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Three blocks solvable lattice models and Birman–Murakami–Wenzl algebra

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Abstract

Birman–Murakami–Wenzl (BMW) algebra was introduced in connection with knot theory. We treat here interaction round the face solvable (IRF) lattice models. We assume that the face transfer matrix obeys a cubic polynomial equation, which is called the three block case. We prove that the three block theories all obey the BMW algebra. We exemplify this result by treating in detail the $SU(2)$ 2×2 fused models, and showing explicitly the BMW structure. We use the connection between the construction of solvable lattice models and conformal field theory. This result is important to the solution of IRF lattice models and the development of new models, as well as to knot theory.

1. Introduction

Solvable lattice models in two dimensions play a major role as models of phase transitions. For a review see Baxter's book [1]. They also have a beautiful role in mathematics. For example, they allow for the definition of knot invariants, which are crucial in the classification of knots. For a review see [2], and ref. therein.

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This paper is concerned with Interaction Round the Face (IRF) solvable lattice models. The first examples are the Ising and the hard hexagon models. These have been vastly generalized, see, e.g. [2]. In particular, in ref. [3], the construction of such models from conformal field theory was described. The conformal field theories in two dimensions were first suggested by Belavin, Polyakov and Zamolodchikov [4], constructing the so called minimal models. They were subsequently extended to Wess–Zumino–Witten (WZW) affine conformal field theories [5–7]. The main examples of solvable IRF models, as discussed here, are related to the WZW theories.

The algebraic structure of IRF models plays a significant role in their solution. It is also important in applications to knot theory. The face transfer matrix obeys some polynomial equation, the order of which we call the number of blocks. This is further elucidated in the following discussion. For two blocks it is known that the algebra is Temperley–Lieb algebra [8], yielding the, so called, graph–state IRF models.

Our aim here is to treat the three block case. We prove that three block IRF models obey the Birman–Murakami–Wenzl (BMW) algebra. This is our main result. We note that the BMW algebra initially arose in connection to knot theory [9,10]. We exemplify by showing that the fused 2×2 $SU(2)$ models obey the BMW algebra.

These results are important to the understanding of IRF models, the construction of new ones, and is mathematically interesting, particularly in knot theory. We hope to further the understanding of the algebraic structure of IRF models in general.

2. Interaction round the face lattice models

Fusion IRF models were constructed in ref. [3]. We consider a square lattice and we fix a rational conformal field theory $\mathcal{O}$. We denote the primary fields by $[p]$ where the unit field is denoted by 1. On the sites of the lattice models we have some primary fields. The interaction is around the faces of the model. The partition function of the model is given by

$$Z = \sum_{\text{configurations}} \prod_{\text{faces}} \omega\begin{pmatrix} a & b \\ c & d \end{matrix}\middle| u \right), \tag{1}$$

where the sum is over all primary fields sitting on the sites and

$$\omega\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right) \tag{2}$$

is some Boltzmann weight. The Boltzmann weights are subject to a set of constraints, which allow to define them explicitly and which will be discussed below. Here u is some parameter called the spectral parameter.

Denote by $f_{a,b}^{c}$ the fusion coefficient of the conformal field theory $\mathcal{O}$, where $[a]$, $[b]$ and $[c]$ are some primary fields. We fix a pair of primary fields, $[h]$ and $[v]$. The model is defined by the admissibility condition that the Boltzmann weights vanish unless the fields around the face obey [3],

$$f_{hb}^{a} f_{vd}^{b} f_{hd}^{c} f_{vc}^{a} > 0. \tag{3}$$

Accordingly, we denote this model as $IRF(\mathcal{O}, h, v)$. We find it convenient to define the face transfer matrix, through its matrix elements,

$$\langle a_1, a_2, \dots, a_n | X_i(u) | a_1', a_2', \dots, a_n' \rangle = \left[\prod_{j \neq i} \delta_{a_j, a_j'}\right] \omega\left(\begin{matrix} a_{i-1} & a_i \\ a_i' & a_{i+1} \end{matrix}\middle| u\right). \tag{4}$$

In this language, the conditions for the solvability of the model, which is the celebrated Yang Baxter equation becomes,

$$X_i(u)X_{i+1}(u+v)X_i(v)=X_{i+1}(v)X_i(u+v)X_{i+1}(u), \tag{5}$$

$$X_i(u)X_j(v)=X_j(v)X_i(u), \quad \text{for } |i-j|\geq 2. \tag{6}$$

The Yang Baxter equation implies that the transfer matrices for all values of u commute with one another.

At the limit $u\to i\infty$ the Yang Baxter equation (YBE) becomes,

$$X_iX_{i+1}X_i=X_{i+1}X_iX_{i+1}, \quad \text{where} \quad X_i=\lim_{u\to i\infty} e^{i(n-1)u}X_i(u). \tag{7}$$

$$X_iX_j=X_jX_i \quad \text{where} \quad |i-j|\geq 2. \tag{8}$$

This is the braid group relation. The number of blocks is denoted by n. The exponent pre-factor is necessary to make the limit finite as is discussed below, eqs. (16)–(17).

We have a natural candidate for the braid group which obeys the admissibility condition eq. (3). This is the braiding matrix of the conformal field theory $\mathcal{O}$, which expresses the braiding of the conformal blocks [11]. This matrix is denoted by C and it obeys the Hexagon relation [11], which for $h=v$ is equivalent to the braid group relations, eqs. (7–8). We define

$$\lim_{u\to i\infty} e^{i(n-1)u}\omega\left(\begin{matrix} a & b \\ c & d\end{matrix}\middle| u\right)=C_{c,d}\begin{bmatrix} h & v \\ a & b\end{bmatrix}. \tag{9}$$

With this definition the Boltzmann weights ω obey the admissibility condition for the lattice model $IRF(\mathcal{O},h,v)$, and the face matrix, X_i, obeys the braid group relations, eqs. (7)–(8). We note that the finiteness of the limit, eq. (9), follows directly from the explicit realization of the Boltzmann weights, eq. (16).

Denote by Δ_p the dimension of the primary field $[p]$. Assume also that $h=v$ and

$$[h]\times[h]=\sum_{i=0}^{n-1}\psi_i, \tag{10}$$

where the product is in terms of the fusion rules, and ψ_i are some primary fields. We denote by Δ_i the dimension of the primary field ψ_i, and n is the number of primary fields in the product. We call this the n block case. The face operator X_i, which is equal to the braiding matrix, has then the eigenvalues,

$$\lambda_i=\epsilon_i e^{i\pi(\Delta_h+\Delta_v-\Delta_i)}, \tag{11}$$

where $\epsilon_i=\pm 1$, according to whether the product is symmetric or anti–symmetric. We shall assume that $\epsilon_i=(-1)^i$.

The matrix X_i then is seen to obey an nth order polynomial equation,

$$\prod_{p=0}^{n-1}(X_i-\lambda_p)=0. \tag{12}$$

The fact that X_i obeys an nth order equation allows us to define projection operators. We define,

$$P_i^a=\prod_{p\neq a}\left[\frac{X_i-\lambda_p}{\lambda_a-\lambda_p}\right]. \tag{13}$$

The projection operators obey the relations,

$$\begin{aligned} \sum_{a=0}^{n-1} P_i^a &= 1 \\ P_i^a P_i^b &= \delta_{ab} P_i^a \\ \sum_{a=0}^{n-1} \lambda_a P_i^a &= X_i \end{aligned} \tag{14}$$

We shall assume that the theory is real. This implies that $h = \bar{h}$. So in the fusion rules $[h] \times [h] = 1 + \ldots$, or $\psi_0 = [1]$. In ref. [3], a conjecture for trigonometric solutions of the YBE, eqs. (5–6), was proposed. For this purpose we introduce the parameters,

$$\zeta_i = \pi(\Delta_{i+1} - \Delta_i)/2, \tag{15}$$

where Δ_i is the dimension of the field ψ_i, and $\Delta_0 = 0$. The trigonometric solution for the Yang Baxter equation is then

$$X_i(u) = \sum_{a=0}^{n-1} f_a(u) P_i^a, \tag{16}$$

where the functions $f_a(u)$ are defined by

$$f_a(u) = \left[\prod_{r=1}^{a} \sin(\zeta_{r-1} - u)\right]\left[\prod_{r=a+1}^{n-1} \sin(\zeta_{r-1} + u)\right] \Big/ \left[\prod_{r=1}^{n-1} \sin(\zeta_{r-1})\right]. \tag{17}$$

In many cases it was verified explicitly that the conjectured Boltzmann weights, eqs. (16)–(17), obey the YBE equation. This was done in examples for models with any number of blocks. See, e.g., refs. [2,12]. In what follows we shall assume that for our conjectured Boltzmann weights, the YBE indeed holds.

The Boltzmann weights, $X_i(u)$ obey several properties [3]. The first one is crossing,

$$\omega\left(\begin{matrix} a & b \\ c & d \end{matrix} \middle| \lambda - u\right) = \left[\frac{S_{b,0} S_{c,0}}{S_{a,0} S_{d,0}}\right]^{1/2} \omega\left(\begin{matrix} c & a \\ d & b \end{matrix} \middle| u\right), \tag{18}$$

where $S_{a,b}$ is the modular matrix, and $\lambda = \zeta_0$ is the crossing parameter.

Define the element of the algebra,

$$E_i = X_i(\lambda). \tag{19}$$

Since $X_i(0) = 1$ we find from the crossing relation an expression for E_i,

$$E\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left[\frac{S_{b,0} S_{c,0}}{S_{a,0} S_{d,0}}\right]^{1/2} \delta_{a,d}. \tag{20}$$

From this expression it easy to verify that E_i obeys the Temperley–Lieb algebra,

$$E_i E_{i\pm 1} E_i = E_i. \tag{21}$$

For the square of E_i we find,

$$E_i^2 = \beta E_i, \quad \text{where} \quad \beta = \prod_{r=1}^{n-1} \frac{\sin(\lambda + \zeta_{r-1})}{\sin(\zeta_{r-1})}. \tag{22}$$

In the case of two blocks $n = 2$ this solves completely for $X_i(u)$ showing that the solution is a graph state IRF [3]. We conclude that for two blocks the algebra is Temperley–Lieb. It is noteworthy that for any number of blocks, E_i obeys the Temperley–Lieb algebra. The only property we used is the crossing symmetry, eq. (18), which is a conjecture. Assuming the crossing symmetry, the Temperley–Lieb algebra directly follows.

Another relation which is evident is the inversion relation,

$$X_i(u)X_i(-u) = \rho(u)\rho(-u)1_i, \tag{23}$$

where

$$\rho(u) = \prod_{r=1}^{n-1} \frac{\sin(\zeta_{r-1} - u)}{\sin(\zeta_{r-1})}. \tag{24}$$

3. Birman Murakami and Wenzl algebra

Our purpose is to generalize the two blocks result to three blocks. Our main result is that every three block theory obeys the Birman–Murakami–Wenzl algebra (BMW) [9,10]. First, for this purpose, we list the relationships of the algebra. There are two generators of the algebra, G_i and E_i. The relations are,

$$\begin{aligned}
&G_i G_j = G_j G_i \text{ if } |i-j| \geq 2, \\
&G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1}, \qquad E_i E_{i\pm1} E_i = E_i, \\
&G_i - G_i^{-1} = m(1 - E_i), \\
&G_{i\pm1} G_i E_{i\pm1} = E_i G_{i\pm1} G_i = E_i E_{i\pm1}, \qquad G_{i\pm1} E_i G_{i\pm1} = G_i^{-1} E_{i\pm1} G_i^{-1}, \\
&G_{i\pm1} E_i E_{i\pm1} = G_i^{-1} E_{i\pm1}, \qquad E_{i\pm1} E_i G_{i\pm1} = E_{i\pm1} G_i^{-1}, \\
&G_i E_i = E_i G_i = l^{-1} E_i, \qquad E_i G_{i\pm1} E_i = l E_i.
\end{aligned} \tag{25}$$

These relations imply the additional relations,

$$E_i E_j = E_j E_i \text{ if } |i-j| \geq 2, \qquad (E_i)^2 = [(l - l^{-1})/m + 1] E_i. \tag{26}$$

Here, l and m are the two parameters of the algebra. The BMW algebra is defined here according to Kauffmann's 'Dubrovnik' version of the algebra [13], which is used in the definition of Kauffmann's polynomial.

The BMW algebra is known to have a canonical Baxterization [9,10]. This is given by two parameters λ and μ. The parameters are related to m and l, as will be described below. The face operator is defined by,

$$U_i(u) = 1 - \frac{i \sin(u)}{2\sin(\lambda)\sin(\mu)} \left[e^{-i(u-\lambda)} G_i - e^{i(u-\lambda)} G_i^{-1} \right]. \tag{27}$$

Here 1 stands for the unit matrix. With the definition of the BMW algebra, eqs. (25)–(26), $U_i(u)$ obeys the Yang Baxter equation,

$$U_i(u) U_{i+1}(u+v) U_i(v) = U_{i+1}(v) U_i(u+v) U_{i+1}(u). \tag{28}$$

We shall concentrate now on the three block case, $n = 3$. In this case, the solution for the face transfer matrix, eqs. (16)–(17), becomes,

$$X_i(u) = \big[P_i^0 \sin(\zeta_0 + u)\sin(\zeta_1 + u) + P_i^1 \sin(\zeta_0 - u)\sin(\zeta_1 + u) \\ + P_i^2 \sin(\zeta_0 - u)\sin(\zeta_1 - u)\big] / [\sin(\zeta_0)\sin(\zeta_1)]\,. \tag{29}$$

We also define

$$X_i^t = \lim_{u \to -i\infty} e^{-2iu} X_i(u). \tag{30}$$

According to the inversion relation, eqs. (23)–(24), $X_i X_i^t$ is proportional to the unit matrix. We wish to scale them, so that they are the inverse of one another. The scale constant can be read from eqs. (23)–(24), the unitarity condition,

$$X_i X_i^t = \left(\lim_{u \to i\infty} e^{2iu}\rho(u)\right)\left(\lim_{u \to -i\infty} e^{-2iu}\rho(u)\right) = 1/w^2, \tag{31}$$

where

$$w = 4\sin(\zeta_0)\sin(\zeta_1), \tag{32}$$

is the necessary scale constant.

Our purpose now is to connect our solution to the YBE $X_i(u)$, eq. (19), with algebraic solution $U_i(u)$, eq. (27),

$$U_i(u) = P_i^0 + P_i^1 + P_i^2 - \frac{i\sin(u)}{2\sin(\lambda)\sin(\mu)}\left(e^{-i(u-\lambda)}G_i - e^{i(u-\lambda)}G_i^{-1}\right). \tag{33}$$

To do this, we identify ζ_0 with λ (as before) and ζ_1 with μ. We also identify the generators G_i and G_i^{-1} to be proportional to X_i and X_i^t respectively,

$$G_i = 4\sin(\lambda)\sin(\mu)e^{-i\lambda}X_i, \\ G_i^{-1} = 4\sin(\lambda)\sin(\mu)e^{i\lambda}X_i^t. \tag{34}$$

The phase is arbitrary, and we fixed it to be compatible with the BMW algebra, eqs. (25–26). Indeed, from eq. (31), we have $G_i G_i^{-1} = 1$, as it should. We identify G_i as the generator of the BMW algebra, eqs. (25)–(26), along with $E_i = X_i(\lambda)$. Our purpose is to show that with this definition the BMW algebra is indeed obeyed.

We substitute the scale definition, eq. (34), into the expression for the Baxterized BMW algebra, $U_i(u)$, eq. (33). We find, after some algebra, that $X_i(u)$ and $U_i(u)$ are identical,

$$U_i(u) = X_i(u), \tag{35}$$

for any P_i^0, P_i^1 and P_i^2, provided only that they obey $P_i^0 + P_i^1 + P_i^2 = 1$, which is true.

We conjectured that $X_i(u)$ obeys the Yang Baxter equation, eqs. (5)–(6). This was proved for many cases (see, e.g. ref. [2]). It then follows that $U_i(u)$ obeys the YBE also. By expanding the face weight, as in eq. (33), this, in turn, implies that G_i and E_i obey the Birman–Murakami–Wenzl algebra, eqs. (25)–(26), which is the main result of this paper. We basically, invert the logic: instead of proving that $U_i(u)$ obeys the Yang Baxter equation, from the BMW relations, we prove the BMW relations from the Yang Baxter equation which is obeyed by $U_i(u)$. In other words, we prove that the YBE implies the correctness of the BMW relations. We see this by expanding the YBE as polynomials in e^{iu} and e^{iv} from which we get three linear relations, which in turn, imply the BMW algebra.

We get now to determining the parameters l and m. We have the equation, eq. (25),

$$G_i - G_i^{-1} = m(1 - E_i), \tag{36}$$

where $E_i = U_i(\lambda)$. This is the Skein relation. In particular E_i obeys the Temperley–Lieb algebra, eqs. (21–22), which is part of the BMW algebra, eqs. (25–26). By calculating $U_i(\lambda)$, from eq. (27), we find

$$m = -2i \sin(\mu). \tag{37}$$

By calculating the equation,

$$G_i E_i = E_i G_i = l^{-1} E_i, \tag{38}$$

we find that

$$l = -e^{i(2\lambda+\mu)}. \tag{39}$$

Finally, we had before that $E_i^2 = \beta E_i$, eq. (22), where

$$\beta = \frac{\sin(2\lambda)\sin(\lambda+\mu)}{\sin(\lambda)\sin(\mu)}. \tag{40}$$

We know from the BMW algebra that we should have,

$$\beta = (l - l^{-1})/m + 1. \tag{41}$$

Substituting l and m we find that this relation is indeed obeyed.

4. The fused $SU(2)$ model

Since our discussion above was somewhat abstract, it is worthwhile to study a concrete example. This example is the model $\mathrm{I}RF(SU(2)_k, [2], [2])$. Namely, the conformal field theory $SU(2)_k$, which is a WZW affine theory, at level k, with the admissibility fields taken to be $h = v = [2]$. We denote by l the weights of $SU(2)_k$ representations, which are twice the isospin, $l = 0, 1, 2, \ldots, k$. So [2] is the field with isospin one.

The fields appearing in the product hv are

$$[2] \times [2] = [0] + [2] + [4]. \tag{42}$$

The dimensions of the fields in the theory are given by

$$\Delta_l = \frac{l(l+2)}{4(k+2)}. \tag{43}$$

So we find for ζ_0 and ζ_1 the values, according to eq. (15),

$$\zeta_0 = \lambda = \frac{\pi}{2}\frac{2 \cdot 4}{4(k+2)} = \frac{\pi}{k+2}, \tag{44}$$

and

$$\zeta_1 = \frac{\pi}{2}\frac{4 \cdot 6 - 2 \cdot 4}{4(k+2)} = \frac{2\pi}{k+2} = 2\lambda. \tag{45}$$

Thus the face transfer matrix, $X_i(u)$, is given by eq. (29), with the substitution of the above values of ζ_0 and ζ_1. In fact, the exact same weights were given in a paper by Pasquier [12]. In his work it is proved that $X_i(u)$ obeys the Yang Baxter equation, with the appropriate choice of P_i^0 and P_i^1. It thus follows, according from the discussion above, that G_i and E_i obeys the BMW algebra, with $\lambda = \pi/(k+2)$ and $\mu = 2\lambda$. The values of l and m for this model are,

$$l = -e^{4i\lambda}, \qquad m = -2i\sin(2\lambda). \tag{46}$$

Explicit expressions for all the Boltzmann weights of the model are listed in the appendix. We checked numerically that all the relations of the BMW algebra, eqs. (25)–(26), are obeyed for this model. We defined G_i and G_i^{-1} as in eq. (34). We checked the BMW relations at levels $k = 8, 9, 10, 12$ and we find a complete agreement with the BMW algebra, with all the relations fulfilled. We conclude that this model obeys the BMW algebra.

5. Discussion

In this paper we proved that any three block IRF model obeys the BMW algebra. We treated explicitly the fused model IRF($SU(2)_k$, [2], [2]), and showed that it obeys the BMW algebra, also by direct computations.

Other important models are the BCD IRF models constructed by Jimbo et al. [14]. These models can be described as IRF(G, $[v]$, $[v]$), where G is either the B_n, C_n or D_n WZW theory at level k, and $[v]$ stands for the vector representation. Since $[v] \times [v]$ contains three fields, this is a three block model. Thus, our analysis in this paper is applicable to these models, proving that they obey the BMW algebra.

Other three block theories were described in ref. [15], at the level of the conformal data, as theories with low number of primary fields. It will be interesting to study the IRF models associated with these models, as well. In particular, from our analysis we expect them to obey the BMW algebra.

An important question, left to future work, is to find the algebras corresponding to IRF models with more than three blocks. We believe that our methods can be useful to study these models, as well. Some of the relations of the general models are known [3], already. Finding these algebras will be of considerable importance to knot theory.

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Appendix A. Weights of 2 × 2 fused model

These are the weights of the model IRF(SU(2), [2], [2]). We define

$$\lambda = \frac{\pi}{k+2} \tag{A.1}$$

and

$$s[x] = \frac{\sin(x)}{\sin(\lambda)}. \tag{A.2}$$

The weights are taken from ref. [16]. The weights were originally computed in ref. [17], by the fusion procedure. We shifted the primary field $a \to a + 1$ so the weights range over $a = 1, 2, \ldots, k + 1$, giving the dimension of the representation.

The 19 Boltzmann weights of the model are as follows:

$$\omega\left(\begin{matrix} a\pm 2 & a \\ a & a\mp 2\end{matrix}\middle| u\right) = \frac{s(u-2\lambda)s(u-\lambda)}{s(2\lambda)}$$

$$\omega\left(\begin{matrix} a & a \\ a & a\pm 2\end{matrix}\middle| u\right) = \omega\left(\begin{matrix} a\pm 2 & a \\ a & a\end{matrix}\middle| u\right) = -\frac{s(u-\lambda)s((a\pm 1)\lambda\mp u)}{s((a\pm 1)\lambda)}$$

$$\omega\left(\begin{matrix} a & a\pm 2 \\ a & a\end{matrix}\middle| u\right) = -\frac{s((a\mp 1)\lambda)s(u)s(a\lambda\pm u)}{s(2\lambda)s(a\lambda)s((a\pm 1)\lambda)}$$

$$\omega\left(\begin{matrix} a & a \\ a\pm 2 & a\end{matrix}\middle| u\right) = -\frac{s(2\lambda)s((a\pm 2)\lambda)s(u)s(a\lambda\pm u)}{s((a-1)\lambda)s((a+1)\lambda)}$$

$$\omega\left(\begin{matrix} a & a\mp 2 \\ a\pm 2 & a\end{matrix}\middle| u\right) = \frac{s((a\mp 2)\lambda)s((a\mp 1)\lambda)s(u)s(\lambda+u)}{s(2\lambda)s(a\lambda)s((a\pm)\lambda)}$$

$$\omega\left(\begin{matrix} a & a\pm 2 \\ a\pm 2 & a\end{matrix}\middle| u\right) = \frac{s(a\lambda\pm u)s((a\pm 1)\lambda\pm u)}{s(a\lambda)s((a\pm 1)\lambda)}$$

$$\omega\left(\begin{matrix} a & a \\ a & a\end{matrix}\middle| u\right) = \frac{s(a\lambda\pm u)s((a\pm 1)\lambda\mp u)}{s(a\lambda)s((a\pm 1)\lambda)} + \frac{s((a\pm 1)\lambda)s((a\mp 2)\lambda)s(u)s(u-\lambda)}{s(2\lambda)s(a\lambda)s((a\mp 1)\lambda)}$$

$$\omega\left(\begin{matrix} a & a\pm 2 \\ a & a\pm 2\end{matrix}\middle| u\right) = \omega\left(\begin{matrix} a\pm 2 & a\pm 2 \\ a & a\end{matrix}\middle| u\right) = \frac{s((a\pm 3)\lambda)s(u)s(u-\lambda)}{s(2\lambda)s((a\pm 1)\lambda)} \tag{A.3}$$

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