Thermodynamics of novel scalar–tensor-Born–Infeld black holes

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Abstract Two novel classes of four-dimensional exact black hole (BH) solutions have been obtained in the scalar–tensor (ST) theory which are coupled to Born–Infeld (BI) electrodynamics. To this end, a conformal transformation (CT) has been applied which transforms the action of ST–BI gravity to that of Einstein–dilaton–BI theory. The scalar-coupled BI theory, which has been introduced here, slightly differs from those have been used, previously. The analytical solutions have been obtained in the Einstein frame (EF) and two classes of charged dilatonic BHs, with unusual asymptotic behaviors, have been presented. All the solutions coincide with the corresponding values of Einstein–dilaton–Maxwell theory, in the limit of large BI parameter. By calculating thermodynamic parameters and, noting the Smarr mass relation, we showed that the first law of BH thermodynamics (FLT) is valid for the novel dilatonic BHs. Stability of the BHs has been investigated in EF, making use of the canonical ensemble method and noting the signature of the BH heat capacity (HC). Next, by use of the inverse CT, the solutions of ST theory have been obtained from their EF counterparts. Although, the entropy of ST BHs violates entropy-area law, the thermodynamic and conserved quantities have been obtained noting their conformal invariance property. It has been found that the ST BHs have the same thermodynamic and stability properties as the Einstein–dilaton ones.

1 Introduction

Despite the great and outstanding predictions of Einstein's gravity theory, such as existence and detection of the BHs [1–3], the related failures imply that it is needed to be completed. Disablement of Friedman equations, as the direct consequences of general relativity, in describing the positive acceleration of the expanding universe is an important and famous problem. Extending this theory to the successful theories of modified gravity is an attempt to address the existing challenges [4–9]. The ST gravity is an alternative theory which was established with the aim of explaining the problems of initial gravity theory. Nowadays, ST modified gravity is known as an interesting theory, because in one side by introducing an inflaton scalar field, it can explain the inflation phase of the universe [10–13] and, in other side the fundamental string theory, in the low-energy regime, reduces to an effective ST one [14–16]. The Jordan frame (JF) field equations, obtained by varying the ST action, are strongly coupled and obtaining the exact analytical solutions is too difficult. It is possible to translate the JF ST action to that of EF, known as the action of Einstein–dilaton (Ed) gravity theory, by utilizing CTs [17–20]. The Ed theory, in which Einstein's original action is coupled to a scalar field, also comes from the low-energy limit of string theory. It is customary to solve the equations of motion in the Ed theory and to obtain the JF solutions by applying inverse CTs to the Ed counterparts [21,22]. According to the entropy-area law, the BH entropy is equal to one-fourth of the horizon area, but this approximately universal law is violated by the ST BHs. Through use of the Euclidean action method, it has been proved that BHs' thermodynamic quantities such as temperature, entropy and electric potential on the horizon and, BHs' mass and electric charge, as the conserved quantities, remain invariant under CTs. Therefore, thermodynamic and conserved quantities of ST BHs are the same as already obtained for Ed BHs [23,24].

It is worth mentioning that ST gravity recovers the Brans–Dicke (BD) theory as an especial case. The BD gravity was initially proposed in 1961 by Brans and Dicke as an extension of original gravity theory by adding a scalar degree of freedom to include Mach's principle. This scalar field which is minimally coupled to gravitation is also inversely proportional to the gravitational constant G [25–27]. It has been found that the exact four-dimensional solutions of BD–Maxwell theory, due tho the conformal invariance of

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Maxwell Lagrangian, are just the Reissner–Nordstrom BHs with a trivial constant scalar field. While in the higher-dimensional spacetimes, where the Maxwell theory is no longer conformal-invariant and through coupling with the scalar field plays the role of source, the BH solutions have been obtained with the nontrivial scalar field [28–30]. Some other studies of BD charged exact solutions in the presence of non(linear) electrodynamics and its cosmological applications can be seen in Refs. [31–36]. This theory is conformally related to the Ed modified gravity, which has been studied in the presence of self-interacting Liouville-type potentials [37–43].

Maxwell's theory of electromagnetism, known as the linear classical electrodynamics, leads to the infinite electric field at the position of charged point-like particles. Different models of non-linear electrodynamics, such as power-law, Euler–Lagrange, Born–Infeld, exponential and logarithmic ones, have been proposed with the aim of solving the problems of linear electrodynamics. Dynamics of the universe have been studied through different models of cosmology by taking into account the impacts of non-linear electromagnetic fields [44–50]. Application of these theories for obtaining non-linear charged black branes and BHs in various spacetime dimensions have generated many interesting results [51–58]. Thermodynamics and thermal stability of the charged BHs have been investigated, under the influence of non-linear electrodynamic models, in three-, four- and higher-dimensional spacetimes [59–65]. Recently, these studies have been extended to the three- and four-dimensional ST BHs by using BI and Maxwell models of electrodynamics in Refs. [66] and [67], respectively. Here, we tend to obtain exact ST–BI BH solutions and, to study thermodynamics and thermal stability of the four-dimensional JF BHs.

This paper has been outlined as follows: In Sect. 2, by utilizing CTs the JF action of ST–BI theory has been translated to that of EF, named as the Ed–BI theory. Section 3 has been devoted to solving the Ed–BI field equations in a static and spherically symmetric geometry. Consequently, two classes of novel Ed–BI BHs, with unusual asymptotic behaviors, have been obtained. Our solutions reduce to the Ed–Maxwell ones if the nonlinearity parameter of BI theory goes to infinity. Thermodynamic variables, mass and charge of Ed–BI BHs have been calculated in Sect. 4. It has been shown that the FLT is valid for novel Ed–BI BH solutions. Thermal stability of Ed–BI BHs have been analyzed based on the heat capacity (HC) of dilaton BHs in Sect. 5. The exact ST BH solutions have been obtained from those of Ed theory by use of the inverse CTs. It leads to the new ST BH solutions which can produce horizon-less, one-horizon and two-horizon ST–BI BHs. In Sect. 6, thermodynamic and stability properties of the ST–BI BHs have been discussed based on their similar ones in the EF. The paper will be finished by highlighting the results in Sect. 7.

2 The general formalism

The four-dimensional JF action of ST gravity theory can be written as [20–22]

$$I^{(ST)} = -\frac{1}{16\pi} \int \sqrt{-\bar{g}} \left[f_1(\psi)\bar{\mathcal{R}} - f_2(\psi)(\bar{\nabla}\psi)^2 - f_3(\psi) + L(\bar{\mathcal{F}}) \right] d^4x. \quad (2.1)$$

The bar sign has been used over to introduce the JF quantities. Thus, $\bar{\mathcal{R}}$ and $\bar{\nabla}$ are the Ricci scalar and covariant derivative i the JF with metric $\bar{g}_{\mu\nu}$. The arbitrary functions $f_1(\psi)$, $f_2(\psi)$ and $f_3(\psi)$, with ψ as the JF scalar field, will be calculated later. The electromagnetic Lagrangian $L(\bar{\mathcal{F}})$ is a function of $\bar{\mathcal{F}} = \bar{F}^{\alpha\beta}\bar{F}_{\alpha\beta}$, which we write in the form of BI non-linear electrodynamics [57,58,68]. That is

$$L(\bar{\mathcal{F}}) = 2a^2 \left(1 - \sqrt{1 + \frac{\bar{\mathcal{F}}}{a^2}} \right), \quad (2.2)$$

and a is the nonlinearity/ or BI parameter. By expanding (2.2), we have

$$L(\bar{\mathcal{F}}) = -\bar{\mathcal{F}} + \frac{\bar{\mathcal{F}}^2}{4a^2} - \frac{\bar{\mathcal{F}}^3}{8a^4} + \dots \quad (2.3)$$

It shows that Maxwell's classical theory is recovered if the electromagnetic fields are taken very weak or a is chosen very large.

By utilizing variational principle to the action (2.1), the electromagnetic field equation is obtained as

$$\bar{\nabla}_\alpha \left(\frac{\bar{\mathcal{F}}^{\alpha\beta}}{\sqrt{1 + \frac{\bar{\mathcal{F}}}{a^2}}} \right) = 0, \quad (2.4)$$

and the gravitational field equation is written the following form

$$f_1(\psi) \left(\bar{\mathcal{R}}_{\alpha\beta} - \frac{1}{2}\bar{\mathcal{R}}\bar{g}_{\alpha\beta} \right) - \left(\bar{\nabla}_\alpha \bar{\nabla}_\beta - \bar{g}_{\alpha\beta}\bar{\Box} \right) f_1(\psi)$$
$$= T_{\alpha\beta}^{(s)} + T_{\alpha\beta}^{(em)}, \quad (2.5)$$

where, $\bar{\Box} = \bar{\nabla}_\mu \bar{\nabla}^\mu$, $T_{\alpha\beta}^{(s)}$ and $T_{\alpha\beta}^{(em)}$ are the energy-momentum tensor of scalar and electromagnetic fields, and

$$T_{\alpha\beta}^{(s)} = f_2(\psi)\bar{\nabla}_\alpha\psi\bar{\nabla}_\beta\psi - \frac{1}{2}\left[f_3(\psi) + f_2(\psi)(\bar{\nabla}\psi)^2 \right]\bar{g}_{\alpha\beta}, \quad (2.6)$$

$$T_{\alpha\beta}^{(em)} = a^2 \left(1 - \sqrt{1 + \frac{\bar{\mathcal{F}}}{a^2}} \right)\bar{g}_{\alpha\beta} + \frac{2\bar{\mathcal{F}}_{\alpha\nu}\bar{\mathcal{F}}_\beta{}^\nu}{\sqrt{1 + \frac{\bar{\mathcal{F}}}{a^2}}}. \quad (2.7)$$

Also, for the scalar field equation, we have

$$2f_2(\psi)\bar{\Box}\psi = f_3'(\psi) - f_2'(\psi)(\bar{\nabla}\psi)^2 - \frac{f_1'(\psi)}{f_1(\psi)}\left[3\bar{\Box}f_1(\psi)\right.$$

$$+f_2(\psi)(\bar{\nabla}\psi)^2 + 2f_3(\psi)$$

$$\left.-4a^2\left(1 - \sqrt{1 + \frac{\bar{\mathcal{F}}}{a^2}}\right) - \frac{2\bar{\mathcal{F}}}{\sqrt{1 + \frac{\bar{\mathcal{F}}}{a^2}}}\right].$$

$$\tag{2.8}$$

Noting the strong coupling between gravitational and scalar field equations, one cannot solve them directly. This problem can be solved by applying CTs to translate the ST action (2.1) to the Ed one [20,22]. Thus, in terms of a well-behavior function $\Omega(\psi)$, we proceed with the following relation as the suitable CTs [18,21]

$$\bar{g}_{\mu\nu} \to \bar{g}_{\mu\nu} = (\Omega(\psi))^2 g_{\mu\nu}, \tag{2.9}$$

which relates the components of JF metric $\bar{g}_{\mu\nu}$ to those of EF $g_{\mu\nu}$. Under these transformations, for the Ricci scalar, we have [69]

$$\bar{\mathcal{R}} \to \bar{\mathcal{R}} = (\Omega(\psi))^{-2}\mathcal{R} - 6(\Omega(\psi))^{-3}\Box\Omega(\psi). \tag{2.10}$$

Also, we obtain

$$\sqrt{-\bar{g}} \to \sqrt{-\bar{g}} = \Omega^4\sqrt{-g}. \tag{2.11}$$

The electromagnetic tensor transformation relation is $\bar{F}_{\mu\nu} \to F_{\mu\nu}$ and $\bar{F}^{\rho\lambda} \to \bar{F}^{\rho\lambda} = \bar{g}^{\rho\mu}\bar{g}^{\lambda\nu}\bar{F}_{\mu\nu} = (\Omega(\psi))^{-4}F^{\rho\lambda}$, and for the Maxwell-invariant we have

$$\bar{\mathcal{F}} \to \bar{\mathcal{F}} = (\Omega(\psi))^{-4}\mathcal{F}, \tag{2.12}$$

where, $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ is the EF Maxwell-invariant with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is the electromagnetic potential.

The EF scalar field analogues to the JF field ψ, which we may call as ϕ, can be considered such that $\psi = \psi(\phi)$. Therefore, we have

$$(\bar{\nabla}\psi)^2 \to \Omega^{-2}\left(\frac{d\psi}{d\phi}\right)^2(\nabla\phi)^2. \tag{2.13}$$

Now, making use of Eqs. (2.10), (2.11), (2.12) and (2.13) in (2.1), we have

$$I^{(ST)} = \frac{-1}{16\pi}\int\sqrt{-g}\left[\Omega^2 f_1(\psi)\mathcal{R}\right.$$

$$-6f_1(\psi)\left(\frac{d\Omega}{d\phi}\right)^2(\nabla\phi)^2 - f_2(\psi)\Omega^2\left(\frac{d\psi}{d\phi}\right)^2(\nabla\phi)^2$$

$$\left.-\Omega^4 f_3(\psi) + \Omega^4 L(\Omega^{-4}\mathcal{F})\right]d^4x. \tag{2.14}$$

It means that we have transformed the ST action (2.1) to that of Ed provided that the following relations are fulfilled

$$f_1(\psi) = \Omega^{-2}(\phi), \tag{2.15}$$

$$f_3(\psi)\Omega^4 = V(\phi), \tag{2.16}$$

$$6\Omega^{-2}\left(\frac{d\Omega}{d\phi}\right)^2 + f_2(\psi)\Omega^2\left(\frac{d\psi}{d\phi}\right)^2 = 2, \tag{2.17}$$

$$\Omega^4 L(\Omega^{-4}\mathcal{F}) = L(\mathcal{F}, \phi). \tag{2.18}$$

Thus, for the action of Ed gravity, one can write

$$I^{(Ed)} = -\frac{1}{16\pi}\int\sqrt{-g}\,d^4x\left[\mathcal{R} - 2g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)\right.$$

$$\left.+L(\mathcal{F}, \phi)\right]. \tag{2.19}$$

Note that, $\mathcal{R} = g^{\alpha\beta}\mathcal{R}_{\alpha\beta}$, and ∇_α is the covariant derivative in the EF with metric tensor $g_{\mu\nu}$. $V(\phi)$ is the self-interacting scalar potential and, the scalar-coupled Lagrangian of BI electrodynamics is given by $L(\mathcal{F}, \phi)$. Its explicit form, by taking $\Omega(\phi) = e^{\alpha\phi}$ [70], is

$$L(\mathcal{F}, \phi) = 2a^2 e^{4\alpha\phi}\left(1 - \sqrt{1 + X}\right), \quad X = \frac{\mathcal{F}}{a^2}e^{-4\alpha\phi}, \tag{2.20}$$

and α denotes the strength of scalar-electromagnetic coupling. Note that the scalar dependency of $L(\mathcal{F}, \phi)$ is slightly different from those previously presented by many authors [71–74]. Expansion of the Lagrangian (2.20) in powers of $\mathcal{F}$ reads

$$L(\mathcal{F}, \phi) = -\mathcal{F} + \frac{\mathcal{F}^2 e^{-4\alpha\phi}}{4a^2} - \frac{\mathcal{F}^3 e^{-8\alpha\phi}}{8a^4} + \dots \tag{2.21}$$

In the case $a \to \infty$, or $\mathcal{F}$ is assumed very small, it recovers the $L(\mathcal{F}, \phi) = -\mathcal{F}$, which reflects conformal-invariant property of the four-dimensional Lagrangian of Maxwell's electrodynamics [67].

The exact solution of the Ed gravity will be obtained in the following section based on the action (2.19).

3 The exact Ed solutions

Through varying the action (2.19), the related field equations can be obtained. Thus we have

$$\partial_\alpha\left[\sqrt{-g}\partial_\mathcal{F}L(\mathcal{F}, \phi)F^{\alpha\beta}\right] = 0, \tag{3.1}$$

$$\mathcal{R}_{\alpha\beta} = 2\partial_\alpha\phi\partial_\beta\phi + \frac{1}{2}g_{\alpha\beta}\left[V(\phi) + 2\mathcal{F}\partial_\mathcal{F}L(\mathcal{F}, \phi)\right.$$

$$\left.-L(\mathcal{F}, \phi)\right] - 2\partial_\mathcal{F}L(\mathcal{F}, \phi)F_{\alpha\sigma}F_\beta^{\,\sigma}, \tag{3.2}$$

$$4\Box\phi = \frac{dV(\phi)}{d\phi} - \partial_\phi L(\mathcal{F}, \phi), \tag{3.3}$$

for the electromagnetic, gravitational and scalar field equations, respectively.

We are aimed to obtain the exact solutions by choosing a spherically symmetric line element, which in terms of the unknown functions $W(r)$ and $R(r)$, can be written as [75,76]

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -W(r)dt^2$$
$$+ \frac{1}{W(r)}dr^2 + r^2 R^2(r)\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \quad (3.4)$$

Note that $W(r)$ may be named as metric function and, $R(r)$ reflects the impacts of dilaton field ϕ on the geometry of spacetime. It is a dimensionless function and reduces to unity when the dilaton field disappears.

In the spacetime identified by (3.4), we have $\mathcal{F} = -2F_{tr}^2$ and, the solution of (3.1) reads

$$F_{tr} = \frac{q}{r^2 R^2(r)\sqrt{1+\xi}}, \qquad \text{with} \qquad \xi = \frac{2q^2 e^{-4\alpha\phi}}{a^2 r^4 R^4(r)}. \quad (3.5)$$

In the limiting case $a \to \infty$, we obtain

$$F_{tr} = \frac{q}{r^2 R^2(r)}, \quad (3.6)$$

which is nothing but the electric field of Einstein–Maxwell–dilaton gravity [67,77].

Also, for the various components of (3.2), we obtained

$$C_{tt} \equiv W''(r) + 2\left[\frac{1}{r} + \frac{R'(r)}{R(r)}\right]W'(r) + V(\phi)$$
$$- 2a^2 e^{4\alpha\phi}\left(1 - \frac{1}{\sqrt{1+\xi}}\right) = 0, \quad (3.7)$$

$$C_{rr} \equiv C_{tt} + 4\left[\frac{R''(r)}{R(r)} + \frac{2R'(r)}{rR(r)} + \phi'^2(r)\right]W(r)$$
$$= 0, \quad (3.8)$$

$$C_{\theta\theta} \equiv C_{\varphi\varphi} \equiv 2\left[\frac{1}{r} + \frac{R'(r)}{R(r)}\right]W'(r)$$
$$+ 2\left[\frac{1}{r^2} + \frac{R''(r)}{R(r)} + \frac{4R'(r)}{rR(r)} + \frac{R'^2(r)}{R^2(r)}\right]W(r)$$
$$- \frac{2}{r^2 R^2(r)} + V(\phi) + 2a^2 e^{4\alpha\phi}\left(\sqrt{1+\xi} - 1\right) = 0. \quad (3.9)$$

Noting Eqs. (3.3), (3.7), (3.8) and (3.9), we have

$$\left[\frac{1}{r} + \frac{R'(r)}{R(r)}\right](C_{tt} - 2C_{\theta\theta}) = \frac{dC_{\theta\theta}}{dr}. \quad (3.10)$$

This mean that Eqs. (3.7) and (3.9) are not unique and, we have five unknowns (i.e. ϕ, $V(\phi)$, F_{tr}, $R(r)$ and $W(r)$), while the number of unique equations is equal to four. Thus, we have not enough unique equations to obtain the full exact solutions and, mathematically we are in-fronted with the problem of indeterminacy. One can solve this problem by use of an ansatz function. Since $R(r)$, which reflects the direct impact of scalar hair on the geometry, hasn't dimension a power-law ansatz is a useful one [78–80]. Thus we consider

$$R(r) = \left(\frac{r}{r_0}\right)^\sigma, \quad (3.11)$$

where r_0 and σ are dimensional and dilaton parameters, respectively [67].

By combining Eqs. (3.8) and (3.11) and noting (3.7), the related equation can be solved for the scalar field $\phi(r)$. It leads to the following relation

$$\phi(r) = \gamma \ln\left(\frac{b}{r}\right), \qquad \text{with} \qquad \gamma = \sqrt{-\sigma(\sigma+1)}. \quad (3.12)$$

Note that b is assumed to be positive and σ must be restricted to $-1 < \sigma \leq 0$.

By replacing these quantities into Eqs. (3.3) and (3.9), we showed that the differential equations governed by the unknowns, $V(\phi)$ and $W(r)$, are as follows

$$\frac{dV(\phi)}{d\phi} - \frac{2\gamma}{\sigma+1}V(\phi) + \frac{4\gamma r_0^{2\sigma}}{(\sigma+1)r^{2+2\sigma}}$$
$$+ 4a^2\left[(\alpha - \frac{\gamma}{\sigma+1})\left(\sqrt{1+\xi} - 1\right)\right.$$
$$+ \alpha\left(\frac{1}{\sqrt{1+\xi}} - 1\right)\left.\right]e^{4\alpha\phi} = 0, \quad (3.13)$$

$$\frac{2(\sigma+1)}{r}\left[W'(r) + \frac{2\sigma+1}{r}W(r)\right] - \frac{2}{r^2 R(r)^2}$$
$$+ V(\phi) + 2a^2\left(\sqrt{1+\xi} - 1\right)e^{4\alpha\phi} = 0. \quad (3.14)$$

Hereafter, without loss of generality, we set $\frac{\gamma}{\sigma+1} = 2\alpha$ for simplicity. It reduces the number of independent arbitrary constants.

Now, based on the relation $F_{tr} = -\partial_r A_t(r)$, the temporal component of A_μ can be calculated. Noting Eq. (3.5), we have

$$A_t(r) = -\int \frac{q\,dr}{r^2 R(r)^2 \sqrt{1+\xi}} + \text{const.}.$$

By considering constant of integration equal to zero and introducing the new variable $\xi_1 = \frac{2q^2}{a^2 b^2 r_0^2}\left(\frac{b}{r}\right)$, we obtain

$$A_t(r) = \begin{cases} \frac{q}{(2\sigma+1)r_0}\left(\frac{r_0}{r}\right)^{2\sigma+1} {}_2F_1\left[\frac{1}{2}, \frac{1+2\sigma}{2(3\sigma+2)}, \right. \\ \left. 1 + \frac{1+2\sigma}{2(3\sigma+2)}, -\xi\right], \quad -\frac{1}{2} < \sigma \leq 0, \\ -\frac{2q}{r_0}\ln\left[2\sqrt{\frac{r}{\ell}}\left(1 + \sqrt{1+\xi_1}\right)\right], \quad \sigma = -\frac{1}{2}. \end{cases} \quad (3.15)$$

Note that, with the constraints on σ, A_t converges at infinity. By expanding $A_t(r)$, we have

$$A_t(r) = \begin{cases} \frac{q}{(2\sigma+1)r_0^2}\left(\frac{r_0}{r}\right)^{2\sigma+1}\left[1 - \frac{(1+2\sigma)\xi}{2(8\sigma+5)} + \mathcal{O}\left(\xi^2\right)\right], \\ -\frac{1}{2} < \sigma \leq 0, \\ -\frac{q}{r_0}\ln\left(\frac{r}{\ell}\right) - \frac{4q}{r_0}\ln 2 - \frac{q}{r_0}\xi_1 + \mathcal{O}\left(\xi_1^2\right), \quad \sigma = -\frac{1}{2}. \end{cases} \quad (3.16)$$

which recovers the results of [67] if the parameter a is chosen very large. The solution of (3.13), after fixing the constant of

integration through $V(\phi = 0) = 2\Lambda$, reads

$$
V(\phi) = \begin{cases}
\left(2\Lambda + \frac{2\sigma a^2}{3\sigma+2}\right) e^{4\alpha\phi} + \frac{2\sigma}{b^2(2\sigma+1)} \left(\frac{r_0}{b}\right)^{2\sigma} e^{\frac{\phi}{\alpha}} \\
\quad - \frac{2\sigma a^2}{2+3\sigma} \sqrt{1+\xi}\, e^{4\alpha\Phi}, \quad -\frac{1}{2} < \sigma \le 0, \\[2mm]
\left(2\Lambda - 2a^2\right) e^{2\phi} - \frac{4\phi}{br_0} e^{2\phi} + 2a^2\sqrt{1+\xi_1}\, e^{2\phi}, \\
\quad \sigma = -\frac{1}{2}.
\end{cases}
\tag{3.17}
$$

By expanding the last terms in powers o ξ (ξ_1) one can show that

$$
V(\phi) = \begin{cases}
2\Lambda e^{4\alpha\phi} + \frac{2\sigma}{b^2(2\sigma+1)} \left(\frac{r_0}{b}\right)^{2\sigma} e^{\frac{\phi}{\alpha}} \\
\quad - \frac{2\sigma q^2}{b^4(2+3\sigma)} \left(\frac{r_0}{b}\right)^{4\sigma} e^{\frac{2\phi}{\alpha}}, \quad -\frac{1}{2} < \sigma \le 0, \\[2mm]
2\Lambda e^{2\phi} - \frac{4\phi}{br_0} e^{2\phi} + \frac{2q^2}{b^2 r_0^2} e^{4\phi}, \quad\quad \sigma = -\frac{1}{2},
\end{cases}
\tag{3.18}
$$

which coincides with the result of Ref. [67].

By replacing (3.17) into (3.14) and solving for $W(r)$, in terms of the integration constant m, we have

$$
W(r) = \begin{cases}
-\frac{m}{r^{2\sigma+1}} + \frac{1}{2\sigma+1} \left(\frac{r_0}{r}\right)^{2\sigma} + \eta \left(\frac{r}{b}\right)^{2(\sigma+1)} \\
\quad - \frac{2a^2 b^{-2\sigma}}{(3\sigma+2)r^{2\sigma+1}} \int r^{4\sigma+2}\sqrt{1+\xi}\, dr, \quad -\frac{1}{2} < \sigma \le 0, \\[2mm]
-m + \frac{2r}{r_0} \left[2 + \ln\left(\frac{b}{r}\right)\right] \eta_1 \frac{r}{b} \\
\quad - 4a^2 b \int \sqrt{1+\xi_1}\, dr, \quad\quad \sigma = -\frac{1}{2},
\end{cases}
\tag{3.19}
$$

where, we have used the definitions $\eta = \frac{2a^2 b^2}{(3\sigma+2)(4\sigma+3)} - \frac{\Lambda b^2}{(\sigma+1)(4\sigma+3)}$ and $\eta_1 = 2\left(2a^2 - \Lambda\right) b^2$. After working the integrals, we have

$$
W(r) = \begin{cases}
-\frac{m}{r^{2\sigma+1}} + \frac{1}{2\sigma+1} \left(\frac{r_0}{r}\right)^{2\sigma} + \eta \left(\frac{r}{b}\right)^{2(\sigma+1)} \\
\quad - \frac{2a^2 b^2}{(\sigma+1)(3\sigma+2)} \left[\sqrt{1+\xi} - \frac{(3\sigma+2)H}{4\sigma+3}\right] \left(\frac{r}{b}\right)^{2(\sigma+1)}, \\
\quad\quad -\frac{1}{2} < \sigma \le 0, \\[2mm]
-m + \frac{2r}{r_0} \left[2 + \ln\left(\frac{b}{r}\right)\right] \\
\quad + \eta_1 \frac{r}{b} - 4a^2 br \left\{\sqrt{1+\xi_1}\right. \\
\quad \left. + \xi_1 \ln\left[2\sqrt{\frac{r}{\ell}}\left(1 + \sqrt{1+\xi_1}\right)\right]\right\}, \quad \sigma = -\frac{1}{2},
\end{cases}
\tag{3.20}
$$

where, $H = {}_2F_1\left[\frac{1}{2}, -\frac{4\sigma+3}{6\sigma+4}, 1 - \frac{4\sigma+3}{6\sigma+4}, -\xi\right]$ is a hypergeometric function. The last terms can be expanded in powers of ξ (or ξ_1) for obtaining

$$
W(r) = \begin{cases}
-\frac{m}{r^{2\sigma+1}} + \frac{1}{2\sigma+1} \left(\frac{r_0}{r}\right)^{2\sigma} \\
\quad - \frac{\Lambda r^2}{(\sigma+1)(4\sigma+3)} \left(\frac{r}{b}\right)^{2\sigma} \\
\quad + \frac{2q^2 r^{-2}}{(2\sigma+1)(3\sigma+2)} \left(\frac{r_0}{r}\right)^{4\sigma} + \mathcal{O}\left(\frac{1}{a^2}\right), \quad -\frac{1}{2} < \sigma < 0 \\[2mm]
-m + \frac{2r}{r_0} \left[2 + \ln\left(\frac{b}{r}\right)\right] - 2\Lambda br \\
\quad - \frac{4q^2}{r_0^2} \ln\left(\frac{r}{\ell}\right) + \mathcal{O}\left(\frac{1}{a^2}\right), \quad \sigma = -\frac{1}{2}.
\end{cases}
\tag{3.21}
$$

which are just the same as obtained in [67] when the limit $a \to \infty$ is taken.

Note that the real root(s) of $W(r = r_+) = 0$, as the black hole horizon radius, can be determined by use of the plots. In Figs. 1 and 2, we have depicted $W(r)$ vs r for $\sigma \ne -\frac{1}{2}$ and $\sigma = -\frac{1}{2}$ cases, respectively. The figures illustrate that two-horizon, one-horizon, extreme and horizon-less BHs can occur if the parameters are fixed properly.

It is worth mentioning that the solutions presented in Eq. (3.20) can be considered as BHs, if both of the following requirements are fulfilled. (a) The metric functions are necessary to have at least one horizon radius. Existence of the horizon radii is confirmed by diagrams of Figs. 1 and 2. (b) At least one physical singularity is required to exist. It can be explored by calculating the Ricci ($\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$) and Kretschmann ($\mathcal{K} = \mathcal{R}^{\alpha\beta\rho\lambda}\mathcal{R}_{\alpha\beta\rho\lambda}$) scalars. As a matter of calculation, one can show that

$$
\mathcal{R} = -W'' - 4(\sigma+1)\left(\frac{W'}{r}\right)
$$
$$
\quad - 2(\sigma+1)(3\sigma+1)\left(\frac{W}{r^2}\right) + \frac{2}{r^2}\left(\frac{r_0}{r}\right)^{2\sigma},
\tag{3.22}
$$

$$
\mathcal{K} = \left(W''\right)^2 + 4\left(\sigma+1\right)^2 \left(\frac{W'}{r}\right)^2
$$
$$
\quad + 8\sigma\left(1+\sigma\right)^2 \frac{WW'}{r^3} + \frac{4}{r^4}\left(\frac{b}{r}\right)^{4\sigma}
$$
$$
\quad + 4\left(\sigma+1\right)^2 \left(3\sigma^2 + 2\sigma + 1\right)\left(\frac{W}{r^2}\right)^2
$$
$$
\quad - 8\left(\sigma+1\right)^2 \left(\frac{r_0}{r}\right)^{2\sigma}\left(\frac{W}{r^4}\right).
\tag{3.23}
$$

By replacing $W(r)$ and its derivatives into Eqs. (3.22) and (3.23), we found that the Ricci and Kretschmann scalars diverge in the limit $r \to 0^+$. As the result, the spacetime under consideration posses a physical singularity at the origin. These facts confirm that our solutions can be interpreted as BHs. Thermodynamic properties of our novel Ed BHs will be investigated in the following sections.

4 Thermodynamic quantities and FLT

Now, we calculate thermodynamic quantities of our novel Ed BHs and, investigate validity of the FLT. At first, using the concept of surface gravity, we calculate the Hawking temperature on the BH's horizon. It is proportional to the surface gravity κ, via $T = \frac{\kappa}{2\pi}$ with $\kappa = \sqrt{-\frac{1}{2}(\nabla_\alpha\chi_\beta)(\nabla^\alpha\chi^\beta)}$ and $\chi^\alpha = (-1, 0, 0, 0)$, is obtained as $T = \frac{1}{4\pi}\left(\frac{dW(r)}{dr}\right)_{r=r_+}$ [81–83]. In our case, one obtains
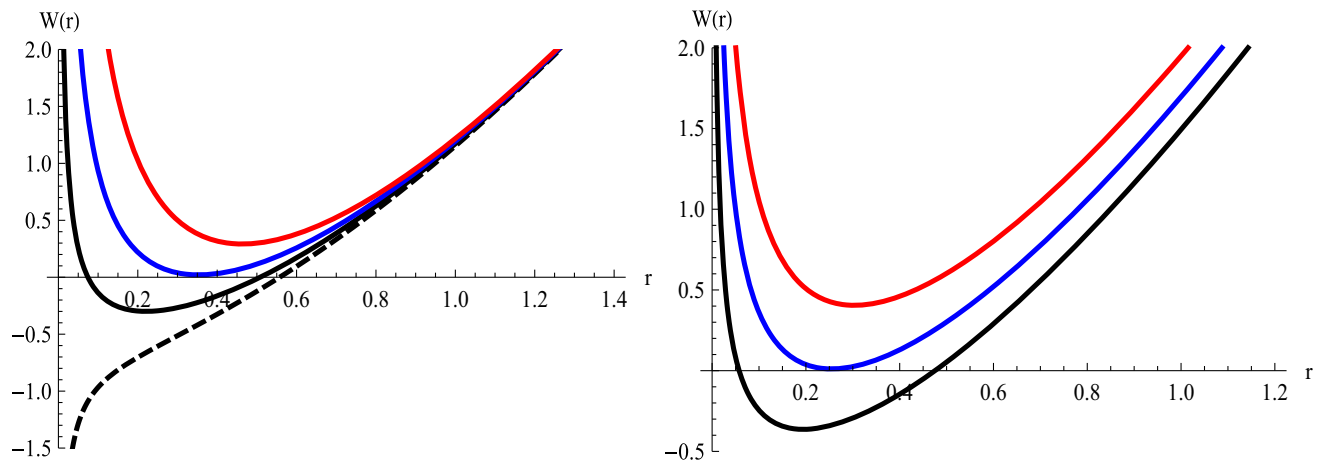
Fig. 1 $W(r)$ vs r, Eq. (3.20) for $\Lambda = -3$, $q = 1$, $r_0 = 1$, $b = 1$: Left: $m = 3$, $\sigma = -0.12$, $a = 0.9$ (*dashed*), 1.02 (black), 1.19 (blue), 1.5 (red), Right: $m = 4$, $a = 1$, $\sigma = -0.195$ (black), -0.2038 (blue), -0.214 (red)
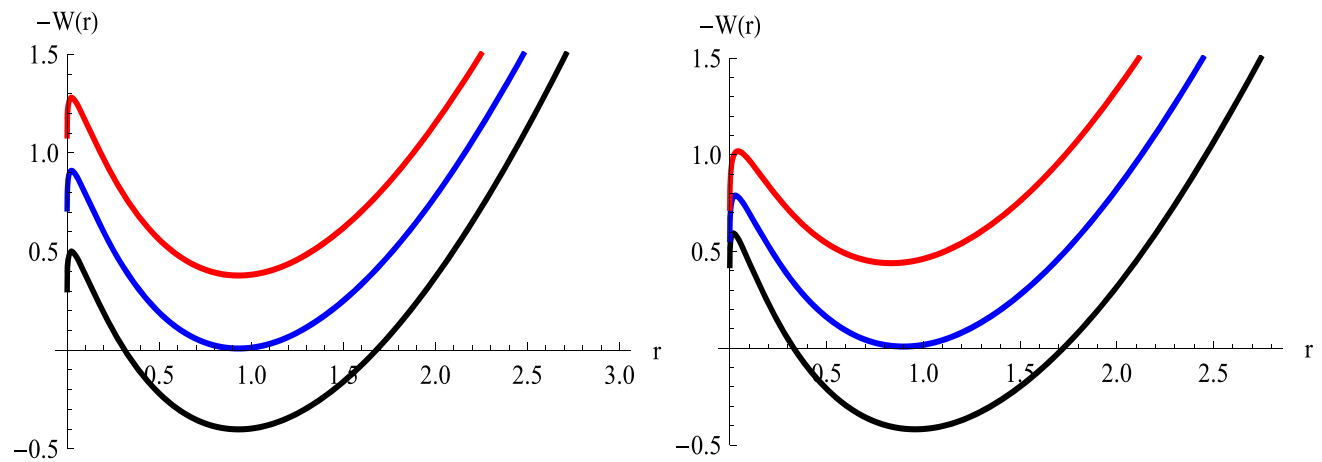

Fig. 2 $-W(r)$ vs r, Eq. (3.20) for $\Lambda = -3$, $b = 0.22$, $r_0 = 1$, $a = 1.8$: Left: $q = 0.33$, $m = 0.22$ (black), 0.63 (blue), 1 (red), Right: $m = 0.4$, $q = 0.31$, (black), 0.352 (blue), 0.39 (red)

$$
T = \begin{cases}
\frac{1}{4\pi(\sigma+1)r_+}\left[\frac{\sigma+1}{2\sigma+1}\left(\frac{r_0}{r_+}\right)^{2\sigma} - \Lambda b^2\left(\frac{r_+}{b}\right)^{2\sigma+2}\right. \\
\left. - \frac{2a^2 b^2(\sigma+1)}{3\sigma+2}\left(\sqrt{1+\xi_+}-1\right)\left(\frac{r_+}{b}\right)^{2\sigma+2}\right], \quad \sigma \neq -\frac{1}{2} \\[2ex]
\frac{1}{2\pi}\left[\frac{1}{r_0} + \frac{1}{r_0}\ln\left(\frac{b}{r_+}\right) - \Lambda b\right. \\
\left. -2a^2 b\left(\sqrt{1+\xi_{1+}}-1\right)\right], \quad \sigma = -\frac{1}{2}.
\end{cases}
\tag{4.1}
$$

Here, r_+ is the radius of BH horizon and, the subscript $+$ means that the corresponding quantity has been calculated on the BH horizon. The last terms can be expanded in powers of ξ (ξ_1) then in the limiting case $a \to \infty$, we have

$$
T = \begin{cases}
\frac{1}{4\pi(\sigma+1)r_+}\left[\frac{\sigma+1}{2\sigma+1}\left(\frac{r_0}{r_+}\right)^{2\sigma} - \Lambda b^2\left(\frac{r_+}{b}\right)^{2\sigma+2}\right. \\
\left. - \frac{2(\sigma+1)q^2}{(3\sigma+2)r_+^2}\left(\frac{r_0}{r_+}\right)^{4\sigma} + \mathcal{O}\left(\frac{1}{a^2}\right)\right], \quad \sigma \neq -\frac{1}{2} \\[2ex]
\frac{1}{2\pi}\left[\frac{1}{r_0} + \frac{1}{r_0}\ln\left(\frac{b}{r_+}\right) - \Lambda b\right. \\
\left. - \frac{2q^2}{r_0^2 r_+} + \mathcal{O}\left(\frac{1}{a^2}\right)\right], \quad \sigma = -\frac{1}{2},
\end{cases}
\tag{4.2}
$$

which is just the same as obtained in [67]. We have eliminated m from Eq. (4.1) by using the condition $W(r_+) = 0$. Note that a zero temperature BH, named as the extreme BH, may exist if its charge and horizon radius, labeled by $q = q_{ext}$ $r_+ = r_{ext}$, are fixed to give $T(r_{ext}, q_{ext}) = 0$. The horizon of extreme BHs can be fund by use of diagrams. The blue curves of Fig. 3 shows T versus r_+. Noting the left and middle panels, the temperature of BHs with $\sigma \neq -\frac{1}{2}$, vanishes at $r_+ = r_{ext}$. The BHs with horizon radii greater than r_{ext}, have positive temperature and, are physically acceptable. The right panel of Fig. 3, which has been depicted for $\sigma = -\frac{1}{2}$, shows that there are two vanishing points which we label by r_{1ext} and r_{2ext} with $r_{1ext} < r_{2ext}$. The BHs with radius of horizon equal to r_{1ext} or r_{2ext} are known as the extreme ones. The BH with the horizon radii smaller than r_{1ext} and greater than r_{2ext} are not physically reasonable, because they have negative temperature. The BHs with horizon radii in the
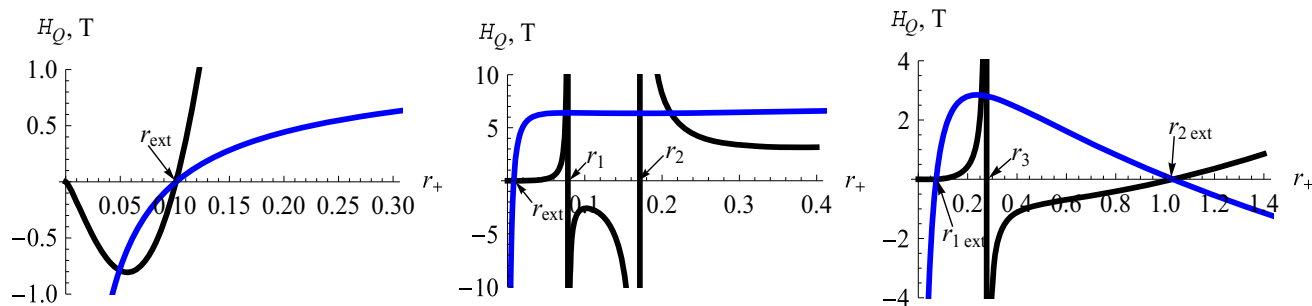
Fig. 3 $\mathcal{H}_Q$ (black)) and T (blue) vs r_+, for $\Lambda = -3$, $a = 0.8$, $r_0 = 1$: Left: $q = 1$, $b = 0.4$, $\sigma = -0.32$, $20\mathcal{H}_Q$, $3T$, Middle: $q = 0.4$, $b = 0.5$, $\sigma = -0.25$, $0.2\mathcal{H}_Q$, $20T$, Right: $q = 0.8$, $b = 0.3$, $\sigma = -0.5$, $\mathcal{H}_Q$, $40T$

range $r_{1ext} < r_+ < r_2$ have positive temperature and, may be named as the physical BHs.

The BH entropy, according to the area law, is equal to one-fourth of BH's surface area. In our case it can be written as

$$S = \frac{4\pi r_+^2 R^2(r_+)}{4} = \pi r_+^2 \left(\frac{r_+}{r_0}\right)^{2\sigma}. \tag{4.3}$$

Note that when the dilaton field is absent, by letting $\sigma = 0$, it recovers the corresponding value for the BHs in Einstein gravity theory.

The horizon electric potential, relative to an appropriate point of reference, can be determined by use of the following standard relation [84–86]

$$U(r_+) = A_\mu \chi^\mu|_{\text{ref.}} - A_\mu \chi^\mu|_{r=r_+}. \tag{4.4}$$

Noting Eq. (3.15) we obtained [62,87]

$$U(r_+) = \begin{cases} \frac{cq}{(2\sigma+1)r_0}\left(\frac{r_0}{r_+}\right)^{2\sigma+1} {}_2F_1\left[\frac{1}{2}, \frac{1+2\sigma}{2(3\sigma+2)}, \right. \\ \left. 1 + \frac{1+2\sigma}{2(3\sigma+2)}, -\xi_+\right], \quad -\frac{1}{2} < \sigma \leq 0, \\ \\ -\frac{2cq}{r_0}\ln\left[2\sqrt{\frac{r_+}{\ell}}\left(1 + \sqrt{1+\xi_{1+}}\right)\right], \quad \sigma = -\frac{1}{2}. \end{cases} \tag{4.5}$$

Note that the arbitrary constant coefficient c, will be fixed later.

The BH mass can be calculated based on the method proposed by Brown and York. Thus, the line element must be written in the form of [78,80,88,89]

$$ds^2 = -X^2(\mathbb{R})dt^2 + \frac{d\mathbb{R}^2}{Y^2(\mathbb{R})} + \mathbb{R}^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{4.6}$$

There is no derivative of metric in the matter field, thus the quasilocal mass $\mathcal{M}$ can be calculated by use of the following equation

$$\mathcal{M} = 2X(\mathbb{R})[Y_0(\mathbb{R}) - Y(\mathbb{R})]. \tag{4.7}$$

The $Y_0(\mathbb{R})$ is obtained from $Y(\mathbb{R})$ by letting $m = 0$. Let's introduce a new variable $\mathbb{R} = r R(r)$, then

$$dr^2 = \frac{d\mathbb{R}^2}{(1+\sigma)^2 R^2}, \tag{4.8}$$

and comparison of (3.4) and (4.6), shows that

$$X^2(\mathbb{R}) = W(r(\mathbb{R})), \tag{4.9}$$

$$Y^2(\mathbb{R}) = (1+\sigma)^2 (R(\mathbb{R}))^2 W(r(\mathbb{R})). \tag{4.10}$$

By replacing $X(\mathbb{R})$ and $Y(\mathbb{R})$ into Eq. (4.7), and taking limit $\mathbb{R} \to \infty$ the BH mass M is obtained as

$$M = \frac{\sigma + 1}{2r_0^{2\sigma}}m. \tag{4.11}$$

It is just the same as that achieved in [67], and recovers the mass of R-N BHs in the case of $\sigma = 0$.

The electric charge of BHs can be obtained by use of the Gauss's law, that is

$$Q = q. \tag{4.12}$$

At this stage, the Smaar mass formula can be achieved if the condition $W(r_+) = 0$ is imposed on the metric functions of Eq. (3.20) then replacing m into Eq. (4.11). That is

$$M(r_+, q) = \begin{cases} \frac{r_+(\sigma+1)}{2(2\sigma+1)} + \frac{r_+}{2}\left(\frac{r_+}{r_0}\right)^{2\sigma}\left(\frac{r_+}{b}\right)^{2\sigma+2} \\ \times \left\{\eta(\sigma+1) - \frac{2a^2b^2}{3\sigma+2}\left[\sqrt{1+\xi_+}\right.\right. \\ \left.\left. -\frac{(3\sigma+2)H_+}{4\sigma+3}\right]\right\}, \quad \sigma \neq -\frac{1}{2}, \\ \\ \frac{r_+}{2}\left[2 + \ln\left(\frac{b}{r_+}\right)\right] + \eta_1\frac{r_0 r_+}{4b} \\ -a^2 b r_0 r_+\left\{\sqrt{1+\xi_{1+}} + \xi_{1+}\ln\left[2\sqrt{\frac{r_+}{\ell}}(1\right.\right. \\ \left.\left. +\sqrt{1+\xi_{1+}})\right]\right\}, \quad \sigma = -\frac{1}{2}. \end{cases} \tag{4.13}$$

Using Eqs. (4.1), (4.3), (4.5), (4.12) and (4.13) one can show that $\left(\frac{\partial M}{\partial S}\right)_Q = T$, and

$$\left(\frac{\partial M}{\partial Q}\right)_S = \begin{cases} \frac{2(\sigma+1)}{3\sigma+2}A_t(r_+), & \sigma \neq -\frac{1}{2}, \\ \\ 2A_t(r_+), & \sigma = -\frac{1}{2}, \end{cases} \tag{4.14}$$

It means that by fixing the constant c, introduced in Eq. (4.5), as $c = \frac{2(\sigma+1)}{3\sigma+2}$, the standard form of FLT is valid for our new Ed BHs. That is

$$dM(S, Q) = TdS + UdQ. \tag{4.15}$$

5 Stability of Ed BHs

Here, we analyze thermal stability of Ed–BI BHs in the canonical ensemble method, by using the BH's HC. It is well-known that physical BHs with positive HC are stable. The BHs with negative HC, can be stable through phase transition (PT). Type-one PT occurs for the BHs with zero HC, while those with singular HC undergo type-two PT [90–93]. To investigate stability or PT properties of the BHs, it is required to calculate the HC. We obtain the HC based on the following standard definition [94,95]

$$\mathcal{H}_Q = T \left(\frac{\partial S}{\partial T} \right)_Q. \tag{5.1}$$

Noting the fact that $T = \left(\frac{\partial M}{\partial S} \right)_Q$, we can write

$$\mathcal{H}_Q = \frac{T}{M_{SS}}, \quad \text{and} \quad M_{SS} = \left(\frac{\partial T}{\partial S} \right)_Q = \left(\frac{\partial^2 M}{\partial S^2} \right)_Q, \tag{5.2}$$

where, the subscript Q reminds that the BH's charge must be treated as a constant. Now, we calculate HC and, then by use of the plots, study PT or thermal stability of BHs.

As it can be seen from Eq. (5.2), the numerator of HC is the BH temperature and it is given by Eq. (4.1). Also, we calculate the denominator by use of Eqs. (4.1) and (4.3). It leads to

$$M_{SS} = \frac{1}{2\pi^2 r_0} \left[\frac{a^2 b\, \xi_{1+}}{r_+ \sqrt{1+\xi_{1+}}} - \frac{1}{r_0 r_+} \right], \quad \sigma = -\frac{1}{2}, \tag{5.3}$$

$$M_{SS} = \frac{-1}{8\pi^2(\sigma+1)r_+^3} \left(\frac{r_0}{r_+} \right)^{2\sigma} \left[\left(\frac{r_0}{r_+} \right)^{2\sigma} \right. $$
$$+ \frac{\Lambda b^2(2\sigma+1)}{\sigma+1} \left(\frac{r_+}{b} \right)^{2\sigma+2} $$
$$+ \frac{2a^2 b^2(2\sigma+1)}{3\sigma+2} \left(\frac{r_+}{b} \right)^{2\sigma+2} \left(\sqrt{1+\xi_+} - 1 \right) $$
$$\left. - 2a^2 b^2 \left(\frac{r_+}{b} \right)^{2\sigma+2} \frac{\xi_+}{\sqrt{1+\xi_+}} \right], \quad \sigma \neq -\frac{1}{2}. \tag{5.4}$$

Note that M_{SS} presented in Eqs. (5.3) and (5.4) coincide with the corresponding quantity in the scalar–tensor–Maxwell theory, as a grows to infinity [67].

Diagrams of T and $\mathcal{H}_Q$ versus r_+ have been given simultaneously in Fig. 3, to discuss PT or stability of the BHs. Noting the left and middle panels with $\sigma \neq -\frac{1}{2}$, temperature vanishes at $r_+ = r_{ext}$ and type-one PT occurs at this

point. The HC diverges at $r_+ = r_1$ and $r_+ = r_2$ thus the BHs with horizon radius equal to r_1 or r_2 experience type-two PT. The BHs with $r_+ > r_1$ (left panel) and those with $r_{ext} < r_+ > r_1$ and $r_+ > r_2$ (middle panel) are stable. The right panel, which has been depicted for $\sigma = -\frac{1}{2}$, shows that there are two points of type-one PT which we label by r_{1ext} and r_{2ext} with $r_{1ext} < r_{2ext}$. The BHs with horizon radius equal to r_{1ext} or r_{2ext} undergo type-one PT. The BH HC diverges at $r_+ = r_3$ and the BHs with the horizon radius equal to r_3 undergo type-two PT. The BHs with horizon radii in the range $r_{1ext} < r_+ < r_3$, with positive T and $\mathcal{H}_Q$, are locally stable.

6 ST–BI BHs in the JF

Up to now, the exact BH solutions have been introduced and the related properties have been studied in the four-dimensional Ed–BI gravity theory. Now, it is the time to continue these studies in the ST–BI gravity theory. To this end, making use of the inverse CTs, one is able to obtain the metric function of ST–BI theory from its Ed–BI counterpart. Therefore, noting Eq. (2.9) for the metric $\bar{g}_{\mu\nu}$ we have $\bar{g}_{\mu\nu} = \left(\frac{b}{r} \right)^{2\alpha\gamma} g_{\mu\nu}$ and, by assuming the line element

$$d\bar{s}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu $$
$$= -\mathcal{A}(r)dt^2 + \frac{1}{\mathcal{B}(r)} dr^2 $$
$$+ r^2 \mathcal{C}^2(r) \left(d\theta^2 + \sin^2\theta d\varphi^2 \right), \tag{6.1}$$

in the JF, one obtains

$$\mathcal{A}(r) = \left(\frac{b}{r} \right)^{2\alpha\gamma} W(r), $$
$$\mathcal{B}(r) = \left(\frac{b}{r} \right)^{-2\alpha\gamma} W(r), \tag{6.2}$$
$$\mathcal{C}(r) = \left(\frac{b}{r} \right)^{\alpha\gamma} R(r). $$

Note that, through exact solutions of Ed–BI theory, $R(r)$ and $W(r)$ have been presented in Eqs. (3.9) and (3.20), respectively. The plots of $\mathcal{B}(r)$ vs r have been depicted, for both of $\sigma = -\frac{1}{2}$ and $\sigma \neq -\frac{1}{2}$ cases in Figs. 4 and 5, respectively. Clearly, the horizon-less, extreme, one-horizon and two-horizon ST BHs can occur.
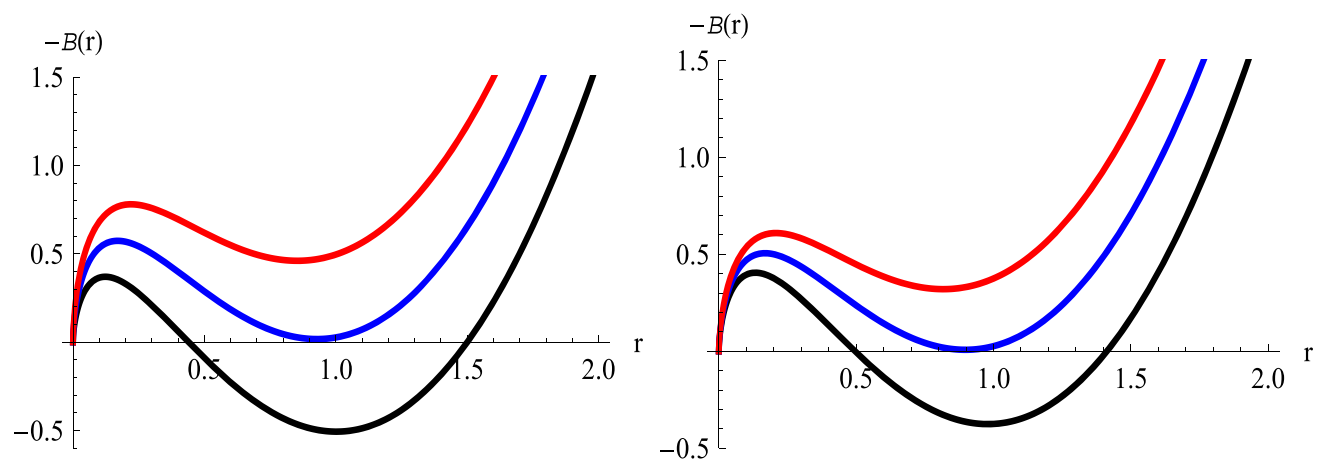
Now, the arbitrary functions f_1 and f_2 and the scalar potential f_3, appeared in the JF action (2.1), can be calculated. From Eqs. (2.15) and (2.17), with the assumption $\psi = Exp(2\alpha\phi)$ [23], we have

$$f_1(\phi) = e^{-2\alpha\phi}, $$
$$f_2(\phi) = \frac{1}{2} \left(\alpha^{-2} - 3 \right) e^{-6\alpha\phi}, \tag{6.3}$$
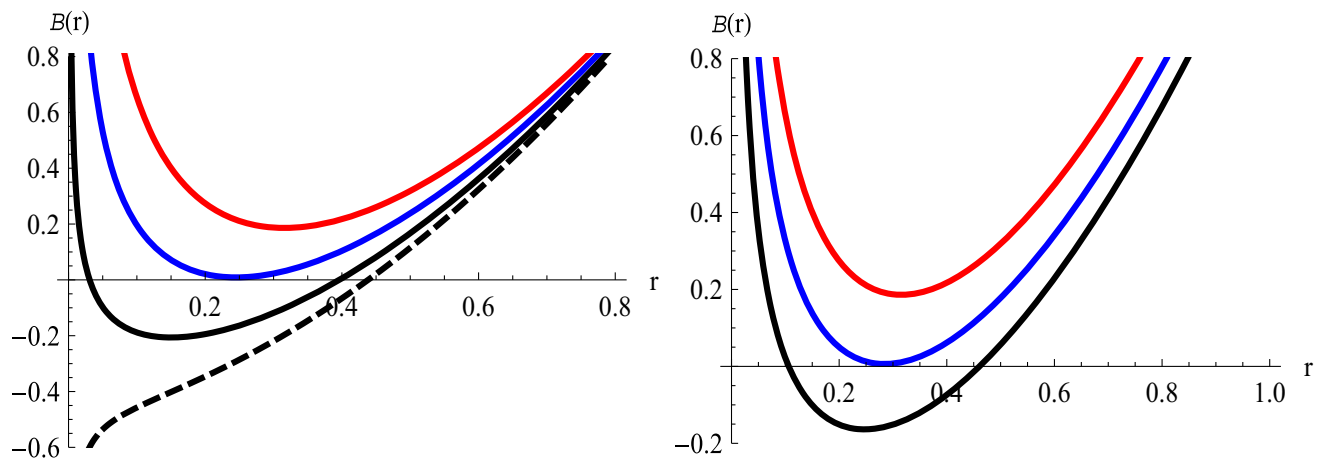
Fig. 4 $\mathcal{B}(r)$ vs r, for $m = 3$, $\Lambda = -3$, $q = 1$, $b = 1$, $r_0 = 1$: Left: $\sigma = -0.14$, $a = 0.78$ (*dashed*), 0.83 (black), 0.905 (blue), 1(red), Right: $a = 1$, $\sigma = -0.125$ (black), -0.132 (blue), -0.14 (red)

Fig. 5 $-\mathcal{B}(r)$ vs r, for $\Lambda = -3$, $b = 0.22$, $a = 1.8$, $r_0 = 1$, $\sigma = -0.5$: Left: $q = 0.33$, $m = 0.38$ (black), 0.63 (blue), 1 (red), Right: $m = 0.4$, $q = 0.334$ (black), 0.3515 (blue), 0.336 (red)

and noting Eqs. (2.16) and (3.17), one obtains

$$f_3(\psi) = \begin{cases} 2\Lambda + \frac{2\sigma}{b^2(2\sigma+1)} \left(\frac{r_0}{b}\right)^{2\sigma} \psi^{\frac{1}{\alpha^2}-4} + \frac{2\sigma a^2}{2+3\sigma} \\ \left(1 - \sqrt{1 + \frac{2q^2}{a^2 b^4} \left(\frac{r_0}{b}\right)^{4\sigma} \psi^{\frac{2}{\alpha^2}-4}}\right); \quad -\frac{1}{2} < \sigma \leq 0, \\ \\ 2\Lambda - \frac{8}{br_0} \ln\psi \\ -2a^2 \left(1 - \sqrt{1 + \frac{2q^2}{a^2 b^2 r_0^2} \psi^4}\right), \quad \sigma = -\frac{1}{2}, \end{cases} \tag{6.4}$$

which can be expanded to obtain

$$f_3(\psi) = \begin{cases} 2\Lambda + \frac{2\sigma}{b^2(2\sigma+1)} \left(\frac{r_0}{b}\right)^{2\sigma} \psi^{\frac{1}{\alpha^2}-4} \\ -\frac{2\sigma q^2}{b^4(2+3\sigma)} \left(\frac{r_0}{b}\right)^{4\sigma} \psi^{\frac{2}{\alpha^2}-4} \\ +\mathcal{O}\left(\frac{1}{a^2}\right); \qquad -\frac{1}{2} < \sigma \leq 0, \\ \\ 2\Lambda - \frac{8}{br_0} \ln\psi + \frac{2q^2}{b^2 r_0^2} \psi^4 \\ +\mathcal{O}\left(\frac{1}{a^2}\right), \qquad \sigma = -\frac{1}{2}, \end{cases} \tag{6.5}$$

It means that our calculations present the JF scalar potential in the form of a typical power law function. It has been shown that application of power law scalar potential can produce acceptable results in the context of ST cosmology [96,97]. Note that, in addition to the scalar term, the Λ and charge terms are exist here.

Thermodynamics of ST–BI BHs can be studied through those of Ed–BI ones. For example, the horizon temperature of ST–BI BHs, which we denote by $\bar{T}$, is

$$\begin{aligned} \bar{T} &= \frac{1}{4\pi} \left(\sqrt{\frac{\mathcal{B}(r)}{\mathcal{A}(r)}} \frac{d\mathcal{A}(r)}{dr}\right)_{r=r_+} \\ &= \frac{1}{4\pi} \left(\frac{b}{r_+}\right)^{-2\alpha\gamma} \frac{d}{dr} \left[\left(\frac{b}{r}\right)^{2\alpha\gamma} W(r)\right]_{r=r_+} \\ &= \frac{1}{4\pi} \frac{dW(r)}{dr}\Big|_{r=r_+} = T. \end{aligned} \tag{6.6}$$

It means that the Hawking temperature remains unchanged for both of the ST and Ed theories [98].

As mentioned in [23,24,28], the entropy of ST–BI BHs cannot be obtained by use of the entropy-area law, but the BH's temperature, mass, entropy, charge and electric poten-

tial remain unchanged under CTs. Thus the Smarr relation is also the same for both of the Ed and ST–BI BHs. As the result we have

$$\bar{U} = \frac{\partial \bar{M}(\bar{S}, \bar{Q})}{\partial \bar{Q}}, \qquad \text{and} \qquad \bar{T} = \frac{\partial \bar{M}(\bar{S}, \bar{Q})}{\partial \bar{S}}, \qquad (6.7)$$

from which one can conclude that, just like the Ed BHs, FLT is valid for the ST–BI BHs. As a matter of calculation, one is able to show that HCs of the ST and Ed BHs are identical. Therefore, the stability properties of our novel BHs are identical in the Einstein and Jordan frames, as described by use of Fig. 3.

7 Conclusions

Due to the strong coupling between various field equations of ST gravity theory, it is not easy to obtain the analytical solutions, directly. Here, we have removed the mentioned problem by utilizing CTs and transforming the ST action to that of Ed theory. It has been found that the scalar-coupled form of the BI electrodynamics differs from that has been used frequently by many other authors. The equations of motion have been solved in a four-dimensional EF and, by applying the inverse CTs, the JF exact analytical solutions have been obtained from their EF counterparts. Finally, two novel classes of BH solutions have been introduced in the Ed and ST theories of gravity, which are related to each other by CTs. The metric functions in both of the Einstein and Jordan frames can produce the horizon-less, extreme and two-horizon BHs. By expanding the solutions, it has been proved that the exact solutions of Ed gravity reduce to the corresponding quantities in the Einstein–Maxwell–dilaton theory, as the BI parameter is taken very large. Asymptotic behavior of the solutions, in the presence of scalar field, is neither flat nor A(dS). Thus, the spacetime geometry, in the Jordan and Einstein frames, has been affected by the scalar fields.

The BH temperature, mass, charge, entropy and electric potential, as the conserved and thermodynamic quantities, have been calculated in the EF and under the influence of dilaton and BI fields. Then by introducing the BH mass as a function of S and Q, we showed that the TFL is valid for our novel Ed BHs. Also, as illustrated in Fig. 3, the BHs with positive temperature named as the physical BHs and those with negative temperature, which we call un-physical ones, can occur. In the case of $\sigma \neq -\frac{1}{2}$ extreme BHs with $r_+ = r_{ext}$ exist and, the physical and un-physical BHs appear with horizon radii greater and smaller than r_{ext}, respectively. For the BHs with $\sigma = -\frac{1}{2}$ the extreme BHs can exist with $r_+ = r_{1ext}$ and $r_+ = r_{2ext}$. They are physically reasonable (i.e. have positive temperature) provided that the horizon radii are in the interval $r_{1ext} < r_+ < r_{2ext}$. The HC diverges at $r_+ = r_1$ and $r_+ = r_2$ for the case $\sigma \neq -\frac{1}{2}$ and at $r_+ = r_3$

for $\sigma = -\frac{1}{2}$. It means that the BHs experience type-two PT at the points $r_+ = r_1$, $r_+ = r_2$ and $r_+ = r_3$. The type-one PT can occur for the BHs having horizon radii equal to r_{ext}, r_{1ext}, and r_{2ext}. Also, those with horizon radii in the intervals $r_+ > r_{ext}$ (left panel), $r_{ext} < r_+ < r_1$ and $r_+ > r_2$ (middle panel) and $r_{1ext} < r_+ < r_3$ are locally stable.

Although the entropy-area law is not valid in the ST gravity, but the thermodynamic quantities such as temperature, mass, charge, entropy and electric potential remain invariant under CTs. With these facts in mind, it has been shown that the thermodynamical first law is valid for the ST BHs too. The physical and un-physical ST BHs exist with the same properties mentioned for the Ed BHs. Also, the ST BHs have the same stability properties as the Ed ones.

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