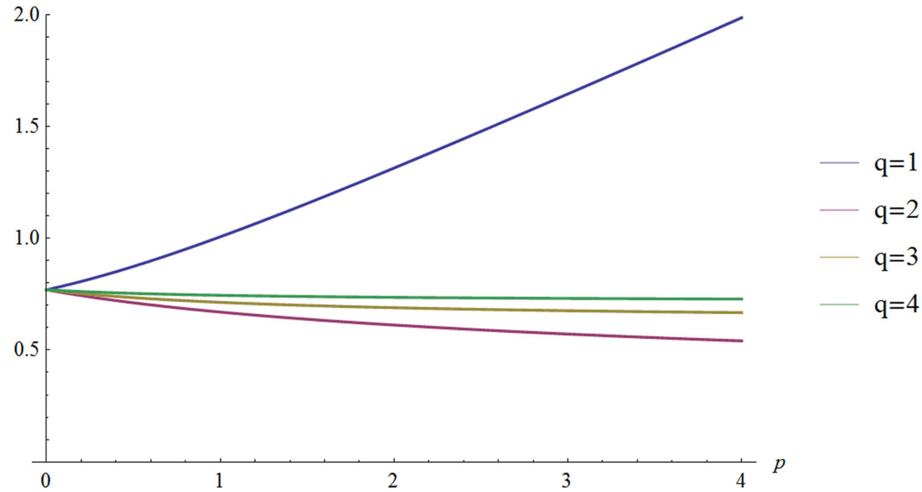

Fig. 5 The graphical representation of the function $y(p, q)$ for $q = 1, 2, 3, 4$

For y_1 we obtain from (7.3)

$$y_1 = y_1(p, q) = \left[\frac{1}{1 - \frac{p}{p+1} z_0^q}\right]^{2/q} < \left[\frac{1}{1 - z_0^q}\right]^{2/q} < \left[\frac{1}{1 - (0.4)^q}\right]^{2/q}, \tag{7.7}$$

for all $p > 0$ and $q > 2$. Now, we use the following fact about the function

$$\tilde{f}(q) = \left[\frac{1}{1 - u^q}\right]^{2/q}, \tag{7.8}$$

where $0 < u < 1$ and $q > 0$. Namely, the function $\tilde{f}(q)$ is monotonicall decreasing in $(0, +\infty)$. This follows from the relation

$$\frac{d\tilde{f}(q)}{dq} = \tilde{f}(q) \frac{2}{q^2(1 - x)} [(1 - x) \ln(1 - x) + x \ln x] < 0, \tag{7.9}$$

where $x = u^q$ and $0 < x < 1$. This fact imlies for $u = 0.4$ the following bound

$$y_1 = y_1(p, q) < \left[\frac{1}{1 - (0.4)^q}\right]^{2/q} \leq \left[\frac{1}{1 - (0.4)^3}\right]^{2/3} \approx 1.04507975. \tag{7.10}$$

for all $p > 0$ and $q \geq 3$. Hence, we get

$$y_1 = y_1(p, q) < 1.0451, \tag{7.11}$$

for all $p > 0$ and $q > 2$.

For y_2 we obtain from (7.3)

$$y_2 = y_2(p, q) = (1 - z_0)^2 < \frac{4}{9}, \tag{7.12}$$

for all $p > 0, q > 2$.

The last bound

$$y_3 = y_3(p, q) = \sqrt{\mathcal{B}(z_0)} < \sqrt{\mathcal{B}(1/3)} = \sqrt{3}, \tag{7.13}$$

is also valid for all $p > 0$ and $q > 2$. It follows from monotonical decreasing of the function $\mathcal{B}(z)$ in $(0, 1)$ and $1/3 < z_0 < z_* < z_1$. Here $\mathcal{B}(z) > 0$ for $z \in (0, z_1)$ and $z_* = z_*(q)$, $z_1 = z_1(q)$ are defined in Appendix.

Plugging the bounds (7.11)–(7.13) into (7.5) we find

$$y = y(p, q) < 1.0451 \times (4/9) \times \sqrt{3} \approx 0.804518, \tag{7.14}$$

and hence

$$y = y(p, q) < 0.80452 < 1, \tag{7.15}$$

for all $p > 0, q > 2$.

This result can be illustrated by a numerical plot of the function $y(p, q)$ for a particular set of values of q, depicted on Fig. 5. For $q = 2$ the validity of Proposition 2 was verified numerically.

We note that recently, some examples of the violation of the Hod conjecture have been discussed for certain black hole solutions in supergravity and other theories [21].

Remark Let us comment also on the case $q = 1$ which gives us the Reissner–Nordström metric. It may be readily verified that in this case the inequality (7.2) is not satisfied for all values of p: it is valid only for $0 < p < p_{cr}$, where p_{cr} is some critical value of parameter p [27]. As it was pointed out in [27] the violation of the Hod inequality in the eikonal regime for certain p (and $n = 0$) does not close the possibility for the obeying this relation for exact values of QNM for certain $l = 0, 1, 2, \ldots$ and all values of parameter p.

8 Conclusion

Here we have studied a non-extremal black hole solutions in a four-dimensional gravitational model with anisotropic fluid proposed in Ref. [1]. The equations of state for the fluid (1.4) contains a parameter q which is natural number $q = 1, 2, 3 \ldots$. We have outlined the global structure of solutions under consideration: for odd $q = 2k + 1$ the

Carter–Penrose diagram is coinciding with that of Reissner–Nordström metric (the case of time-like singularity hidden by two horizons) while for even $q = 2k$ it is coinciding with that of Schwarzschild metric (the case of space-like singularity hidden by one horizon). For $q = 1$ the metric of the solution [1] coincides with the metric of the Reissner–Nordström solution while in the limit $q = +\infty$, we get the metric of the Schwarzschild solution. We have also presented certain physical parameters corresponding to BH solutions: gravitational mass M, Hawking temperature, black hole area entropy.

We have examined the solutions to massless Klein–Fock–Gordon equation in the background of our static BH metric for given $q = 1, 2, 3 \ldots$. By using the tortoise coordinate we have reduced this equation to radial one governed by certain effective potential. This potential contains the parameters of solution such as $P > 0$, $\mu > 0$, natural parameter q and also l which is the multipole quantum number, $l = 0, 1, \ldots$.

Here we have studied the eikonal part of the effective potential for large l and have found a master equation for the value $z_0 = 1 - 2\mu/r_0$, where r_0 is the value of the radial coordinate (radius) r_0 corresponding to the maximum of the eikonal part of the effective potential. By using the maximum value of (the eikonal part) of the effective potential $\mathcal{V}_0$ and r_0, we have calculated the cyclic frequencies of the QNMs in the eikonal approximation up to solution of the master equation in z_0. Since the master equation is an algebraic equation of order $q + 1$ in z_0 we were able to find analytical exact solutions for $q = 1, 2, 3$. For obtained values of eikonal QNMs we have also considered special cases $q = 1, 2, 3$ and a limiting cases $q = +\infty$. For $q = 1$ our (eikonal) relations are compatible with the well-known result for Reissner–Nordström solution [24] (for $n = 0$), while for $q = +\infty$ they in an agreement with the well-known result for the Schwarzschild solution [5].

We have also tested the validity of the Hod conjecture for our solutions by considering QNMs (eikonal) frequences with the lowest value of the overtone number $n = 0$. We have shown that the Hod conjecture is valid in the range of $q > 1$. This assumption is valid for these values of $q > 1$ since it is supported by examples of states with large enough values of the multipole number l.

We note, that the results obtained here for eikonal QMN modes of test massless (non-charged) scalar field are also valid for some other test fields, e.g. for electromagnetic one. This may be considered (by product) in a separate publication. (The results of Refs. [28,29] may be also used in future work.)

Acknowledgements This paper has been supported by the RUDN University Strategic Academic Leadership Program (recipients: V.D.I.—mathematical model development and S.V.B.—simulation model development). The reported study was partially funded by RFBR, project number 19-02-00346 (recipients S.V.B. and V.D.I.—physical model development).

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: All data generated or analysed during this study are included in this published article.]

Funded by SCOAP^3. SCOAP^3 supports the goals of the International Year of Basic Sciences for Sustainable Development.

Appendix A

Here we prove the Proposition 1. Since the extremality condition (5.18) for the effective potential $\mathcal{V}$ $(r > 2\mu)$ is equivalent to the master equation (5.20) $(z = 1 - 2\mu/r)$, and the second derivative $\mathcal{V}_0''$ at the point of extremum is given by relation (5.26) with $\mathcal{V}_0 > 0$ (see (5.21)), the Proposition 1 is equivalent to the following Lemma.

Lemma *For any* $p > 0$ *and* $q \in \mathbf{N} = \{1, 2, 3 \ldots\}$*, the master equation*

$$p z^{q+1} - 3 p z^q + (1 + p)(3z - 1) = 0 \tag{A.1}$$

has only one solution $z_0 = z_0(p, q)$*, belonging to interval* $(0, 1)$. *This solution obeys the inequality*

$$\mathcal{B}(z_0) = \frac{3}{2} q - \frac{2(q-2) z_0}{(1 - z_0)^2} > 0 \tag{A.2}$$

for all $p > 0$ *and* $q \in \mathbf{N}$.

Proof Since $z = 1/3$ is not a solution to Eq. (A.1) we present the master equation in the following form

$$F(z) = F(z, q) = z^q \frac{z - 3}{3z - 1} = -b = -1 - \frac{1}{p} < 0, \tag{A.3}$$

$p > 0$. The functions $F(z) = F(z, q)$, $q = 1, 2, 3, 4$, are presented at Fig. 6. It follows from the definition (A.3) that

$$F(z, q) > 0, \tag{A.4}$$

for $z \in (0, 1/3)$, $q \in \mathbf{N}$ and

$$\lim_{z \to 1/3 \pm 0} F(z, q) = \mp\infty, \tag{A.5}$$

$$\lim_{z \to 1 - 0} F(z, q) = -1, \tag{A.6}$$

for all $q \in \mathbf{N}$. Hence the seminterval $(0, 1/3]$ should be excluded in our search the solution to Eq. (A.3).

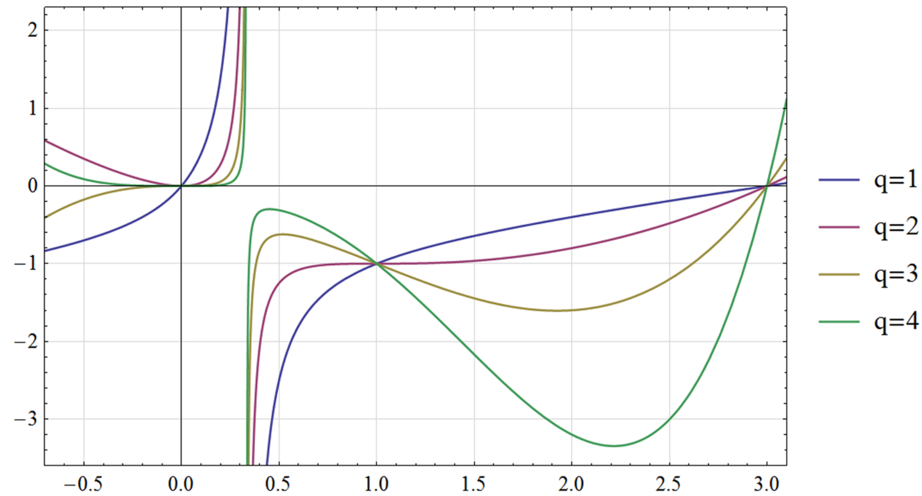

Fig. 6 The graphical representation of the functions $F(z) = F(z, q)$ for $q = 1, 2, 3, 4$

Let us analyze behavior of the function $F(z) = F(z, q)$ for $z \in (1/3, 1)$ and fixed $q \in \mathbf{N} = \{1, 2, 3 \dots\}$. The first derivative reads

$$\frac{dF(z)}{dz} = \frac{\partial F(z,q)}{\partial z} = z^{q-1}\frac{[3qz^2 + (8-10q)z + 3q]}{(3z-1)^2}. \tag{A.7}$$

For $q = 1, 2$, we have $\frac{dF(z)}{dz} > 0$ for $z \in (1/3, 1)$ and hence the function $F(z)$ is monotonically increasing from $-\infty$ to -1, when $z \in (1/3, 1)$. By applying the Intermediate Value Theorem to our continuous monotonically increasing function $F(z) = F(z, q)$, $q = 1, 2$, we get that for any $p > 0$ there exist unique $z_0(p, q) \in (0, 1)$, with $z_0(p, q) > 1/3$, which obeys Eq. (A.1). [1] Inequality (A.2) is obviously satisfied for $q = 1, 2$. That means that the Lemma is valid for $q = 1, 2$.

Now we consider the case $q > 2$. From (A.7) we obtain that the there exists a unique point of extremum of the function $F(z, q)$ in the interval $(1/3, 1)$

$$z_1 = z_1(q) = \frac{10q - 8 - \sqrt{(16q-8)(4q-8)}}{6q}, \tag{A.8}$$

$1/3 < z_1(q) < 1$, which is the first root of the quadratic equation $3qz^2 + (8 - 10q)z + 3q = 0$. The second root $z_2(q) = 1/z_1(q) \in (1, 3)$ is irrelevant for our consideration.

The calculations give us: $z_1(3) = (11 - 2\sqrt{10})/9 \approx 0,5195$, $z_1(4) = (4 - \sqrt{7})/3 \approx 0,4514$, $z_1(5) = (7 - 2\sqrt{6})/5 \approx 0,4202$ and $F(z_1(3)) \approx -0,6227$, $F(z_1(4)) \approx -0,2987$, $F(z_1(5)) \approx -0,1297$. We note that

$$z_1(q+1) < z_1(q), \tag{A.9}$$

for all $q > 2$. This follows from monotonical decreasing of the function $z_1(q)$ for $q > 2$, since $z_1(q) = 1/z_2(q)$ and

$$z_2(q) = \frac{10 - 8/q + \sqrt{(16-8/q)(4-8/q)}}{6}, \tag{A.10}$$

is monotonically increasing in q for $q > 2$.

It may be verified that

$$z_1(q) \to \frac{1}{3}, \quad F(z_1(q)) \to 0, \tag{A.11}$$

for $q \to +\infty$. Indeed, it follows from (A.8) that

$$z_1(q) = \frac{1}{3} + \frac{1}{3q} + O(q^{-2}), \tag{A.12}$$

and

$$F(z_1(q)) \sim \frac{1}{3^q}\left(1 + \frac{1}{q}\right)^q \left(-\frac{8}{3}\right) q \sim -\frac{8e}{3^{q+1}} q \to 0 \tag{A.13}$$

as $q \to +\infty$.

The function $F(z) = F(z, q)$ (for $q > 2$) is monotonically increasing in the interval $(1/3, z_1)$, since $\frac{dF(z)}{dz} > 0$ in this interval, see (A.7), while it is monotonically decreasing in the interval $(z_1, 1)$ due to inequality $\frac{dF(z)}{dz} < 0$ which is valid there. Hence we get

$$F(z_1(q), q) > F(z, q) > F(1, q) = -1 \tag{A.14}$$

for all $z \in (z_1, 1)$ and $q > 2$. This implies that the semi-interval $[z_1(q), 1)$ should be excluded in our search of solution to Eq. (A.3) for a given $q > 2$. Thus, we restrict our consideration to $z \in (1/3, z_1(q))$.

Let us define $z_*(q) \in (1/3, z_1(q))$, which obeys the following equation

$$F(z_*(q), q) = -1, \tag{A.15}$$

$q > 2$. By applying the Intermediate Value Theorem for a continuos monotonically increasing function $F(z(q), q)$ defined on $(1/3, z_1(q))$ and using (A.5) and (A.14) one can

[1] We remind that the Intermediate Value Theorem states that if F is a continuous function defined on the interval $[a, b]$, then it takes on any given value between $F(a)$ and $F(b)$ at some point of this interval.

readily prove that such point does exist and is unique for any $q > 2$.

The calculations give us

$$z_*(3) = \frac{3-\sqrt{5}}{2} \approx 0,382, \quad z_*(4) \approx 0,346, \quad z_*(5) \approx 0,337. \tag{A.16}$$

It may be proved that

$$z_*(q+1) < z_*(q), \tag{A.17}$$

for any natural $q > 2$. Indeed, if we suppose that $z_*(q+1) \geq z_*(q)$ for some q we get from monotonical increasing of the function $F(z, q+1)$ in $(1/3, z_1(q+1))$ and obvious inequality $F(z, q+1) > F(z, q)$ for $z \in (1/3, 1)$ that

$$-1 = F(z_*(q+1), q+1) \geq F(z_*(q), q+1) > F(z_*(q), q) = -1$$

and hence we come to a contradiction. Thus, the chain of inequalities (A.17) is correct.

Now we return to our original equation (A.3). From monotonical increasing of the function $F(z, q)$ in $(1/3, z_1(q))$ we get that $F(z) \geq F(z_*(q)) = -1$ for $z \in [z_*(q), z_1(q))$ and hence the semi-interval $[z_*(q), z_1(q))$ should be excluded for our consideration of (A.3). By applying once more the Intermediate Value Theorem for a continuos monotonically increasing function $F(z(q), q)$ defined on $(1/3, z_*(q))$ and using (A.5) and (A.15) we can find that the point z_0 which obeys the equation (A.3) does exist, belongs to $(1/3, z_*(q))$ and is unique for any $q > 2$ and $p > 0$. We denote this point as $z_0 = z_0(p, q)$. Thus, we have

$$1/3 < z_0(p, q) < z_*(q) < z_1(q), \tag{A.18}$$

for all $q > 2$ and $p > 0$. It follows from (A.11) and (A.18)

$$z_0(p, q) \to \frac{1}{3}, \tag{A.19}$$

as $q \to +\infty$ uniformly in $p \in (0, +\infty)$.

We note that one can present the solution as

$$z_0(p, q) = F_q^{-1}\left(-1 - \frac{1}{p}\right), \tag{A.20}$$

where F_q^{-1} is the function which is inverse to the function $F_q : (1/3, z_*(q)) \longrightarrow (-\infty, -1)$, defined as $F_q(z) = F(z, q)$. The function F_q^{-1} is a continuos and monotonically increasing one (due to a proper theorem on inverse function). It may be readily verified that

$$\lim_{p \to +\infty} z_0(p, q) = z_*(q), \tag{A.21}$$

and

$$\lim_{p \to +0} z_0(p, q) = 1/3. \tag{A.22}$$

Thus, the first part of the Lemma is proved for all $q \in \mathbf{N}$. Now, we should prove the second part of the Lemma for $q > 2$ (for $q = 1, 2$ it was checked above). Let us consider the function

$$\mathcal{B}(z) = \frac{3}{2}q - \frac{2(q-2)z}{(1-z)^2} \tag{A.23}$$

for $z \in (0, 1)$ and $q = 3, 4, \ldots$. We get

$$\mathcal{B}(z) = \frac{3qz^2 + (8 - 10q)z + 3q}{2(1-z)^2} = \frac{3q(z - z_1(q))(z - z_2(q))}{2(1-z)^2}, \tag{A.24}$$

where $z_1(q) < 1$ and $z_2(q) > 1$ are defined by relations (A.8) and (A.10), respectively. We find that $\mathcal{B}(z) > 0$ for all $z \in (0, z_1(q))$ and hence for $z = z_0(p, q)$ with $q > 2$ and $p > 0$. We remind that $1/3 < z_0(p, q) < z_*(q) < z_1(q)$ for all $q > 2$ and $p > 0$. Thus, the inequality (A.2) is satisfied. The Lemma is proved.

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