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Do the Einstein-matter field equations always predict the existence of light rings in black-hole spacetimes?

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Abstract We are used to think of null circular geodesics as an integral part of black-hole spacetimes. In the present compact paper we reveal the physically interesting fact that, contrary to the general belief, the non-linearly coupled Einstein-matter field equations do not a priori guarantee the existence of external light rings in extremal black-hole spacetimes. This observation raises the intriguing possibility of having extremal black-hole spacetimes that possess no external light rings. We prove that these unique black holes, if exist, are characterized by non-positive tangential pressures on their surfaces.

1 Introduction

The existence of closed light-like orbits around highly compact objects is one of the most remarkable characteristics of curved spacetimes. In particular, the existence of null circular geodesics in black-hole spacetimes seems to be a generic mathematical prediction of the non-linearly coupled Einstein-mater field equations [1–7].

The presence of photonspheres around black holes has many important implications on the physical properties of these fascinating compact objects. For instance, the instability properties of light-like circular orbits around central black holes are known to determine the characteristic relaxation rates of the corresponding perturbed black-hole spacetimes (see [8–16] and references therein). In addition, the physically interesting phenomenon of strong gravitational lensing by black holes is closely related to the presence of null circular geodesics around the central compact objects [17–19].

Intriguingly, it has been explicitly proved that, in spherically symmetric hairy black-hole spacetimes, the radial location of the innermost null circular geodesic provides a lower bound on the effective lengths of the externally supported hairy field configurations [4,10–14,20,21]. In addition, it has been revealed in [22] (see also [23]) that, among all equatorial trajectories around a central black hole, the null circular geodesic is unique in the sense that it determines the shortest possible orbital period around the black hole as measured by asymptotic observers.

In the present compact paper we raise the following physically important question regarding the presence of null circular geodesics in black-hole spacetimes: Do the non-linearly coupled Einstein-matter field equations always guarantee the existence of photonspheres in the external regions of black-hole spacetimes? In order to address this interesting question, we shall analyze the physical and mathematical properties of null circular geodesics in curved black-hole spacetimes.

In particular, using analytical techniques, we shall reveal the physically interesting fact that the Einstein-matter field equations seem to fail to provide a generic proof for the existence of external null circular geodesics in extremal black-hole spacetimes which are characterized by non-positive tangential pressures on their surfaces. This important observation leaves open the intriguing possibility of obtaining extremal black-hole solutions of the Einstein-matter field equations that possess no external light rings.

Before proceeding, it is important to emphasize that the existence of null circular geodesics in generic (*non*-extremal) asymptotically flat black-hole spacetimes has been proved in [4] for spherically symmetric hairy configurations and in the physically important work [5] for stationary axi-symmetric black-hole spacetimes. In the present paper we shall focus our attention on extremal spherically symmetric hairy black-hole spacetimes.

2 Description of the system

We shall study the physical and mathematical properties of null circular geodesics in black-hole spacetimes whose spher-

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ically symmetric curved line element is given by the compact expression [22,24–28]

$$ds^2 = -e^{-2\delta}\mu dt^2 + \mu^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

with the near-horizon ($r \to r_H$) functional behaviors [29]

$$\mu(r \to r_H) = O\Big(\frac{r - r_H}{r_H}\Big) \quad \text{for non-extremal black holes}\ , \quad (2)$$

$$\mu(r \to r_H) = O\Big[\Big(\frac{r - r_H}{r_H}\Big)^2\Big] \quad \text{for extremal black holes}\ , \quad (3)$$

and

$$\delta(r = r_H) < \infty \quad ; \quad [d\delta/dr]_{r=r_H} < \infty \quad (4)$$

of the metric functions. In addition, the black-hole spacetime (1) is assumed to be asymptotically flat, in which case the metric functions are characterized by the simple large-r functional relations

$$\mu(r \to \infty) \to 1 \quad \text{and} \quad \delta(r \to \infty) \to 0\ . \quad (5)$$

The non-linearly coupled Einstein-matter field equations $G^{\mu}_{\nu} = 8\pi T^{\mu}_{\nu}$ can be expressed in the form [22,25–27]

$$\frac{d\mu}{dr} = -8\pi r\rho + \frac{1-\mu}{r} \quad (6)$$

and

$$\frac{d\delta}{dr} = -\frac{4\pi r(\rho + p)}{\mu}\ , \quad (7)$$

where the radially-dependent energy density ρ, radial pressure p, and tangential pressure p_T of the external matter fields are related to the components of the energy-momentum tensor [30]:

$$\rho \equiv -T^t_t \quad , \quad p \equiv T^r_r \quad , \quad p_T \equiv T^{\theta}_{\theta} = T^{\phi}_{\phi}\ . \quad (8)$$

Below we shall use the characteristic functional relations [29]

$$1 + 8\pi r_H^2 p(r_H) > 0 \quad \text{for non-extremal black holes} \quad (9)$$

and

$$1 + 8\pi r_H^2 p(r_H) = 0 \quad \text{for extremal black holes}\ . \quad (10)$$

In addition, taking cognizance of Eqs. (3), (4), and (7), one finds the near-horizon functional behavior [29]

$$[\rho + p](r \to r_H) = O\Big[\Big(\frac{r - r_H}{r_H}\Big)^2\Big] \quad (11)$$

for extremal black-hole spacetimes.

We shall assume that the external matter fields satisfy the dominant energy condition which implies that the non-negative energy density is bounded from below by the inequalities [29]:

$$\rho \geq |p| \quad \text{and} \quad \rho \geq |p_T|\ . \quad (12)$$

From the inequality (12) and the assumption that the total gravitational mass of the spacetime,

$$M = \frac{r_H}{2} + \int_{r_H}^{\infty} 4\pi r^2 \rho(r) dr\ , \quad (13)$$

is finite [31], one finds the asymptotic functional behavior

$$r^3 p(r) \to 0 \quad \text{as} \quad r \to \infty \quad (14)$$

of the radial pressure function. In addition, from Eqs. (6) and (13) one finds the radial functional relation

$$\mu(r) = 1 - \frac{2m(r)}{r}\ , \quad (15)$$

where $m(r) = r_H/2 + \int_{r_H}^{r} 4\pi r'^2 \rho(r') dr'$ is the mass contained within a sphere of radius r.

3 Null circular geodesics in black-hole spacetimes

In the present section we shall reveal the physically important fact that, while the Einstein-matter field equations guarantee that generic (non-extremal) black holes possess at least one light ring outside the outermost black-hole horizon, they do not a priori guarantee the existence of external light rings in extremal black-hole spacetimes.

To this end, it proves useful to define the radial function

$$\mathcal{N}(r) \equiv 3\mu - 1 - 8\pi r^2 p\ , \quad (16)$$

whose roots determine the radial location of the black-hole null circular geodesic(s) [20]:

$$\mathcal{N}(r = r_{\gamma}) = 0\ . \quad (17)$$

Taking cognizance of Eqs. (5) and (14), one deduces that the dimensionless function (16) is characterized by the simple asymptotic relation

$$\mathcal{N}(r \to \infty) \to 2\,. \tag{18}$$

In addition, from Eqs. (2), (3), (9), and (10), one finds that the radial function (16) is characterized by the horizon boundary condition

$$\mathcal{N}(r = r_{\rm H}) < 0 \quad \text{for non-extremal black holes} \tag{19}$$

and

$$\mathcal{N}(r = r_{\rm H}) = 0 \quad \text{for extremal black holes}\,. \tag{20}$$

Taking cognizance of the characteristic functional relations (17), (18), and (19), one immediately deduces that all spherically symmetric non-extremal black holes possess at least one external ($r_\gamma > r_{\rm H}$) null circular geodesic which is characterized by the property $\mathcal{N}(r = r_\gamma) = 0$ [32].

On the other hand, the corresponding relations (17), (18), and (20) for extremal black holes do not guarantee the existence of external null circular geodesics. In particular, the functional relations (18) and (20) leave open the possibility of having extremal black-hole spacetimes that are characterized by the positive definite functional relation

$$\mathcal{N}(r > r_{\rm H}) > 0\,. \tag{21}$$

4 A necessary condition for the existence of extremal black holes with no external light rings

In the present section we raise the following physically interesting question: What does it take to create extremal black holes that possess no external photonspheres? In order to address this important question, we shall analyze the spatial functional behavior of the dimensionless radial function (16) in extremal black-hole spacetimes. In particular, below we shall derive a necessary condition for the existence of extremal black holes that possess no external null circular geodesics.

To this end, it proves useful to substitute the Einstein field Eqs. (6) and (7) into the conservation equation

$$T^{\mu}_{r;\mu} = 0\,, \tag{22}$$

which yields the radial differential equation

$$\frac{d}{dr}(r^2 p) = \frac{r}{2\mu}\Big[(3\mu - 1 - 8\pi r^2 p)(\rho + p) + 2\mu(-\rho - p + 2p_T)\Big]. \tag{23}$$

From Eqs. (13), (15), (16), and (23) one obtains the gradient relation

$$\frac{d\mathcal{N}}{dr} = \frac{3}{r}(1-8\pi r^2\rho-\mu) - \frac{4\pi r}{\mu}\mathcal{N}(\rho+p) + 8\pi r(\rho+p-2p_T)\,. \tag{24}$$

Taking cognizance of Eqs. (3), (10), and (11), one finds the characteristic near-horizon $[(r - r_{\rm H})/r_{\rm H} \ll 1]$ functional relations

$$1 - 8\pi r^2\rho - \mu = O\Big(\frac{r - r_{\rm H}}{r_{\rm H}}\Big)\,, \tag{25}$$

$$\frac{\mathcal{N}(\rho + p)}{\mu} = O\Big(\frac{r - r_{\rm H}}{r_{\rm H}}\Big)\,, \tag{26}$$

and

$$\rho + p - 2p_T = -2p_T + O\Big[\Big(\frac{r - r_{\rm H}}{r_{\rm H}}\Big)^2\Big]\,, \tag{27}$$

which yield the near-horizon gradient behavior [see Eq. (24)]

$$\Big[\frac{d\mathcal{N}}{dr}\Big]_{r=r_{\rm H}} = -16\pi r_{\rm H} p_T(r_{\rm H}) \tag{28}$$

for extremal black holes.

Interestingly, one deduces from Eqs. (3), (10), (16), and (28) that extremal black holes with the property $p_T(r_{\rm H}) > 0$ are characterized by the near-horizon functional relations

$$\Big\{\mathcal{N}(r = r_{\rm H}) = 0 \ \text{ and } \ \Big[\frac{d\mathcal{N}}{dr}\Big]_{r=r_{\rm H}} < 0\Big\} \ \text{ for } \ p_T(r_{\rm H}) > 0\,, \tag{29}$$

which, together with the asymptotic relation (18), guarantee the existence of at least one external ($r > r_{\rm H}$) null circular geodesic with the property $\mathcal{N}(r = r_\gamma) = 0$ in the extremal black-hole spacetime.

On the other hand, from Eqs. (3), (10), (16), and (28) one learns that extremal black holes with the horizon property $p_T(r_{\rm H}) < 0$ are characterized by the functional relations

$$\Big\{\mathcal{N}(r = r_{\rm H}) = 0 \ \text{ and } \ \Big[\frac{d\mathcal{N}}{dr}\Big]_{r=r_{\rm H}} \geq 0\Big\} \ \text{ for } \ p_T(r_{\rm H}) \leq 0\,, \tag{30}$$

in which case one intriguingly deduces that the non-linearly coupled Einstein-matter field equations do *not* a priori guarantee the general existence of external null circular geodesics [with the property $\mathcal{N}(r = r_\gamma > r_{\rm H}) = 0$] in the extremal black-hole spacetime (1).

5 Summary and discussion

The existence of photonspheres in highly curved spacetimes, on which massless particles can perform closed orbital motions, has many implications on the physical properties of the corresponding central compact objects. In particular, null circular geodesics play an important role in observational as well as in theoretical studies of black-hole physics (see [1–23] and references therein).

In the present compact paper we have raised the physically intriguing question: Do the non-linearly coupled Einstein-matter field equations always predict the existence of light rings in black-hole spacetimes? Using analytical techniques, it has been explicitly proved that spherically symmetric non-extremal black holes possess at least one external light ring [32].

Intriguingly, however, we have revealed the fact that, contrary to the general belief, the Einstein-matter field equations seem to fail to provide a general proof for the existence of light rings in the external regions of *extremal* black holes which are characterized by non-positive tangential pressures on their surfaces:

$$p_T(r=r_{\rm H})\leq 0 \Longrightarrow \text{No guarantee for the existence of external null circular geodesics.} \tag{31}$$

This important observation leaves open the physically intriguing possibility of finding solutions of the non-linearly coupled Einstein-matter field equations that describe extremal black-hole spacetimes with the physical property (31) that possess no external light rings. It is worth stressing the fact that the horizon property (31) provides a necessary condition for the existence of these unique black-hole spacetimes [33].

It is worth mentioning that an example of a non-asymptotically flat black-hole spacetime that has no external light rings has been given in the physically interesting work [34]. In particular, it has been explicitly proved in [34] that black holes immersed in uniform magnetic fields, which are described by the curved Schwarzschild-Melvin spacetime, have no external light rings in the dimensionless super-critical magnetic regime $BM > (BM)_{\rm c} \simeq 0.189$ [here $\{M, B\}$ are respectively the mass and the magnetic field of the spacetime]. Interestingly, it has been shown that the absence of external equatorial null circular geodesics implies that the black-hole spacetime is characterized by a panoramic shadow [34].

It is important to emphasize the fact that, as opposed to non-extremal black holes which are generally characterized by an *odd* number of light rings, if extremal black-hole spacetimes with the horizon property (31) do exist, then these black holes would be special in the sense that they are expected to be characterized by an *even* (possibly zero) number of external light rings [35–37].

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32. It is worth emphasizing again that the existence of null circular geodesics in *non*-extremal asymptotically flat black-hole spacetimes has been proved in [4] for spherically symmetric hairy configurations and in the highly interesting work [5] for stationary axi-symmetric black-hole spacetimes
33. It is worth emphasizing that we do not rule out the possible existence of external light rings in extremal black-hole spacetimes. In particular, our analysis reveals the fact that the Einstein-matter field equations seem not to a priori guarantee the existence of a general proof (which is valid for generic matter models) for the existence of external light rings in extremal black-hole spacetimes which are characterized by the property $p_T(r_{\rm H}) \leq 0$
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