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THE LARGE-N LIMIT WITH VANISHING LEADING ORDER CONDENSATE FOR ZERO PION MASS*

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It is conventionally assumed that the negative mass squared term in the linear sigma model version of the pion Lagrangian is $M^2 \sim \Lambda^2_{\rm QCD}$ in powers of $N_{\rm c}$. We consider the case where $M^2 \sim \Lambda^2_{\rm QCD}/N_{\rm c}$ so that to leading order in $N_{\rm c}$, this symmetry breaking term vanishes. We present some arguments why this might be plausible. One might think that such a radical assumption would contradict lattice Monte Carlo data on QCD as a function of $N_{\rm c}$. We show that the linear sigma model gives a fair description of the data of DeGrand and Liu both for $N_{\rm c} = 3$ and for variable $N_{\rm c}$. The values of quark masses considered by DeGrand and Liu, and by Bali *et al.* turn out to be too large to resolve the case we consider from that of the conventional large-$N_{\rm c}$ limit. We argue that for quark masses $m_q \ll \Lambda_{\rm QCD}/N_{\rm c}^{3/2}$, both the baryon mass and nucleon size scale as $\sqrt{N_{\rm c}}$. For $m_q \gg \Lambda_{\rm QCD}/N_{\rm c}^{3/2}$, the conventional large-$N_{\rm c}$ counting holds. The physical values of quark masses for QCD ($N_{\rm c} = 3$) correspond to the small quark-mass limit. We find pion nucleon coupling strengths are reduced to the order $\mathcal{O}(1)$ rather than $\mathcal{O}(N_{\rm c})$. Under the assumption that in the large-$N_{\rm c}$ limit, the sigma meson mass is larger than that of the omega, and that the omega–nucleon coupling constant is larger than that of the sigma, we argue that the nucleon–nucleon large-range potential is weakly attractive and admits an interaction energy of the order of $\Lambda_{\rm QCD}/N_{\rm c}^{5/2} \sim 10$ MeV. With these assumptions on coupling and masses, there is no strong long-range attractive channel for nucleon–nucleon interactions, so that nuclear matter at densities much smaller than that where nucleons strongly interact is a weakly interacting configuration of nucleons with strongly interacting localized cores. This situation is unlike the case in the conventional large-N_c limit, where nuclear matter is bound with binding energies of the order of the nucleon mass and forms a Skyrme crystal.

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1. Introduction

It is standard lore that the large-N_c limit predicts a vacuum expectation value for the sigma field, f_π, which scales as $\sqrt{N_c}$, and that baryons have a mass of the order of N_c [1]. A model for baryons is provided by the Skyrme model; within the Skyrme model, the meson–nucleon interactions are strong, of the order of N_c [2]. This includes pion–nucleon interactions, and for massless pions leads to very strong long-range forces [1].

The long-range force associated with pion exchange leads to the conclusion that the binding energy of nuclear matter is of the order of the nucleon mass, and that a nucleon liquid would be unstable, forming a Skyrme crystal of tightly bound nucleons [3]. In nature, nuclear matter is weakly bound, with binding energy of the order of 15 MeV. In spite of the great successes of large-N_c phenomenology for mesons, it appears that the large-N_c limit dramatically fails to describe the most basic feature of nuclear matter, that it is weakly bound [4].

The large strength of meson forces is impossible to evade in the standard Skyrme model of nucleons. The Skyrme model describes a two-baryon solution, and as a function of position, the energy scales as N_c. This has led some to modify the Skyrme model, by ignoring the kinetic energy term in the non-linear sigma model [5]. It is then argued that the Wess–Zumino–Witten term [6, 7] generates a baryon–baryon interaction, and this combined with the mass term for the pions generates stable skyrmions. Remarkably, such skyrmions saturate the BPS bound that means to leading order in large N_c, they are non-interacting. The essential feature of such consideration is that the ordinary sigma model kinetic energy term vanishes corresponding to vanishing vacuum expectation values for the sigma field.

While the BPS Skyrme model has the attractive feature that the interaction energy of nucleons is generically weak [8, 9], it has the unattractive feature that the baryon number density squared is stabilized by the mass term associated with explicit breaking of chiral symmetry. As we shall see, this results in the baryon density being of the order of $N_c m_\pi$, and the massless pion limit of the theory corresponds to a baryon of infinite extent. We can see this from elementary scaling arguments. The baryon self interaction associated with omega meson exchange is of the order of

$$\int \mathrm{d}^3x \frac{N_\mathrm{c}}{M_\omega^2}\rho_B(x)^2 \sim \frac{N_\mathrm{c}}{M_B^2 R^3}\,, \tag{1}$$

where R is the nucleon radius. The stabilizing mass generating by the "sigma term" is of the order of

$$\sim N_\mathrm{c} m_q \Lambda_\mathrm{QCD}^3 R^3\,. \tag{2}$$

Extremization gives

$$R_\mathrm{nuc} \sim m_q^{-1/6}\Lambda_\mathrm{QCD}^{-5/6} \tag{3}$$

and the nucleon mass of the order of

$$M_\mathrm{nuc} \sim N_\mathrm{c} m_q^{1/2}\Lambda_\mathrm{QCD}^{1/2} \sim N_\mathrm{c} m_\pi\,. \tag{4}$$

The baryon number density is of the order of

$$1/R^3 \sim m_q^{1/2}\Lambda^{5/2} \sim m_\pi \Lambda_\mathrm{QCD}^2\,. \tag{5}$$

The vanishing of the nucleon mass in the chiral limit is an unfortunate consequence of this theory. Nevertheless, the idea that the vacuum expectation value of the sigma field is small, and as well the corresponding kinetic term in the non-linear sigma model, might have merit and we will consider this assumption in this paper. We will see that if we assume the kinetic term is suppressed by one order of N_c, we find a different dependence on quark mass for the nucleon mass and radius, and that for the values for which there is lattice data from DeGrand and Liu [10], the resulting nucleon mass is consistent with the data.

A vanishing vacuum expectation value for the scalar field in leading order in N_c corresponds to a vanishing mass for the sigma particle in this order. A vanishing mass for the scalar sigma field does happen in the large-N_c limit of QCD in two dimensions [11]. It is also not so implausible in four dimensions. In leading order in N_c, the four-dimensional theory is a non-interacting theory. With a negative mass squared for the sigma field, this large-N_c theory would be unstable, since to leading order in N_c, interaction terms which would stabilize the theory vanish. A positive mass squared term would not allow for chiral symmetry breaking. Assuming a negative mass squared term generated in non-leading order in N_c would allow for a stable theory in the strict large-N_c limit. The potential would be flat in this limit, allowing for symmetry breaking as required by the Coleman–Witten theorem [12]. The value of the condensate would be determined by the next to leading order corrections to the sigma model.

It is difficult to provide an explicit mechanism for how the symmetry breaking negative mass squared term might vanish. It is interesting that the low-energy sector that results for massless quarks is scale invariant at the mean field level for the effective pion-sigma action. Scale invariance is associated with critical phenomena. Perhaps such scale invariance might be argued by renormalization group methods in the large-$N_{\rm c}$ limit. As we will see, this scale invariance argument is also useful for arguing the form of the effective action that generate the nucleon mass. In any case, we will take throughout this paper an unproven assumption that this masslessness exists and shall explore the consequences of this assumption.

One can ask whether this $1/N_{\rm c}$ behavior of the squared mass of the sigma field is preserved by higher order radiative corrections. This is equivalent to ask if the $1/N_{\rm c}$ behavior is natural. Indeed, when one computes the radiative correction to the sigma mass, the lowest order correction arises from a tadpole diagram. The four-meson interaction in the diagram is of the order of $1/N_{\rm c}$ and the overall quadratic divergence is cutoff at the QCD scale so this gives a contribution to the mass squared of the order of $\Lambda_{\rm QCD}^2/N_{\rm c}$, and indeed the behavior of the scalar mass term is natural. Higher order corrections indeed maintain this behavior.

Another implication of these results is that $f_\pi \sim \mathcal{O}(1)$ in powers of $N_{\rm c}$. On the other hand, standard counting in powers of $N_{\rm c}$ using quark counting gives $f_\pi \sim \sqrt{N_{\rm c}}$. In the considerations below, we will limit ourselves to momentum scales which are less than $\Lambda_{\rm QCD}/\sqrt{N_{\rm c}}$. If, for example, we compute the matrix elements of the vector current squared, it is given by a vacuum polarization diagram involving a quark loop, and this appears to be of the order of $N_{\rm c}$ corresponding to a current with a typical value of the order of $\sqrt{N_{\rm c}}$. However, if we look at low momentum scales, of the order of $q \sim \Lambda_{\rm QCD}/\sqrt{N_{\rm c}}$, the contribution to the quark loop involving gluons must be summed. The leading order contribution in powers of $N_{\rm c}$ are one vector meson states such as the ρ meson, and this state decouples at zero momentum transfer as q^2/M_ρ^2, suppressing the contribution by the order of $1/N_{\rm c}$. Now, if a vector meson coupling to this channel shrinks to zero mass in the large-$N_{\rm c}$ limit, then the ordinary counting can be maintained.

We shall later argue that the sigma meson mass becomes small in the large-$N_{\rm c}$ limit. If this is the case, the ordinary $N_{\rm c}$ counting should work for vector current matrix elements. The axial vector channel is however different. In the axial isoscalar vector channel, there is a U(1) anomaly, and we will argue this will result in a pseudo-scalar particle with mass of the order of $\Lambda_{\rm QCD}$.

Let us consider the two axial vector current correlation function. This correlation function involves a quark–antiquark pair, the interactions of which can lead to a pole corresponding to an axial vector meson. We assume that, as is the case for the scalar axial vector mesons, also have masses of the order of $\Lambda_{\rm QCD}$. In this case, axial vector meson contributions will be suppressed at small q^2. This leaves two meson intermediate and the pion states which are suppressed by powers of N_c. Note that the contribution from the scalar sigma and pion to the axial vector isotriplet current can be written in terms of the low momentum degrees of freedom as

$$J_{\mu a5} \sim \sigma \left\{ \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu \right\} \pi^a \,. \tag{6}$$

For the cases where $f_\pi = \langle \sigma \rangle \sim \mathcal{O}(1)$ in powers of N_c, this current has matrix elements of the order of 1, consistent with the reasoning above. A necessary condition for this to be maintained, therefore, is that the isovector axial vector mesons have a mass of the order of Λ_{QCD}.

We can easily see why the η' mass is of order one in our scenario. It follows simply by the assumption that the energy density dependence of the θ angle of QCD is of order one because

$$m_{\eta'}^2 \sim \frac{1}{f_\pi^2} \frac{\mathrm{d}^2 E}{\mathrm{d}\theta^2} \sim \Lambda_{\mathrm{QCD}}^2 \,, \tag{7}$$

where we have assumed the U(1) current is of the form of

$$J_{\mu 5} \sim \sigma \left\{ \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu \right\} \eta' \,. \tag{8}$$

Since the U(1) axial symmetry is explicitly broken, there is no spontaneous symmetry breaking so the parity doublet of the pion, the scalar isovector mesons presumable have masses of the order of Λ_{QCD}.

Strictly speaking, Eq. (7) for the η' mass is derived when $m_{\eta'}$ is much less than the QCD scale. There are corrections of order one in powers of N_c for this relation when the mass is of the order of the QCD scale. In the conventional argument due to Witten, one assumes the matrix element of the axial vector current that connects the vacuum to the η' state is suppressed relative to the topological charge changing matrix elements. The η' mass is small so that it compensates for the small values of the axial current matrix elements. If, however, the topological charge change is not suppressed in large N_c, the axial vector anomaly requires that the matrix element for producing the η' may be large, and *a priori* a small η' mass is not necessary. The small η' mass in the conventional argument arises because the smallness of the mass compensates for the smallness of the matrix element.

It is useful to be a little more explicit and show how Witten's argument modifies when applied to this case [7]. Witten considers the contribution to the topological susceptibility from gluonic contribution and meson states composed of quarks. When added together, they should yield zero, since the topological susceptibility vanishes in a theory with massless quarks. This leads to a sum rule, as the sum is over meson states

$$\sum_n \frac{|\, \langle 0 | Q_{\mathrm{top}} | n \rangle \,|^2}{m_n^2} = \frac{\mathrm{d}^2 E}{\mathrm{d}^2 \theta} \,, \tag{9}$$

where $\mathrm{d}^2E/\mathrm{d}^2\theta$ is the topological susceptibility in a pure gluon theory. We then use the anomaly

$$Q_{\mathrm{top}} = 2N_{\mathrm{f}} \int \mathrm{d}^4x \partial_\mu J_5^\mu \tag{10}$$

relating the topological charge to twice the number of light quark flavors times the change in the U(1) axial vector charge.

Let us concentrate on the contribution of the η' state to the sum over mesonic states. If this matrix element is of the right magnitude, then it can cancel the non-zero contribution of the gluons to the topological susceptibility. Low-energy current algebra theorems give

$$\langle 0|\partial_\mu J_5^\mu|\eta'\rangle = m_\eta^2 f_\eta \,. \tag{11}$$

We can see this in the low-energy Lagrangian considered above, where in this effective theory

$$J_5^\mu = \phi \partial^\mu \eta' \tag{12}$$

and ϕ is the expectation value of the scalar sigma field so that $f_\pi = f_\eta$ to leading order in N_{c}.

In the ordinary large-N_{c} argument of Witten, $f_\pi \sim \sqrt{N_{\mathrm{c}}}$ and $m_\eta^2 \sim 1/N_{\mathrm{c}}$ so that the matrix element is of the order of $1/\sqrt{N_{\mathrm{c}}}$. The factor of m_η^2 in the denominator of Eq. (9) cancels the factor $1/N_{\mathrm{c}}$ originating from the matrix element squared, and makes the overall η' contribution of order 1.

Now consider the vacuum expectation value of order one, $f_\pi \sim N_{\mathrm{c}}^0$; in order to satisfy the sum rule (9) and have the desired cancellation, we have to have the η' mass of order 1, $m_{\eta'} \sim N_{\mathrm{c}}^0$. Additionally, since the divergence of the axial current is already of order in this case, a plethora of massive states, not just single η' contribute.

We also comment that sometime it is argued that the η' mass is of the order of $1/N_{\mathrm{c}}$ based on the *perturbative* counting of interactions with the gluon field. The difficulty with this argument is that in order to have topological charge changing effects, the gluon field has to be strong, that is *non-perturbative*, of the order of $\sqrt{N_{\mathrm{c}}}$ rendering inconsistency in this argument.

There is no reasonable theory of nuclear matter if the scale σ becomes massless and there is no other small mass isosinglet vector, since this generates an attractive force on baryons of the order of N_{c} and will cause nuclear matter to collapse. One needs also that the ω meson becomes massless also, with $m_\omega \leq m_\sigma$ and that in this limit, ω nucleon coupling is larger than that of the σ meson. This guarantees that in the isosinglet channel at distances less than $R \leq \sqrt{N_{\mathrm{c}}}/\Lambda_{\mathrm{QCD}}$, the overall isosinglet force is repulsive. In various theories such as those with a hidden local gauge symmetry, the omega meson mass is of the order of $f_\pi/\sqrt{N_{\mathrm{c}}}$. We will need to have this possibility here as well. Presumably, there is a similar behavior for the ρ meson. So the theory we propose would have a Goldstone pion, a small mass sigma and omega, $m_\omega \leq m_\sigma \sim 1/\sqrt{N_\mathrm{c}}$. We will make such assumptions in this paper, and explore the consequences.

The reason why ρ-meson mass goes to zero in the large-N_c limit if $f_\pi \sim \mathcal{O}(\mathcal{N}_\mathrm{c}')$ is due to the KSRF relation of current algebra [13]. It states that

$$m_\rho^2 = 2g_{\rho\pi\pi}^2 f_\pi^2\,. \tag{13}$$

Using $g_{\rho\pi\pi}^2 \sim 1/N_\mathrm{c}$, we get $m_\rho^2 \sim 1/N_\mathrm{c}$. In theories with a Hidden Local Gauge symmetry, this mass relation is a consequence of gauge bosons of the effective field theory eliminating the unphysical Goldstone mode, and the non-zero expectation value of the σ field. In such theories, if there is a SU(2) flavor symmetry, the ω meson acquires a mass with parametrically similar dependence upon N_c. In the large-N_c limit, if we assume f_π is of order 1, these vector meson masses vanish.

Hidden Local gauge symmetries provide an explicit realization of the KRSF relation for masses and maintain global symmetries. It is amusing that the limit we consider has a massless sector of scalar, pseudo-scalar and vector bosons that breaks the scale invariance by the appearance of one scale, the vacuum expectations value of the σ field.

In the next section, we write out the explicit action for the pion–nucleon sigma model. We then use this action to compute f_π, the sigma meson mass and the pion mass. We compute these quantities as a function of the quark mass, the number of colors N_c, the four meson coupling strength λ and the negative mass squared which drive the symmetry breaking. We show that our results are in good accord with those of DeGrand and Liu for $N_\mathrm{c} = 3$ and determine the values of the underlying parameters of our sigma model [10]. The values for quark masses in the computation of Bali *et al.* [14] are quite large, and they work in the quenched approximation, and we do not compare with their data. We find that there are two possible cases: In the first one, the quark mass is generically $m_q \gg \Lambda_\mathrm{QCD}/N_\mathrm{c}^{3/2}$. In this limit, symmetry breaking is driven by the explicit symmetry breaking of the quark mass. The large-N_c counting is conventional for physical quantities. The data of DeGrand and Liu is in this range [10]. The other case is $m_q \ll \Lambda_\mathrm{QCD}/N_\mathrm{c}^{3/2}$. In this range, physical quantities have unanticipated dependences on N_c and there is no Monte Carlo data for N_c dependence for this case. For QCD, ($N_\mathrm{c} = 3$), the physical value of quark masses is in this region.

In the fourth section, we compare the sigma model action prediction to the computations of DeGrand and Liu for variable N_c [10]. We find fair agreement with their results, although the quark masses considered are sufficiently large so that deviations from the naive large-N_c scaling predictions are small.

In the fifth section, we consider baryons. We argue that the baryon mass for the small quark mass limit scales as $M_B \sim \sqrt{N_\mathrm{c}}\Lambda_\mathrm{QCD}$, and that the radius $R \sim \sqrt{N_\mathrm{c}}/\Lambda_\mathrm{QCD}$. These scaling relations imply that the baryon mass is of the order of $\sqrt{N_\mathrm{c}}f_\pi$, and this dependence is consistent with the data of DeGrand and Liu for large quark masses [10]. The data available are not in the small quark mass limit where the unexpected behavior as a function of N_c is found. Nevertheless, the agreement with our N_c dependence is only fair for the range of parameters considered by DeGrand and Liu.

In the last section, we consider general properties of the nucleon–nucleon force. We find the pion to be couple not with a coupling of the order of $\sqrt{N_\mathrm{c}}$ to the nucleon but of order 1. The pion force generates a long-range tail for nucleon–nucleon interaction.

To achieve a theory of the nucleon that has a reasonable small interaction strength for nuclear matter, we needed to make drastic assumptions about the behavior of QCD in the large-N_c limit. In the future, we intend to investigate various Lagrangian for scalar and vector mesons to see under what set of assumptions, if any, such a description is internally consistent. It is also true that the behavior we predict makes testable predictions for Monte Carlo simulations of the large-N_c limit of QCD. Unfortunately, the restriction to very small mass quarks makes explicit computation very difficult.

2. The low-energy linear pion sigma model as a function of N_c

We begin with the effective action

$$S = \frac{1}{2}\left[(\partial\phi)^2 + (\partial\pi)^2\right] - \frac{1}{2}\frac{m^2}{N_\mathrm{c}}\left(\phi^2+\pi^2\right) + \frac{\lambda}{4N_\mathrm{c}}\left(\phi^2+\pi^2\right)^2 - \sqrt{N_\mathrm{c}}m_q\mu^2\phi\,. \tag{14}$$

Here, ϕ is the sigma field and π is the pion field. The quark mass is $m_q = (m_u+m_d)/2 \sim 3.5$ MeV. The parameters m, λ, μ will be determined from the sigma meson mass, the pion decay constant f_π which is the vacuum expectation value of ϕ and by the pion mass.

We are here assuming that the negative mass squared term in our action is of the order of $1/N_\mathrm{c}$. The conventional large-N_c limit is obtained if $m^2 \sim N_\mathrm{c}$. The $\sqrt{N_\mathrm{c}}$ in front of the last term (the sigma term) is of the correct order for the large-N_c limit.

The vacuum expectation values of the sigma field is determined by the roots of

$$\phi^3 - \phi m^2/\lambda - N_\mathrm{c}^{3/2}m_q\mu^2/\lambda = 0\,. \tag{15}$$

We denote this root by ϕ_0.

One can solve for the values of the parameters m^2, μ^2 and λ in terms of the masses of the sigma meson, the pion and the expectation value of the scalar field ϕ_0. The vacuum values for $N_\mathrm{c} = 3$ are taken to be

$$m_\pi = 140\,\mathrm{MeV}\,, \qquad \phi_0 \sim f_\pi \sim 130\,\mathrm{MeV}\,, \qquad M_\sigma \sim 950\text{–}1050\,\mathrm{MeV}\,. \quad (16)$$

The solutions are

$$\lambda = N_\mathrm{c}\frac{m_\sigma^2 - m_\pi^2}{2\phi_0^2}\,, \quad (17)$$

$$\mu^2 = \frac{m_\pi^2\phi_0}{m_q\sqrt{N_\mathrm{c}}}\,, \quad (18)$$

$$m^2 = N_\mathrm{c}\frac{m_\sigma^2 - 3m_\pi^2}{2}\,. \quad (19)$$

Note that if we were to take $m^2 \sim N_\mathrm{c}$ for the conventional large-N_c limit, we would have $\phi_o \sim \sqrt{N_\mathrm{c}}$, and the values of λ, μ^2 and m^2 would be N_c-independent. Our philosophy will be to take these values from the case of $N_\mathrm{c} = 3$, fix them, and then determine the variation of m_π, m_σ and $\phi_0 = f_\pi$ predicted by the variation of N_c.

Notice that there are two regimes for solution to this equation. We will get these values more precisely later, but for now if we assume that $\mu \sim m \sim \Lambda_\mathrm{QCD}$ and $\lambda \sim 1$, these regimes are the small and large quark mass limits which are separated by $m_q \sim \Lambda/N_\mathrm{c}^{3/2}$. When we put in properly determined numbers, we will see that the small quark mass regime is characteristic of the physical values for the real world of pions, sigma mesons and f_π. However, we will see that for almost all of the range covered in the work of DeGrand, the large quark mass region dominates. Notice that this also implies that as $N_\mathrm{c} \to \infty$, there is a non-uniformity of the small quark mass limit. The small mass limit is what is required for good phenomenology, so we must always take N_c large but finite and consider masses in the range of $m_q \ll \Lambda/N_\mathrm{c}^{3/2}$ for the correct physical limit.

Let us explore solutions in these different limits. In the small mass limit (Region I), the VEV is determined by ignoring the sigma term (term proportional to the quark mass)

$$\phi_0 = m/\sqrt{\lambda}\,. \quad (20)$$

Because of the extra $1/N_\mathrm{c}$ in the mass term, the VEV for case one is N_c-independent.

In the large quark mass limit (Region II),

$$\phi_0 = \sqrt{N_\mathrm{c}}\left(m_q\mu^2/\lambda\right)^{1/3}\,. \quad (21)$$

In this latter case, the scalar field has the canonical dependence on $N_{\rm c}$ as expected in the ordinary large-$N_{\rm c}$ limit. We will see that this will guarantee the expected large-$N_{\rm c}$ behavior for physical quantities in this limit.

We can now compute the sigma mass

$$m_\sigma^2 = -\frac{m^2}{N_{\rm c}} + \frac{3\lambda}{N_{\rm c}}\phi_0^2\,. \tag{22}$$

For Region I,

$$m_\sigma^2 = \frac{2m^2}{N_{\rm c}}\,. \tag{23}$$

In this small-mass region, the sigma mass shrinks to zero as $N_{\rm c} \to \infty$. Of course, for any fixed quark mass, we are only in the region up till some large N and then we move into the effective large mass region. In Region II,

$$m_\sigma^2 = 3\left(m_q\mu^2\sqrt{\lambda}\right)^{2/3}\,. \tag{24}$$

The pion mass can also be computed in these two limits:
Region I

$$m_\pi^2 = \sqrt{N_{\rm c}}\lambda m_q\frac{\mu^2}{m^2} \tag{25}$$

and Region II

$$m_\pi^2 = \left(m_q\mu^2\sqrt{\lambda}\right)^{2/3}\,. \tag{26}$$

Let us now determine parameters for the world we live in[1], see Eqs. (16)

$$m = 1112.3~\text{MeV}\,, \qquad \mu = 648.3~\text{MeV}\,, \qquad \lambda = 76.7\,. \tag{27}$$

The obtained value of the coupling constant is somewhat large even if we take into account that the expansion parameter that controls perturbative computations is $\lambda/(4\pi^2 N_{\rm c}) \approx 0.7$. This either means that the model at hand is too simplistic or that the lattice artifacts dominate in the observables. Here, our strategy is to be pragmatic and to take the fitted values at their face value.

The edge of the region where the large limit dominates is when the quartic term in the potential dominates over the quadratic. This is when

$$\lambda\phi^2 \sim m^2 \tag{28}$$

or when

$$m_q \sim m\frac{m^2}{\mu^2}\frac{1}{\sqrt{\lambda}}\frac{1}{N_{\rm c}^{3/2}} = 72~\text{MeV}\,. \tag{29}$$

[1] For certainty, we use $M_\sigma = 1000$ MeV.

3. Mesons at $N_c = 3$: Comparison to LQCD results

Using the model described in Section 2, we attempt to fit the dependence of the mesonic properties on the quark mass obtained from the lattice data [10] for $N_c = 3$. We would like to point out that the model has a few parameters and fixing the parameters at the physical quark mass leaves us with only one parameter to vary — the sigma mass. This over-constrains the model and we fail to achieve a good fit of the lattice data. We thus allow small variation of the physical parameter m_π. This variation can be attributed to either systematics of the lattice calculations or, which is more plausible, to relevant physics our model fails to capture. We get a good description of the lattice if we increase m_π to 180 MeV.

In Fig. 1, we show the dependence of the pion decay constant and the pion mass on the quark mass in the model; the results are compared to the lattice QCD calculations from Ref. [10]. The model results are obtained by solving the stationarity conditions, Eq. (15); the pion mass is then computed. The stars in Fig. 1 denote the physical values of the observables.

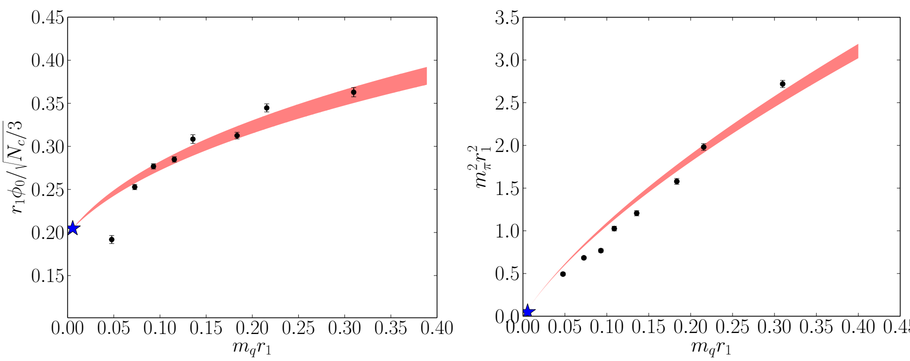

Fig. 1. The pion decay constant (left panel) and the pion mass (right panel) as a function of the quark mass for $N_c = 3$. The shaded region corresponds to the variation of the sigma mass in the range from 950 to 1050 MeV. The symbols show the lattice data from Ref. [10]. The stars show the physical values of the observables.

Although the model misses the first LQCD data point for the pion decay constant, the overall description of the lattice results is quite remarkable given the simplicity of the model.

Additionally, we compare our results for the vector meson mass, assuming $m_V \propto f_\pi$, see Fig. 2.

Using this fit, we can proceed by computing the observables for any number of colors.

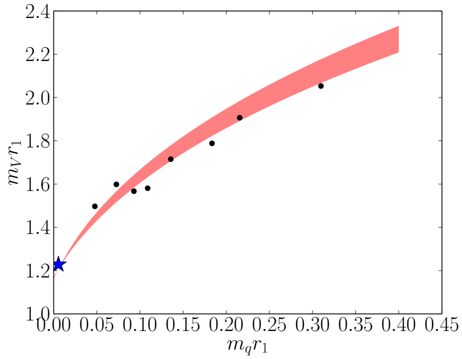

Fig. 2. The vector meson mass as a function of the quark mass for $N_c = 3$. The symbols show the lattice data from Ref. [10]. The stars show the physical values of the observables.

4. Variable N_c

Recent calculations by DeGrand and Liu show the dependence of various observables on N_c [10]. Here, we want to demonstrate that, in their range of the quark masses (in the model, it is Region II described in Section 2), our model follows the data and exhibits the conventional dependence on the number of colors. In Fig. 3, we show the pion decay coupling constant as a function of the mass for various N_c. For the model, we also show the limit of $N_c \to \infty$. As seen from the figure, for these values of the quark masses the large-N_c limit describes lower values N_c quite well, which demonstrates the conventional scaling of Region II, see Eq. (21).

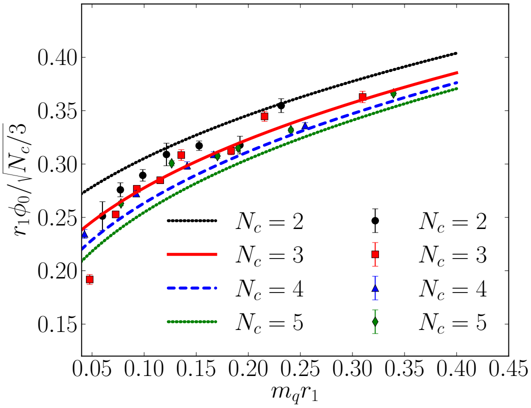

Fig. 3. The pion decay constant as a function of the quark mass for $N_c = 2, 3, 4, 5, \infty$. The model results are shown by the curves. The sigma mass in the model is fixed to 1000 MeV. The lattice data (symbols) is from Ref. [10].

The important question this exercise allows us to address is: Are the lattice data in a range where we can rule out our $1/N_{\mathrm{c}}$ hypothesis for the negative mass-square term of action (14)? The answer is that no, we cannot rule this hypothesis out based on the *available* data.

As we argued in the introduction, the relevant region for the physical quark masses is Region I. Indeed, by applying the argument presented in Eq. (29) to the physical quark mass and computing the critical $N_{\mathrm{c}}^{\mathrm{cr}}$ where the transition from Region I to Region II takes place, we get $N_{\mathrm{c}}^{\mathrm{cr}} \approx 22$. We also demonstrate this in Fig. 4, where the dependence of the pion decay coupling constant on the number of colors for the physical quark mass is plotted. The two asymptotic regimes, see Eq. (20) and Eq. (21), are also shown. This comparison shows that at the physical pion mass, the relevant approach to the limit $N_{\mathrm{c}} \to \infty$ should be considered in the case of a small quark mass, Region I.

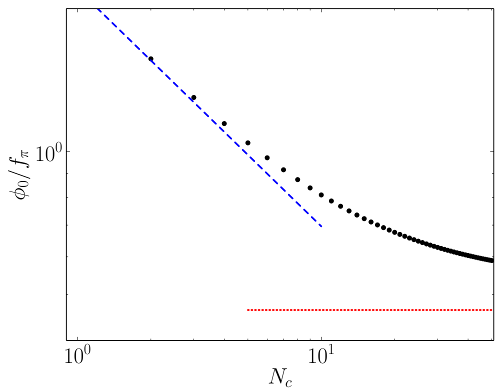

Fig. 4. Illustration of the different regions in approaching the large-N_{c} limit: the pion decay constant as a function of N_{c}. The dashed line shows the Region I, see Eq. (20); the dotted line shows the Region II, see Eq. (21).

5. Baryons

Baryons in the large-N_{c} limit are conventionally assumed to be given by non-perturbative solutions to the classical equation of motion of the non-linear sigma model [2]. These skyrmions have a topological charge corresponding to baryon number. Let us first consider general features of such non-perturbative solutions.

In the small quark mass region where the large-$N_{\rm c}$ behavior is unconventional, the potential is generically of the order of $1/N_{\rm c}$. Therefore, the kinetic energy term is of the order of potential energy if $R \sim \sqrt{N_{\rm c}}/\Lambda_{\rm QCD}$. This implies the mass of the skyrmion is of the order of $M \sim \sqrt{N_{\rm c}}\Lambda_{\rm QCD}$. More generally, even in the large-$N_{\rm c}$ region, the kinetic energy will trade off against the potential energy when $R \sim \sqrt{N_{\rm c}}/f_\pi$, and $M \sim \sqrt{N_{\rm c}} f_\pi$. The lattice data seems to support it, see our representation of results by DeGrand and Liu in Fig. 5. For the conventional Skyrme model treatment, this would imply $M \sim N_{\rm c}\Lambda_{\rm QCD}$ and $R \sim 1/\Lambda_{\rm QCD}$.

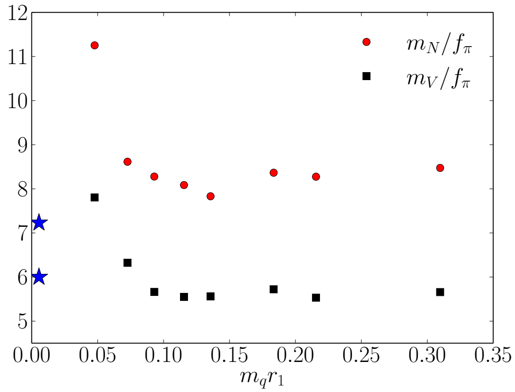

Fig. 5. The ratios of the nucleon and vector meson mass to the pion coupling constant as a function of the quark mass. This is a representation of the results by DeGrand and Liu, see Ref. [10].

For large m_q, we would indeed find the correct $N_{\rm c}$ counting for the mass and radius of any solitonic solution, however, the dependence upon quark mass is non-trivial. As the scale is set by f_π, for large quark mass, we have the radius

$$R \sim \sqrt{N_{\rm c}}/\phi_0 \sim \left(\lambda/m_q\mu^2\right)^{-1/3} . \tag{30}$$

In order to be in the large quark mass limit appropriate for this expression, we must have $R \ll \sqrt{N_{\rm c}}/\Lambda_{\rm QCD}$. Correspondingly, the mass is $M \gg \sqrt{N_{\rm c}}\Lambda_{\rm QCD}$.

In general, the only scale in our theory of pions and sigma mesons is f_π so if there is a non-perturbative skyrmionic solution for the theory, its size will generically be of the order of $\sqrt{N_{\rm c}}/f_\pi$ and mass of the order of $\sqrt{N_{\rm c}} f_\pi$. We might have thought that a derivative expansion for an effective Lagrangian for the skyrmion would be well-behaved because of the large distances involved. This is not the case as the mass scale associated with this derivative expansion is the σ mass, which is of this same size.

A light mass sigma meson will cause problems for baryonic matter. The sigma meson generates an attractive self-interaction. This will show up in a skyrmion solution for two-particle interactions that will have a long-distance attractive interaction that will generate a binding energy for the two-skyrmion solution that is of the order of the skyrmion mass. The sigma mass cuts this interaction off at large distances, so large binding is generated at the size scale of the skyrmion and is a hard core attraction. In order to have a sensible theory, one needs a compensating repulsion. This might be given by the omega meson. The coupling of the omega to the nucleon would need to be larger than that of the sigma. This is phenomenologically the case. More importantly, the omega mass would have to shrink to zero like the omega in the large-$N_{\rm c}$ limit, and $m_\omega \leq m_\sigma$. In models with a hidden local gauge symmetry, this is plausible, since the omega meson's mass is proportional to $f_\pi/\sqrt{N_{\rm c}}$. It is also not so implausible if the large-$N_{\rm c}$ limit corresponds to critical behavior since both the sigma meson and omega meson couple to isospin singlet density fluctuations. With this added assumption, the omega and sigma mesons combine together to generate a short-distance repulsive core. If the typical separation of nucleon is large compared to this core's size, it should not affect much the energy density of nuclear matter. The hard core interaction will generate effects of the order of $\sqrt{N_{\rm c}} R^3_{\rm nucleon} \rho_{\rm baryon}$, so if the density of nuclear matter is sufficiently low, the effects are small. This will be discussed more in the next section when we discuss pion interactions. At this point, we note that the typical distance scale for the pion interactions is of the order of $1/m_\pi$, and since $R_{\rm nucleon} m_\pi \sim \sqrt{N_{\rm c}} m_\pi / \Lambda_{\rm QCD} \ll N_{\rm c}^{-1/4}$, so that the effects of the hard core relative to the mass contributions to the energy are suppressed by at least a factor of $N_{\rm c}^{-3/4}$. In the case of QCD for $N_{\rm c} = 3$, the bound that $m_q \leq \Lambda_{\rm QCD}/N_{\rm c}^{3/2}$ is satisfied with about an order of magnitude to spare, and for general $N_{\rm c}$, there is always some sufficiently small quark mass where this will give an acceptably small correction from the core.

An explicit form for the skyrmion solution is difficult to argue, since one will have all order in derivatives, as is really the case for the standard skyrmion solution, and because the ω meson will play an essential role in its structure. Nevertheless, the dependence of the mass of the skyrmion upon f_π can be compared to the lattice data, see Fig. 6. The figure demonstrates quite a good agreement affirming our discussion based on the skyrmion argument.

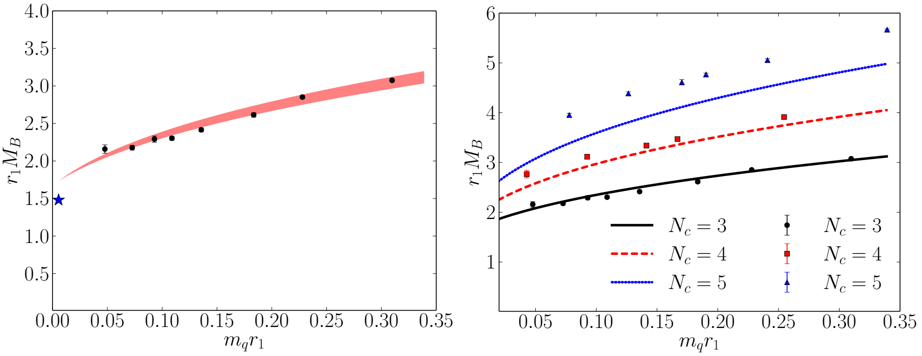

Fig. 6. The nucleon mass as a function of the quark mass for $N_c = 3$ (left) and its N_c dependence (right). The model results are shown by the shaded region which represents the variation of the sigma mass from 950 to 1050 MeV; and by the curves, in this case, the sigma mass is taken to be 1000 MeV. The lattice data (symbols) are from Ref. [10]. The star shows the physical value of the observables.

6. Strength of interaction

As described above, the short-distance repulsive core is associated with omega and sigma exchange. What about the pion long range tail? The conventional pion–nucleon interaction of the sigma model, $g\,\overline{\psi}\tau\cdot\pi\gamma^5\psi$, has an interaction strength naively of the order of $\sqrt{N_c}$ from the coupling. However, for a non-relativistic nucleon, this is of order $g/2M_{\rm nucleon}\overline{\psi}\gamma^\mu\gamma^5\psi$ which is of order 1 in powers of N_c. There is also a potentially dangerous term that arises from the axial current interaction

$$\frac{g^2}{\Lambda_{\rm QCD}^2}\left\{\overline{\psi}_{\rm L}\chi\gamma\cdot\partial\chi^\dagger\psi_{\rm L}+\overline{\psi}_{\rm R}\chi^\dagger\gamma\cdot\partial\chi\psi_{\rm R}\right\}. \tag{31}$$

Here, g^2 is of order 1 in powers of N_c, and if this interaction is generated by the exchange of an axial vector meson, then the scale is $\Lambda_{\rm QCD}$. For $f_\pi \sim \mathcal{O}(\infty)$ in powers of N_c again this generates an interaction of order 1. Note that the basic pion nucleon interaction strength is reduced by a factor of $\sqrt{N_c}$ relative to the naive counting, where $f_\pi \sim \sqrt{N_c}$.

This counting means that the one pion exchange isospin-dependent interaction is of the order of $1/\Lambda_{\rm QCD}^2R^3$, which at the typical length scale of the nucleon is of the order of $1/N_c^{3/2}$. For isopsin singlet interaction which should be typical of nuclear matter, two pion exchanges are important, and these are of strength of $1/\Lambda_{\rm QCD}^4R^5 \sim 1/N_c^{5/2}$. So the picture naturally arises that the nucleon has a strongly repulsive core with a weak long-scale interaction generated by pion exchange. There is no strong long-range force. This is consistent with the phenomenology of nuclear matter.

One can, of course, also include the effects of massive mesons $M \sim \Lambda_{\mathrm{QCD}}$, but such mesons when convoluted over the large size scale of the nucleon give small effects.

The issue of the binding of nuclear matter is a very subtle one and may be special to the case of intermediate values of N_{c} and the details of the pion, sigma and omega meson masses. Nevertheless, the basic outline of our description seems reasonable.

Concluding this section, we want to mention that the effective theory with the required properties was discussed in the literature before, see *e.g.* [15]. First of all, the vector meson (rho and omega) masses satisfy the KSRF relation $m_V^2 \propto f_\pi^2$. Second, at least for three colors, the parameters of the model are consistent with the phenomenologically reasonable hierarchy of the interaction ranges; one may expect that the associated hierarchy of the coupling constants between matter fields scalar/vector mesons is preserved at larger N_{c}.

7. Summary

In this article, we tried to resolve the issue of the interaction strength for the nuclear matter. For this, we needed to make assumptions on the QCD behaviour in the large-N_{c} limit; namely, in contrast to the conventional scaling N_{c}^0, we considered that the negative mass term of the associated linear sigma meson Lagrangian is inversely proportional to N_{c} and thus vanishes in the large-N_{c} limit. We showed that this radical assumption does not contradict the existent lattice QCD data, which provides results for the quark masses in the range where, in our approach, the conventional scaling still holds $m_q \gg \Lambda_{\mathrm{QCD}}/N_{\mathrm{c}}^{3/2}$. With some modest assumptions for the values of the scalar–nucleon and vector–nucleon coupling constants, we were able to get a weakly attractive nucleon–nucleon potential admitting an interaction energy of the order of the physical scale.

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