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$\widehat{\mathfrak{sl}}(n)_N$ WZW conformal blocks from $SU(N)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_n$

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To Prof Rodney J Baxter on the occasion of his 80th birthday

Abstract

Generalizations of the AGT correspondence between 4D $\mathcal{N}=2$ $SU(2)$ supersymmetric gauge theory on $\mathbb{C}^2$ with Ω-deformation and 2D Liouville conformal field theory include a correspondence between 4D $\mathcal{N}=2$ $SU(N)$ supersymmetric gauge theories, $N=2,3,\ldots$, on $\mathbb{C}^2/\mathbb{Z}_n$, $n=2,3,\ldots$, with Ω-deformation and 2D conformal field theories with $\mathcal{W}^{para}_{N,n}$ (n-th parafermion $\mathcal{W}_N$) symmetry and $\widehat{\mathfrak{sl}}(n)_N$ symmetry. In this work, we trivialize the factor with $\mathcal{W}^{para}_{N,n}$ symmetry in the 4D $SU(N)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_n$ (by using specific choices of parameters and imposing specific conditions on the N-tuples of Young diagrams that label the states), and extract the 2D $\widehat{\mathfrak{sl}}(n)_N$ WZW conformal blocks, $n=2,3,\ldots$, $N=1,2,\ldots$.

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1. Introduction

1.1. Algebras on the equivariant cohomology of instanton moduli spaces

In [1], Alday, Gaiotto and Tachikawa conjectured a profound correspondence between $SU(2)$ instanton partition functions in $\mathcal{N}=2$ supersymmetric gauge theories on $\mathbb{C}^2$, with Ω-deformation [2], and Virasoro conformal blocks on the sphere and on the torus (see [3] for a proof[1]).

Their conjecture was further generalized to correspondences between $SU(N)$ instanton partition functions on $\mathbb{C}^2$ and $\mathcal{W}_N$ conformal blocks [7,8], $SU(2)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathcal{N}=1$ super-Virasoro conformal blocks [9–14], $SU(2)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_4$ and conformal blocks of S_3 parafermion algebra [15,16], *etc.*

In [17], by considering N M5-branes compactified on $\mathbb{C}^2/\mathbb{Z}_n$ with Ω-deformation, Nishioka and Tachikawa, following a proposal in [9], suggested that $\mathcal{N}=2$ $SU(N)$ supersymmetric gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ are in correspondence with 2D CFTs with n-th parafermion $\mathcal{W}_N$ symmetry, which we refer to as $\mathcal{W}^{para}_{N,n}$, and affine $\widehat{\mathfrak{sl}}(n)_N$ symmetry.

In [18], it was proposed that the AGT correspondence for $U(N)$ supersymmetric gauge theory on $\mathbb{C}^2/\mathbb{Z}_n$ can be understood in terms of a 2D CFT based on the algebra

$$\mathcal{A}(N,n;\,p)=\mathcal{H}\oplus\widehat{\mathfrak{sl}}(n)_N\oplus\frac{\widehat{\mathfrak{sl}}(N)_n\oplus\widehat{\mathfrak{sl}}(N)_{p-N}}{\widehat{\mathfrak{sl}}(N)_{n+p-N}}, \tag{1.1}$$

which acts on the equivariant cohomology of the moduli space of $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$, $n=2,3,\ldots$. Here, the first factor $\mathcal{H}\cong\mathfrak{u}(1)$ is the affine Heisenberg algebra, the second factor is the affine $\mathfrak{sl}(n)$ level-N algebra, and the third (coset) factor is the $\mathcal{W}^{para}_{N,n}$ algebra, whose parameter p, which controls the central charge,[2] is related to the Ω-deformation parameters ϵ_1,ϵ_2 by

$$\frac{\epsilon_1}{\epsilon_2}=-1-\frac{n}{p}\,. \tag{1.2}$$

The coset factor gives a Virasoro algebra when $(N,n)=(2,1)$, a $\mathcal{W}_N$ algebra when $(N,n)=(N,1)$, an $\mathcal{N}=1$ super-$\mathcal{W}_N$ algebra when $(N,n)=(N,N)$, and an S_3 parafermion algebra when $(N,n)=(2,4)$.

1.2. Burge conditions

Let $p\geq N$ be a positive integer. For $n=1$, the $\widehat{\mathfrak{sl}}(n)_N$ factor in the algebra $\mathcal{A}(N,n;\,p)$ is trivialised, while the coset (third) factor describes the $\mathcal{W}_N$ $(p,p+1)$-minimal model. In [19,20] for $(N,n)=(2,1)$ and further in [21] for $(N,1)$, $N=3,4,\ldots$, it was shown that to obtain minimal model conformal blocks from the $SU(N)$ instanton partition functions on $\mathbb{C}^2$ with Ω-deformation (1.2), we need to remove the *non-physical poles*, corresponding to $\mathcal{W}_N$ minimal model null states, from the instanton partition functions. These non-physical poles emerge when the Coulomb and mass parameters of the gauge theory take special values labeled by integers

[1] In the context of geometric representation theory, the AGT correspondence for pure $SU(N)$ supersymmetric gauge theory on $\mathbb{C}^2$ was proved in [4,5] (see [6] for a generalization to all simply-laced gauge groups).

[2] In general $p\in\mathbb{C}$.

r_I, s_I, $I = 1, \ldots, N-1$ with $N-1 \leq \sum_{I=1}^{N-1} r_I \leq p-1$, $N-1 \leq \sum_{I=1}^{N-1} s_I \leq p$. The conditions that exclude the non-physical poles were shown to be *(N-)Burge conditions* (see [22,23] for $N=2$ and [24–26] for general N)

$$Y_{I,i} \geq Y_{I+1,i+r_I-1} - s_I + 1 \quad \text{for } i \geq 1,\ 0 \leq I < N, \tag{1.3}$$

for N-tuples of Young diagrams $(Y_1, \ldots, Y_N)$ which define the instanton partition functions, where $Y_0 = Y_N$, and $r_0 = p - \sum_{I=1}^{N-1} r_I$, $s_0 = p + 1 - \sum_{I=1}^{N-1} s_I$. For $n \geq 2$, the coset factor in the algebra $\mathcal{A}(N, n; p)$ is considered to describe a $\mathcal{W}_{N,n}^{para}(p, p+n)$-minimal model. In this paper, we show that the same $(N$-)Burge conditions above also remove the non-physical poles from the $SU(N)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_n$ with Ω-deformation (1.2).

1.3. Trivialization of the coset factor

For $p = N$, the coset factor in the algebra $\mathcal{A}(N, n; p)$ is trivialized (the partition function reduces to 1),

$$\mathcal{A}(N, n; N) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N, \tag{1.4}$$

and the $SU(N)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_n$ provide the $\widehat{\mathfrak{sl}}(n)_N$ WZW conformal blocks. Since all parameters are now integral (or at least non-generic), the affine factor will include non-physical poles due to null states. To remove these, we need to impose the appropriate Burge conditions (1.3) on the gauge theory side. In the present work, we show that the integrable $\widehat{\mathfrak{sl}}(n)_N$ WZW conformal blocks can be extracted from the instanton partition functions by an appropriate choice of the parameters and imposing the appropriate Burge conditions.

1.4. Plan of the paper

In Section 2 we briefly recall the generating functions of coloured Young diagrams and the instanton partition functions in $\mathcal{N}=2$ $U(N)$ supersymmetric gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ with Ω-deformation. The relevant AGT-corresponding 2D CFTs are reviewed in Section 3. In Section 4 we derive the Burge conditions (Proposition 4.3) from the requirement that the $SU(N)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_n$, with Ω-deformation (1.2), labeled by a positive integer p, do not have non-physical poles of the type described in Section 1.2. In subsequent sections, we only consider the Burge conditions that correspond taking $p = N$, which we need to trivialize the coset factor. In Section 5, by imposing the Burge conditions, we introduce what we refer to as *Burge-reduced generating functions* of the coloured Young diagrams. In Proposition 5.1 we show, using the crystal graph theory developed by the Kyoto group [27], that these coincide with the integrable $\widehat{\mathfrak{sl}}(n)_N$ WZW characters. In Section 6, we introduce what we refer to as *Burge-reduced instanton partition functions*, by imposing the appropriate Burge conditions, and find that specific integrable $\widehat{\mathfrak{sl}}(n)_N$ WZW conformal blocks are obtained from them (Conjectures 6.5, 6.6 and 6.7). Our proposal, for computing WZW conformal blocks from models based on the $\mathcal{A}(N, n; N)$ algebra, is tested in Section 7 for $(N, n) = (2, 2)$, $(2, 3)$ and $(3, 2)$. Finally, in Section 8 we make some remarks. In Appendix A, we summarize the notation of Lie algebras that is used in this paper, in Appendix B, we review some AGT correspondences to confirm our conventions, and in Appendix C, we recall a class of integrable $\widehat{\mathfrak{sl}}(n)_N$ WZW 4-point conformal blocks computed in [28], which we compare in Section 7 with our results.

2. $U(N)$ instanton counting on $\mathbb{C}^2/\mathbb{Z}_n$

We review how the moduli space of $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$ with Ω-deformation is characterized by coloured Young diagrams, and define the generating functions of the coloured Young diagrams and the instanton partition functions.

2.1. Characterisation of the instanton moduli space by coloured Young diagrams

Consider the $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$, where $\mathbb{Z}_n$ acts on $(z_1, z_2) \in \mathbb{C}^2$ by

$$\mathbb{Z}_n: \quad \lfloor z_1, z_2 \rfloor \ \rightarrow\ \left\lfloor e^{\frac{2\pi i}{n}\sigma} z_1,\, e^{-\frac{2\pi i}{n}\sigma} z_2 \right\rfloor, \qquad \sigma = 0, 1, \ldots, n-1, \tag{2.1}$$

and introduce the Ω-deformation parameters (ϵ_1, ϵ_2) [2,29], by

$$U(1)^2: \quad \lfloor z_1, z_2 \rfloor \rightarrow \lfloor e^{\epsilon_1} z_1, e^{\epsilon_2} z_2 \rfloor\ . \tag{2.2}$$

Using localization, the $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$ are described by the fixed points, on the instanton moduli space, of the $U(1)^2 \times U(1)^N$ torus generated by e^{ϵ_1}, e^{ϵ_2} and e^{a_I}, where a_I, $I = 1, \ldots, N$, are the Coulomb parameters in the $U(N)$ gauge theory. The Coulomb parameters have charges $\sigma_I \in \{0, 1, \ldots, n-1\}$ under the $\mathbb{Z}_n$ action

$$\mathbb{Z}_n: \quad a_I \ \rightarrow\ e^{\frac{2\pi i}{n}\sigma_I} a_I\,. \tag{2.3}$$

Let Y^σ be a coloured Young diagram, with a $\mathbb{Z}_n$ charge $\sigma \in \{0, 1, \ldots, n-1\}$, in other words, Y^σ is composed of boxes such that the box at position $(i, j) \in Y^\sigma$ is assigned the colour $\sigma - i + j$ (mod n). The fixed points of $U(N)$ k-instanton moduli space on $\mathbb{C}^2/\mathbb{Z}_n$ are labeled by N-tuples of coloured Young diagrams $\boldsymbol{Y}^{\boldsymbol{\sigma}} = (Y_1^{\sigma_1}, \ldots, Y_N^{\sigma_N})$ with

$$k = \left|\boldsymbol{Y}^{\boldsymbol{\sigma}}\right| := \sum_{I=1}^{N} \left|Y_I^{\sigma_I}\right| \tag{2.4}$$

total number of boxes [30,31], where $Y_I^{\sigma_I}$ are charged by (2.3), and $\left|Y_I^{\sigma_I}\right|$ denotes the number of boxes in $Y_I^{\sigma_I}$. Let N_σ and k_σ be the number of Young diagrams with charge σ and the total number of boxes with colour σ, respectively. Then,

$$\sum_{\sigma=0}^{n-1} N_\sigma = N, \qquad \sum_{\sigma=0}^{n-1} k_\sigma = k\,. \tag{2.5}$$

In what follows, we use $\boldsymbol{N}$ to denote the sequence $[N_0, \ldots, N_{n-1}]$.[3] Fig. 1 shows an example of a coloured Young diagram.

Remark 2.1. In this paper, without less of generality, we assume

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_N, \tag{2.6}$$

by arranging the ordering of the Coulomb parameters. Then, given non-negative integers $\boldsymbol{N} = [N_0, \ldots, N_{n-1}]$, the charges $\sigma_1, \ldots, \sigma_N$ are uniquely fixed by the above prescription. This implies that $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N, 0, 0, \ldots)$ is a partition having at most N non-zero parts and $\sigma_1 < n$.

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[3] It is often convenient to regard $\boldsymbol{N}$ as a vector on the basis of fundamental weights of $\widehat{\mathfrak{sl}}(n)$. That each $N_\sigma \geq 0$ with $\sum_{\sigma=0}^{n-1} N_\sigma = N$ then implies that $\boldsymbol{N} \in P_{n,N}^{+}$. Moreover, if we set $\lambda = \mathrm{par}(\boldsymbol{N})$ using (A.8), then $(\sigma_1, \sigma_2, \ldots) = \lambda^T$, the partition conjugate to λ, by Lemma A.1.

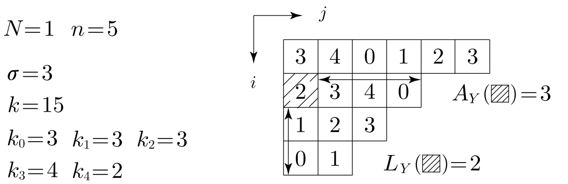


Fig. 1. An example of a coloured Young diagram $Y = Y^\sigma$ with charge $\sigma = 3$, $k = 15$, in the case of $(N, n) = (1, 5)$. For $\square = (2, 1)$, the arm length and the leg length defined in (2.19) are $A_Y(\square) = 3$ and $L_Y(\square) = 2$, respectively.

Using the above notation of coloured Young diagrams, the $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$ are further characterized by the first Chern class of the gauge bundle

$$c_1 = \sum_{\sigma=0}^{n-1} \mathfrak{c}_\sigma\, c_1(\mathcal{T}_\sigma), \tag{2.7}$$

where

$$\begin{aligned} \mathfrak{c}_\sigma &= N_\sigma + \delta k_{\sigma-1} - 2\delta k_\sigma + \delta k_{\sigma+1} \\ &= N_\sigma - \sum_{i=0}^{n-1} A_{\sigma i}\, \delta k_i, \qquad \sigma = 0, 1, \dots, n-1, \qquad \delta k_\sigma := k_\sigma - k_0, \end{aligned} \tag{2.8}$$

with $k_n = k_0$ and $k_{-1} = k_{n-1}$. Here, $c_1(\mathcal{T}_\sigma)$ is the first Chern class of the vector bundle $\mathcal{T}_\sigma$ on the ALE space with holonomy $\mathrm{e}^{2\pi i\sigma/n}$, and A is the Cartan matrix of $\widehat{\mathfrak{sl}}(n)$. Note that $c_1(\mathcal{T}_0) = 0$, and the instanton moduli space is labelled by the $n-1$ integers $\mathfrak{c} = (\mathfrak{c}_1, \dots, \mathfrak{c}_{n-1})$. The inverse of the Cartan matrix $\overline{A}$ of the finite dimensional $\mathfrak{sl}(n)$ (see Appendix A.1) allows the relations (2.8) to be inverted, giving

$$\delta k_\sigma = \sum_{i=1}^{n-1} \left(\overline{A}^{-1}\right)_{\sigma i} \left(N_i - \mathfrak{c}_i\right), \qquad \left(\overline{A}^{-1}\right)_{\sigma i} = \min\{\sigma, i\} - \frac{\sigma i}{n} \tag{2.9}$$

for $1 \le \sigma < n$.

2.2. Generating functions of coloured Young diagrams

Let $\mathcal{P}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}$ be the set of N-tuples of coloured Young diagrams $\boldsymbol{Y}^{\boldsymbol{\sigma}}$ labelled by the charges $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$ and $\boldsymbol{\delta k} = (\delta k_1, \dots, \delta k_{n-1})$. We introduce a generating function of the coloured Young diagrams, that counts the number of torus fixed points of the $U(N)$ instanton moduli space on $\mathbb{C}^2/\mathbb{Z}_n$, as

$$X_{\boldsymbol{\sigma};\boldsymbol{\delta k}}(\mathfrak{q}) = \sum_{\boldsymbol{Y}^{\boldsymbol{\sigma}} \in \mathcal{P}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}} \mathfrak{q}^{\frac{1}{n}|\boldsymbol{Y}^{\boldsymbol{\sigma}}|}. \tag{2.10}$$

Example 2.2 *($n = 1$).* For $n = 1$, $\boldsymbol{\delta k} = \emptyset$, the generating function (2.10) is

$$X_{\boldsymbol{0};\emptyset}(\mathfrak{q}) = \chi_{\mathcal{H}}(\mathfrak{q})^N := \frac{1}{(\mathfrak{q};\mathfrak{q})_\infty^N}, \tag{2.11}$$

where

$$(a;\mathfrak{q})_\infty = \prod_{n=0}^{\infty}\left(1 - a\,\mathfrak{q}^n\right). \tag{2.12}$$

Example 2.3 *($N = 1$, see [32,33]).* For $N = 1$, the generating function (2.10) with a charge $\sigma \in \{0, 1, \dots, n-1\}$ and $\boldsymbol{\delta k} = (\delta k_1, \dots, \delta k_{n-1})$ is

$$X_{(\sigma);\boldsymbol{\delta k}}(\mathfrak{q}) = \frac{1}{(\mathfrak{q};\mathfrak{q})_\infty^n}\,\mathfrak{q}^{\sum_{i=1}^{n-1}\left[\delta k_i^2 + \frac{\delta k_i}{n} - \delta k_{i-1}\,\delta k_i - \delta_{\sigma i}\,\delta k_i\right]}. \tag{2.13}$$

For example, for $(N, n) = (1, 2)$, the generating functions are

$$X_{(0);(\ell)}(\mathfrak{q}) = \frac{\mathfrak{q}^{\frac{1}{2}\ell(2\ell+1)}}{(\mathfrak{q};\mathfrak{q})_\infty}, \quad X_{(1);(\ell)}(\mathfrak{q}) = \frac{\mathfrak{q}^{\frac{1}{2}\ell(2\ell-1)}}{(\mathfrak{q};\mathfrak{q})_\infty}. \tag{2.14}$$

Example 2.4 *($N = 2, n = 2$, see [14]).* For $(N, n) = (2, 2)$, the generating functions (2.10) are

$$\begin{aligned} X_{(0,0);(\ell)}(\mathfrak{q}) + X_{(1,1);(1+\ell)}(\mathfrak{q}) &= \frac{\mathfrak{q}^{\frac{1}{2}\ell(\ell+1)}}{(\mathfrak{q};\mathfrak{q})_\infty}\,\chi_{\mathrm{NS}}(\mathfrak{q})^2, \\ X_{(0,1);(\ell)}(\mathfrak{q}) = X_{(1,0);(\ell)}(\mathfrak{q}) &= \frac{\mathfrak{q}^{\frac{1}{2}\ell^2}}{(\mathfrak{q};\mathfrak{q})_\infty}\,\chi_{\mathrm{R}}(\mathfrak{q})^2, \end{aligned} \tag{2.15}$$

where

$$\chi_{\mathrm{NS}}(\mathfrak{q}) = \frac{\left(-\mathfrak{q}^{\frac{1}{2}};\mathfrak{q}\right)_\infty}{(\mathfrak{q};\mathfrak{q})_\infty}, \qquad \chi_{\mathrm{R}}(\mathfrak{q}) = \frac{(-\mathfrak{q};\mathfrak{q})_\infty}{(\mathfrak{q};\mathfrak{q})_\infty}, \tag{2.16}$$

are, respectively, the NS sector and Ramond sector characters in $\mathcal{N} = 1$ super-Virasoro algebra.

2.3. *Instanton partition functions*

To define instanton partition function, we introduce a fundamental building block, which is associated with $U(N) \times U(N)$ gauge symmetry, with coloured Young diagrams $\boldsymbol{Y}^{\boldsymbol{\sigma}} = (Y_1^{\sigma_1}, \dots, Y_N^{\sigma_N})$ and $\boldsymbol{W}^{\boldsymbol{\sigma}'} = (W_1^{\sigma'_1}, \dots, W_N^{\sigma'_N})$ by[4]

$$\begin{aligned} Z_{\mathrm{bif}}\left(\boldsymbol{a}, \boldsymbol{Y}^{\boldsymbol{\sigma}}; \boldsymbol{a}', \boldsymbol{W}^{\boldsymbol{\sigma}'}\right) = \prod_{I,J=1}^{N} \prod_{\square\in Y_I^{\sigma_I}}^{*} &E\left(-a_I + a'_J, Y_I^{\sigma_I}(\square), W_J^{\sigma'_J}(\square)\right) \\ &\times \prod_{\square\in W_J^{\sigma'_J}}^{*} \left[\epsilon_1 + \epsilon_2 - E\left(a_I - a'_J, W_J^{\sigma'_J}(\square), Y_I^{\sigma_I}(\square)\right)\right], \end{aligned} \tag{2.17}$$

where

$$E\left(P, Y(\square), W(\square)\right) = P - \epsilon_1\, L_W(\square) + \epsilon_2 \left[A_Y(\square) + 1\right]. \tag{2.18}$$

Here the arm length $A_Y(\square)$ and the leg length $L_Y(\square)$ of a Young diagram Y are defined by

[4] By shifting $a'_J \to a'_J - \mu$, it is possible to introduce the mass parameter μ of bifundamental hypermultiplet.

$$A_Y(\square) = Y_i - j, \quad L_Y(\square) = Y_j^T - i, \quad \text{for all } \square = (i,j) \in \mathbb{N}^2, \tag{2.19}$$

where Y_i (resp. Y_j^T) is the length of the i-row in Y (resp. the j-row in the transposed Young diagram Y^T of Y, *i.e.* the j-column in Y). The product $\prod_{\square\in Y}^{*}$ in (2.17) means to take the $\mathbb{Z}_n$ invariant factors in the product, modulo $2\pi i$, under the shift of parameters following (2.1) and (2.3),

$$\epsilon_1 \to \epsilon_1 + \frac{2\pi i}{n}, \quad \epsilon_2 \to \epsilon_2 - \frac{2\pi i}{n}, \quad a_I \to a_I + \sigma_I \frac{2\pi i}{n}, \quad a'_J \to a'_J + \sigma'_J \frac{2\pi i}{n}. \tag{2.20}$$

Thus, the factors in the first and second products of (2.17) are constrained, respectively, by

$$\begin{aligned} -\sigma_I + \sigma'_J - L_{W_J^{\sigma'_J}}(\square) - A_{Y_I^{\sigma_I}}(\square) - 1 &\equiv 0 \pmod n, \\ \sigma_I - \sigma'_J - L_{Y_I^{\sigma_I}}(\square) - A_{W_J^{\sigma'_J}}(\square) - 1 &\equiv 0 \pmod n. \end{aligned} \tag{2.21}$$

Definition 2.5. Using the building block (2.17), the $U(N)$ instanton partition function on $\mathbb{C}^2/\mathbb{Z}_n$ with N fundamental and N anti-fundamental hypermultiplets, which is defined by an equivariant integration over the moduli space of instantons [2] (see also [29,34,35]), is [16] (see also [31,14]),

$$Z^{\boldsymbol{b},\boldsymbol{b}'}_{\sigma;\delta k}\left(\boldsymbol{a},\boldsymbol{m},\boldsymbol{m}';\mathfrak{q}\right) = \sum_{\boldsymbol{Y}^{\sigma}\in\mathcal{P}_{\sigma;\delta k}} \frac{Z_{\text{bif}}\left(\boldsymbol{m},\boldsymbol{\emptyset}^{\boldsymbol{b}};\boldsymbol{a},\boldsymbol{Y}^{\sigma}\right) Z_{\text{bif}}\left(\boldsymbol{a},\boldsymbol{Y}^{\sigma};-\boldsymbol{m}',\boldsymbol{\emptyset}^{\boldsymbol{b}'}\right)}{Z_{\text{vec}}\left(\boldsymbol{a},\boldsymbol{Y}^{\sigma}\right)}\, \mathfrak{q}^{\frac{1}{n}|\boldsymbol{Y}^{\sigma}|}, \tag{2.22}$$

where $\boldsymbol{m} = (m_1, \ldots, m_N)$ and $\boldsymbol{m}' = (m'_1, \ldots, m'_N)$ are the mass parameters, associated with $U(N)^2$ flavor symmetry, of N fundamental and N anti-fundamental hypermultiplets, respectively. The denominator, which is the contribution from the $U(N)$ vector multiplet with Coulomb parameters $\boldsymbol{a} = (a_1, \ldots, a_N)$, is

$$Z_{\text{vec}}\left(\boldsymbol{a},\boldsymbol{Y}^{\sigma}\right) = Z_{\text{bif}}\left(\boldsymbol{a},\boldsymbol{Y}^{\sigma};\boldsymbol{a},\boldsymbol{Y}^{\sigma}\right). \tag{2.23}$$

The instanton partition function (2.22) depends on not only the Chern classes $\mathfrak{c} = (\mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})$, but also the $\mathbb{Z}_n$ boundary charges $\boldsymbol{b} = (b_1, \ldots, b_N)$ and $\boldsymbol{b}' = (b'_1, \ldots, b'_N)$, which take values in $\{0, 1, \ldots, n-1\}$, assigned to the empty Young diagrams. Similar to (2.6), we assume

$$b_1 \geq b_2 \geq \ldots \geq b_N, \qquad b'_1 \geq b'_2 \geq \ldots \geq b'_N, \tag{2.24}$$

by arranging the ordering of the mass parameters.

3. 2D CFT for $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$

We recall various versions of the AGT correspondence, focusing on the algebra acting on the equivariant cohomology of the moduli space of $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$, and on explicit parameter relations.

3.1. Algebra on the moduli space of instantons and 2D CFT

In [9,18], it has been proposed that the algebra

$$\mathcal{A}(N,n;p) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N \oplus \frac{\widehat{\mathfrak{sl}}(N)_n \oplus \widehat{\mathfrak{sl}}(N)_{p-N}}{\widehat{\mathfrak{sl}}(N)_{p'-N}}, \qquad p' = p + n, \tag{3.1}$$

naturally acts on the equivariant cohomology of the moduli space of $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$ with Ω-deformation, where $\mathcal{H} \cong \mathfrak{u}(1)$ is the Heisenberg algebra.[5] The parameter p is identified with the Ω-deformation parameters ϵ_1, ϵ_2 by the relation

$$\frac{\epsilon_1}{\epsilon_2} = -\frac{p'}{p} = -1 - \frac{n}{p}. \tag{3.2}$$

This proposal implies that there exists a combined system of 2D CFTs, one with $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N$ symmetry and the other with $\mathcal{W}^{para}_{N,n}$ symmetry, corresponding to 4D $\mathcal{N}=2$ $U(N)$ supersymmetric gauge theory on $\mathbb{C}^2/\mathbb{Z}_n$ with Ω-deformation [17]. The central charges of these CFTs are

$$\begin{aligned} c\left(\mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N\right) &= 1 + \frac{N\,(n^2-1)}{n+N}, \\ c\left(\mathcal{W}^{para}_{N,n}\right) &= \frac{n\,(N^2-1)}{n+N} + \frac{N\,(N^2-1)}{n}\,\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\,\epsilon_2} \\ &\overset{(3.2)}{=} \frac{n\,(N^2-1)}{n+N} - \frac{n\,N\,(N^2-1)}{p\,(p+n)}. \end{aligned} \tag{3.3}$$

In (3.1), the first and second factors are realized by the $\widehat{\mathfrak{sl}}(n)_N$ WZW model with an additional $\mathfrak{u}(1)$ symmetry, and the third (coset) factor is realized by a $\mathcal{W}^{para}_{N,n}$ $(p, p+n)$-minimal model,[6] where p is taken to be a positive integer with $p \geq N$.

3.2. Instanton partition functions as 4-point conformal blocks

We now provide the relations between the parameters of the instanton partition function (2.22) for $N \geq 2$ and those of the conformal blocks of the 4-point function on $\mathbb{P}^1$ of primary fields $\psi_{\boldsymbol{\mu}_r}$ with momenta $\boldsymbol{\mu}_r$, $r = 1, 2, 3, 4$ (see Remark 3.2),

$$\left\langle \psi_{\boldsymbol{\mu}_1}(\infty)\, \psi_{\boldsymbol{\mu}_2}(1)\, \psi_{\boldsymbol{\mu}_3}(\mathfrak{q})\, \psi_{\boldsymbol{\mu}_4}(0) \right\rangle^{\mathcal{W}^{para}_{N,n}}_{\mathbb{P}^1} \tag{3.4}$$

in the $\mathcal{W}^{para}_{N,n}$ CFT described by the coset factor in (3.1). Using the notation of the finite dimensional Lie algebra $\mathfrak{sl}(N)$ in Appendix A.1 for $M = N$, we propose that the mass parameters $\boldsymbol{m}$ and $\boldsymbol{m}'$ in (2.22) are related to the external momenta $\boldsymbol{\mu}_r$ of the four primary fields by

$$\begin{aligned} 2\,\boldsymbol{\mu}_1 &= \Big[\epsilon_1+\epsilon_2\Big]\,\overline{\rho} + \sum_{I=1}^{N-1} \Big[m_I - m_{I+1}\Big]\,\overline{\Lambda}_I, \quad 2\,\boldsymbol{\mu}_2 = \left[\sum_{I=1}^{N} m_I\right] \overline{\Lambda}_{N-1}, \\ 2\,\boldsymbol{\mu}_4 &= \Big[\epsilon_1+\epsilon_2\Big]\,\overline{\rho} - \sum_{I=1}^{N-1} \Big[m'_I - m'_{I+1}\Big]\,\overline{\Lambda}_I, \quad 2\,\boldsymbol{\mu}_3 = \left[\sum_{I=1}^{N} m'_I\right] \overline{\Lambda}_1. \end{aligned} \tag{3.5}$$

We consider this as a generalisation of the $n=1$ case in [1,7,8,39,40] to positive integer n. By writing $\boldsymbol{\mu}_2 = \mu_2\,\overline{\Lambda}_{N-1}$, $\boldsymbol{\mu}_3 = \mu_3\,\overline{\Lambda}_1$, and $\boldsymbol{\mu}_r = \sum_{I=1}^{N-1} \mu_{r,I}\,\overline{\Lambda}_I$ for $r = 1, 4$, the relations (3.5) are equivalent to

[5] The first works on this subject, in the absence of an Ω-deformation, are by Nakajima [36,37].

[6] While the $\mathcal{W}^{para}_{N,n}$ $(p, p+n)$-minimal models are in general not well-understood except in special cases, see [38] and references therein, in this work, we only need to assume that they exist.

$$
\begin{aligned}
&2\,\mu_{1,I} = \left(\epsilon_1+\epsilon_2\right) + \left(m_I - m_{I+1}\right), \quad 2\,\mu_2 = \sum_{I=1}^{N} m_I, \\
&2\,\mu_{4,I} = \left(\epsilon_1+\epsilon_2\right) - \left(m'_I - m'_{I+1}\right), \quad 2\,\mu_3 = \sum_{I=1}^{N} m'_I, \\
\iff \quad & m_I = \left(I - \frac{N+1}{2}\right)\left(\epsilon_1+\epsilon_2\right) \\
&\quad + \frac{2}{N}\left[-\sum_{J=1}^{I-1} J\,\mu_{1,J} + \sum_{J=I}^{N-1}\left(N-J\right)\mu_{1,J} + \mu_2\right], \\
& m'_I = -\left(I - \frac{N+1}{2}\right)\left(\epsilon_1+\epsilon_2\right) \\
&\quad + \frac{2}{N}\left[\sum_{J=1}^{I-1} J\,\mu_{4,J} - \sum_{J=I}^{N-1}\left(N-J\right)\mu_{4,J} + \mu_3\right].
\end{aligned}
\tag{3.6}
$$

Note that the momenta $\boldsymbol{\mu}_2$ and $\boldsymbol{\mu}_3$ of two of the primary fields are taken to be proportional to $\overline{\Lambda}_1$ or $\overline{\Lambda}_{N-1}$, *i.e.* $\mathcal{W}$-null, which ensures the matching of the number of free parameters $\{m_I, m'_I\}_{I=1,\dots,N}$ and $\{\mu_{1,I}, \mu_2, \mu_3, \mu_{4,I}\}_{I=1,\dots,N-1}$ [7,8,39]. The Coulomb parameters $\boldsymbol{a}$ in (2.22) are related to the internal momenta $\boldsymbol{\mu}^v = \sum_{I=1}^{N-1} \mu_I^v\,\overline{\Lambda}_I$ by

$$
\begin{aligned}
&2\,\boldsymbol{\mu}^v = \left(\epsilon_1+\epsilon_2\right)\overline{\rho} + \sum_{I=1}^{N} a_I\,\boldsymbol{\varepsilon}_I, \qquad \boldsymbol{\varepsilon}_I := \mathbf{e}_I - \mathbf{e}_0, \\
\iff \quad & a_I - \frac{1}{N}\sum_{I=1}^{N} a_I = \left\langle 2\,\boldsymbol{\mu}^v - \left(\epsilon_1+\epsilon_2\right)\overline{\rho}, \mathbf{e}_I\right\rangle.
\end{aligned}
\tag{3.7}
$$

Remark 3.1 *($U(1)$ factor).* The $U(N)$ instanton partition function (2.22) contains a $U(1)$ factor coming from the Heisenberg algebra $\mathcal{H}$ in the algebra $\mathcal{A}(N, n;\, p)$. To obtain it, we need to impose the traceless condition

$$
\sum_{I=1}^{N} a_I = 0. \tag{3.8}
$$

Then, following [1,7,8,11,16], we find an overall $U(1)$ factor for general N and n in the instanton partition function (2.22),

$$
Z_{\mathcal{H}}\left(\boldsymbol{m}, \boldsymbol{m}';\, \mathfrak{q}\right) := (1-\mathfrak{q})^{\frac{\left(\sum_{I=1}^{N} m_I\right)\left(\epsilon_1+\epsilon_2-\frac{1}{N}\sum_{I=1}^{N} m'_I\right)}{n\,\epsilon_1\,\epsilon_2}}. \tag{3.9}
$$

In Appendix B, we confirm the above parameter relations and the $U(1)$ factor by checking some AGT correspondences.

By analogy with known minimal model CFTs, we propose that, in the $\mathcal{W}^{para}_{N,n}$ $(p, p+n)$-minimal models, the momenta should take the degenerate values

$$
2\,\boldsymbol{\mu}^{\boldsymbol{r},\boldsymbol{s}} = -\sum_{I=1}^{N-1}\left(\left(r_I - 1\right)\epsilon_1 + \left(s_I - 1\right)\epsilon_2\right)\overline{\Lambda}_I, \tag{3.10}
$$

where $\boldsymbol{r} = [r_0, r_1, \ldots, r_{N-1}]$ and $\boldsymbol{s} = [s_0, s_1, \ldots, s_{N-1}]$ are sequences of positive integers for which

$$\sum_{I=0}^{N-1} r_I = p\,, \qquad \sum_{I=0}^{N-1} s_I = p' = p + n. \tag{3.11}$$

It will be useful to note that if $\boldsymbol{r}$ and $\boldsymbol{s}$ are regarded as vectors on the basis of fundamental weights of $\widehat{\mathfrak{sl}}(N)$ then $\boldsymbol{r} \in P^{++}_{N,p}$ and $\boldsymbol{s} \in P^{++}_{N,p'}$.

Remark 3.2 *(Free field realization).* We check our normalization conventions by focusing on the well-understood $n = 1$ CFT with $\mathcal{W}_N$ symmetry. In this case, one can introduce the energy-momentum tensor by

$$T(z) = \frac{1}{2} \sum_{I=1}^{N} : \partial\phi_I(z)^2 : + Q \left\langle \overline{\rho}, \partial^2 \boldsymbol{\phi}(z) \right\rangle, \qquad Q = \frac{\epsilon_1 + \epsilon_2}{g_s}, \qquad g_s^2 = -\epsilon_1\, \epsilon_2, \tag{3.12}$$

where $: \cdots :$ is the normal ordered product,

$$\boldsymbol{\phi}(z) = \sum_{I=1}^{N} \phi_I(z)\, \boldsymbol{\varepsilon}_I, \qquad \sum_{I=1}^{N} \phi_I(z) = 0, \qquad \boldsymbol{\varepsilon}_I = \mathbf{e}_I - \mathbf{e}_0, \tag{3.13}$$

are N free chiral bosons with

$$\partial\phi_I(z)\, \phi_J(w) = \frac{\langle \boldsymbol{\varepsilon}_I, \boldsymbol{\varepsilon}_J \rangle}{z - w} + : \partial\phi_I(z)\, \phi_J(w) :, \qquad \langle \boldsymbol{\varepsilon}_I, \boldsymbol{\varepsilon}_J \rangle = \delta_{IJ} - \frac{1}{N}, \tag{3.14}$$

and g_s is introduced as a mass parameter just for a convention. Then, the Virasoro central charge

$$c = (N-1) - N\left(N^2 - 1\right) Q^2 = (N-1) + N\,(N^2 - 1)\, \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1\, \epsilon_2}, \tag{3.15}$$

which is the one in (3.3) for $n = 1$, is obtained. One can also introduce the primary field with momenta $\boldsymbol{\mu}$ by

$$\psi_{\boldsymbol{\mu}}(z) =: \mathrm{e}^{\left\langle 2\, \frac{\boldsymbol{\mu}}{g_s}, \boldsymbol{\phi}(z) \right\rangle} :, \qquad \boldsymbol{\mu} = \sum_{I=1}^{N-1} \mu_I\, \overline{\Lambda}_I, \tag{3.16}$$

which has the conformal dimension

$$\Delta_{\boldsymbol{\mu}} = 2 \left\langle \frac{\boldsymbol{\mu}}{g_s}, \frac{\boldsymbol{\mu}}{g_s} - Q\, \overline{\rho} \right\rangle = -\frac{2}{\epsilon_1\, \epsilon_2} \left\langle \boldsymbol{\mu}, \boldsymbol{\mu} - \Big[\epsilon_1 + \epsilon_2 \Big]\, \overline{\rho} \right\rangle, \tag{3.17}$$

under the action of the energy-momentum tensor (3.12). For example, when $N = 2$ with the Ω-background $\frac{\epsilon_1}{\epsilon_2} = -\frac{p'}{p}$ (Virasoro (p, p')-minimal model case), the conformal dimension of the primary field with degenerate momentum $2\mu^{r,s} = -(r-1)\epsilon_1 - (s-1)\epsilon_2$ is

$$\Delta_{\mu^{r,s}} = \frac{\mu^{r,s}\,(\epsilon_1 + \epsilon_2 - \mu^{r,s})}{\epsilon_1\, \epsilon_2} = \frac{\left(r\, p' - s\, p\right)^2 - \left(p' - p\right)^2}{4\, p\, p'}. \tag{3.18}$$

Similarly, for general n the central charge and the conformal dimension of the primary field $\psi_{\boldsymbol{\mu}}(z)$ are found to be $c(\mathcal{W}^{para}_{N,n})$ in (3.3) and $\Delta_{\boldsymbol{\mu}}/n$ in (3.17), respectively.

4. Burge conditions from $SU(N)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_n$

We deduce the Burge conditions in Proposition 4.3 by looking at the non-physical poles of the $SU(N)$ instanton partition function (2.22), with $\sum_{I=1}^{N} a_I = 0$, on $\mathbb{C}^2/\mathbb{Z}_n$ with the rational Ω-deformation (3.2).

For the rational Ω-background (3.2), *i.e.* $p\,\epsilon_1 + p'\,\epsilon_2 = 0$, $p \geq N$, we see that the instanton partition function (2.22) with $\sum_{I=1}^{N} a_I = 0$ has poles at the values

$$
\begin{aligned}
a_I = a_I^{\boldsymbol{r},\boldsymbol{s}} &:= -\sum_{J=1}^{N-1} \langle \overline{\Lambda}_J, \mathbf{e}_I \rangle \left[r_J\,\epsilon_1 + s_J\,\epsilon_2 \right] \\
&= -\sum_{J=I}^{N-1} \left[r_J\,\epsilon_1 + s_J\,\epsilon_2 \right] + \frac{1}{N} \sum_{J=1}^{N-1} J \left[r_J\,\epsilon_1 + s_J\,\epsilon_2 \right]
\end{aligned}
\tag{4.1}
$$

of the Coulomb parameters (3.7) corresponding to the degenerate momenta (3.10). These poles correspond to the propagation of null-states and need to be removed. Taking a shift of the central $U(1)$ factor in the $U(N)$ gauge symmetry, from (2.1) and (2.3), into account, the $\mathbb{Z}_n$ charges σ_I assigned to the a_I are related to $\boldsymbol{r}$ and $\boldsymbol{s}$ by

$$
\sigma_I - \sigma_{I+1} \equiv -r_I + s_I \pmod{n}, \qquad I = 1, \ldots, N-1. \tag{4.2}
$$

We refer to (4.2) as the $\mathbb{Z}_n$ *charge conditions*.

Definition 4.1 *(Burge Conditions).* For sequences $\boldsymbol{r} = [r_0, r_1, \ldots, r_{N-1}]$ and $\boldsymbol{s} = [s_0, s_1, \ldots, s_{N-1}]$ of positive integers, the N-tuple $\boldsymbol{Y} = (Y_1, \ldots, Y_N)$ of Young diagrams is said to satisfy the Burge conditions if

$$
Y_{I,i} \geq Y_{I+1, i+r_I-1} - s_I + 1 \quad \text{for } i \geq 1,\ 0 \leq I < N, \tag{4.3}
$$

where we set $Y_0 = Y_N$. Then define $\mathcal{C}^{\boldsymbol{r},\boldsymbol{s}}$ to be the set of N-tuples $\boldsymbol{Y} = (Y_1, \ldots, Y_N)$ of Young diagrams that satisfy (4.3).[7]

The following result is easily obtained by exchanging the roles of the rows and columns in (4.3):

Lemma 4.2 *([19]). Let $\boldsymbol{Y} = (Y_1, \ldots, Y_N)$ and, by conjugating the Young diagrams therein, define $\boldsymbol{Y}^T = (Y_1^T, \ldots, Y_N^T)$. Then*

$$
\boldsymbol{Y} \in \mathcal{C}^{\boldsymbol{r},\boldsymbol{s}} \quad \Longleftrightarrow \quad \boldsymbol{Y}^T \in \mathcal{C}^{\boldsymbol{s},\boldsymbol{r}}. \tag{4.4}
$$

As with the Young diagrams in Section 2.1, we colour those in the N-tuples in $\mathcal{C}^{\boldsymbol{r},\boldsymbol{s}}$. Given a Young diagram Y, and $0 \leq \sigma < N$, let Y^σ denote a Young diagram in which the box (i, j) of Y^σ is coloured $(\sigma - i + j)$ modulo n. The value of σ is known as the charge of Y^σ. By Remark 2.1, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots)$ is a partition that has at most N non-zero parts with $\sigma_1 < n$. For

[7] The Burge conditions specify sets of tuples of partitions that are equivalent to certain cylindric (plane) partitions defined in [24] (what we define as $\mathcal{C}^{\boldsymbol{r},\boldsymbol{s}}$ here is equivalent to the set of cylindric partitions denoted $\mathcal{C}^N_{\boldsymbol{s}-\rho,\boldsymbol{r}-\rho}$ in [41, Section 3.4]).

$\boldsymbol{Y} = (Y_1, \dots, Y_N) \in \mathcal{C}^{\boldsymbol{r},\boldsymbol{s}}$, we define $\boldsymbol{Y}^{\boldsymbol{\sigma}} = (Y_1^{\sigma_1}, \dots, Y_N^{\sigma_N})$. Set $\mathcal{C}_{\boldsymbol{\sigma}}^{\boldsymbol{r},\boldsymbol{s}}$ to be the set of all such N-tuples. We will often drop the superscripts on $\boldsymbol{Y}^{\boldsymbol{\sigma}}$ or Y^{σ} if these can be determined from the context.

Proposition 4.3. *If $\boldsymbol{Y}^{\boldsymbol{\sigma}} \in \mathcal{C}_{\boldsymbol{\sigma}}^{\boldsymbol{r},\boldsymbol{s}}$ then the instanton partition function (2.22) at $a_I = a_I^{\boldsymbol{r},\boldsymbol{s}}$, in the background $p\,\epsilon_1 + p'\,\epsilon_2 = 0$ where $p' = p + n$, does not have poles.*

Proof. We follow the proof of [21] for $n = 1$ (see also [19,20] for $N = 2$, $n = 1$). At $a_I = a_I^{\boldsymbol{r},\boldsymbol{s}}$ with (3.2), the instanton partition function (2.22) has poles if and only if the denominator vanishes, *i.e.* there exists $\square \in Y_I$ such that

$$E_{I,J}^{\boldsymbol{r},\boldsymbol{s}}(\square) + \eta = 0, \qquad \eta = 0 \ \text{ or } \ n, \tag{4.5}$$

where $n = p' - p$ and

$$\begin{aligned} E_{I,J}^{\boldsymbol{r},\boldsymbol{s}}(\square) &= \frac{p}{\epsilon_2}\, E\Big(a_J^{\boldsymbol{r},\boldsymbol{s}} - a_I^{\boldsymbol{r},\boldsymbol{s}}, Y_I(\square), Y_J(\square)\Big) \\ &= \sum_{K=1}^{N-1} \langle \overline{\Lambda}_K, \mathbf{e}_J - \mathbf{e}_I \rangle \Big[r_K\, p' - s_K\, p \Big] + p'\, L_{Y_J}(\square) + p\, \Big[A_{Y_I}(\square) + 1 \Big] \,. \end{aligned} \tag{4.6}$$

Because $E_{I,I}^{\boldsymbol{r},\boldsymbol{s}}(\square) \neq 0$ for $\square \in Y_I$, to find $\square \in Y_I$ which satisfies (4.5) we only need to consider the case (i) $I > J$ and case (ii) $I < J$.

Case (i) $I > J$

In this case, the zero-condition (4.5) is $E_{I+\ell,I}^{\boldsymbol{r},\boldsymbol{s}}(\square) + \eta = 0$ for $\square \in Y_{I+\ell}$, where $1 \le I \le N-1$ and $1 \le \ell \le N - I$. By $\sum_{K=1}^{N-1} \langle \overline{\Lambda}_K, \mathbf{e}_I - \mathbf{e}_{I+\ell} \rangle = \sum_{K=1}^{N-1} \sum_{J=1}^{\ell} \delta_{K,I+J-1}$, this zero-condition is written as

$$\sum_{J=1}^{\ell} \Big[r_{I+J-1}\, p' - s_{I+J-1}\, p \Big] + p'\, L_{Y_I}(\square) + p\, \Big[A_{Y_{I+\ell}}(\square) + 1 \Big] + \eta = 0, \quad \square \in Y_{I+\ell} \,. \tag{4.7}$$

Let $d = \gcd(p, p')$, $p = d\, p_d$ and $p' = d\, p'_d$, then the zero-condition (4.7) is equivalent to

$$\begin{aligned} L_{Y_I}(\square) &= -\sum_{J=1}^{\ell} r_{I+J-1} - \gamma\, p_d - \delta_{\eta n}, \\ A_{Y_{I+\ell}}(\square) &= \sum_{J=1}^{\ell} s_{I+J-1} + \gamma\, p'_d - 1 + \delta_{\eta n}, \quad \square \in Y_{I+\ell}, \end{aligned} \tag{4.8}$$

where γ is an indeterminate integer. For $\square = (i, j) \in Y_{I+\ell}$, using $L_{Y_I}(\square) = Y_{I,j}^T - i$, the zero-conditions (4.8) imply that an obvious condition for any Young diagrams,

$$Y_{I+\ell,\, j + A_{Y_{I+\ell}}(\square)}^T \ge i, \tag{4.9}$$

yields

$$Y_{I+\ell,\, j+\sum_{J=1}^{\ell} s_{I+J-1} + \gamma\, p'_d - 1 + \delta_{\eta n}}^T \ge Y_{I,j}^T + \sum_{J=1}^{\ell} r_{I+J-1} + \gamma\, p_d + \delta_{\eta n} \,. \tag{4.10}$$

For the above zero-conditions, $\square \in Y_{I+\ell}$ needs to be restricted by the $\mathbb{Z}_n$ condition like (2.21)

$$\sigma_I - \sigma_{I+\ell} - L_{Y_I}(\square) - A_{Y_{I+\ell}}(\square) - 1 \equiv 0 \pmod n. \tag{4.11}$$

By the $\mathbb{Z}_n$ charge conditions (4.2) and the zero-conditions (4.8), the $\mathbb{Z}_n$ condition (4.11) yields

$$0 \equiv \gamma\,(p_d - p'_d) \equiv -\frac{n}{d}\gamma \pmod n \quad \Longleftrightarrow \quad \gamma = d\,\gamma_d, \tag{4.12}$$

where the indeterminate integer γ_d should be $\gamma_d \geq 0$ by $A_{Y_{I+\ell}}(\square) \geq 0$ and (3.11). As a result, the zero-condition (4.10) yields

$$Y^T_{I+\ell,\,j+\sum_{J=1}^{\ell} s_{I+J-1}+\gamma_d\,p'-1+\delta_{\eta n}} \geq Y^T_{I,j} + \sum_{J=1}^{\ell} r_{I+J-1} + \gamma_d\,p + \delta_{\eta n}. \tag{4.13}$$

Therefore, if conditions

$$Y^T_{I,j} \geq Y^T_{I+\ell,\,j+\sum_{J=1}^{\ell} s_{I+J-1}+\gamma_d\,p'-1+\delta_{\eta n}} - \sum_{J=1}^{\ell} r_{I+J-1} - \gamma_d\,p + 1 - \delta_{\eta n}, \quad \gamma_d \geq 0 \tag{4.14}$$

are satisfied, there does not exist $\square \in Y_{I+\ell}$ such that $E^{\boldsymbol{r},\boldsymbol{s}}_{I+\ell,I}(\square) + \eta = 0$. These non-zero conditions follow from the ones for $\gamma_d = 0$ and $\eta = 0$:

$$Y^T_{I,j} \geq Y^T_{I+\ell,\,j+\sum_{J=1}^{\ell} s_{I+J-1}-1} - \sum_{J=1}^{\ell} r_{I+J-1} + 1. \tag{4.15}$$

All these non-zero conditions (4.15) for $1 \leq \ell \leq N - I$ are obtained from the ones for $\ell = 1$, *i.e.* we arrive at the strongest non-zero conditions among them as

$$Y^T_{I,j} \geq Y^T_{I+1,\,j+s_I-1} - r_I + 1, \quad I = 1, \dots, N-1, \tag{4.16}$$

which are the $I = 1, \dots, N-1$ cases of (4.3) with $\boldsymbol{r}$ and $\boldsymbol{s}$ interchanged.

Case (ii) $I < J$

In this case, the zero-condition (4.5) is $E^{\boldsymbol{r},\boldsymbol{s}}_{I,I+\ell}(\square) + \eta = 0$ for $\square \in Y_I$, where $1 \leq I \leq N - 1$ and $1 \leq \ell \leq N - I$. We repeat the proof of case (i). As (4.7) and (4.8), the zero-condition $E^{\boldsymbol{r},\boldsymbol{s}}_{I,I+\ell}(\square) + \eta = 0$ is

$$-\sum_{J=1}^{\ell} \Big[r_{I+J-1}\,p' - s_{I+J-1}\,p \Big] + p'\,L_{Y_{I+\ell}}(\square) + p\,\Big[A_{Y_I}(\square) + 1 \Big] + \eta = 0, \quad \square \in Y_I, \tag{4.17}$$

which is equivalent to

$$\begin{aligned} L_{Y_{I+\ell}}(\square) &= \sum_{J=1}^{\ell} r_{I+J-1} - \gamma_d\,p - \delta_{\eta n}, \\ A_{Y_I}(\square) &= -\sum_{J=1}^{\ell} s_{I+J-1} + \gamma_d\,p' - 1 + \delta_{\eta n}, \quad \square \in Y_I, \end{aligned} \tag{4.18}$$

where we have used (4.12) obtained from the $\mathbb{Z}_n$ condition. From $A_{Y_I}(\square) \geq 0$ and (3.11), the indeterminate integer γ_d should be $\gamma_d \geq 1$. As in (4.10), these conditions yield a zero-condition

$$Y^T_{I,j-\sum_{J=1}^{\ell} s_{I+J-1}+\gamma_d\, p'-1+\delta_{\eta n}} \geq Y^T_{I+\ell,j} - \sum_{J=1}^{\ell} r_{I+J-1} + \gamma_d\, p + \delta_{\eta n}. \tag{4.19}$$

Therefore, if conditions

$$Y^T_{I+\ell,j} \geq Y^T_{I,j-\sum_{J=1}^{\ell} s_{I+J-1}+\gamma_d\, p'-1+\delta_{\eta n}} + \sum_{J=1}^{\ell} r_{I+J-1} - \gamma_d\, p + 1 - \delta_{\eta n}, \quad \gamma_d \geq 1 \tag{4.20}$$

are satisfied, there does not exist $\square \in Y_I$ such that $E^{\boldsymbol{r},\boldsymbol{s}}_{I,I+\ell}(\square) + \eta = 0$. Among the non-zero conditions (4.20), the strongest ones are $\gamma_d = 1$ and $\eta = 0$:

$$Y^T_{I+\ell,j} \geq Y^T_{I,j-\sum_{J=1}^{\ell} s_{I+J-1}+p'-1} + \sum_{J=1}^{\ell} r_{I+J-1} - p + 1. \tag{4.21}$$

In particular, for $\ell = N - I$, one obtains

$$Y^T_{N,j} \geq Y^T_{1,j+s_0-1} - r_0 + 1, \tag{4.22}$$

which is the $I = 0$ case of (4.3) with r_0 and s_0 exchanged. Together with (4.16) from case (i) we thus obtain all cases of (4.3) with $\boldsymbol{r}$ and $\boldsymbol{s}$ exchanged. It is straightforward to see that together the conditions (4.16) and (4.22) are stronger than the non-zero conditions (4.21). Use of Lemma 4.2 then completes the proof. $\square$

5. Burge-reduced generating functions of coloured Young diagrams and $\widehat{\mathfrak{sl}}(n)_N$ WZW characters

In this and in subsequent sections, we concentrate on the case of $p = N$ in the algebra $\mathcal{A}(N, n; p)$. This choice of parameters trivializes the coset factor,[8] *and we obtain $\mathcal{A}(N, n; N) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N$. In this case, on imposing the Burge conditions (4.3) on the generating functions of coloured Young diagrams*[9] *and instanton partition functions, the $\widehat{\mathfrak{sl}}(n)_N$ WZW characters and conformal blocks emerge. In this section we discuss these generating functions, while the instanton partition functions are discussed in Section 6. We make use of notation and results pertaining to the representation theory of $\widehat{\mathfrak{sl}}(n)$ that are described in Appendix A: for the current purposes the symbols M and m in the appendix are replaced by n and N, respectively.*

In the case of the algebra $\mathcal{A}(1, n; p) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_1$ for $U(1)$ ($N = 1$) instantons on $\mathbb{C}^2/\mathbb{Z}_n$, the highest-weight representations have no null-states, and thus there is no restriction on the tuples of partitions $\boldsymbol{Y}$. However, for $N > 1$, eliminating the null states requires the Burge conditions (4.3) to be imposed.

Throughout this section, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots)$ is a partition for which $\sigma_1 < n$ and $\sigma_{N+1} = 0$. Thus, in particular, $\boldsymbol{\sigma}$ has at most N non-zero parts. In addition, because we are restricting to the $p = N$ case, the condition (3.11) dictates that $r_0 = r_1 = \ldots = r_{N-1} = 1$. Thus, we define $\mathbf{1} = [1, 1, \ldots, 1]$ (N components), and use $\boldsymbol{r} = \mathbf{1}$ throughout this section. The generating functions that we define below then depend on a sequence $\boldsymbol{s} = [s_0, s_1, \ldots, s_{N-1}]$ of positive integers which

[8] When $p = N$, the central charge $c(\mathcal{W}^{para}_{N,n}) = 0$ in (3.3).

[9] Without the Burge conditions, the generating functions correspond to the partition functions of $\mathcal{N} = 4$ topologically twisted $U(N)$ supersymmetric gauge theories [36,42]. A description of the generating functions/WZW characters in terms of *'orbifold partitions'*, and a realization in terms of intersecting D4 and D6-branes can be found in [43,44].

satisfies (3.11) with $p' = n + N$, and which satisfies the $\mathbb{Z}_n$ charge conditions (4.2). With $r_0 = r_1 = \ldots = r_{N-1} = 1$, these requirements are readily accomplished by setting

$$s_I = \sigma_I - \sigma_{I+1} + 1 \tag{5.1}$$

for $1 \leq I < N$. Note then that (3.11) gives[10]

$$s_0 = \sigma_N - \sigma_1 + n + 1\,. \tag{5.2}$$

Using the above $\boldsymbol{r} = \mathbf{1}$ and $\boldsymbol{s}$, we now define sets of N-tuples of coloured Young diagrams $\boldsymbol{Y}^{\boldsymbol{\sigma}} = (Y_1^{\sigma_1}, \ldots, Y_N^{\sigma_N})$ that respect the Burge conditions (4.3), which now take the simplified form

$$Y_{I,i}^{\sigma_I} \geq Y_{I+1,i}^{\sigma_{I+1}} - s_I + 1 \quad \text{for } i \geq 1,\ 0 \leq I < N, \tag{5.3}$$

where we set $Y_0^{\sigma_0} = Y_N^{\sigma_N}$. So define $\mathcal{C}_{\boldsymbol{\sigma}}^{\boldsymbol{s}}$ to be the set of all N-tuples of coloured Young diagrams $\boldsymbol{Y}$ that respect (5.3). In order to impose the Chern relations (2.8), for each $\boldsymbol{Y} \in \mathcal{C}_{\boldsymbol{\sigma}}^{\boldsymbol{s}}$, define $k_\sigma(\boldsymbol{Y})$ to be the number of boxes in $\boldsymbol{Y}$ that are coloured σ for each $\sigma \in \{0, 1, \ldots, n-1\}$, and then set $\delta k_\sigma(\boldsymbol{Y}) = k_\sigma(\boldsymbol{Y}) - k_0(\boldsymbol{Y})$.

Now, to generalise $X_{\boldsymbol{\sigma};\boldsymbol{\ell}}(\mathfrak{q})$ in (2.10) and be able to relate the Chern classes (2.8) to the representation theory of $\widehat{\mathfrak{sl}}(n)$, we define two generating functions $X_{\boldsymbol{\sigma}}^{\boldsymbol{s}}(\mathfrak{q}, \mathbf{t})$ and $X_{\boldsymbol{\sigma};\boldsymbol{\ell}}^{\boldsymbol{s}}(\mathfrak{q})$. For fugacities $\mathbf{t} = (\mathfrak{t}_1, \ldots, \mathfrak{t}_{n-1})$, define the $U(N)$ $\mathfrak{t}$-refined Burge-reduced generating function

$$X_{\boldsymbol{\sigma}}^{\boldsymbol{s}}(\mathfrak{q}, \mathbf{t}) = \sum_{\boldsymbol{Y} \in \mathcal{C}_{\boldsymbol{\sigma}}^{\boldsymbol{s}}} \mathfrak{q}^{\frac{1}{n}|\boldsymbol{Y}|} \prod_{i=1}^{n-1} \mathfrak{t}_i^{\mathfrak{c}_i(\boldsymbol{Y})}, \tag{5.4}$$

where the $\mathfrak{c}_i(\boldsymbol{Y})$ are given by the Chern classes (*cf.* (2.8))

$$\mathfrak{c}_i(\boldsymbol{Y}) = N_i + \delta k_{i-1}(\boldsymbol{Y}) - 2\delta k_i(\boldsymbol{Y}) + \delta k_{i+1}(\boldsymbol{Y}) \tag{5.5}$$

for each $i \in \overline{\mathcal{I}}_n$, with $\delta k_0(\boldsymbol{Y}) = \delta k_n(\boldsymbol{Y}) = 0$. In addition, for a vector $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_{n-1}) \in \mathbb{Z}^{n-1}$, define $\mathcal{C}_{\boldsymbol{\sigma};\boldsymbol{\ell}}^{\boldsymbol{s}} \subset \mathcal{C}_{\boldsymbol{\sigma}}^{\boldsymbol{s}}$ to be the set of all $\boldsymbol{Y} \in \mathcal{C}_{\boldsymbol{\sigma}}^{\boldsymbol{s}}$ for which $\delta k_i(\boldsymbol{Y}) = \ell_i$ for each $i \in \overline{\mathcal{I}}_n$. Then define the Burge-reduced generating function

$$X_{\boldsymbol{\sigma};\boldsymbol{\ell}}^{\boldsymbol{s}}(\mathfrak{q}) = \sum_{\boldsymbol{Y} \in \mathcal{C}_{\boldsymbol{\sigma};\boldsymbol{\ell}}^{\boldsymbol{s}}} \mathfrak{q}^{\frac{1}{n}|\boldsymbol{Y}|}\,. \tag{5.6}$$

Because the values of $\mathfrak{c}_i(\boldsymbol{Y})$ are constant on each set $\mathcal{C}_{\boldsymbol{\sigma};\boldsymbol{\ell}}^{\boldsymbol{s}}$, the generating function (5.4) can be written as

$$X_{\boldsymbol{\sigma}}^{\boldsymbol{s}}(\mathfrak{q}, \mathbf{t}) = \sum_{\boldsymbol{\ell} \in \mathbb{Z}^{n-1}} X_{\boldsymbol{\sigma};\boldsymbol{\ell}}^{\boldsymbol{s}}(\mathfrak{q}) \prod_{i=1}^{n-1} \mathfrak{t}_i^{N_i + \ell_{i-1} - 2\ell_i + \ell_{i+1}} \tag{5.7}$$

with $\ell_0 = \ell_n = 0$. We also introduce the generating function $X_{\boldsymbol{\sigma}}^{\boldsymbol{s}}(\mathfrak{q})$ defined as the specialisation $\mathfrak{t}_1 = \mathfrak{t}_2 = \cdots = \mathfrak{t}_{n-1} = 1$ of $X_{\boldsymbol{\sigma}}^{\boldsymbol{s}}(\mathfrak{q}, \mathbf{t})$:

$$X_{\boldsymbol{\sigma}}^{\boldsymbol{s}}(\mathfrak{q}) = \sum_{\boldsymbol{Y} \in \mathcal{C}_{\boldsymbol{\sigma}}^{\boldsymbol{s}}} \mathfrak{q}^{\frac{1}{n}|\boldsymbol{Y}|} = \sum_{\boldsymbol{\ell} \in \mathbb{Z}^{n-1}} X_{\boldsymbol{\sigma};\boldsymbol{\ell}}^{\boldsymbol{s}}(\mathfrak{q})\,. \tag{5.8}$$

[10] In fact, this solution is unique except in the case where $\sigma_1 = \sigma_2 = \cdots = \sigma_N$. In that particular case, any other solution leads to the given solution by cyclic permutation of the indices $I \in \{0, 1, \ldots, N-1\}$.

We now give expressions for $X^{s}_{\sigma}(\mathfrak{q},\mathfrak{t})$ in terms of (graded) WZW characters $\chi^{\widehat{\mathfrak{sl}}(n)_N}_{\Lambda}(\mathfrak{q},\hat{\mathfrak{t}})$ that arise from level-N irreducible highest weight modules $L(\Lambda)$ of $\widehat{\mathfrak{sl}}(n)$. The characters of these $\widehat{\mathfrak{sl}}(n)$-modules are described in Appendix A.7. Let $\Lambda \in P^{+}_{n,N}$. As also described in Appendix A.7, the Virasoro algebra *Vir* also acts on $L(\Lambda)$. The central charge c and conformal dimension h_Λ of this *Vir*-module are given by

$$c = \frac{N(n^2-1)}{N+n}, \qquad h_\Lambda = \frac{\langle \Lambda, \Lambda + 2\rho\rangle}{2(N+n)}, \tag{5.9}$$

respectively.

For indeterminates $\hat{\mathfrak{t}} = (\hat{\mathfrak{t}}_1, \dots, \hat{\mathfrak{t}}_{n-1})$, we define the (graded) character of $L(\Lambda)$ to be:

$$\chi^{\widehat{\mathfrak{sl}}(n)_N}_{\Lambda}(\mathfrak{q},\hat{\mathfrak{t}}) = \mathrm{Tr}_{L(\Lambda)}\, \mathfrak{q}^{L_0} \prod_{i=1}^{n-1} \hat{\mathfrak{t}}_i^{H_i}, \tag{5.10}$$

where L_0 is a Virasoro generator and H_i are Chevalley elements in the Cartan subalgebra of $\widehat{\mathfrak{sl}}(n)$. Making use of the crystal graph description of characters of $\widehat{\mathfrak{sl}}(n)$ (see Appendix A.6) then leads to the following:

Proposition 5.1. *For a partition $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots)$ for which $\sigma_1 < n$ and $\sigma_{N+1} = 0$, define $\boldsymbol{s} = [s_0, s_1, \dots, s_{N-1}]$ by (5.1) and (5.2), and set $\Lambda = \sum_{i=1}^{N} \Lambda_{\sigma_i}$. Then*

$$X^{s}_{\sigma}(\mathfrak{q},\mathfrak{t}) = \frac{\mathfrak{q}^{w_\Lambda - h_\Lambda}}{(\mathfrak{q};\mathfrak{q})_\infty}\, \chi^{\widehat{\mathfrak{sl}}(n)_N}_{\Lambda}(\mathfrak{q},\hat{\mathfrak{t}}), \tag{5.11}$$

where $\hat{\mathfrak{t}} = (\hat{\mathfrak{t}}_1, \dots, \hat{\mathfrak{t}}_{n-1})$ is related to $\mathfrak{t} = (\mathfrak{t}_1, \dots, \mathfrak{t}_{n-1})$ by

$$\hat{\mathfrak{t}}_i = \mathfrak{q}^{-\frac{1}{2n} i(n-i)}\, \mathfrak{t}_i \tag{5.12}$$

for $1 \le i < n$, and

$$w_\Lambda = \frac{1}{2n} \sum_{i=1}^{n-1} i(n-i) N_i, \tag{5.13}$$

when $\Lambda = [N_0, N_1, \dots, N_{n-1}]$.

Proof. Comparison of the conditions (A.18) with (4.3) shows that there is a bijection $\mathcal{M}^{\sigma} \to \mathcal{C}^{s,\mathbf{1}}$, with the map $\boldsymbol{Y} \mapsto \boldsymbol{X}$ from the former to the latter being obtained by ignoring the colours. Combining this with the bijection described by (4.4) then yields a bijection $\mathcal{M}^{\sigma} \to \mathcal{C}^{\mathbf{1},s}$ described by $\boldsymbol{Y} \mapsto \boldsymbol{X} \mapsto \boldsymbol{X}^T$. Moreover, because of the differing ways in which the colours are ordered in $\mathcal{M}^{\sigma}$ and $\mathcal{C}^{\mathbf{1},s}_{\sigma}$, colouring $\boldsymbol{X}^T$ to give an element of $\mathcal{C}^{\mathbf{1},s}_{\sigma}$, results in $\boldsymbol{Y}^T$. Thus, in the expression (5.4), $\mathcal{C}^{s}_{\sigma} \equiv \mathcal{C}^{\mathbf{1},s}_{\sigma}$ can be replaced by $\mathcal{M}^{\sigma}$. Noting that $|\boldsymbol{Y}| = \sum_{i=0}^{n-1} k_i(\boldsymbol{Y}) = nk_0(\boldsymbol{Y}) + \sum_{i=1}^{n-1} \delta k_i(\boldsymbol{Y})$, and using (5.5), then gives

$$\begin{aligned} X^{s}_{\sigma}(\mathfrak{q},\mathfrak{t}) &= \sum_{\boldsymbol{Y}\in\mathcal{M}^{\sigma}} \mathfrak{q}^{k_0(\boldsymbol{Y})} \prod_{i=1}^{n-1} \mathfrak{q}^{\frac{1}{n}\delta k_i(\boldsymbol{Y})} \mathfrak{t}_i^{N_i + \delta k_{i-1}(\boldsymbol{Y}) - 2\delta k_i(\boldsymbol{Y}) + \delta k_{i+1}(\boldsymbol{Y})} \\ &= \sum_{\boldsymbol{Y}\in\mathcal{M}^{\sigma}} \mathfrak{q}^{k_0(\boldsymbol{Y})} \prod_{i=1}^{n-1} \mathfrak{t}_i^{N_i} \left(\mathfrak{q}^{\frac{1}{n}} \frac{\mathfrak{t}_{i-1}\mathfrak{t}_{i+1}}{\mathfrak{t}_i^2} \right)^{\delta k_i(\boldsymbol{Y})}, \end{aligned} \tag{5.14}$$

where we set $\mathfrak{t}_0 = \mathfrak{t}_n = 1$. Substituting for each $\mathfrak{t}_i$ using (5.12) then shows that

$$X^{s}_{\sigma}(\mathfrak{q}, \mathbf{t}) = \mathfrak{q}^{w_\Lambda} \sum_{Y\in\mathcal{M}^{\sigma}} \mathfrak{q}^{k_0(Y)} \prod_{i=1}^{n-1} \hat{\mathfrak{t}}_i^{N_i} \left(\frac{\hat{\mathfrak{t}}_{i-1}\hat{\mathfrak{t}}_{i+1}}{\hat{\mathfrak{t}}_i^2}\right)^{\delta k_i(Y)} \tag{5.15}$$

which yields (5.11) using (A.31). □

This result enables $X^{s}_{\sigma;\ell}(\mathfrak{q})$ to be expressed in terms of the $\widehat{\mathfrak{sl}}(n)$ string functions $\bar{a}^{\Lambda}_{\ell}(\mathfrak{q})$ in (A.24):

Corollary 5.2. *For a partition $\sigma = (\sigma_1, \sigma_2, \ldots)$ for which $\sigma_1 < n$ and $\sigma_{N+1} = 0$, define $s = [s_0, s_1, \ldots, s_{N-1}]$ by (5.1) and (5.2), and set $\Lambda = \sum_{i=1}^{N} \Lambda_{\sigma_i}$. Then for each $\ell = (\ell_1, \ldots, \ell_{n-1}) \in \mathbb{Z}^{n-1}$,*

$$X^{s}_{\sigma;\ell}(\mathfrak{q}) = \frac{\mathfrak{q}^{\frac{1}{n}|\ell|}}{(\mathfrak{q};\mathfrak{q})_\infty} \bar{a}^{\Lambda}_{\ell}(\mathfrak{q}), \tag{5.16}$$

where we set $|\ell| = \sum_{i\in\mathcal{I}_n} \ell_i$.

Proof. This results from reexpressing the left and right sides of (5.11) using (5.7) and (A.27) respectively, using (5.12), and then using the fact that the matrix $\overline{A}$ is invertible. □

Combining (5.11) with (A.17) enables a product expression to be given for $X^{s}_{\sigma}(\mathfrak{q})$:

Corollary 5.3. *For a partition $\sigma = (\sigma_1, \sigma_2, \ldots)$ for which $\sigma_1 < n$ and $\sigma_{N+1} = 0$, let $s = [s_0, s_1, \ldots, s_{N-1}]$ be defined by (5.1) and (5.2). Then*

$$X^{s}_{\sigma}(\mathfrak{q}) = \frac{1}{\left(\mathfrak{q}^{1+\frac{N}{n}};\mathfrak{q}^{1+\frac{N}{n}}\right)_\infty} \prod_{\substack{1\le i<j\le n+N\\ i\notin\Omega,\, j\in\Omega}} \frac{1}{\left(\mathfrak{q}^{\frac{j-i}{n}};\mathfrak{q}^{1+\frac{N}{n}}\right)_\infty} \prod_{\substack{1\le i<j\le n+N\\ i\in\Omega,\, j\notin\Omega}} \frac{1}{\left(\mathfrak{q}^{1+\frac{N-j+i}{n}};\mathfrak{q}^{1+\frac{N}{n}}\right)_\infty}, \tag{5.17}$$

where $\Omega = \{N + j - \sigma^T_j \mid j = 1, \ldots, n\}$.

Proof. From Appendix A.5, we have $\Omega_\Lambda = \{N + j - \lambda_j \mid j = 1, \ldots, n\}$, where $\lambda = \mathrm{par}(\Lambda)$ for $\Lambda = \sum_{i=1}^{N} \Lambda_{\sigma_i}$. Lemma A.1 shows that $\lambda = \sigma^T$ and thus $\Omega_\Lambda = \Omega$.

From (5.8), by (5.16), (A.23), and (A.16), we obtain

$$\begin{aligned} X^{s}_{\sigma}(\mathfrak{q}) &= \frac{1}{(\mathfrak{q};\mathfrak{q})_\infty} \sum_{\ell\in\mathbb{Z}^{n-1}} \mathfrak{q}^{\frac{1}{n}|\ell|} \bar{a}^{\Lambda}_{\ell}(\mathfrak{q}) = \frac{e^{-\Lambda}}{(\mathfrak{q};\mathfrak{q})_\infty} \bar{\chi}^{\widehat{\mathfrak{sl}}(n)}_{\Lambda}(\mathfrak{q}, \boldsymbol{x})\Big|_{\{x_i\to\mathfrak{q}^{-i/n},1\le i\le n\}} \\ &= \frac{1}{(\mathfrak{q};\mathfrak{q})_\infty} \mathrm{Pr}\,\chi^{\widehat{\mathfrak{sl}}(n)}_{\Lambda}(\mathfrak{q}). \end{aligned}$$

Use of (A.17) then gives (5.17). □

In the cases in which $N = 1$, the Burge conditions (4.3) are vacuous (assuming that r_1 and s_1 are both positive). Then $X^{s}_{\sigma;\ell}(\mathfrak{q})$ coincides with $X_{\sigma;\ell}(\mathfrak{q})$ defined by (2.10). In this $N = 1$ case, we also set $X_{\sigma}(\mathfrak{q}, \mathbf{t}) = X^{s}_{\sigma}(\mathfrak{q}, \mathbf{t})$ and $X_{\sigma}(\mathfrak{q}) = X^{s}_{\sigma}(\mathfrak{q})$. Because Proposition 5.1 and Corollaries 5.2 and 5.3 remain valid in this case, when combined with expressions from Appendix A.7, they lead to the following:

Corollary 5.4. *Let $0 \le k < n$. Then*

$$X_{(k)}(\mathfrak{q}, \mathfrak{t}) = \frac{1}{(\mathfrak{q}; \mathfrak{q})_\infty^n} \sum_{\boldsymbol{\ell} \in \mathbb{Z}^{n-1}} \mathfrak{q}^{-\ell_k + \sum_{i=1}^{n-1} (\ell_i^2 - \ell_i \ell_{i-1} + \frac{1}{n}\ell_i)} \prod_{i=1}^{n-1} \mathfrak{t}_i^{\delta_{ik} + \ell_{i-1} - 2\ell_i + \ell_{i+1}}, \tag{5.18}$$

$$X_{(k);\boldsymbol{\ell}}(\mathfrak{q}) = \frac{1}{(\mathfrak{q}; \mathfrak{q})_\infty^n} \mathfrak{q}^{-\ell_k + \sum_{i=1}^{n-1} (\ell_i^2 - \ell_i \ell_{i-1} + \frac{1}{n}\ell_i)}, \tag{5.19}$$

$$X_{(k)}(\mathfrak{q}) = \frac{1}{\left(\mathfrak{q}^{\frac{1}{n}}; \mathfrak{q}^{\frac{1}{n}}\right)_\infty}, \tag{5.20}$$

where $\boldsymbol{\ell} = (\ell_1, \ell_2, \ldots, \ell_{n-1})$ and $\ell_0 = \ell_n = 0$.

Proof. Set $\Lambda = \Lambda_k$. Firstly, combining (5.16) with (A.32) gives (5.19). Substituting (5.19) into (5.7) then gives (5.18). For (5.20) we use Corollary 5.3 with $\boldsymbol{\sigma} = (k)$. Then $\Omega = \{1, 2, \ldots, n\} \backslash \{k+1\}$, whereupon (5.17) yields (5.20). □

Note that (5.19) accords with (2.13).

Let $\boldsymbol{N} = [N_0, N_1, \ldots, N_{n-1}]$ be such that each $N_i \ge 0$ with $\sum_{i=0}^{n-1} N_i = N$. If we regard $\boldsymbol{N}$ as an $\widehat{\mathfrak{sl}}(n)$ weight, then $\boldsymbol{N} = \sum_{i=0}^{n-1} N_i \Lambda_i \in P_{n,N}^+$. Let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots)$ be the partition with $\sigma_{N+1} = 0$ such that $\boldsymbol{N} = \sum_{j=1}^{N} \Lambda_{\sigma_j}$. Then $\boldsymbol{\sigma} = \lambda^T$, the partition conjugate to $\lambda = \mathrm{par}(\boldsymbol{N})$ defined by (A.8). Now define $\boldsymbol{s} = [s_0, s_1, \ldots, s_{N-1}]$ by (5.1) and (5.2), and define the $SU(N)$ $\mathfrak{t}$-refined Burge-reduced generating function of coloured Young diagrams, by subtracting the Heisenberg factor $\mathcal{H}$ whose character is $\chi_{\mathcal{H}}(\mathfrak{q}) = (\mathfrak{q}; \mathfrak{q})_\infty^{-1}$ in (2.11),

$$\widehat{X}_{\boldsymbol{N}}^{\mathrm{red}}(\mathfrak{q}, \mathfrak{t}) = (\mathfrak{q}; \mathfrak{q})_\infty \times X_{\boldsymbol{\sigma}}^{\boldsymbol{s}}(\mathfrak{q}, \mathfrak{t}). \tag{5.21}$$

Proposition 5.1 immediately shows that:

Corollary 5.5. *If $\boldsymbol{N} \in P_{n,N}^+$, then*

$$\widehat{X}_{\boldsymbol{N}}^{\mathrm{red}}(\mathfrak{q}, \mathfrak{t}) = \mathfrak{q}^{w_{\boldsymbol{N}} - h_{\boldsymbol{N}}} \chi_{\boldsymbol{N}}^{\widehat{\mathfrak{sl}}(n)_N}(\mathfrak{q}, \hat{\mathfrak{t}}), \tag{5.22}$$

where $\hat{\mathfrak{t}}$ is related to $\mathfrak{t}$ by (5.12), and $h_{\boldsymbol{N}}$ and $w_{\boldsymbol{N}}$ are given by (5.9) and (5.13).

This corollary implies that the Chern classes (2.8) on the gauge side are identified with the eigenvalues of Cartan elements H_i of $\widehat{\mathfrak{sl}}(n)$ on the CFT side.

Example 5.6. In the case of $N = 1$, (5.22) is particularly simple, because then $h_{\boldsymbol{N}} = w_{\boldsymbol{N}}$. For instance, for $(N, n) = (1, 2)$,

$$\begin{aligned}
\widehat{X}_{[1,0]}(\mathfrak{q}, \mathfrak{t}) &= (\mathfrak{q}; \mathfrak{q})_\infty \sum_{\ell \in \mathbb{Z}} X_{(0);(-\ell)}(\mathfrak{q})\, \mathfrak{t}^{2\ell} = \frac{1}{(\mathfrak{q}; \mathfrak{q})_\infty} \sum_{j \in \mathbb{Z}} \mathfrak{q}^{j^2}\, \hat{\mathfrak{t}}^{2j} = \chi_{[1,0]}^{\widehat{\mathfrak{sl}}(n)_N}(\mathfrak{q}, \hat{\mathfrak{t}}), \\
\widehat{X}_{[0,1]}(\mathfrak{q}, \mathfrak{t}) &= (\mathfrak{q}; \mathfrak{q})_\infty \sum_{\ell \in \mathbb{Z}} X_{(1);(-\ell)}(\mathfrak{q})\, \mathfrak{t}^{2\ell+1} = \frac{1}{(\mathfrak{q}; \mathfrak{q})_\infty} \sum_{j \in \mathbb{Z}+\frac{1}{2}} \mathfrak{q}^{j^2}\, \hat{\mathfrak{t}}^{2j} = \chi_{[0,1]}^{\widehat{\mathfrak{sl}}(n)_N}(\mathfrak{q}, \hat{\mathfrak{t}}),
\end{aligned} \tag{5.23}$$

where $\hat{\mathfrak{t}} = \mathfrak{q}^{-\frac{1}{4}}\, \mathfrak{t}$.

In Section 7, we will give explicit examples of Corollary 5.5 for $(N,n)=(2,2),(2,3)$ and $(3,2)$ by comparing with the $\widehat{\mathfrak{sl}}(n)_N$ WZW characters computed using the Weyl-Kac character formula (A.35).

Note that in the principally specialised case $\mathbf{t}=(1,\dots,1)$,

$$\widehat{X}^{\mathrm{red}}_{\boldsymbol{N}}(\mathfrak{q},(1,\dots,1))=(\mathfrak{q};\mathfrak{q})_\infty\times X^{s}_{\boldsymbol{\sigma}}(\mathfrak{q}) \tag{5.24}$$

is immediately evaluated using the right side of (5.17) with $\Omega=\{j+\sum_{i=0}^{j-1}N_i\mid j=1,\dots,n\}$, and gives the $\widehat{\mathfrak{sl}}(n)$ principally specialised character $\mathrm{Pr}\,\chi_{\boldsymbol{N}}^{\widehat{\mathfrak{sl}}(n)}(\mathfrak{q})$ in Appendix A.5.

6. Burge-reduced instanton partition functions and $\widehat{\mathfrak{sl}}(n)_N$ WZW conformal blocks

We discuss how the integrable $\widehat{\mathfrak{sl}}(n)_N$ WZW conformal blocks are extracted from the $SU(N)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_n$ with $\sum_{I=1}^{N}a_I=0$.

6.1. $U(1)$ instanton partition function

In the $U(1)$ case, as was mentioned in Section 5, for generic p (generic Ω-background) one obtains the algebra $\mathcal{A}(1,n;p)=\mathcal{H}\oplus\widehat{\mathfrak{sl}}(n)_1$ acting on the equivariant cohomology of the moduli space of $U(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$. Let us consider the instanton partition function (2.22) for $N=1$ with vanishing Coulomb parameter $a=0$ and labelled by $\boldsymbol{N}_\sigma=[N_0,\dots,N_{n-1}]$, $N_i=\delta_{i\sigma}$, and $\delta k_i=0$. Following Corollary 5.5, the corresponding module in $\widehat{\mathfrak{sl}}(n)_1$ is the highest-weight module with $\Lambda=\Lambda_\sigma$. We define

$$Z^{b,b'}_{\boldsymbol{N}_\sigma}(m,m';\mathfrak{q})=Z^{b,b'}_{(\sigma);\mathbf{0}}(0,m,m';\mathfrak{q}), \tag{6.1}$$

and make the following conjecture.

Conjecture 6.1. *The $U(1)$ instanton partition function (6.1) on $\mathbb{C}^2/\mathbb{Z}_n$ with $b'=b$ and $\boldsymbol{N}_0=[1,0,\dots,0]$ is*

$$Z^{b,b}_{\boldsymbol{N}_0}(m,m';\mathfrak{q})=(1-\mathfrak{q})^{\frac{m\left[\epsilon_1+\epsilon_2-m'\right]}{n\epsilon_1\epsilon_2}}(1-\mathfrak{q})^{-2h_b}, \tag{6.2}$$

where $h_b=h_{\boldsymbol{N}_b}=\frac{b(n-b)}{2n}$ is the conformal dimension of the highest-weight state $|N_b\rangle$ in the $\widehat{\mathfrak{sl}}(n)_1$ WZW model. The first factor is the $U(1)$ factor $Z_{\mathcal{H}}\left(m,m';\mathfrak{q}\right)$ in (3.9) for $N=1$, and the second factor is the 2-point function of $\widehat{\mathfrak{sl}}(n)_1$ WZW primary fields with highest-weights Λ_b and Λ_{n-b}

6.2. $SU(N)$ Burge-reduced instanton partition functions

For $N\geq 2$, in the same way that we defined the Burge-reduced generating function (5.6) of coloured Young diagrams, we now introduce a reduced version of the instanton partition function (2.22) by imposing the specialized ones (5.3) for the Burge conditions (4.3) with $\boldsymbol{r}=\mathbf{1}\in P^{++}_{N,N}$ and $\boldsymbol{s}\in P^{++}_{N,N+n}$,

$$\mathcal{Z}^{\boldsymbol{s};\boldsymbol{b},\boldsymbol{b}'}_{\boldsymbol{\sigma};\boldsymbol{\ell}}\left(\boldsymbol{a},\boldsymbol{m},\boldsymbol{m}';\mathfrak{q}\right)=\sum_{\boldsymbol{Y}^\sigma\in\mathcal{C}^{s}_{\sigma;\ell}}\frac{Z_{\mathrm{bif}}\left(\boldsymbol{m},\boldsymbol{\emptyset}^{\boldsymbol{b}};\boldsymbol{a},\boldsymbol{Y}^{\boldsymbol{\sigma}}\right)Z_{\mathrm{bif}}\left(\boldsymbol{a},\boldsymbol{Y}^{\boldsymbol{\sigma}};-\boldsymbol{m}',\boldsymbol{\emptyset}^{\boldsymbol{b}'}\right)}{Z_{\mathrm{vec}}\left(\boldsymbol{a},\boldsymbol{Y}^{\boldsymbol{\sigma}}\right)}\mathfrak{q}^{\frac{1}{n}|\boldsymbol{Y}^{\sigma}|}, \tag{6.3}$$

where $\sum_{I=1}^{N} a_I = 0$ is imposed. The Coulomb parameters $\boldsymbol{a} = (a_1, \ldots, a_N)$, and the mass parameters $\boldsymbol{m} = (m_1, \ldots, m_N)$, $\boldsymbol{m}' = (m'_1, \ldots, m'_N)$, are related to the internal momenta $\boldsymbol{\mu}^v$, and the external momenta $\boldsymbol{\mu}_{r=1,2,3,4}$, of a 4-point conformal block in a $\mathcal{W}^{para}_{N,n}$ CFT, by the relations (3.7) and (3.6), respectively. The gauge theory in the rational Ω-background (3.2) for $p = N$,

$$\frac{\epsilon_1}{\epsilon_2} = -1 - \frac{n}{N}, \tag{6.4}$$

is expected to describe a minimal model CFT whose momenta take values in the degenerate momenta (3.10) for $r_I = 1$,

$$\begin{aligned} 2\,\boldsymbol{\mu}^v &= -\sum_{I=1}^{N-1} \left[s_I - 1 \right] \epsilon_2\, \overline{\Lambda}_I, \\ \overset{(3.7)}{\Longrightarrow} \quad a_I = a_I^{\boldsymbol{s}} &:= -\sum_{J=1}^{N-1} \left\langle \overline{\Lambda}_J, \mathbf{e}_I \right\rangle \left[s_J - 1 - \frac{n}{N} \right] \epsilon_2 \\ &= -\sum_{J=I}^{N-1} \left[s_J - 1 - \frac{n}{N} \right] \epsilon_2 + \frac{1}{N} \sum_{J=1}^{N-1} J \left[s_J - 1 - \frac{n}{N} \right] \epsilon_2, \end{aligned} \tag{6.5}$$

parametrized by $\boldsymbol{s} = [s_0, s_1, \ldots, s_{N-1}] \in P^{++}_{N,N+n}$, and

$$\begin{aligned} 2\,\boldsymbol{\mu}_1 &= -\sum_{I=1}^{N-1} \left[s_{1,I} - 1 \right] \epsilon_2\, \overline{\Lambda}_I, \quad 2\,\boldsymbol{\mu}_2 = -\left[s_{2,N-1} - 1 \right] \epsilon_2\, \overline{\Lambda}_{N-1}, \\ 2\,\boldsymbol{\mu}_4 &= -\sum_{I=1}^{N-1} \left[s_{4,I} - 1 \right] \epsilon_2\, \overline{\Lambda}_I, \quad 2\,\boldsymbol{\mu}_3 = -\left[s_{3,1} - 1 \right] \epsilon_2\, \overline{\Lambda}_1, \\ \overset{(3.6)}{\Longrightarrow} \quad m_I = m_I^{s_1,s_2} &:= -\left[I - \frac{N+1}{2} \right] \frac{n}{N} \epsilon_2 \\ &+ \frac{1}{N} \left[\sum_{J=1}^{I-1} J \left[s_{1,J} - 1 \right] \right. \\ &\left. - \sum_{J=I}^{N-1} \lfloor N - J \rfloor \left[s_{1,J} - 1 \right] - \left[s_{2,N-1} - 1 \right] \right] \epsilon_2, \\ m'_I = m'^{\,s_3,s_4}_I &:= \left[I - \frac{N+1}{2} \right] \frac{n}{N} \epsilon_2 \\ &+ \frac{1}{N} \left[-\sum_{J=1}^{I-1} J \left[s_{4,J} - 1 \right] \right. \\ &\left. + \sum_{J=I}^{N-1} \lfloor N - J \rfloor \left[s_{4,J} - 1 \right] - \left[s_{3,1} - 1 \right] \right] \epsilon_2, \end{aligned} \tag{6.6}$$

parametrized by $\boldsymbol{s}_1 = [s_{1,0}, s_{1,1}, \ldots, s_{1,N-1}] \in P^{++}_{N,N+n}$, $\boldsymbol{s}_4 = [s_{4,0}, s_{4,1}, \ldots, s_{4,N-1}] \in P^{++}_{N,N+n}$, and

$$\begin{aligned} \boldsymbol{s}_2 &= [s_{2,0}, s_{2,1}, \ldots, s_{2,N-1}] = [s_{2,0}, 1, \ldots, 1, s_{2,N-1}] \in P^{++}_{N,N+n}, \\ \boldsymbol{s}_3 &= [s_{3,0}, s_{3,1}, \ldots, s_{3,N-1}] = [s_{3,0}, s_{3,1}, 1, \ldots, 1] \in P^{++}_{N,N+n}, \end{aligned} \tag{6.7}$$

following $\boldsymbol{\mu}_2 \propto \overline{\Lambda}_{N-1}$ and $\boldsymbol{\mu}_3 \propto \overline{\Lambda}_1$.

Remark 6.2 *(Fixing $\boldsymbol{s}_1, \boldsymbol{s}_4$).* By (5.1), the (Coulomb) parameters in $\boldsymbol{s}$ are determined as $s_I = \sigma_I - \sigma_{I+1} + 1$, from the ordered charges $\sigma_1 \geq \ldots \geq \sigma_N$. Similarly, we fix the (mass) parameters in $\boldsymbol{s}_1$ and $\boldsymbol{s}_4$. Taking a shift by the central $U(1)$ factor in the $U(N)$ flavor symmetry, from (3.6), into account, one obtains the $\mathbb{Z}_n$ boundary charge conditions

$$s_{1,I} - 1 \equiv b_I - b_{I+1} \pmod n, \qquad s_{4,I} - 1 \equiv b'_I - b'_{I+1} \pmod n, \qquad I = 1, \ldots, N-1. \tag{6.8}$$

We can then determine the independent parameters in $\boldsymbol{s}_1$ and $\boldsymbol{s}_4$ as

$$s_{1,I} = b_I - b_{I+1} + 1, \quad s_{4,I} = b'_I - b'_{I+1} + 1, \quad I = 1, \ldots, N-1. \tag{6.9}$$

The remaining independent parameters $s_{2,N-1}$ and $s_{3,1}$ in (6.7) are determined in Remark 6.4.

By subtracting the $U(1)$ factor (3.9), as in the case of the $\mathfrak{t}$-refined Burge-reduced generating function (5.21), we define a Burge-reduced instanton partition function labelled by $\boldsymbol{N} = [N_0, \ldots, N_{n-1}] \in P^+_{n,N}$, $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_{n-1}) \in \mathbb{Z}^{n-1}$, and $\mathbb{Z}_n$ boundary charges $\boldsymbol{b} = (b_1, \ldots, b_N)$ and $\boldsymbol{b}' = (b'_1, \ldots, b'_N)$.

Definition 6.3. The $SU(N)$ Burge-reduced instanton partition function is defined by

$$\widehat{\mathcal{Z}}^{\boldsymbol{b},\boldsymbol{b}'}_{\boldsymbol{N};\boldsymbol{\ell}}(\mathfrak{q}) = Z_{\mathcal{H}}\left(\boldsymbol{m}^{s_1,s_2}, \boldsymbol{m}'^{s_3,s_4}; \mathfrak{q}\right)^{-1} \times \mathcal{Z}^{s;\boldsymbol{b},\boldsymbol{b}'}_{\boldsymbol{\sigma};\boldsymbol{\ell}}\left(\boldsymbol{a}^s, \boldsymbol{m}^{s_1,s_2}, \boldsymbol{m}'^{s_3,s_4}; \mathfrak{q}\right). \tag{6.10}$$

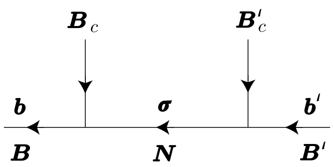
Here the Coulomb parameters $\boldsymbol{a}^s = (a^s_1, \ldots, a^s_N)$ are given by (6.5) with $s_I = \sigma_I - \sigma_{I+1} + 1$ in (5.1), and the mass parameters $\boldsymbol{m}^{s_1,s_2} = (m^{s_1,s_2}_1, \ldots, m^{s_1,s_2}_N)$ and $\boldsymbol{m}'^{s_3,s_4} = (m'^{s_3,s_4}_1, \ldots, m'^{s_3,s_4}_N)$ are given by (6.6) with $s_{1,I}$, $s_{4,I}$ in (6.9) and $s_{2,N-1}$, $s_{3,1}$ determined in Remark 6.4.

By Corollary 5.5, the set $\boldsymbol{N}$, determined from the $\mathbb{Z}_n$ charges $\boldsymbol{\sigma}$, indicates level-N dominant integral highest-weight in $\widehat{\mathfrak{sl}}(n)_N$ WZW model. We propose that, in the $\widehat{\mathfrak{sl}}(n)_N$ WZW 4-point conformal blocks, the integrable representations of two of the four external primary fields are also determined from the $\mathbb{Z}_n$ boundary charges $\boldsymbol{b} = (b_1, \ldots, b_N)$ and $\boldsymbol{b}' = (b'_1, \ldots, b'_N)$ by

$$\boldsymbol{B} = \sum_{I=1}^{N} \Lambda_{b_I} = [B_0, B_1, \ldots, B_{n-1}], \qquad \boldsymbol{B}' = \sum_{I=1}^{N} \Lambda_{b'_I} = [B'_0, B'_1, \ldots, B'_{n-1}]. \tag{6.11}$$

We now represent the Burge-reduced instanton partition function (6.10), graphically, as

$$\widehat{\mathcal{Z}}^{\boldsymbol{b},\boldsymbol{b}'}_{\boldsymbol{N};\boldsymbol{\ell}}(\mathfrak{q}) \quad = \tag{6.12}$$



We also represent (6.12) schematically by $\boldsymbol{B} - \boldsymbol{B}_c - (\boldsymbol{N}) - \boldsymbol{B}'_c - \boldsymbol{B}'$. The representations $\boldsymbol{B}_c$ and $\boldsymbol{B}'_c$ of the remaining two of the four external primary fields need to be taken so that they respect the fusion rules, which apply from right to left in (6.12), of the $\widehat{\mathfrak{sl}}(n)_N$ WZW model when $\boldsymbol{N}$, $\boldsymbol{B}$ and $\boldsymbol{B}'$ are fixed (see *e.g.* Chapter 16 of [45]). Then, the choice of the integers $\boldsymbol{\ell}$ on the left hand side of (6.12), which indicate the states of internal channel following Corollary 5.5, is also restricted by the fusion rules of $\boldsymbol{B}'$ and $\boldsymbol{B}'_c$.

Remark 6.4 *(Fixing the remaining parameters $s_{2,N-1}, s_{3,1}$).* In Remark 6.2, the parameters in $\boldsymbol{s}_1$ and $\boldsymbol{s}_4$ were fixed using the $\mathbb{Z}_n$ boundary charge conditions. We now fix the remaining parameters $s_{2,N-1}, s_{3,1}$ in (6.7) using the fusion rules. Let $\boldsymbol{b}_c = (b_{c,1}, \dots, b_{c,N})$ and $\boldsymbol{b}'_c = (b'_{c,1}, \dots, b'_{c,N})$ be boundary charges associated with $\boldsymbol{B}_c$ and $\boldsymbol{B}'_c$, respectively.[11] We propose that they satisfy the same type of boundary charge conditions with (6.8) as $s_{2,I} - 1 \equiv b_{c,I+1} - b_{c,I} \pmod n$ and $s_{3,I} - 1 \equiv b'_{c,I} - b'_{c,I+1} \pmod n$ for the parameters in (6.7). As a result, these boundary charges are

$$\begin{aligned} \boldsymbol{b}_c &\equiv (b_c, b_c, \dots, b_c, b_c + s_{2,N-1} - 1) \pmod n, \\ \boldsymbol{b}'_c &\equiv (b'_c + s_{3,1} - 1, b'_c, b'_c, \dots, b'_c) \pmod n, \end{aligned} \tag{6.13}$$

where $b_c, b'_c \in \{0, 1, \dots, n-1\}$, and $s_{2,N-1}, s_{3,1}$ should be determined by the fusion rules. For definiteness, we restrict $s_{2,N-1}, s_{3,1} \in \{1, \dots, n\}$, and if $N = 2$ we take $b_c + s_{2,1} \le n$, $b'_c + s_{3,1} \le n$ so that the boundary charges are $\boldsymbol{b}_c = (b_c, b_c + s_{2,1} - 1)$ and $\boldsymbol{b}'_c = (b'_c + s_{3,1} - 1, b'_c)$.

6.3. Conjectures

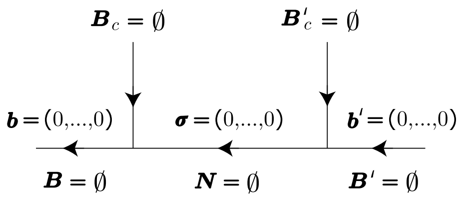
We propose the following conjectures on the relation between the $SU(N)$ Burge-reduced instanton partition functions (6.10) on $\mathbb{C}^2/\mathbb{Z}_n$ and the $\widehat{\mathfrak{sl}}(n)_N$ WZW conformal blocks. To describe our conjectures, we represent $\Lambda = [d_0, d_1, \dots, d_{n-1}] \in P^+_{n,N}$ as a Young diagram by a partition $\lambda = \mathrm{par}(\Lambda)$ using (A.8).

Conjecture 6.5 *($\emptyset - \emptyset - (\emptyset) - \emptyset - \emptyset$). The $\widehat{\mathfrak{sl}}(n)_N$ WZW 2-point conformal block of the type*

$$\langle \emptyset(1)\, \emptyset(\mathfrak{q}) \rangle^{\widehat{\mathfrak{sl}}(n)_N}_{\mathbb{P}^1}$$

agrees with the following Burge-reduced instanton partition function

$$\widehat{\mathcal{Z}}^{\boldsymbol{0},\boldsymbol{0}}_{[N,0,\dots,0];\boldsymbol{0}}(\mathfrak{q}) \quad =$$



$$= \quad (1-\mathfrak{q})^{-2h_\emptyset} = 1. \tag{6.14}$$

Here $\boldsymbol{s} = \boldsymbol{s}_1 = \boldsymbol{s}_2 = \boldsymbol{s}_3 = \boldsymbol{s}_4 = [n+1, 1, \dots, 1]$ *are fixed by* (5.1), (6.9) *and* (6.13), *and* $h_\emptyset = 0$ *is the conformal dimension for the representation* $\emptyset = [N, 0, \dots, 0]$.

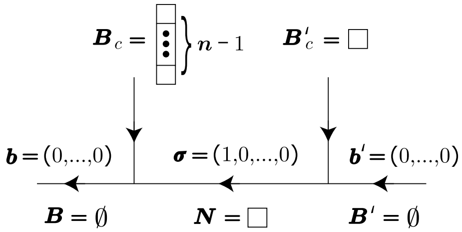
Conjecture 6.6 *($\emptyset - [N-1, 0, \dots, 0, 1] - (\Box) - \Box - \emptyset$). The $\widehat{\mathfrak{sl}}(n)_N$ WZW 2-point conformal block of the type*

$$\langle \overline{\Box}(1)\, \Box(\mathfrak{q}) \rangle^{\widehat{\mathfrak{sl}}(n)_N}_{\mathbb{P}^1}$$

agrees with the following Burge-reduced instanton partition function

[11] We will not assume the ordering of the boundary charges $\boldsymbol{b}_c$ and $\boldsymbol{b}'_c$.

$$\widehat{\mathcal{Z}}^{\mathbf{0},\mathbf{0}}_{[N-1,1,0\dots,0];\mathbf{0}}(\mathfrak{q}) \quad = $$



$$= \quad (1-\mathfrak{q})^{-2h_\square}. \tag{6.15}$$

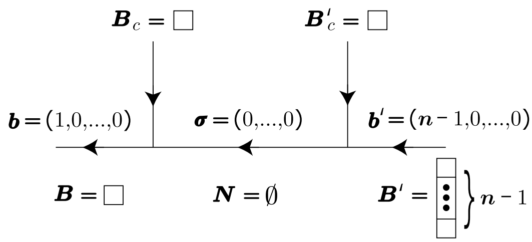
Here $s = s_3 = [n, 2, 1, \dots, 1]$, $s_1 = s_4 = [n+1, 1, \dots, 1]$ and $s_{2,N-1} = n$ are fixed by (5.1), (6.9) *and* (6.13), *and $h_\square = \frac{n^2-1}{2n(n+N)}$ is the conformal dimension for the representation $\square = [N-1, 1, 0 \dots, 0]$.*

Conjecture 6.7 *($\square - \square -$ ($\emptyset$ or $[N-2, 1, 0, \dots, 0, 1]$) $- \square - [N-1, 0, \dots, 0, 1]$). The $\widehat{\mathfrak{sl}}(n)_N$ WZW 4-point conformal blocks of the type*

$$\left\langle \overline{\square}(\infty)\,\square(1)\,\square(\mathfrak{q})\,\overline{\square}(0)\right\rangle^{\widehat{\mathfrak{sl}}(n)_N}_{\mathbb{P}^1},$$

which are (C.5) *in Appendix* C, *agree with, up to certain overall factors, the following Burge-reduced instanton partition functions,*[12]

$$\widehat{\mathcal{Z}}^{(1,0,\dots,0),(n-1,0,\dots,0)}_{[N,0,\dots,0];\boldsymbol{\ell}}(\mathfrak{q}) \quad = $$

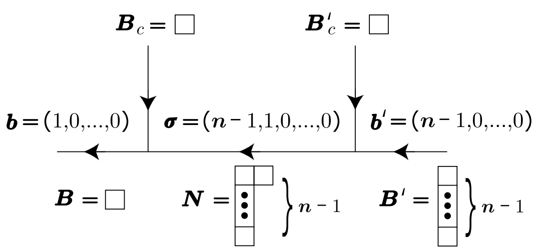


$$\tag{6.16}$$

$$= \begin{cases} (1-\mathfrak{q})^{2h_\square - \frac{n+1}{n+N}}\, {}_2F_1\left(-\frac{1}{n+N}, \frac{N-1}{n+N}; \frac{N}{n+N}; \mathfrak{q}\right), \\ \quad \textit{for } \boldsymbol{\ell} = \mathbf{0}, \\ \frac{1}{N}\, \mathfrak{q}^{\frac{1}{n}}\, (1-\mathfrak{q})^{2h_\square - \frac{n+1}{n+N}}\, {}_2F_1\left(\frac{N-1}{n+N}, 1 - \frac{1}{n+N}; 1 + \frac{N}{n+N}; \mathfrak{q}\right), \\ \quad \textit{for } \boldsymbol{\ell} = (-1, \dots, -1), \end{cases}$$

and

$$\widehat{\mathcal{Z}}^{(1,0,\dots,0),(n-1,0,\dots,0)}_{[N-2,1,0,\dots,0,1];\boldsymbol{\ell}}(\mathfrak{q}) \quad = $$



$$\tag{6.17}$$

[12] (6.16) and (6.17) correspond to, respectively, the 4-point WZW conformal blocks $\widehat{\mathcal{F}}^{(0)}_{i=1,2}(\mathfrak{q})$ and $\widehat{\mathcal{F}}^{(1)}_{i=2,1}(\mathfrak{q})$ in (C.5).

$$= \begin{cases} (1-\mathfrak{q})^{2h_{\square}-\frac{n+1}{n+N}}\, {}_2F_1\left(-\frac{1}{n+N}, \frac{n-1}{n+N}; \frac{n}{n+N}; \mathfrak{q}\right), & \textit{for } \boldsymbol{\ell}=\mathbf{0}, \\ \frac{1}{n}\,\mathfrak{q}^{1-\frac{1}{n}}\,(1-\mathfrak{q})^{2h_{\square}-\frac{n+1}{n+N}}\, {}_2F_1\left(\frac{n-1}{n+N}, 1-\frac{1}{n+N}; 1+\frac{n}{n+N}; \mathfrak{q}\right), & \textit{for } \boldsymbol{\ell}=(1,\ldots,1). \end{cases}$$

Here, by (5.1)*,* (6.9) *and* (6.13)*, for* (6.16) $\boldsymbol{s}=[n+1,1,\ldots,1]$*,* $\boldsymbol{s}_1=\boldsymbol{s}_3=[n,2,1,\ldots,1]$*,* $\boldsymbol{s}_4=[2,n,1,\ldots,1]$ *and* $s_{2,N-1}=2$ *are fixed, and for* (6.17) $\boldsymbol{s}=[2,n-1,2,1,\ldots,1]$*,* $\boldsymbol{s}_1=\boldsymbol{s}_3=[n,2,1,\ldots,1]$*,* $\boldsymbol{s}_4=[2,n,1,\ldots,1]$ *and* $s_{2,N-1}=2$ *are fixed, where when* $N=2$*,* $[2,n-1,2,1,\ldots,1]$ *means* $[3,n-1]$*. The integers* $\boldsymbol{\ell}=\boldsymbol{\delta k}$ *are taken so that the corresponding modules on the CFT side, following Corollary 5.5, are in the fundamental chamber under the action of affine Weyl group of* $\widehat{\mathfrak{sl}}(n)$*, and the second ones in* (6.16) *and* (6.17) *respect the fusion rules by*

$$\begin{aligned} &\boldsymbol{N}=[N,0,\ldots,0]=\emptyset \quad \xrightarrow{\boldsymbol{\delta k}=(-1,\ldots,-1)} \quad \mathfrak{c}=[N-2,1,0,\ldots,0,1], \\ &\boldsymbol{N}=[N-2,1,0,\ldots,0,1] \quad \xrightarrow{\boldsymbol{\delta k}=(1,\ldots,1)} \quad \mathfrak{c}=[N,0,\ldots,0]=\emptyset, \end{aligned} \tag{6.18}$$

where $\mathfrak{c}=[\mathfrak{c}_0,\mathfrak{c}_1,\ldots,\mathfrak{c}_{n-1}]$ *are defined by the Chern classes* (2.8)*. When* $n=2$*,* $[N-2,1,0,\ldots,0,1]$ *means* $[N-2,2]=\square\square$ *and then* $\boldsymbol{\sigma}=(1,1,0,\ldots,0)$*.*

7. Examples of $SU(N)$ Burge-reduced instanton counting on $\mathbb{C}^2/\mathbb{Z}_n$

We illustrate the statement of Corollary 5.5 and check Conjectures 6.5, 6.6 and 6.7 for $(N,n)=(2,2),(2,3)$ *and* $(3,2)$*. In particular we demonstrate how one can extract their* $\widehat{\mathfrak{sl}}(n)_N$ *WZW conformal blocks from the Burge-reduced instanton partition functions.*[13]

7.1. $(N,n)=(2,2)$ *and* $\widehat{\mathfrak{sl}}(2)_2$ *WZW model*

For $(N,n)=(2,2)$, there are three highest-weight representations

$$\emptyset=[2,0], \quad \square=[1,1], \quad \square\square=[0,2], \tag{7.1}$$

with conformal dimensions

$$h_{[k_0,k_1]}=\frac{k_1\,(k_1+2)}{16}: \quad h_{\emptyset}=0, \quad h_{\square}=\frac{3}{16}, \quad h_{\square\square}=\frac{1}{2}. \tag{7.2}$$

7.1.1. Burge-reduced generating functions of coloured Young diagrams

The $\mathfrak{t}$-refined Burge-reduced generating functions (5.21) for $(N,n)=(2,2)$ are obtained as

$$\begin{aligned} \widehat{X}^{\mathrm{red}}_{[2,0]}(\mathfrak{q},\mathfrak{t}) &= (\mathfrak{q};\mathfrak{q})_\infty \sum_{\ell\in\mathbb{Z}} X^{[3,1]}_{(0,0);(-\ell)}(\mathfrak{q})\,\mathfrak{t}^{2\ell} = X^{[2,0]}_{[2,0]}(\mathfrak{q})\,f_0(\mathfrak{q},\hat{\mathfrak{t}}) + X^{[2,0]}_{[0,2]}(\mathfrak{q})\,f_1(\mathfrak{q},\hat{\mathfrak{t}}), \\ \widehat{X}^{\mathrm{red}}_{[0,2]}(\mathfrak{q},\mathfrak{t}) &= (\mathfrak{q};\mathfrak{q})_\infty \sum_{\ell\in\mathbb{Z}} X^{[3,1]}_{(1,1);(-\ell)}(\mathfrak{q})\,\mathfrak{t}^{2\ell+2} = X^{[0,2]}_{[0,2]}(\mathfrak{q})\,f_1(\mathfrak{q},\hat{\mathfrak{t}}) + X^{[0,2]}_{[2,0]}(\mathfrak{q})\,f_0(\mathfrak{q},\hat{\mathfrak{t}}), \\ \widehat{X}^{\mathrm{red}}_{[1,1]}(\mathfrak{q},\mathfrak{t}) &= (\mathfrak{q};\mathfrak{q})_\infty \sum_{\ell\in\mathbb{Z}} X^{[2,2]}_{(1,0);(-\ell)}(\mathfrak{q})\,\mathfrak{t}^{2\ell+1} = X^{[1,1]}_{[1,1]}(\mathfrak{q})\,g(\mathfrak{q},\hat{\mathfrak{t}}), \end{aligned} \tag{7.3}$$

[13] The computations in this section heavily rely on Mathematica. We have also checked Conjectures 6.5, 6.6 and 6.7 for $(N,n)=(2,4)$ up to $O(\mathfrak{q}^5)$.

where $\hat{\mathfrak{t}} = \mathfrak{q}^{-\frac{1}{4}}\,\mathfrak{t}$,

$$\begin{aligned} X^{[2,0]}_{[2,0]}(\mathfrak{q}) &= 1 + \mathfrak{q} + 3\mathfrak{q}^2 + 5\mathfrak{q}^3 + 10\mathfrak{q}^4 + 16\mathfrak{q}^5 + 28\mathfrak{q}^6 + 43\mathfrak{q}^7 + 70\mathfrak{q}^8 + 105\mathfrak{q}^9 \\ &\quad + 161\mathfrak{q}^{10} + \cdots, \\ X^{[2,0]}_{[0,2]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{2}} + 2\mathfrak{q}^{\frac{3}{2}} + 4\mathfrak{q}^{\frac{5}{2}} + 7\mathfrak{q}^{\frac{7}{2}} + 13\mathfrak{q}^{\frac{9}{2}} + 21\mathfrak{q}^{\frac{11}{2}} + 35\mathfrak{q}^{\frac{13}{2}} + 55\mathfrak{q}^{\frac{15}{2}} + 86\mathfrak{q}^{\frac{17}{2}} \\ &\quad + 130\mathfrak{q}^{\frac{19}{2}} + \cdots, \\ X^{[1,1]}_{[1,1]}(\mathfrak{q}) &= 1 + 2\mathfrak{q} + 4\mathfrak{q}^2 + 8\mathfrak{q}^3 + 14\mathfrak{q}^4 + 24\mathfrak{q}^5 + 40\mathfrak{q}^6 + 64\mathfrak{q}^7 + 100\mathfrak{q}^8 + \cdots, \\ X^{[0,2]}_{[0,2]}(\mathfrak{q}) &= X^{[2,0]}_{[2,0]}(\mathfrak{q}), \quad X^{[0,2]}_{[2,0]}(\mathfrak{q}) = X^{[2,0]}_{[0,2]}(\mathfrak{q}), \end{aligned} \tag{7.4}$$

and

$$f_\sigma(\mathfrak{q}, \hat{\mathfrak{t}}) = \sum_{j \in 4\mathbb{Z}+2\sigma} \mathfrak{q}^{\frac{1}{8}j^2}\,\hat{\mathfrak{t}}^{\,j}, \quad \sigma = 0, 1, \quad g(\mathfrak{q}, \hat{\mathfrak{t}}) = \sum_{j \in 2\mathbb{Z}+1} \mathfrak{q}^{\frac{1}{8}j^2 + \frac{1}{8}}\,\hat{\mathfrak{t}}^{\,j}. \tag{7.5}$$

The Burge-reduced generating functions (7.3) agree with the $\widehat{\mathfrak{sl}}(2)_2$ WZW characters computed by (A.35),

$$\widehat{X}^{\text{red}}_{[2,0]}(\mathfrak{q}, \mathfrak{t}) = \chi^{\widehat{\mathfrak{sl}}(2)_2}_{[2,0]}(\mathfrak{q}, \hat{\mathfrak{t}}), \quad \widehat{X}^{\text{red}}_{[0,2]}(\mathfrak{q}, \mathfrak{t}) = \chi^{\widehat{\mathfrak{sl}}(2)_2}_{[0,2]}(\mathfrak{q}, \hat{\mathfrak{t}}), \quad \widehat{X}^{\text{red}}_{[1,1]}(\mathfrak{q}, \mathfrak{t}) = \mathfrak{q}^{\frac{1}{16}}\,\chi^{\widehat{\mathfrak{sl}}(2)_2}_{[1,1]}(\mathfrak{q}, \hat{\mathfrak{t}}), \tag{7.6}$$

and Corollary 5.5 is confirmed. Up to an overall factor, the functions (7.4) are the $\widehat{\mathfrak{sl}}(2)$ string functions of level-2 in [46] and given by (*cf.* Corollary 5.2),

$$\begin{aligned} &X^{[2,0]}_{[2,0]}(\mathfrak{q}) + X^{[2,0]}_{[0,2]}(\mathfrak{q}) = \frac{\left(-\mathfrak{q}^{\frac{1}{2}}; \mathfrak{q}\right)_\infty}{(\mathfrak{q};\mathfrak{q})_\infty}, \quad X^{[2,0]}_{[2,0]}(\mathfrak{q}) - X^{[2,0]}_{[0,2]}(\mathfrak{q}) = \frac{\left(\mathfrak{q}^{\frac{1}{2}}; \mathfrak{q}^{\frac{1}{2}}\right)_\infty}{(\mathfrak{q};\mathfrak{q})^2_\infty}, \\ &X^{[1,1]}_{[1,1]}(\mathfrak{q}) = \frac{\left(\mathfrak{q}^2; \mathfrak{q}^2\right)_\infty}{(\mathfrak{q};\mathfrak{q})^2_\infty}. \end{aligned} \tag{7.7}$$

Note that they are related to the NS sector and Ramond sector characters in (2.16) by $X^{[2,0]}_{[2,0]}(\mathfrak{q}) + X^{[2,0]}_{[0,2]}(\mathfrak{q}) = \chi_{\text{NS}}(\mathfrak{q})$ and $X^{[1,1]}_{[1,1]}(\mathfrak{q}) = \chi_{\text{R}}(\mathfrak{q})$. Using the Jacobi triple product identity

$$\sum_{\ell \in \mathbb{Z}} x^\ell\, y^{\frac{1}{2}\ell(\ell-1)} = (-x; y)_\infty \left(-\frac{y}{x}; y\right)_\infty (y;y)_\infty, \tag{7.8}$$

one can easily obtain (5.24) for the principal characters of $\widehat{\mathfrak{sl}}(2)$,

$$\begin{aligned} &\widehat{X}^{\text{red}}_{[2,0]}(\mathfrak{q}, 1) = \widehat{X}^{\text{red}}_{[0,2]}(\mathfrak{q}, 1) = \Pr \chi^{\widehat{\mathfrak{sl}}(2)}_{[2,0]}(\mathfrak{q}) = \left(-\mathfrak{q}^{\frac{1}{2}}; \mathfrak{q}^{\frac{1}{2}}\right)_\infty (-\mathfrak{q};\mathfrak{q})_\infty, \\ &\widehat{X}^{\text{red}}_{[1,1]}(\mathfrak{q}, 1) = \Pr \chi^{\widehat{\mathfrak{sl}}(2)}_{[1,1]}(\mathfrak{q}) = \left(-\mathfrak{q}^{\frac{1}{2}}; \mathfrak{q}^{\frac{1}{2}}\right)_\infty \left(-\mathfrak{q}^{\frac{1}{2}}; \mathfrak{q}\right)_\infty. \end{aligned} \tag{7.9}$$

7.1.2. Burge-reduced instanton partition functions

For $N = 2$ with general n, the Burge-reduced instanton partition functions (6.10) are determined by the parameters in $\boldsymbol{s} = [s_0, s_1] \in P^{++}_{2,n+2}$ and $\boldsymbol{s}_r = [s_{r,0}, s_{r,1}] \in P^{++}_{2,n+2}$, $r = 1, 2, 3, 4$, fixed by the relations (5.1), (6.9):

$$s_1 = \sigma_1 - \sigma_2 + 1, \quad s_{1,1} = b_1 - b_2 + 1, \quad s_{4,1} = b'_1 - b'_2 + 1, \tag{7.10}$$

and (6.13) from the ordered charges $\sigma_1 \geq \sigma_2$, $b_1 \geq b_2$ and $b'_1 \geq b'_2$. The Coulomb parameters are then determined from the parameter $s := s_1$ by (6.5):

$$a_1 = -\frac{1}{2}\left[s - 1 - \frac{n}{2}\right]\epsilon_2, \quad a_2 = \frac{1}{2}\left[s - 1 - \frac{n}{2}\right]\epsilon_2, \tag{7.11}$$

and the mass parameters $\boldsymbol{m} = (m_1, m_2)$ and $\boldsymbol{m}' = (m'_1, m'_2)$ are determined from the parameters in $\boldsymbol{s}_1, \boldsymbol{s}_2$ and $\boldsymbol{s}_3, \boldsymbol{s}_4$, respectively, by (6.6).

Let us consider the case of $(N, n) = (2, 2)$ with the rational Ω-background $\epsilon_1/\epsilon_2 = -2$ in (6.4).[14]

Example 7.1 *($\emptyset - \emptyset - (\emptyset) - \emptyset - \emptyset$).* Consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[2,0];(\ell)}(\mathfrak{q})$ and take $\ell = 0$ in the fundamental chamber, which respects the fusion rules, as in Conjecture 6.5. Here $\boldsymbol{s} = \boldsymbol{s}_1 = \boldsymbol{s}_2 = \boldsymbol{s}_3 = \boldsymbol{s}_4 = [3, 1]$ are fixed. Then, the Burge-reduced instanton partition function is obtained as

$$\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[2,0];(0)}(\mathfrak{q}) = (1-\mathfrak{q})^{-2h_\emptyset} = 1, \qquad h_\emptyset = 0, \tag{7.12}$$

and Conjecture 6.5 is confirmed.

Example 7.2 *($\emptyset - \square - (\square) - \square - \emptyset$).* Consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[1,1];(\ell)}(\mathfrak{q})$ and take $\ell = 0$ in the fundamental chamber as in Conjecture 6.6. Here $\boldsymbol{s} = \boldsymbol{s}_2 = \boldsymbol{s}_3 = [2, 2]$ and $\boldsymbol{s}_1 = \boldsymbol{s}_4 = [3, 1]$ are fixed. Then we see that the Burge-reduced instanton partition function is

$$\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[1,1];(0)}(\mathfrak{q}) = (1-\mathfrak{q})^{-2h_\square} = 1 + \frac{3\mathfrak{q}}{8} + \frac{33\mathfrak{q}^2}{128} + \frac{209\mathfrak{q}^3}{1024} + \frac{5643\mathfrak{q}^4}{32768} + \frac{39501\mathfrak{q}^5}{262144} + \cdots, \tag{7.13}$$

where $h_\square = 3/16$, and Conjecture 6.6 is confirmed.

Example 7.3 *($\square - \square - (\emptyset) - \square - \square$ and $\square - \square - (\square\square) - \square - \square$).* For Conjecture 6.7, consider, first, the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[2,0];(\ell)}(\mathfrak{q})$, where $\boldsymbol{s} = [3, 1]$ and $\boldsymbol{s}_1 = \boldsymbol{s}_2 = \boldsymbol{s}_3 = \boldsymbol{s}_4 = [2, 2]$ are fixed. Then we find that the Burge-reduced instanton partition functions for $\ell = 0, -1$ in the fundamental chamber are

$$\begin{aligned}
\widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[2,0];(0)}(\mathfrak{q}) &= (1-\mathfrak{q})^{2h_\square - \frac{3}{4}}\, {}_2F_1\left(-\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; \mathfrak{q}\right) \\
&= 1 + \frac{\mathfrak{q}}{4} + \frac{11\mathfrak{q}^2}{64} + \frac{35\mathfrak{q}^3}{256} + \frac{949\mathfrak{q}^4}{8192} + \frac{3333\mathfrak{q}^5}{32768} + \frac{47909\mathfrak{q}^6}{524288} + \cdots, \\
\widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[2,0];(-1)}(\mathfrak{q}) &= \frac{\mathfrak{q}^{\frac{1}{2}}}{2}(1-\mathfrak{q})^{2h_\square - \frac{3}{4}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{3}{2}; \mathfrak{q}\right) \\
&= \frac{\mathfrak{q}^{\frac{1}{2}}}{2} + \frac{\mathfrak{q}^{\frac{3}{2}}}{4} + \frac{23\mathfrak{q}^{\frac{5}{2}}}{128} + \frac{37\mathfrak{q}^{\frac{7}{2}}}{256} + \frac{2013\mathfrak{q}^{\frac{9}{2}}}{16384} + \frac{3537\mathfrak{q}^{\frac{11}{2}}}{32768} + \cdots,
\end{aligned} \tag{7.14}$$

where $h_\square = 3/16$, and the second one respects the fusion rules by (6.18). Consider, next, the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[0,2];(\ell)}(\mathfrak{q})$, where $\boldsymbol{s} = [3, 1]$ and $\boldsymbol{s}_1 = \boldsymbol{s}_2 = \boldsymbol{s}_3 =$

[14] Examples 7.1, 7.2 and 7.3 are confirmed up to $O(\mathfrak{q}^6)$.

$\boldsymbol{s}_4 = [2, 2]$ are fixed. Then we obtain the Burge-reduced instanton partition functions for $\ell = 0, 1$ in the fundamental chamber as

$$\begin{aligned}
\widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[0,2];(0)}(\mathfrak{q}) &= (1-\mathfrak{q})^{2h_{\square}-\frac{3}{4}}\,{}_2F_1\left(-\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; \mathfrak{q}\right) = \widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[2,0];0}(\mathfrak{q}),\\
\widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[0,2];(1)}(\mathfrak{q}) &= \frac{\mathfrak{q}^{\frac{1}{2}}}{2}(1-\mathfrak{q})^{2h_{\square}-\frac{3}{4}}\,{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{3}{2}; \mathfrak{q}\right) = \widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[2,0];-1}(\mathfrak{q}),
\end{aligned} \tag{7.15}$$

where the second one respects the fusion rules by (6.18). The above results (7.14) and (7.15) support Conjecture 6.7. By

$${}_2F_1\left(-\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; \mathfrak{q}\right) = \left[\frac{1+\sqrt{1-\mathfrak{q}}}{2}\right]^{\frac{1}{2}}, \quad \frac{\mathfrak{q}^{\frac{1}{2}}}{2}\,{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{3}{2}; \mathfrak{q}\right) = \left[\frac{1-\sqrt{1-\mathfrak{q}}}{2}\right]^{\frac{1}{2}}, \tag{7.16}$$

they are also consistent with the results in [14] by Belavin and Mukhametzhanov.[15]

7.2. $(N, n) = (2, 3)$ *and* $\widehat{\mathfrak{sl}}(3)_2$ *WZW model*

For $(N, n) = (2, 3)$, there are six highest-weight representations

$$\begin{aligned}
&\emptyset = [2, 0, 0], \quad \ydiagram{1} = [1, 1, 0], \quad \ydiagram{2} = [0, 2, 0], \quad \ydiagram{1,1} = [1, 0, 1],\\
&\ydiagram{2,1} = [0, 1, 1], \quad \ydiagram{2,2} = [0, 0, 2],
\end{aligned} \tag{7.17}$$

with conformal dimensions

$$\begin{aligned}
h_{[k_0,k_1,k_2]} &= \frac{k_1^2 + k_2^2 + k_1 k_2 + 3k_1 + 3k_2}{15}:\\
h_{\emptyset} &= 0, \quad h_{\ydiagram{1}} = h_{\ydiagram{1,1}} = \frac{4}{15}, \quad h_{\ydiagram{2}} = h_{\ydiagram{2,2}} = \frac{2}{3}, \quad h_{\ydiagram{2,1}} = \frac{3}{5}.
\end{aligned} \tag{7.18}$$

[15] More precisely, in [14], the generic Ω-background, without the Burge conditions, was discussed. Then the first one of (7.14) and the second one of (7.15), with $\mathfrak{c} = 0$, were obtained as prefactors combined with the $\mathcal{N} = 1$ super-Virasoro Ramond conformal blocks $H_{\pm}(\mathfrak{q})$, $F_{\pm}(\mathfrak{q})$, $\widetilde{H}_{\pm}(\mathfrak{q})$ and $\widetilde{F}_{\pm}(\mathfrak{q})$. What we found is that, when we impose the specific Burge conditions, the conformal blocks are trivialized as $H_{\pm}(\mathfrak{q}), F_{\pm}(\mathfrak{q}) \to 1$ and $\widetilde{H}_{\pm}(\mathfrak{q}), \widetilde{F}_{\pm}(\mathfrak{q}) \to 0$, and only the prefactors are obtained.

7.2.1. Burge-reduced generating functions of coloured Young diagrams

The $\mathfrak{t}$-refined Burge-reduced generating functions (5.21) for $(N,n)=(2,3)$ are obtained as

$$\begin{aligned}
\widehat{X}^{\mathrm{red}}_{[2,0,0]}(\mathfrak{q},(\mathfrak{t}_1,\mathfrak{t}_2)) &= (\mathfrak{q};\mathfrak{q})_\infty \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2} X^{[4,1]}_{(0,0);(-\ell_1,-\ell_2)}(\mathfrak{q})\, \mathfrak{t}_1^{2\ell_1-\ell_2}\, \mathfrak{t}_2^{-\ell_1+2\ell_2} \\
&= X^{[2,0,0]}_{[2,0,0]}(\mathfrak{q})\, f_{00}(\mathfrak{q},\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) + X^{[2,0,0]}_{[0,1,1]}\, g_{00}(\mathfrak{q},\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2), \\
\widehat{X}^{\mathrm{red}}_{[0,2,0]}(\mathfrak{q},(\mathfrak{t}_1,\mathfrak{t}_2)) &= (\mathfrak{q};\mathfrak{q})_\infty \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2} X^{[4,1]}_{(1,1);(-\ell_1,-\ell_2)}(\mathfrak{q})\, \mathfrak{t}_1^{2+2\ell_1-\ell_2}\, \mathfrak{t}_2^{-\ell_1+2\ell_2} \\
&= X^{[0,2,0]}_{[0,2,0]}(\mathfrak{q})\, f_{10}(\mathfrak{q},\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) + X^{[0,2,0]}_{[1,0,1]}(\mathfrak{q})\, g_{10}(\mathfrak{q},\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2), \\
\widehat{X}^{\mathrm{red}}_{[0,0,2]}(\mathfrak{q},(\mathfrak{t}_1,\mathfrak{t}_2)) &= (\mathfrak{q};\mathfrak{q})_\infty \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2} X^{[4,1]}_{(2,2);(-\ell_1,-\ell_2)}(\mathfrak{q})\, \mathfrak{t}_1^{2\ell_1-\ell_2}\, \mathfrak{t}_2^{2-\ell_1+2\ell_2} \\
&= X^{[0,0,2]}_{[0,0,2]}(\mathfrak{q})\, f_{01}(\mathfrak{q},\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) + X^{[0,0,2]}_{[1,1,0]}(\mathfrak{q})\, g_{01}(\mathfrak{q},\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2), \\
\widehat{X}^{\mathrm{red}}_{[1,1,0]}(\mathfrak{q},(\mathfrak{t}_1,\mathfrak{t}_2)) &= (\mathfrak{q};\mathfrak{q})_\infty \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2} X^{[3,2]}_{(1,0);(-\ell_1,-\ell_2)}(\mathfrak{q})\, \mathfrak{t}_1^{1+2\ell_1-\ell_2}\, \mathfrak{t}_2^{-\ell_1+2\ell_2} \\
&= X^{[1,1,0]}_{[1,1,0]}\, g_{01}(\mathfrak{q},\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) + X^{[1,1,0]}_{[0,0,2]}(\mathfrak{q})\, f_{01}(\mathfrak{q},\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2), \\
\widehat{X}^{\mathrm{red}}_{[0,1,1]}(\mathfrak{q},(\mathfrak{t}_1,\mathfrak{t}_2)) &= (\mathfrak{q};\mathfrak{q})_\infty \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2} X^{[3,2]}_{(2,1);(-\ell_1,-\ell_2)}(\mathfrak{q})\, \mathfrak{t}_1^{1+2\ell_1-\ell_2}\, \mathfrak{t}_2^{1-\ell_1+2\ell_2} \\
&= X^{[0,1,1]}_{[0,1,1]}(\mathfrak{q})\, g_{00}(\mathfrak{q},\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) + X^{[0,1,1]}_{[2,0,0]}(\mathfrak{q})\, f_{00}(\mathfrak{q},\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2), \\
\widehat{X}^{\mathrm{red}}_{[1,0,1]}(\mathfrak{q},(\mathfrak{t}_1,\mathfrak{t}_2)) &= (\mathfrak{q};\mathfrak{q})_\infty \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2} X^{[2,3]}_{(2,0);(-\ell_1,-\ell_2)}(\mathfrak{q})\, \mathfrak{t}_1^{2\ell_1-\ell_2}\, \mathfrak{t}_2^{1-\ell_1+2\ell_2} \\
&= X^{[1,0,1]}_{[1,0,1]}(\mathfrak{q})\, g_{10}(\mathfrak{q},\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) + X^{[1,0,1]}_{[0,2,0]}(\mathfrak{q})\, f_{10}(\mathfrak{q},\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2),
\end{aligned} \tag{7.19}$$

where $\hat{\mathfrak{t}}_1 = \mathfrak{q}^{-\frac{1}{3}}\,\mathfrak{t}_1$, $\hat{\mathfrak{t}}_2 = \mathfrak{q}^{-\frac{1}{3}}\,\mathfrak{t}_2$,

$$\begin{aligned}
X^{[2,0,0]}_{[2,0,0]}(\mathfrak{q}) &= 1 + 2\mathfrak{q} + 8\mathfrak{q}^2 + 20\mathfrak{q}^3 + 52\mathfrak{q}^4 + 116\mathfrak{q}^5 + 256\mathfrak{q}^6 + 522\mathfrak{q}^7 + \cdots, \\
X^{[2,0,0]}_{[0,1,1]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{3}} + 4\mathfrak{q}^{\frac{4}{3}} + 12\mathfrak{q}^{\frac{7}{3}} + 32\mathfrak{q}^{\frac{10}{3}} + 77\mathfrak{q}^{\frac{13}{3}} + 172\mathfrak{q}^{\frac{16}{3}} + 365\mathfrak{q}^{\frac{19}{3}} \\
&\quad + 740\mathfrak{q}^{\frac{22}{3}} + \cdots, \\
X^{[0,1,1]}_{[0,1,1]}(\mathfrak{q}) &= 1 + 4\mathfrak{q} + 13\mathfrak{q}^2 + 36\mathfrak{q}^3 + 89\mathfrak{q}^4 + 204\mathfrak{q}^5 + 441\mathfrak{q}^6 + 908\mathfrak{q}^7 + \cdots, \\
X^{[0,1,1]}_{[2,0,0]}(\mathfrak{q}) &= 2\mathfrak{q}^{\frac{2}{3}} + 7\mathfrak{q}^{\frac{5}{3}} + 22\mathfrak{q}^{\frac{8}{3}} + 56\mathfrak{q}^{\frac{11}{3}} + 136\mathfrak{q}^{\frac{14}{3}} + 300\mathfrak{q}^{\frac{17}{3}} + 636\mathfrak{q}^{\frac{20}{3}} \\
&\quad + 1280\mathfrak{q}^{\frac{23}{3}} + \cdots, \\
X^{[0,2,0]}_{[0,2,0]}(\mathfrak{q}) &= X^{[0,0,2]}_{[0,0,2]}(\mathfrak{q}) = X^{[2,0,0]}_{[2,0,0]}(\mathfrak{q}), \quad X^{[0,2,0]}_{[1,0,1]}(\mathfrak{q}) = X^{[0,0,2]}_{[1,1,0]}(\mathfrak{q}) = X^{[2,0,0]}_{[0,1,1]}(\mathfrak{q}), \\
X^{[1,1,0]}_{[1,1,0]}(\mathfrak{q}) &= X^{[1,0,1]}_{[1,0,1]}(\mathfrak{q}) = X^{[0,1,1]}_{[0,1,1]}(\mathfrak{q}), \quad X^{[1,1,0]}_{[0,0,2]}(\mathfrak{q}) = X^{[1,0,1]}_{[0,2,0]}(\mathfrak{q}) = X^{[0,1,1]}_{[2,0,0]}(\mathfrak{q}),
\end{aligned} \tag{7.20}$$

and f_{00}, f_{10}, f_{01}, g_{00}, g_{10}, g_{01} are

$$
\begin{aligned}
f_{\sigma_1\sigma_2}(\mathfrak{q}, \hat{\mathfrak{t}}_1, \hat{\mathfrak{t}}_2) &= \sum_{\substack{(j_1, j_2)\in\{2\mathbb{Z}\}^2 \\ j_1-j_2\in 6\mathbb{Z}+2(\sigma_1-\sigma_2)}} \mathfrak{q}^{\frac{1}{6}\left[j_1^2+j_2^2+j_1 j_2\right]} \hat{\mathfrak{t}}_1^{j_1} \hat{\mathfrak{t}}_2^{j_2}, \\
g_{\sigma_1\sigma_2}(\mathfrak{q}, \hat{\mathfrak{t}}_1, \hat{\mathfrak{t}}_2) &= \sum_{\substack{(j_1, j_2)\in\{2\mathbb{Z}+1\}^2 \\ j_1-j_2\in 6\mathbb{Z}+2(\sigma_1-\sigma_2)}} \mathfrak{q}^{\frac{1}{6}\left[j_1^2+j_2^2+j_1 j_2\right]+\frac{1}{6}} \hat{\mathfrak{t}}_1^{j_1} \hat{\mathfrak{t}}_2^{j_2} \\
&\quad + \sum_{\substack{(j_1, j_2)\in\{2\mathbb{Z}\}\times\{2\mathbb{Z}+1\} \\ j_1-j_2\in 6\mathbb{Z}+1+2(\sigma_1-\sigma_2)}} \mathfrak{q}^{\frac{1}{6}\left[j_1^2+j_2^2+j_1 j_2\right]+\frac{1}{6}} \left(\hat{\mathfrak{t}}_1^{j_1} \hat{\mathfrak{t}}_2^{j_2} + \hat{\mathfrak{t}}_1^{j_2} \hat{\mathfrak{t}}_2^{j_1}\right).
\end{aligned}
\tag{7.21}
$$

The Burge-reduced generating functions (7.19) agree with the $\widehat{\mathfrak{sl}}(3)_2$ WZW characters computed by (A.35),

$$
\begin{aligned}
&\widehat{X}^{\text{red}}_{[2,0,0]}(\mathfrak{q}, (\mathfrak{t}_1, \mathfrak{t}_2)) = \chi^{\widehat{\mathfrak{sl}}(3)_2}_{[2,0,0]}(\mathfrak{q}, (\hat{\mathfrak{t}}_1, \hat{\mathfrak{t}}_2)), \quad \widehat{X}^{\text{red}}_{[0,2,0]}(\mathfrak{q}, (\mathfrak{t}_1, \mathfrak{t}_2)) = \chi^{\widehat{\mathfrak{sl}}(3)_2}_{[0,2,0]}(\mathfrak{q}, (\hat{\mathfrak{t}}_1, \hat{\mathfrak{t}}_2)), \\
&\widehat{X}^{\text{red}}_{[0,0,2]}(\mathfrak{q}, (\mathfrak{t}_1, \mathfrak{t}_2)) = \chi^{\widehat{\mathfrak{sl}}(3)_2}_{[0,0,2]}(\mathfrak{q}, (\hat{\mathfrak{t}}_1, \hat{\mathfrak{t}}_2)), \quad \widehat{X}^{\text{red}}_{[1,1,0]}(\mathfrak{q}, (\mathfrak{t}_1, \mathfrak{t}_2)) = \mathfrak{q}^{\frac{1}{15}}\, \chi^{\widehat{\mathfrak{sl}}(3)_2}_{[1,1,0]}(\mathfrak{q}, (\hat{\mathfrak{t}}_1, \hat{\mathfrak{t}}_2)), \\
&\widehat{X}^{\text{red}}_{[0,1,1]}(\mathfrak{q}, (\mathfrak{t}_1, \mathfrak{t}_2)) = \mathfrak{q}^{\frac{1}{15}}\, \chi^{\widehat{\mathfrak{sl}}(3)_2}_{[0,1,1]}(\mathfrak{q}, (\hat{\mathfrak{t}}_1, \hat{\mathfrak{t}}_2)), \\
&\widehat{X}^{\text{red}}_{[1,0,1]}(\mathfrak{q}, (\mathfrak{t}_1, \mathfrak{t}_2)) = \mathfrak{q}^{\frac{1}{15}}\, \chi^{\widehat{\mathfrak{sl}}(3)_2}_{[1,0,1]}(\mathfrak{q}, (\hat{\mathfrak{t}}_1, \hat{\mathfrak{t}}_2)),
\end{aligned}
\tag{7.22}
$$

and Corollary 5.5 is confirmed. Up to an overall factor, the functions (7.20) are the $\widehat{\mathfrak{sl}}(3)$ string functions of level-2 in [46] and given by (*cf.* Corollary 5.2),

$$
\begin{aligned}
X^{[2,0,0]}_{[2,0,0]}(\mathfrak{q}) - \mathfrak{q}^{\frac{1}{6}} X^{[2,0,0]}_{[0,1,1]}(\mathfrak{q}) &= \frac{\left(\mathfrak{q}^{\frac{1}{2}}; \mathfrak{q}^{\frac{1}{2}}\right)_\infty \left(\mathfrak{q}; \mathfrak{q}^{\frac{5}{2}}\right)_\infty \left(\mathfrak{q}^{\frac{3}{2}}; \mathfrak{q}^{\frac{5}{2}}\right)_\infty \left(\mathfrak{q}^{\frac{5}{2}}; \mathfrak{q}^{\frac{5}{2}}\right)_\infty}{(\mathfrak{q}; \mathfrak{q})^4_\infty}, \\
X^{[2,0,0]}_{[0,1,1]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{3}} \frac{(\mathfrak{q}^2; \mathfrak{q}^2)_\infty (\mathfrak{q}^2; \mathfrak{q}^{10})_\infty (\mathfrak{q}^8; \mathfrak{q}^{10})_\infty (\mathfrak{q}^{10}; \mathfrak{q}^{10})_\infty}{(\mathfrak{q}; \mathfrak{q})^4_\infty}, \\
X^{[0,1,1]}_{[0,1,1]}(\mathfrak{q}) &= \frac{(\mathfrak{q}^2; \mathfrak{q}^2)_\infty (\mathfrak{q}^4; \mathfrak{q}^{10})_\infty (\mathfrak{q}^6; \mathfrak{q}^{10})_\infty (\mathfrak{q}^{10}; \mathfrak{q}^{10})_\infty}{(\mathfrak{q}; \mathfrak{q})^4_\infty}, \\
\mathfrak{q}^{\frac{1}{6}} X^{[0,1,1]}_{[0,1,1]}(\mathfrak{q}) - X^{[0,1,1]}_{[2,0,0]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{6}} \frac{\left(\mathfrak{q}^{\frac{1}{2}}; \mathfrak{q}^{\frac{1}{2}}\right)_\infty \left(\mathfrak{q}^{\frac{1}{2}}; \mathfrak{q}^{\frac{5}{2}}\right)_\infty \left(\mathfrak{q}^2; \mathfrak{q}^{\frac{5}{2}}\right)_\infty \left(\mathfrak{q}^{\frac{5}{2}}; \mathfrak{q}^{\frac{5}{2}}\right)_\infty}{(\mathfrak{q}; \mathfrak{q})^4_\infty}.
\end{aligned}
\tag{7.23}
$$

By taking $\mathfrak{t}_1 = \mathfrak{t}_2 = 1$, the principal characters of $\widehat{\mathfrak{sl}}(3)$ are obtained as in (5.24):

$$
\begin{aligned}
\widehat{X}^{\text{red}}_{[2,0,0]}(\mathfrak{q}, (1,1)) = \widehat{X}^{\text{red}}_{[0,2,0]}(\mathfrak{q}, (1,1)) = \widehat{X}^{\text{red}}_{[0,0,2]}(\mathfrak{q}, (1,1)) &= \Pr \chi^{\widehat{\mathfrak{sl}}(3)}_{[2,0,0]}(\mathfrak{q}) \\
&= \frac{(\mathfrak{q}; \mathfrak{q})_\infty}{\left(\mathfrak{q}^{\frac{1}{3}}; \mathfrak{q}^{\frac{1}{3}}\right)_\infty \left(\mathfrak{q}^{\frac{2}{3}}; \mathfrak{q}^{\frac{5}{3}}\right)_\infty \left(\mathfrak{q}; \mathfrak{q}^{\frac{5}{3}}\right)_\infty}, \\
\widehat{X}^{\text{red}}_{[1,1,0]}(\mathfrak{q}, (1,1)) = \widehat{X}^{\text{red}}_{[0,1,1]}(\mathfrak{q}, (1,1)) = \widehat{X}^{\text{red}}_{[1,0,1]}(\mathfrak{q}, (1,1)) &= \Pr \chi^{\widehat{\mathfrak{sl}}(3)}_{[1,1,0]}(\mathfrak{q}) \\
&= \frac{(\mathfrak{q}; \mathfrak{q})_\infty}{\left(\mathfrak{q}^{\frac{1}{3}}; \mathfrak{q}^{\frac{1}{3}}\right)_\infty \left(\mathfrak{q}^{\frac{1}{3}}; \mathfrak{q}^{\frac{5}{3}}\right)_\infty \left(\mathfrak{q}^{\frac{4}{3}}; \mathfrak{q}^{\frac{5}{3}}\right)_\infty}.
\end{aligned}
\tag{7.24}
$$

7.2.2. Burge-reduced instanton partition functions

For $(N, n) = (2, 3)$, the rational Ω-background (6.4) yields $\epsilon_1/\epsilon_2 = -5/2$. The parameters in $\boldsymbol{s} = [s_0, s_1] \in P^{++}_{2,5}$ and $\boldsymbol{s}_r = [s_{r,0}, s_{r,1}] \in P^{++}_{2,5}$, $r = 1, 2, 3, 4$, which determine the Burge-reduced instanton partition functions, are fixed as in (7.10).[16]

Example 7.4 *($\emptyset - \emptyset - (\emptyset) - \emptyset - \emptyset$).* Consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[2,0,0];(\ell_1,\ell_2)}(\mathfrak{q})$ and take $(\ell_1, \ell_2) = (0, 0)$ in the fundamental chamber as in Conjecture 6.5. Here $\boldsymbol{s} = \boldsymbol{s}_1 = \boldsymbol{s}_2 = \boldsymbol{s}_3 = \boldsymbol{s}_4 = [4, 1]$ are fixed. Then we see that the Burge-reduced instanton partition function is

$$\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[2,0,0];(0,0)}(\mathfrak{q}) = (1-\mathfrak{q})^{-2h_\emptyset} = 1, \qquad h_\emptyset = 0, \tag{7.25}$$

and Conjecture 6.5 is confirmed.

Example 7.5 *($\emptyset - \begin{smallmatrix}\square\\\square\end{smallmatrix} - (\square) - \square - \emptyset$).* Consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[1,1,0];(\ell_1,\ell_2)}(\mathfrak{q})$ and take $(\ell_1, \ell_2) = (0, 0)$ in the fundamental chamber as in Conjecture 6.6. Here $\boldsymbol{s} = \boldsymbol{s}_3 = [3, 2]$, $\boldsymbol{s}_1 = \boldsymbol{s}_4 = [4, 1]$ and $\boldsymbol{s}_2 = [2, 3]$ are fixed. Then the Burge-reduced instanton partition function is

$$\begin{aligned}\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[1,1,0];(0,0)}(\mathfrak{q}) = (1-\mathfrak{q})^{-2h_\square} = 1 + \frac{8\mathfrak{q}}{15} + \frac{92\mathfrak{q}^2}{225} + \frac{3496\mathfrak{q}^3}{10125} + \frac{46322\mathfrak{q}^4}{151875} \\ + \frac{3149896\mathfrak{q}^5}{11390625} + \cdots,\end{aligned} \tag{7.26}$$

where $h_\square = 4/15$, and Conjecture 6.6 is confirmed.

Example 7.6 *($\square - \square - (\emptyset) - \square - \begin{smallmatrix}\square\\\square\end{smallmatrix}$ and $\square - \square - (\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}) - \square - \begin{smallmatrix}\square\\\square\end{smallmatrix}$).* For Conjecture 6.7, consider, first, the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(1,0),(2,0)}_{[2,0,0];(\ell_1,\ell_2)}(\mathfrak{q})$, where $\boldsymbol{s} = [4, 1]$, $\boldsymbol{s}_1 = \boldsymbol{s}_2 = \boldsymbol{s}_3 = [3, 2]$ and $\boldsymbol{s}_4 = [2, 3]$ are fixed. Then we find that the Burge-reduced instanton partition functions for $(\ell_1, \ell_2) = (0, 0)$ and $(-1, -1)$ in the fundamental chamber are

$$\begin{aligned}\widehat{\mathcal{Z}}^{(1,0),(2,0)}_{[2,0,0];(0,0)}(\mathfrak{q}) &= (1-\mathfrak{q})^{2h_\square - \frac{4}{5}}\, {}_2F_1\left(-\frac{1}{5}, \frac{1}{5}; \frac{2}{5}; \mathfrak{q}\right) \\ &= 1 + \frac{\mathfrak{q}}{6} + \frac{34\mathfrak{q}^2}{315} + \frac{67\mathfrak{q}^3}{810} + \frac{49309\mathfrak{q}^4}{722925} + \frac{254267\mathfrak{q}^5}{4337550} + \cdots, \\ \widehat{\mathcal{Z}}^{(1,0),(2,0)}_{[2,0,0];(-1,-1)}(\mathfrak{q}) &= \frac{\mathfrak{q}^{\frac{1}{3}}}{2}(1-\mathfrak{q})^{2h_\square - \frac{4}{5}}\, {}_2F_1\left(\frac{1}{5}, \frac{4}{5}; \frac{7}{5}; \mathfrak{q}\right) \\ &= \frac{\mathfrak{q}^{\frac{1}{3}}}{2} + \frac{4\mathfrak{q}^{\frac{4}{3}}}{21} + \frac{79\mathfrak{q}^{\frac{7}{3}}}{630} + \frac{4619\mathfrak{q}^{\frac{10}{3}}}{48195} + \frac{16237\mathfrak{q}^{\frac{13}{3}}}{206550} + \cdots,\end{aligned} \tag{7.27}$$

where $h_\square = 4/15$, and the second one respects the fusion rules by (6.18). Consider, next, the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(1,0),(2,0)}_{[0,1,1];(\ell_1,\ell_2)}(\mathfrak{q})$, where $\boldsymbol{s} = \boldsymbol{s}_1 = \boldsymbol{s}_2 = \boldsymbol{s}_3 = [3, 2]$ and $\boldsymbol{s}_4 = [2, 3]$ are fixed. Then we see that the Burge-reduced instanton partition functions for $(\ell_1, \ell_2) = (0, 0)$ and $(1, 1)$ in the fundamental chamber are

[16] Examples 7.4, 7.5 and 7.6 are confirmed up to $O(\mathfrak{q}^5)$.

$$\begin{aligned}\widehat{\mathcal{Z}}^{(1,0),(2,0)}_{[0,1,1];(0,0)}(\mathfrak{q}) &= (1-\mathfrak{q})^{2h_{\square}-\frac{4}{5}}\,{}_2F_1\left(-\frac{1}{5},\frac{2}{5};\frac{3}{5};\mathfrak{q}\right)\\ &= 1+\frac{2\mathfrak{q}}{15}+\frac{13\mathfrak{q}^2}{150}+\frac{8792\mathfrak{q}^3}{131625}+\frac{218507\mathfrak{q}^4}{3948750}+\frac{54190157\mathfrak{q}^5}{1135265625}+\cdots,\\ \widehat{\mathcal{Z}}^{(1,0),(2,0)}_{[0,1,1];(1,1)}(\mathfrak{q}) &= \frac{\mathfrak{q}^{\frac{2}{3}}}{3}(1-\mathfrak{q})^{2h_{\square}-\frac{4}{5}}\,{}_2F_1\left(\frac{2}{5},\frac{4}{5};\frac{8}{5};\mathfrak{q}\right)\\ &= \frac{\mathfrak{q}^{\frac{2}{3}}}{3}+\frac{7\mathfrak{q}^{\frac{5}{3}}}{45}+\frac{1867\mathfrak{q}^{\frac{8}{3}}}{17550}+\frac{32582\mathfrak{q}^{\frac{11}{3}}}{394875}+\frac{18575621\mathfrak{q}^{\frac{14}{3}}}{272463750}+\cdots,\end{aligned} \tag{7.28}$$

where the second one respects the fusion rules by (6.18). The above results (7.27) and (7.28) support Conjecture 6.7.

7.3. $(N,n)=(3,2)$ and $\widehat{\mathfrak{sl}}(2)_3$ WZW model

For $(N,n)=(3,2)$, there are four highest-weight representations

$$\emptyset=[3,0],\quad \square=[2,1],\quad \square\square=[1,2],\quad \square\square\square=[0,3], \tag{7.29}$$

with conformal dimensions

$$h_{[k_0,k_1]}=\frac{k_1\,(k_1+2)}{20}:\quad h_{\emptyset}=0,\quad h_{\square}=\frac{3}{20},\quad h_{\square\square}=\frac{2}{5},\quad h_{\square\square\square}=\frac{3}{4}. \tag{7.30}$$

7.3.1. Burge-reduced generating functions of coloured Young diagrams

The $\mathfrak{t}$-refined Burge-reduced generating functions (5.21) for $(N,n)=(3,2)$ are obtained as

$$\begin{aligned}\widehat{X}^{\mathrm{red}}_{[3,0]}(\mathfrak{q},\mathfrak{t}) &= (\mathfrak{q};\mathfrak{q})_\infty\sum_{\ell\in\mathbb{Z}}X^{[3,1,1]}_{(0,0,0);(-\ell)}(\mathfrak{q})\,\mathfrak{t}^{2\ell}=X^{[3,0]}_{[3,0]}(\mathfrak{q})\,f_0(\mathfrak{q},\hat{\mathfrak{t}})+X^{[3,0]}_{[1,2]}(\mathfrak{q})\,g_0(\mathfrak{q},\hat{\mathfrak{t}}),\\ \widehat{X}^{\mathrm{red}}_{[0,3]}(\mathfrak{q},\mathfrak{t}) &= (\mathfrak{q};\mathfrak{q})_\infty\sum_{\ell\in\mathbb{Z}}X^{[3,1,1]}_{(1,1,1);(-\ell)}(\mathfrak{q})\,\mathfrak{t}^{2\ell+3}=X^{[0,3]}_{[0,3]}(\mathfrak{q})\,f_1(\mathfrak{q},\hat{\mathfrak{t}})+X^{[0,3]}_{[2,1]}(\mathfrak{q})\,g_1(\mathfrak{q},\hat{\mathfrak{t}}),\\ \widehat{X}^{\mathrm{red}}_{[2,1]}(\mathfrak{q},\mathfrak{t}) &= (\mathfrak{q};\mathfrak{q})_\infty\sum_{\ell\in\mathbb{Z}}X^{[2,2,1]}_{(1,0,0);(-\ell)}(\mathfrak{q})\,\mathfrak{t}^{2\ell+1}=X^{[2,1]}_{[2,1]}(\mathfrak{q})\,g_1(\mathfrak{q},\hat{\mathfrak{t}})+X^{[2,1]}_{[0,3]}(\mathfrak{q})\,f_1(\mathfrak{q},\hat{\mathfrak{t}}),\\ \widehat{X}^{\mathrm{red}}_{[1,2]}(\mathfrak{q},\mathfrak{t}) &= (\mathfrak{q};\mathfrak{q})_\infty\sum_{\ell\in\mathbb{Z}}X^{[2,1,2]}_{(1,1,0);(-\ell)}(\mathfrak{q})\,\mathfrak{t}^{2\ell+2}=X^{[1,2]}_{[1,2]}(\mathfrak{q})\,g_0(\mathfrak{q},\hat{\mathfrak{t}})+X^{[1,2]}_{[3,0]}(\mathfrak{q})\,f_0(\mathfrak{q},\hat{\mathfrak{t}}),\end{aligned} \tag{7.31}$$

where $\hat{\mathfrak{t}}=\mathfrak{q}^{-\frac{1}{4}}\,\mathfrak{t}$,

$$\begin{aligned}X^{[3,0]}_{[3,0]}(\mathfrak{q}) &= 1+\mathfrak{q}+3\mathfrak{q}^2+6\mathfrak{q}^3+12\mathfrak{q}^4+21\mathfrak{q}^5+39\mathfrak{q}^6+64\mathfrak{q}^7+108\mathfrak{q}^8+\cdots,\\ X^{[3,0]}_{[1,2]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{2}}+2\mathfrak{q}^{\frac{3}{2}}+5\mathfrak{q}^{\frac{5}{2}}+9\mathfrak{q}^{\frac{7}{2}}+18\mathfrak{q}^{\frac{9}{2}}+31\mathfrak{q}^{\frac{11}{2}}+55\mathfrak{q}^{\frac{13}{2}}+90\mathfrak{q}^{\frac{15}{2}}+149\mathfrak{q}^{\frac{17}{2}}+\cdots,\\ X^{[1,2]}_{[1,2]}(\mathfrak{q}) &= 1+2\mathfrak{q}+5\mathfrak{q}^2+10\mathfrak{q}^3+20\mathfrak{q}^4+36\mathfrak{q}^5+64\mathfrak{q}^6+108\mathfrak{q}^7+180\mathfrak{q}^8+\cdots,\\ X^{[1,2]}_{[3,0]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{2}}+3\mathfrak{q}^{\frac{3}{2}}+6\mathfrak{q}^{\frac{5}{2}}+13\mathfrak{q}^{\frac{7}{2}}+24\mathfrak{q}^{\frac{9}{2}}+44\mathfrak{q}^{\frac{11}{2}}+76\mathfrak{q}^{\frac{13}{2}}+129\mathfrak{q}^{\frac{15}{2}}\\ &\quad+210\mathfrak{q}^{\frac{17}{2}}+\cdots,\\ X^{[0,3]}_{[0,3]}(\mathfrak{q}) &= X^{[3,0]}_{[3,0]}(\mathfrak{q}),\qquad X^{[0,3]}_{[2,1]}(\mathfrak{q})=X^{[3,0]}_{[1,2]}(\mathfrak{q}),\\ X^{[2,1]}_{[2,1]}(\mathfrak{q}) &= X^{[1,2]}_{[1,2]}(\mathfrak{q}),\qquad X^{[2,1]}_{[0,3]}(\mathfrak{q})=X^{[1,2]}_{[3,0]}(\mathfrak{q}),\end{aligned} \tag{7.32}$$

and

$$f_\sigma(\mathfrak{q},\hat{\mathfrak{t}}) = \sum_{j\in 6\mathbb{Z}+3\sigma} \mathfrak{q}^{\frac{1}{12}j^2}\,\hat{\mathfrak{t}}^j, \quad g_\sigma(\mathfrak{q},\hat{\mathfrak{t}}) = \sum_{j\in 6\mathbb{Z}\pm(2-\sigma)} \mathfrak{q}^{\frac{1}{12}j^2+\frac{1}{6}}\,\hat{\mathfrak{t}}^j, \quad \sigma=0,1. \tag{7.33}$$

The Burge-reduced generating functions (7.31) agree with the $\widehat{\mathfrak{sl}}(2)_3$ WZW characters computed by (A.35),

$$\begin{aligned} &\widehat{X}^{\text{red}}_{[3,0]}(\mathfrak{q},\mathfrak{t}) = \chi^{\widehat{\mathfrak{sl}}(2)_3}_{[3,0]}(\mathfrak{q},\hat{\mathfrak{t}}), \quad \widehat{X}^{\text{red}}_{[0,3]}(\mathfrak{q},\mathfrak{t}) = \chi^{\widehat{\mathfrak{sl}}(2)_3}_{[0,3]}(\mathfrak{q},\hat{\mathfrak{t}}), \\ &\widehat{X}^{\text{red}}_{[2,1]}(\mathfrak{q},\mathfrak{t}) = \mathfrak{q}^{\frac{1}{10}}\,\chi^{\widehat{\mathfrak{sl}}(2)_3}_{[2,1]}(\mathfrak{q},\hat{\mathfrak{t}}), \quad \widehat{X}^{\text{red}}_{[1,2]}(\mathfrak{q},\mathfrak{t}) = \mathfrak{q}^{\frac{1}{10}}\,\chi^{\widehat{\mathfrak{sl}}(2)_3}_{[1,2]}(\mathfrak{q},\hat{\mathfrak{t}}), \end{aligned} \tag{7.34}$$

and Corollary 5.5 is confirmed. Up to an overall factor, the functions (7.32) are the $\widehat{\mathfrak{sl}}(2)$ string functions of level-3 in [46] and given by (*cf.* Corollary 5.2),

$$\begin{aligned} &X^{[3,0]}_{[3,0]}(\mathfrak{q}) - \mathfrak{q}^{\frac{1}{6}}\,X^{[3,0]}_{[1,2]}(\mathfrak{q}) = \frac{\left(\mathfrak{q}^{\frac{2}{3}};\mathfrak{q}^{\frac{5}{3}}\right)_\infty \left(\mathfrak{q};\mathfrak{q}^{\frac{5}{3}}\right)_\infty \left(\mathfrak{q}^{\frac{5}{3}};\mathfrak{q}^{\frac{5}{3}}\right)_\infty}{(\mathfrak{q};\mathfrak{q})^2_\infty}, \\ &X^{[3,0]}_{[1,2]}(\mathfrak{q}) = \mathfrak{q}^{\frac{1}{2}}\,\frac{\left(\mathfrak{q}^3;\mathfrak{q}^{15}\right)_\infty \left(\mathfrak{q}^{12};\mathfrak{q}^{15}\right)_\infty \left(\mathfrak{q}^{15};\mathfrak{q}^{15}\right)_\infty}{(\mathfrak{q};\mathfrak{q})^2_\infty}, \\ &X^{[1,2]}_{[1,2]}(\mathfrak{q}) = \frac{\left(\mathfrak{q}^6;\mathfrak{q}^{15}\right)_\infty \left(\mathfrak{q}^9;\mathfrak{q}^{15}\right)_\infty \left(\mathfrak{q}^{15};\mathfrak{q}^{15}\right)_\infty}{(\mathfrak{q};\mathfrak{q})^2_\infty}, \\ &\mathfrak{q}^{\frac{1}{6}}\,X^{[1,2]}_{[1,2]}(\mathfrak{q}) - X^{[1,2]}_{[3,0]}(\mathfrak{q}) = \mathfrak{q}^{\frac{1}{6}}\,\frac{\left(\mathfrak{q}^{\frac{1}{3}};\mathfrak{q}^{\frac{5}{3}}\right)_\infty \left(\mathfrak{q}^{\frac{4}{3}};\mathfrak{q}^{\frac{5}{3}}\right)_\infty \left(\mathfrak{q}^{\frac{5}{3}};\mathfrak{q}^{\frac{5}{3}}\right)_\infty}{(\mathfrak{q};\mathfrak{q})^2_\infty}. \end{aligned} \tag{7.35}$$

By taking $\mathfrak{t}=1$, the principal characters of $\widehat{\mathfrak{sl}}(2)$ are obtained as in (5.24):

$$\begin{aligned} &\widehat{X}^{\text{red}}_{[3,0]}(\mathfrak{q},1) = \widehat{X}^{\text{red}}_{[0,3]}(\mathfrak{q},1) = \Pr\chi^{\widehat{\mathfrak{sl}}(2)}_{[3,0]}(\mathfrak{q}) = \frac{\left(-\mathfrak{q}^{\frac{1}{2}};\mathfrak{q}^{\frac{1}{2}}\right)_\infty}{\left(\mathfrak{q};\mathfrak{q}^{\frac{5}{2}}\right)_\infty \left(\mathfrak{q}^{\frac{3}{2}};\mathfrak{q}^{\frac{5}{2}}\right)_\infty}, \\ &\widehat{X}^{\text{red}}_{[2,1]}(\mathfrak{q},1) = \widehat{X}^{\text{red}}_{[1,2]}(\mathfrak{q},1) = \Pr\chi^{\widehat{\mathfrak{sl}}(2)}_{[2,1]}(\mathfrak{q}) = \frac{\left(-\mathfrak{q}^{\frac{1}{2}};\mathfrak{q}^{\frac{1}{2}}\right)_\infty}{\left(\mathfrak{q}^{\frac{1}{2}};\mathfrak{q}^{\frac{5}{2}}\right)_\infty \left(\mathfrak{q}^2;\mathfrak{q}^{\frac{5}{2}}\right)_\infty}. \end{aligned} \tag{7.36}$$

Note that, these principal characters are related to the principal characters of $\widehat{\mathfrak{sl}}(3)$ in (7.24) by

$$\frac{\Pr\chi^{\widehat{\mathfrak{sl}}(2)}_{[3,0]}(\mathfrak{q}^2)}{\left(\mathfrak{q}^2;\mathfrak{q}^2\right)_\infty} = \frac{\Pr\chi^{\widehat{\mathfrak{sl}}(3)}_{[2,0,0]}(\mathfrak{q}^3)}{\left(\mathfrak{q}^3;\mathfrak{q}^3\right)_\infty}, \qquad \frac{\Pr\chi^{\widehat{\mathfrak{sl}}(2)}_{[2,1]}(\mathfrak{q}^2)}{\left(\mathfrak{q}^2;\mathfrak{q}^2\right)_\infty} = \frac{\Pr\chi^{\widehat{\mathfrak{sl}}(3)}_{[1,1,0]}(\mathfrak{q}^3)}{\left(\mathfrak{q}^3;\mathfrak{q}^3\right)_\infty}. \tag{7.37}$$

7.3.2. Burge-reduced instanton partition functions

For $N=3$ with general n, the Burge-reduced instanton partition functions (6.10) are determined from the parameters in $\boldsymbol{s}=[s_0,s_1,s_2]$, $\boldsymbol{s}_1=[s_{1,0},s_{1,1},s_{1,2}]$, $\boldsymbol{s}_2=[s_{2,0},1,s_{2,2}]$, $\boldsymbol{s}_3=[s_{3,0},s_{3,1},1]$ and $\boldsymbol{s}_4=[s_{4,0},s_{4,1},s_{4,2}]$ in $P^{++}_{3,n+3}$ that are fixed by the relations (5.1), (6.9):

$$s_I=\sigma_I-\sigma_{I+1}+1, \quad s_{1,I}=b_I-b_{I+1}+1, \quad s_{4,I}=b'_I-b'_{I+1}+1, \quad I=1,2, \tag{7.38}$$

and (6.13) from the ordered charges $\sigma_1\ge\sigma_2\ge\sigma_3$, $b_1\ge b_2\ge b_3$, $b'_1\ge b'_2\ge b'_3$. The Coulomb parameters are then determined from $\boldsymbol{s}$ by (6.5):

$$
\begin{aligned}
a_1 &= \frac{1}{3} \sum_{I=1,2} \Big[I-3 \Big] \Big[s_I - 1 - \frac{n}{3} \Big] \epsilon_2, \\
a_2 &= \frac{1}{3} \sum_{I=1,2} \Big[3-2I \Big] \Big[s_I - 1 - \frac{n}{3} \Big] \epsilon_2, \\
a_3 &= \frac{1}{3} \sum_{I=1,2} I \Big[s_I - 1 - \frac{n}{3} \Big] \epsilon_2,
\end{aligned} \tag{7.39}
$$

and the mass parameters $\boldsymbol{m} = (m_1, \ldots, m_N)$ and $\boldsymbol{m}' = (m'_1, \ldots, m'_N)$ are determined from the parameters in $\boldsymbol{s}_1, \boldsymbol{s}_2$ and $\boldsymbol{s}_3, \boldsymbol{s}_4$, respectively, by (6.6).

We now consider the case of $(N, n) = (3, 2)$ with the rational Ω-background $\epsilon_1/\epsilon_2 = -5/3$ in (6.4).[17]

Example 7.7 *($\emptyset - \emptyset - (\emptyset) - \emptyset - \emptyset$).* Consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(0,0,0),(0,0,0)}_{[3,0];(\ell)}(\mathfrak{q})$ and take $\ell = 0$ in the fundamental chamber, which respects the fusion rules, as in Conjecture 6.5. Here $\boldsymbol{s} = \boldsymbol{s}_1 = \boldsymbol{s}_2 = \boldsymbol{s}_3 = \boldsymbol{s}_4 = [3, 1, 1]$ are fixed. Then we see that the Burge-reduced instanton partition function is

$$
\widehat{\mathcal{Z}}^{(0,0,0),(0,0,0)}_{[3,0];(0)}(\mathfrak{q}) = (1-\mathfrak{q})^{-2h_\emptyset} = 1, \qquad h_\emptyset = 0, \tag{7.40}
$$

and Conjecture 6.5 is confirmed.

Example 7.8 *($\emptyset - \square - (\square) - \square - \emptyset$).* Consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(0,0,0),(0,0,0)}_{[2,1];(\ell)}(\mathfrak{q})$ and take $\ell = 0$ in the fundamental chamber as in Conjecture 6.6, where $\boldsymbol{s} = \boldsymbol{s}_3 = [2, 2, 1]$, $\boldsymbol{s}_1 = \boldsymbol{s}_4 = [3, 1, 1]$ and $\boldsymbol{s}_2 = [2, 1, 2]$ are fixed. Then the Burge-reduced instanton partition function is obtained as

$$
\begin{aligned}
\widehat{\mathcal{Z}}^{(0,0,0),(0,0,0)}_{[2,1];(0)}(\mathfrak{q}) &= (1-\mathfrak{q})^{-2h_\square} \\
&= 1 + \frac{3\mathfrak{q}}{10} + \frac{39\mathfrak{q}^2}{200} + \frac{299\mathfrak{q}^3}{2000} + \frac{9867\mathfrak{q}^4}{80000} + \frac{424281\mathfrak{q}^5}{4000000} + \cdots,
\end{aligned} \tag{7.41}
$$

where $h_\square = 3/20$, and Conjecture 6.6 is confirmed.

Example 7.9 *($\square - \square - (\emptyset) - \square - \square$ and $\square - \square - (\square\square) - \square - \square$).* For Conjecture 6.7, consider, first, the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(1,0,0),(1,0,0)}_{[3,0];(\ell)}(\mathfrak{q})$, where $\boldsymbol{s} = [3, 1, 1]$, $\boldsymbol{s}_1 = \boldsymbol{s}_3 = \boldsymbol{s}_4 = [2, 2, 1]$ and $\boldsymbol{s}_2 = [2, 1, 2]$ are fixed. Then, we find that the Burge-reduced instanton partition functions for $\ell = 0, -1$ in the fundamental chamber are

[17] Examples 7.7, 7.8 and 7.9 are confirmed up to $O(\mathfrak{q}^{\frac{11}{2}})$.

$$\begin{aligned}
\widehat{\mathcal{Z}}^{(1,0,0),(1,0,0)}_{[3,0];(0)}(\mathfrak{q}) &= (1-\mathfrak{q})^{2h_\square-\frac{3}{5}}\,{}_2F_1\left(-\frac{1}{5},\frac{2}{5};\frac{3}{5};\mathfrak{q}\right) \\
&= 1+\frac{\mathfrak{q}}{6}+\frac{13\mathfrak{q}^2}{120}+\frac{87\mathfrak{q}^3}{1040}+\frac{8669\mathfrak{q}^4}{124800}+\frac{344797\mathfrak{q}^5}{5740800}+\cdots, \\
\widehat{\mathcal{Z}}^{(1,0,0),(1,0,0)}_{[3,0];(-1)}(\mathfrak{q}) &= \frac{\mathfrak{q}^{\frac{1}{2}}}{3}(1-\mathfrak{q})^{2h_\square-\frac{3}{5}}\,{}_2F_1\left(\frac{2}{5},\frac{4}{5};\frac{8}{5};\mathfrak{q}\right) \\
&= \frac{\mathfrak{q}^{\frac{1}{2}}}{3}+\frac{\mathfrak{q}^{\frac{3}{2}}}{6}+\frac{61\mathfrak{q}^{\frac{5}{2}}}{520}+\frac{289\mathfrak{q}^{\frac{7}{2}}}{3120}+\frac{222529\mathfrak{q}^{\frac{9}{2}}}{2870400}+\frac{25723\mathfrak{q}^{\frac{11}{2}}}{382720}+\cdots,
\end{aligned} \tag{7.42}$$

where $h_\square = 3/20$, and the second one respects the fusion rules by (6.18). Consider, next, the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(1,0,0),(1,0,0)}_{[1,2];(\ell)}(\mathfrak{q})$, where $\boldsymbol{s}=\boldsymbol{s}_2=[2,1,2]$ and $\boldsymbol{s}_1=\boldsymbol{s}_3=\boldsymbol{s}_4=[2,2,1]$ are fixed. Then we find that the Burge-reduced instanton partition functions for $\ell=0,1$ in the fundamental chamber are

$$\begin{aligned}
\widehat{\mathcal{Z}}^{(1,0,0),(1,0,0)}_{[1,2];(0)}(\mathfrak{q}) &= (1-\mathfrak{q})^{2h_\square-\frac{3}{5}}\,{}_2F_1\left(-\frac{1}{5},\frac{1}{5};\frac{2}{5};\mathfrak{q}\right) \\
&= 1+\frac{\mathfrak{q}}{5}+\frac{183\mathfrak{q}^2}{1400}+\frac{353\mathfrak{q}^3}{3500}+\frac{796073\mathfrak{q}^4}{9520000}+\frac{17182143\mathfrak{q}^5}{238000000}+\cdots, \\
\widehat{\mathcal{Z}}^{(1,0,0),(1,0,0)}_{[1,2];(1)}(\mathfrak{q}) &= \frac{\mathfrak{q}^{\frac{1}{2}}}{2}(1-\mathfrak{q})^{2h_\square-\frac{3}{5}}\,{}_2F_1\left(\frac{1}{5},\frac{4}{5};\frac{7}{5};\mathfrak{q}\right) \\
&= \frac{\mathfrak{q}^{\frac{1}{2}}}{2}+\frac{29\mathfrak{q}^{\frac{3}{2}}}{140}+\frac{393\mathfrak{q}^{\frac{5}{2}}}{2800}+\frac{51949\mathfrak{q}^{\frac{7}{2}}}{476000}+\frac{1725293\mathfrak{q}^{\frac{9}{2}}}{19040000} \\
&\quad+\frac{74432711\mathfrak{q}^{\frac{11}{2}}}{952000000}+\cdots,
\end{aligned} \tag{7.43}$$

where the second one respects the fusion rules by (6.18). The above results (7.42) and (7.43) support Conjecture 6.7.

8. Summary of results and remarks

8.1. Summary of results

The point of this paper is to compute conformal blocks in integral-level WZW models. Starting from the $SU(N)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_n$, with rational Ω-deformation, based on the algebra $\mathcal{A}(N,n;p)$ in (1.1), we proposed (in Conjectures 6.5, 6.6 and 6.7) a way to compute integral-level, integrable $\widehat{\mathfrak{sl}}(n)_N$ WZW conformal blocks, with rational central charges, where one has to deal with the issue of null states. By considering a rational Ω-background $\frac{\epsilon_1}{\epsilon_2}=-1-\frac{n}{N}$ in (6.4) and imposing appropriate Burge conditions in (5.3) to eliminate the null states, we trivialized the coset factor in the algebra $\mathcal{A}(N,n;N)$ as in (1.4), and were left with an integral-level WZW model. Further, we showed, in Corollary 5.5, that the Chern classes (2.8) of the gauge bundle, which labels the instanton partition functions on the gauge side, can be interpreted as the eigenvalues of Chevalley elements in the Cartan subalgebra of $\widehat{\mathfrak{sl}}(n)$ on the CFT side.

8.2. The work of Alday and Tachikawa

In [47], Alday and Tachikawa, using results from [48–51], as well as AGT, found that $SU(2)$ instanton partition functions on $(z_1, z_2) \in \mathbb{C}^2$ with generic Ω-deformation, and in the presence of a *full* surface operator at $z_2 = 0$, agree with $\widehat{\mathfrak{sl}}(2)$ conformal blocks that are modified by a $\mathcal{K}$-operator insertion, at generic-level $k = -2 - \frac{\epsilon_2}{\epsilon_1}$. A generalization to the relation between $SU(N)$ instanton partition functions in the presence of a full surface operator and modified $\widehat{\mathfrak{sl}}(N)$ conformal blocks at generic-level

$$k = -N - \frac{\epsilon_2}{\epsilon_1},$$

was proposed in [52].

In analogy with the moduli space of $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$ without surface operators described in Section 2, to describe the moduli space of $U(N)$ instantons on $\mathbb{C}^2$ in the presence of a full surface operator, one can use the moduli space of $U(N)$ instantons on $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_N)$ [50,53,54]. Unlike the $\widehat{\mathfrak{sl}}(n)_N$ conformal blocks discussed in our work, these conformal blocks are at generic-level, and modified by the $\mathcal{K}$-operator insertion.

8.3. The work of Belavin and Mukhametzhanov

In [14], Belavin and Mukhametzhanov obtained integrable WZW conformal blocks for $(N, n) = (2, 2)$, (see footnote 15). They found that, starting from the $SU(2)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_2$ with generic Ω-deformation, the $\widehat{\mathfrak{sl}}(2)_2$ WZW conformal blocks in Examples 7.1, 7.2 and 7.3 are obtained as prefactors of $\mathcal{N} = 1$ super-Virasoro conformal blocks with generic central charge. In our work, with suitable rational choices of the parameters and by imposing Burge conditions, we trivialized the super-Virasoro conformal blocks (and their higher (N, n) analogues), and computed conformal blocks for rational central charges, for more values of (N, n). We conjecture that our approach works, for rational central charges, for all (N, n), $N, n \in \mathbb{Z}_{\geq 1}$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Lie algebras, affine Lie algebras and notation

Here we describe the notation we use from the structure and representation theories of finite dimensional and affine Lie algebras, as it pertains to $\mathfrak{sl}(M)$ and $\widehat{\mathfrak{sl}}(M)$. For a more comprehensive treatment, see [55].

A.1. The finite dimensional Lie algebra $\mathfrak{sl}(M)$

Define the index set $\overline{\mathcal{I}}_M = \{1, 2, \ldots, M-1\}$. The Lie algebra $\mathfrak{sl}(M)$ has Chevalley generators $\{H_i, E_i, F_i \mid i \in \overline{\mathcal{I}}_M\}$ with $\{H_i \mid i \in \overline{\mathcal{I}}_M\}$ a basis for its Cartan subalgebra $\overline{\mathfrak{h}}$. The Cartan matrix $\overline{A}$ of $\mathfrak{sl}(M)$ is the $(M-1) \times (M-1)$ matrix having entries $\overline{A}_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}$ for $i, j \in \overline{\mathcal{I}}_M$. The dual $\overline{\mathfrak{h}}^*$ of $\overline{\mathfrak{h}}$ has basis $\{\alpha_j \mid j \in \overline{\mathcal{I}}_M\}$ where the simple root α_j is defined by $\alpha_j(H_i) = \overline{A}_{ij}$ for $i, j \in \overline{\mathcal{I}}_M$. For $i \in \overline{\mathcal{I}}_M$, the fundamental weight $\overline{\Lambda}_i \in \overline{\mathfrak{h}}^*$ is uniquely defined by $\overline{\Lambda}_i(H_j) = \delta_{ij}$ for $j \in \overline{\mathcal{I}}_M$. It follows that $\overline{\Lambda}_i = \sum_{j \in \overline{\mathcal{I}}_M} (\overline{A}^{-1})_{ij}\, \alpha_j$, where $\overline{A}^{-1}$, the inverse of $\overline{A}$, has entries

$$\left(\overline{A}^{-1}\right)_{ij} = \min\{i, j\} - \frac{ij}{M} \tag{A.1}$$

for $i, j \in \overline{\mathcal{I}}_M$. Note that $\{\overline{\Lambda}_1, \overline{\Lambda}_2, \ldots, \overline{\Lambda}_{M-1}\}$ is also a basis for $\overline{\mathfrak{h}}^*$. The Weyl vector $\overline{\rho}$ is defined by $\overline{\rho} = \sum_{i \in \overline{\mathcal{I}}_M} \overline{\Lambda}_i$.

It is convenient to embed $\overline{\mathfrak{h}}^*$ in $\mathbb{C}^M$, by choosing an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_M\}$ for $\mathbb{C}^M$, and setting $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i \in \overline{\mathcal{I}}_M$. The standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^M$ then leads to

$$\langle \alpha_i, \alpha_j \rangle = \overline{A}_{ij}, \qquad \langle \alpha_i, \overline{\Lambda}_j \rangle = \delta_{ij}, \qquad \langle \overline{\Lambda}_i, \overline{\Lambda}_j \rangle = \left(\overline{A}^{-1}\right)_{ij} \tag{A.2}$$

for $i, j \in \overline{\mathcal{I}}_M$. Note that $\overline{\mathfrak{h}}^*$ is the $(M-1)$-dimensional subspace of $\mathbb{C}^M$ that is perpendicular to $\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_M$. For convenience, we set $\mathbf{e}_0 = \frac{1}{M}(\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_M)$. It is then easily confirmed that

$$\overline{\Lambda}_i = \sum_{k=1}^{i} \mathbf{e}_k - i\, \mathbf{e}_0 \qquad \text{and} \qquad \overline{\rho} = \frac{1}{2} \sum_{k=1}^{M} (M - 2k + 1)\, \mathbf{e}_k\,. \tag{A.3}$$

A.2. The affine Lie algebra $\widehat{\mathfrak{sl}}(M)$

Define the index set $\mathcal{I}_M = \{0, 1, \ldots, M-1\}$. The affine Lie algebra $\widehat{\mathfrak{sl}}(M)$ has Chevalley generators $\{D, H_i, E_i, F_i \mid i \in \mathcal{I}_M\}$ with $\{D, H_i \mid i \in \mathcal{I}_M\}$ a basis for its Cartan subalgebra $\mathfrak{h}$ (which is $(M+1)$-dimensional). The element $C = \sum_{i \in \mathcal{I}_M} H_i$ is central in $\widehat{\mathfrak{sl}}(M)$. The Cartan matrix A of $\widehat{\mathfrak{sl}}(M)$ is the $M \times M$ matrix having entries $A_{ij} = 2\delta^{(M)}_{ij} - \delta^{(M)}_{i,j+1} - \delta^{(M)}_{i,j-1}$ for $i, j \in \mathcal{I}_M$, where $\delta^{(M)}_{ij} = 1$ if $i \equiv j \pmod M$ and $\delta^{(M)}_{ij} = 0$ otherwise. The dual $\mathfrak{h}^*$ of $\mathfrak{h}$ has basis $\{\Lambda_0, \alpha_j \mid j \in \mathcal{I}_M\}$, where the simple root α_j is defined by $\alpha_j(H_i) = A_{ij}$ for $i, j \in \mathcal{I}_M$ and $\alpha_j(D) = \delta_{j0}$, and the fundamental weight Λ_0 is defined by $\Lambda_0(H_i) = \delta_{i0}$ for $i \in \mathcal{I}_M$ and $\Lambda_0(D) = 0$. For $j \in \overline{\mathcal{I}}_M$ we also define $\Lambda_j \in \mathfrak{h}^*$ by setting $\Lambda_j(H_i) = \delta_{ij}$ for $i \in \mathcal{I}_M$, and $\Lambda_j(D) = 0$. The Weyl vector ρ is defined by $\rho = \sum_{i \in \mathcal{I}_M} \Lambda_i$. The null root δ is defined by $\delta = \sum_{i \in \mathcal{I}_M} \alpha_i$, and is such that $\delta(H_i) = 0$ for $i \in \mathcal{I}_M$ and $\delta(D) = 1$. Note that $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_{M-1}, \delta\}$ is also a basis for $\mathfrak{h}^*$.

Because $\mathfrak{sl}(M)$ appears canonically as a subalgebra of $\widehat{\mathfrak{sl}}(M)$, we may identify the α_i in the two cases for $i \in \overline{\mathcal{I}}_M$. In addition, $\Lambda_i = \overline{\Lambda}_i + \Lambda_0$ for $i \in \overline{\mathcal{I}}_M$. Then $\rho = \overline{\rho} + M\Lambda_0$. We again use the set of orthonormal vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_M$, supplementing it with Λ_0 and δ to give a basis for $\mathfrak{h}^* \cup \mathbb{C}\mathbf{e}_0$. In terms of these,

$$\alpha_0 = \mathbf{e}_M - \mathbf{e}_1 + \delta \qquad \text{and} \qquad \alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1} \tag{A.4}$$

for $i \in \overline{\mathcal{I}}_M$. The inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^* \cup \mathbb{C}\mathbf{e}_0$ is defined by setting

$$\langle \delta, \Lambda_0 \rangle = 1, \qquad \langle \delta, \delta \rangle = \langle \Lambda_0, \Lambda_0 \rangle = \langle \delta, \mathbf{e}_i \rangle = \langle \Lambda_0, \mathbf{e}_i \rangle = 0\,, \tag{A.5}$$

in addition to $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$, for $i, j \in \overline{\mathcal{I}}_M$. Then

$$\langle \alpha_i, \alpha_j \rangle = A_{ij}, \qquad \langle \alpha_i, \Lambda_j \rangle = \delta_{ij}, \qquad \langle \delta, \Lambda_j \rangle = 1, \qquad \langle \Lambda_i, \Lambda_j \rangle = \min\{i, j\} - \frac{ij}{M} \tag{A.6}$$

for $i, j \in \mathcal{I}_M$. It follows that for each $\beta \in \mathfrak{h}^*$, we have $\beta(D) = \langle \beta, \Lambda_0 \rangle$ and $\beta(H_i) = \langle \beta, \alpha_i \rangle$ for $i \in \mathcal{I}_M$, as is easily checked.

The $\widehat{\mathfrak{sl}}(M)$ weight lattice P_M, the level-m weight lattice $P_{M,m}$, the dominant weight lattice P_M^+, the level-m dominant weight lattice $P_{M,m}^+$, the regular dominant weight lattice P_M^{++}, and the level-m regular dominant weight lattice $P_{M,m}^{++}$, are defined by[18]

$$\begin{aligned} P_M &= \bigoplus_{i \in \mathcal{I}_M} \mathbb{Z}\Lambda_i\,, & P_{M,m} &= \{\Lambda \in P_M \mid \langle \delta, \Lambda \rangle = m\}\,, \\ P_M^+ &= \bigoplus_{i \in \mathcal{I}_M} \mathbb{Z}_{\geq 0}\Lambda_i\,, & P_{M,m}^+ &= P_M^+ \cap P_{M,m}\,, \\ P_M^{++} &= \bigoplus_{i \in \mathcal{I}_M} \mathbb{Z}_{>0}\Lambda_i\,, & P_{M,m}^{++} &= P_M^{++} \cap P_{M,m}\,. \end{aligned} \tag{A.7}$$

We will often use $[d_0, d_1, \ldots, d_{M-1}]$ to denote the element $\sum_{i \in \mathcal{I}_M} d_i \Lambda_i$ from any of these sets. Note that if $[d_0, d_1, \ldots, d_{M-1}] \in P_{M,m}$ then $\sum_{i \in \mathcal{I}_M} d_i = m$.

A.3. Affine weights and partitions

For $\Lambda = [d_0, d_1, \ldots, d_{M-1}] \in P_M^+$, it is convenient to define a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ by setting

$$\lambda_i = \begin{cases} \sum_{j=i}^{M-1} d_j & \text{if } 1 \leq i < M, \\ 0 & \text{if } i \geq M\,. \end{cases} \tag{A.8}$$

We will denote the partition so obtained by $\mathrm{par}(\Lambda)$. Note that if $\Lambda \in P_{M,m}^+$ and $\lambda = \mathrm{par}(\Lambda)$ then $\lambda_1 \leq m$ and $\lambda_M = 0$. On the other hand, given a partition λ with $\lambda_1 \leq m$ and $\lambda_M = 0$, there is a unique $\Lambda \in P_{M,m}^+$ such that $\lambda = \mathrm{par}(\Lambda)$. We then write $\Lambda = \mathrm{par}^{-1}(\lambda)$. Note that par^{-1} is well-defined only if m is specified.

Lemma A.1. *Consider a partition $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots)$ for which $\sigma_1 < M$ and $\sigma_{m+1} = 0$, and define $\Lambda \in P_{M,m}^+$ by $\Lambda = \sum_{i=1}^m \Lambda_{\sigma_i}$. If $\lambda = \mathrm{par}(\Lambda)$ then $\lambda = \boldsymbol{\sigma}^T$, the partition conjugate to $\boldsymbol{\sigma}$.*

Proof. This follows after noting that in frequency notation $\boldsymbol{\sigma}$ is expressed $((M-1)^{d_{M-1}}, \ldots, 2^{d_2}, 1^{d_1})$. $\square$

A.4. Representations and characters of $\widehat{\mathfrak{sl}}(M)$

For $\Lambda \in P_M^+$, let $L(\Lambda)$ denote the highest weight $\widehat{\mathfrak{sl}}(M)$-module whose highest weight vector v_Λ is such that $H(v_\Lambda) = \Lambda(H) v_\Lambda$ for all $H \in \mathfrak{h}$. Then, with $\Lambda = [d_0, d_1, \ldots, d_{M-1}]$, we have

[18] Usually, the set $\mathbb{C}\delta$ is adjoined to these sets. The theory then proceeds with little change.

$\Lambda(H_i) = d_i \in \mathbb{Z}_{\geq 0}$ for $i \in \mathcal{I}_M$ and $\Lambda(D) = 0$. If $\Lambda \in P^+_{M,m}$ then the module $L(\Lambda)$ is said to be of level m. The formal character of $L(\Lambda)$ is defined to be

$$\mathrm{ch}\, L(\Lambda) = \sum_{\beta \in \mathfrak{h}^*} (\dim V_\beta)\, \mathrm{e}^\beta, \tag{A.9}$$

where V_β is the subspace of $L(\Lambda)$ for which, for each $v \in V_\beta$, we have $H(v) = \beta(H)v$ for all $H \in \mathfrak{h}$. Instead of using the formal exponentials e^β, it is convenient to set $\mathrm{e}^{-\delta} = \mathfrak{q}$ and $\mathrm{e}^{-\mathbf{e}_i} = x_i$ for $1 \leq i \leq M$, and define

$$\bar{\chi}_\Lambda^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q}, \boldsymbol{x}) = \mathrm{ch}\, L(\Lambda)\Big|_{\{\mathrm{e}^{-\delta}=\mathfrak{q}, \mathrm{e}^{-\mathbf{e}_i}=x_i \,|\, 1\leq i\leq M\}}. \tag{A.10}$$

For $\Lambda \in P^+_M$, the Weyl-Kac character formula ([55, eqn. (10.4.5)]) yields

$$\bar{\chi}_\Lambda^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q}, \boldsymbol{x}) = \mathrm{e}^\Lambda \frac{\mathcal{N}_\Lambda^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q}, \boldsymbol{x})}{\mathcal{N}_0^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q}, \boldsymbol{x})} \tag{A.11}$$

with $\mathcal{N}_\Lambda^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q}, \boldsymbol{x})$ given by

$$\mathcal{N}_\Lambda^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q}, \boldsymbol{x}) = \sum_{\substack{k_1,\ldots,k_M \in \mathbb{Z} \\ k_1+\cdots+k_M=0}} \det_{1\leq i,j\leq M} \left(x_i^{-(M+m)k_i - \lambda_j + j + \lambda_i - i}\, \mathfrak{q}^{(\lambda_j - j)k_i + \frac{1}{2}(M+m)k_i^2} \right), \tag{A.12}$$

where $\lambda = \mathrm{par}(\Lambda)$. Using the $\widehat{\mathfrak{sl}}(M)$ Macdonald identity [55, eqn. (10.4.4)], the denominator of (A.11) may be alternatively expressed:

$$\mathcal{N}_0^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q}, \boldsymbol{x}) = (\mathfrak{q}; \mathfrak{q})_\infty^{M-1} \prod_{1\leq i<j\leq M} \left(\frac{x_i}{x_j}; \mathfrak{q} \right)_\infty \left(\frac{x_j}{x_i}\mathfrak{q}; \mathfrak{q} \right)_\infty. \tag{A.13}$$

(For a more detailed derivation of (A.12) and (A.13), see [41, Appendix B.2]).

In the case in which Λ is of level one, so that $\Lambda = \Lambda_k$ for $k \in \mathcal{I}_M$, the character $\mathrm{ch}\, L(\Lambda)$ has the explicit expression [55, eqn. (12.13.6)]

$$\mathrm{ch}\, L(\Lambda_k) = \frac{\mathrm{e}^{\Lambda_k}}{(\mathfrak{q}; \mathfrak{q})_\infty^{M-1}} \sum_{\eta \in \bigoplus_{i=1}^{M-1} \mathbb{Z}\alpha_i} \mathrm{e}^{\eta + \frac{1}{2}\langle \Lambda_k, \Lambda_k\rangle\delta - \frac{1}{2}\langle \eta + \overline{\Lambda}_k, \eta + \overline{\Lambda}_k\rangle\delta}. \tag{A.14}$$

Then, use of the inner products in Appendix A.2, and noting that $\langle \eta + \overline{\Lambda}_k, \eta + \overline{\Lambda}_k\rangle = \langle \eta + \Lambda_k, \eta + \Lambda_k\rangle = \langle \eta, \eta\rangle + 2\langle \eta, \Lambda_k\rangle + \langle \Lambda_k, \Lambda_k\rangle$, leads to

$$\bar{\chi}_{\Lambda_k}^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q}, \boldsymbol{x}) = \frac{\mathrm{e}^{\Lambda_k}}{(\mathfrak{q}; \mathfrak{q})_\infty^{M-1}} \sum_{\boldsymbol{\ell} \in \mathbb{Z}^{M-1}} \mathfrak{q}^{\ell_k + \sum_{i=1}^{M-1}(\ell_i^2 - \ell_i \ell_{i-1})} \prod_{i=1}^{M-1} \left(\frac{x_{i+1}}{x_i} \right)^{\ell_i}, \tag{A.15}$$

where $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_{M-1})$ with $\ell_0 = 0$.

It will be useful to note that $\bar{\chi}_\Lambda^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q}, \boldsymbol{x})$ is invariant on multiplying each of the x_i by the same non-zero constant. This is obvious in (A.15), and consideration of the determinant in (A.12) shows that it holds also for both the numerator and denominator of (A.11).

A.5. Principally specialised characters of $\widehat{\mathfrak{sl}}(M)$

Although the denominator $\mathcal{N}_0^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q},\boldsymbol{x})$ of (A.11) can be written in product form, this is not the case with the numerator $\mathcal{N}_\Lambda^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q},\boldsymbol{x})$. However, after substituting $\mathrm{e}^{-\alpha_i} \to \mathfrak{q}^{1/M}$ for each simple root α_i, $\mathcal{N}_\Lambda^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q},\boldsymbol{x})$ can be written in product form. The same is then true for $\bar{\chi}_\Lambda^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q},\boldsymbol{x})$ [55, Proposition 10.9]. This specialisation is effected by substituting $x_i \to \mathfrak{q}^{-i/M}$ into (A.12). So define the principally specialised character

$$\operatorname{Pr}\chi_\Lambda^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q}) = e^{-\Lambda}\,\bar{\chi}_\Lambda^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q},\boldsymbol{x})\Big|_{\{x_i\to\mathfrak{q}^{-i/M},1\le i\le M\}}\,. \tag{A.16}$$

The result can be conveniently expressed by, for $\Lambda \in P^+_{M,m}$, setting $\lambda = \operatorname{par}(\Lambda)$ and defining the set $\Omega(\Lambda) = \{m + j - \lambda_j \mid j = 1,\ldots,M\}$. Then

$$\begin{aligned}\operatorname{Pr}\chi_\Lambda^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q}) =& \frac{(\mathfrak{q};\mathfrak{q})_\infty}{\left(\mathfrak{q}^{1+\frac{m}{M}};\mathfrak{q}^{1+\frac{m}{M}}\right)_\infty} \prod_{\substack{1\le i<j\le M+m\\ i\notin\Omega(\Lambda),\,j\in\Omega(\Lambda)}} \frac{1}{\left(\mathfrak{q}^{\frac{j-i}{M}};\mathfrak{q}^{1+\frac{m}{M}}\right)_\infty}\\ &\times \prod_{\substack{1\le i<j\le M+m\\ i\in\Omega(\Lambda),\,j\notin\Omega(\Lambda)}} \frac{1}{\left(\mathfrak{q}^{1+\frac{m-j+i}{M}};\mathfrak{q}^{1+\frac{m}{M}}\right)_\infty}\,.\end{aligned} \tag{A.17}$$

(See [41] for more details: the substitution $q \to \mathfrak{q}^{1/M}$ there gives the normalisation used here.)

A.6. Characters of $\widehat{\mathfrak{sl}}(M)$ *via crystal graphs*

Here we describe the crystal graph enumeration of characters of $\widehat{\mathfrak{sl}}(M)$ that was developed by the Kyoto group [56,27,57]. The formulation that we use is similar to that in [58, Section 2].

For $\Lambda = [d_0, d_1, \ldots, d_{M-1}] \in P^+_{M,m}$, let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots)$ be the partition for which $\sigma_{m+1} = 0$ and $\Lambda = \sum_{i=1}^m \Lambda_{\sigma_i}$.[19] Let $\mathcal{M}^{\boldsymbol{\sigma}}$ be the set of m-tuples of coloured Young diagrams $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$ whose row lengths $Y_{\ell,i}$ are constrained by

$$\begin{aligned} &Y_{\ell,i} \ge Y_{\ell+1,i+\sigma_\ell-\sigma_{\ell+1}} && \text{for } i\ge 1,\ 1\le \ell < m;\\ &Y_{m,i} \ge Y_{1,i+\sigma_m-\sigma_1+M} && \text{for } i \ge 1,\end{aligned} \tag{A.18}$$

and where the box (i, j) of Y_ℓ is coloured $(\sigma_\ell + i - j) \bmod M$.[20] The elements of $\mathcal{M}^{\boldsymbol{\sigma}}$ are called cylindrical multipartitions in [58]. For $\boldsymbol{Y} \in \mathcal{M}^{\boldsymbol{\sigma}}$ and $i \in \mathcal{I}_M$, define $k_i(\boldsymbol{Y})$ to be the number of boxes in $\boldsymbol{Y}$ that are coloured i, and then set $\delta k_i(\boldsymbol{Y}) = k_i(\boldsymbol{Y}) - k_0(\boldsymbol{Y})$.

The crystal graph theory shows that the character $\operatorname{ch} L(\Lambda)$ of the irreducible $\widehat{\mathfrak{sl}}(M)$ representation with highest weight Λ can be expressed as a sum over a subset $\mathcal{M}^{\boldsymbol{\sigma}}_*$ of $\mathcal{M}^{\boldsymbol{\sigma}}$, the elements of $\mathcal{M}^{\boldsymbol{\sigma}}_*$ being known as highest-lift multipartitions. We will not define $\mathcal{M}^{\boldsymbol{\sigma}}_*$ here (the definition can be found in [58, Proposition 2.11]), because we will just use and state its pivotal property. Using, as before, $\mathrm{e}^{-\delta} = \mathfrak{q}$ and $\mathrm{e}^{-\mathbf{e}_i} = x_i$ for $i \in \overline{\mathcal{I}}_M$, this expression is [27, Theorem 1.2]:

[19] By Lemma A.1, if $\lambda = \operatorname{par}(\Lambda)$ then $\boldsymbol{\sigma} = \lambda^T$.

[20] Note that this colouring convention differs from that defined in Section 2.1. It also differs from that used in [58], but is appropriate to the conventions used here and in [41].

$$\begin{aligned}\bar{\chi}_{\Lambda}^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q},\boldsymbol{x}) &= \mathrm{e}^{\Lambda} \sum_{\boldsymbol{Y}\in\mathcal{M}_*^{\boldsymbol{\sigma}}} \mathfrak{q}^{k_0(\boldsymbol{Y})} \prod_{i=0}^{M-1} \left(\frac{x_i}{x_{i+1}}\right)^{k_i(\boldsymbol{Y})} \\ &= \mathrm{e}^{\Lambda} \sum_{\boldsymbol{Y}\in\mathcal{M}_*^{\boldsymbol{\sigma}}} \mathfrak{q}^{k_0(\boldsymbol{Y})} \prod_{i=1}^{M-1} \left(\frac{x_i}{x_{i+1}}\right)^{\delta k_i(\boldsymbol{Y})},\end{aligned} \tag{A.19}$$

the second equality following on using $k_i(\boldsymbol{Y}) = \delta k_i(\boldsymbol{Y}) + k_0(\boldsymbol{Y})$ for $i \in \mathcal{I}_M$, and noting that, in particular, $\delta k_0(\boldsymbol{Y}) = 0$.

The property of $\mathcal{M}_*^{\boldsymbol{\sigma}}$ that is needed is that there is a natural bijection[21]

$$\mathcal{M}^{\boldsymbol{\sigma}} \to \mathcal{M}_*^{\boldsymbol{\sigma}} \times Par, \tag{A.20}$$

where Par is the set of all partitions, such that if $\boldsymbol{Y} \mapsto (\boldsymbol{Y}^*, \lambda)$ then $\delta k_i(\boldsymbol{Y}) = \delta k_i(\boldsymbol{Y}^*)$ for $i \in \mathcal{I}_M$, and $k_0(\boldsymbol{Y}) = k_0(\boldsymbol{Y}^*) + |\lambda|$. Because the generating function for Par is

$$\sum_{\lambda\in Par} \mathfrak{q}^{|\lambda|} = \frac{1}{(\mathfrak{q};\mathfrak{q})_\infty}, \tag{A.21}$$

the expression (A.19) yields

$$\bar{\chi}_{\Lambda}^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q},\boldsymbol{x}) = \mathrm{e}^{\Lambda}\,(\mathfrak{q};\mathfrak{q})_\infty \sum_{\boldsymbol{Y}\in\mathcal{M}^{\boldsymbol{\sigma}}} \mathfrak{q}^{k_0(\boldsymbol{Y})} \prod_{i=1}^{M-1} \left(\frac{x_i}{x_{i+1}}\right)^{\delta k_i(\boldsymbol{Y})}. \tag{A.22}$$

Now for a vector $\boldsymbol{\ell} = (\ell_1, \ell_2, \ldots, \ell_{M-1}) \in \mathbb{Z}^{M-1}$, define $\mathcal{M}^{\boldsymbol{\sigma},\boldsymbol{\ell}} \subset \mathcal{M}^{\boldsymbol{\sigma}}$ to be the set of all $\boldsymbol{Y} \in \mathcal{M}^{\boldsymbol{\sigma}}$ for which $\delta k_i(\boldsymbol{Y}) = \ell_i$ for each $i \in \overline{\mathcal{I}}_M$. Also set $\ell_0 = \ell_M = 0$ for convenience. We can then write (A.22) in the form

$$\begin{aligned}\bar{\chi}_{\Lambda}^{\widehat{\mathfrak{sl}}(M)}(\mathfrak{q},\boldsymbol{x}) &= \mathrm{e}^{\Lambda}\,(\mathfrak{q};\mathfrak{q})_\infty \sum_{\boldsymbol{\ell}\in\mathbb{Z}^{M-1}} \sum_{\boldsymbol{Y}\in\mathcal{M}^{\boldsymbol{\sigma},\boldsymbol{\ell}}} \mathfrak{q}^{k_0(\boldsymbol{Y})} \prod_{i=1}^{M-1} \left(\frac{x_i}{x_{i+1}}\right)^{\ell_i} \\ &= \mathrm{e}^{\Lambda} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^{M-1}} \bar{a}_{\boldsymbol{\ell}}^{\Lambda}(\mathfrak{q}) \prod_{i=1}^{M} x_i^{\ell_i-\ell_{i-1}},\end{aligned} \tag{A.23}$$

where the $\widehat{\mathfrak{sl}}(M)$ (normalised) string function $\bar{a}_{\boldsymbol{\ell}}^{\Lambda}(\mathfrak{q})$ is given by

$$\bar{a}_{\boldsymbol{\ell}}^{\Lambda}(\mathfrak{q}) = (\mathfrak{q};\mathfrak{q})_\infty \sum_{\boldsymbol{Y}\in\mathcal{M}^{\boldsymbol{\sigma},\boldsymbol{\ell}}} \mathfrak{q}^{k_0(\boldsymbol{Y})}. \tag{A.24}$$

Tabulations of the coefficients of the string functions $\bar{a}_{\boldsymbol{\ell}}^{\Lambda}(\mathfrak{q})$ in the cases $2 \le M \le 9$ for weights $\Lambda \in P_{M,m}^{+}$ of various small levels m can be found in [59].[22]

A.7. WZW characters

For any affine Lie algebra $\mathfrak{g}$, the Sugawara construction demonstrates a homomorphism from the Virasoro algebra Vir to $U_c(\mathfrak{g})$, a completion of the universal enveloping algebra of $\mathfrak{g}$, for any

[21] Obtained using an abacus with M rungs — see [41].

[22] This string function $\bar{a}_{\boldsymbol{\ell}}^{\Lambda}(\mathfrak{q})$ is usually denoted as $a_{\gamma}^{\Lambda}(\mathfrak{q})$ where $\gamma = \Lambda - \sum_{i=1}^{M-1} \ell_i \alpha_i$ (see Appendix A.7, especially (A.30)).

level $m \geq 0$ (see [55, §12.8] for details). Consequently, for $\Lambda \in P^+_{M,m}$, the $\widehat{\mathfrak{sl}}(M)$-module $L(\Lambda)$ also serves as a *Vir*-module. The central charge c and conformal dimension h_Λ of this *Vir*-module are given in (5.9).

Through the homomorphism $Vir \to U_c(\widehat{\mathfrak{sl}}(M))$, the Virasoro generator L_0 acts on $L(\Lambda)$ by $L_0 \mapsto h_\Lambda \mathrm{Id} - D$, where Id is the identity operator [55, Corollary 12.8]. Consequently, the definition (5.10) yields

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(M)_m}(\mathfrak{q}, \hat{\mathfrak{t}}) = \mathfrak{q}^{h_\Lambda} \operatorname{Tr}_{L(\Lambda)} \mathfrak{q}^{-D} \prod_{i=1}^{M-1} \hat{\mathfrak{t}}_i^{H_i} . \tag{A.25}$$

Now note that (A.23) can be written as a sum of terms $\exp(\beta)$ with β of the form

$$\beta = \Lambda - k\delta - \sum_{j=1}^{M} \mathrm{e}_j(\ell_j - \ell_{j-1}) = \Lambda - k\delta - \sum_{j=1}^{M} \ell_j \alpha_j \tag{A.26}$$

for some $k \in \mathbb{Z}$. Then, because $\beta(D) = -k$ and $\beta(H_i) = d_i + \ell_{i-1} - 2\ell_i + \ell_{i+1}$, (A.25) yields:

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(M)_m}(\mathfrak{q}, \hat{\mathfrak{t}}) = \mathfrak{q}^{h_\Lambda} \sum_{\boldsymbol{\ell} \in \mathbb{Z}^{M-1}} \bar{a}_{\boldsymbol{\ell}}^\Lambda(\mathfrak{q}) \prod_{i=1}^{M-1} \hat{\mathfrak{t}}_i^{d_i + \ell_{i-1} - 2\ell_i + \ell_{i+1}} . \tag{A.27}$$

Alternatively, this may be expressed as

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(M)_m}(\mathfrak{q}, \hat{\mathfrak{t}}) = \mathfrak{q}^{h_\Lambda} \sum_{\boldsymbol{\ell} \in \mathbb{Z}^{M-1}} a_{\gamma(\boldsymbol{\ell})}^\Lambda(\mathfrak{q}) \prod_{i=1}^{M-1} \hat{\mathfrak{t}}_i^{\gamma(\boldsymbol{\ell})_i} , \tag{A.28}$$

after defining $\gamma(\boldsymbol{\ell}) = [\gamma_0, \gamma_1, \ldots, \gamma_{M-1}] \in P_{M,m}$ by setting

$$\gamma_i = d_i + \ell_{i-1} - 2\ell_i + \ell_{i+1} = d_i - \sum_{j=1}^{M-1} A_{ij} \ell_j \tag{A.29}$$

for each $i \in \mathcal{I}_M$, and defining

$$a_{\gamma(\boldsymbol{\ell})}^\Lambda(\mathfrak{q}) = \bar{a}_{\boldsymbol{\ell}}^\Lambda(\mathfrak{q}) . \tag{A.30}$$

By using (A.24), we can also express (A.27) as

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(M)_m}(\mathfrak{q}, \hat{\mathfrak{t}}) = \mathfrak{q}^{h_\Lambda} (\mathfrak{q}; \mathfrak{q})_\infty \sum_{\boldsymbol{Y} \in \mathcal{M}^\sigma} \mathfrak{q}^{k_0(\boldsymbol{Y})} \prod_{i=1}^{M-1} \hat{\mathfrak{t}}_i^{d_i + \delta k_{i-1}(\boldsymbol{Y}) - 2\delta k_i(\boldsymbol{Y}) + \delta k_{i+1}(\boldsymbol{Y})} , \tag{A.31}$$

where we set $\delta k_0(\boldsymbol{Y}) = \delta k_M(\boldsymbol{Y}) = 0$.

In the level one case where $\Lambda = \Lambda_k$, comparing (A.15) with (A.23) shows that

$$\bar{a}_{\boldsymbol{\ell}}^{\Lambda_k}(\mathfrak{q}) = \frac{1}{(\mathfrak{q}; \mathfrak{q})_\infty^{M-1}} \mathfrak{q}^{-\ell_k + \sum_{i=1}^{M-1} (\ell_i^2 - \ell_i \ell_{i-1})} , \tag{A.32}$$

where $\ell_0 = 0$. Then (A.27) gives

$$\chi_{\Lambda_k}^{\widehat{\mathfrak{sl}}(M)_1}(\mathfrak{q}, \hat{\mathfrak{t}}) = \frac{\mathfrak{q}^{h_{\Lambda_k}}}{(\mathfrak{q}; \mathfrak{q})_\infty^{M-1}} \sum_{\boldsymbol{\ell} \in \mathbb{Z}^{M-1}} \mathfrak{q}^{-\ell_k + \sum_{i=1}^{M-1} (\ell_i^2 - \ell_i \ell_{i-1})} \prod_{i=1}^{M-1} \hat{\mathfrak{t}}_i^{\delta_{ik} + \ell_{i-1} - 2\ell_i + \ell_{i+1}} , \tag{A.33}$$

where $\ell_0 = \ell_M = 0$.

A.8. *Converting between the* $\mathbf{x}$ *and* $\hat{\mathfrak{t}}$ *variables*

In Appendix A.4, in expressing the character of $L(\Lambda)$, the formal exponentials e^{β} were exchanged for $\mathrm{e}^{-\delta} = \mathfrak{q}$ and $\mathrm{e}^{-\mathbf{e}_i} = x_i$ for $1 \leq i \leq M$. However, in Appendix A.7, the same character was expressed using $\mathfrak{q}$ along with $\hat{\mathfrak{t}}_1, \ldots, \hat{\mathfrak{t}}_{M-1}$. Here, we convert between these variables, and give a form of the Weyl-Kac formula that expresses $\chi_{\Lambda}^{\widehat{\mathfrak{sl}}(M)_m}(\mathfrak{q}, \hat{\mathfrak{t}})$ in the $\hat{\mathfrak{t}}$ variables.

In terms of Dynkin components, (A.26) takes the form $\beta = -k\delta + \sum_{i \in \mathcal{I}_M} (d_i + \ell_{i-1} - 2\ell_i + \ell_{i+1})\Lambda_i$. Therefore (see (A.27)), $\chi_{\Lambda}^{\widehat{\mathfrak{sl}}(M)_m}(\mathfrak{q}, \hat{\mathfrak{t}})$ is obtained from $\mathfrak{q}^{h_\Lambda} L(\Lambda)$ by exchanging $\mathrm{e}^{-\delta} = \mathfrak{q}$, $e^{\Lambda_0} = 1$ and $\mathrm{e}^{\Lambda_i} = \hat{\mathfrak{t}}_i$ for $1 \leq i < M$.

Because $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ and $\alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}$, the $\boldsymbol{x}$ and $\hat{\mathfrak{t}}$ variables are related by

$$\frac{x_i}{x_{i+1}} = \frac{\hat{\mathfrak{t}}_{i-1}\hat{\mathfrak{t}}_{i+1}}{\hat{\mathfrak{t}}_i^2} \quad \Longleftrightarrow \quad x_i = \frac{\hat{\mathfrak{t}}_{i-1}}{\hat{\mathfrak{t}}_i}\frac{\hat{\mathfrak{t}}_M}{\hat{\mathfrak{t}}_{M-1}} x_M \tag{A.34}$$

for $1 \leq i < M$, where we set $\hat{\mathfrak{t}}_0 = \hat{\mathfrak{t}}_M = 1$. In view of the last paragraph of Appendix A.4, we then obtain $\mathfrak{q}^{-h_\Lambda}\chi_{\Lambda}^{\widehat{\mathfrak{sl}}(M)_m}(\mathfrak{q}, \hat{\mathfrak{t}})$ by substituting $x_i \to \hat{\mathfrak{t}}_{i-1}/\hat{\mathfrak{t}}_i$ into (A.11). For $\Lambda = [d_0, d_1, \ldots, d_{M-1}] \in P^{+}_{M,m}$ this gives, after also making use of (A.12) and (A.13),

$$\begin{aligned}\chi_{\Lambda}^{\widehat{\mathfrak{sl}}(M)_m}(\mathfrak{q}, \hat{\mathfrak{t}}) =& \mathfrak{q}^{h_\Lambda} \frac{\mathcal{N}_\Lambda(\mathfrak{q}, \hat{\mathfrak{t}})}{(\mathfrak{q};\mathfrak{q})_\infty^{M-1} \prod_{1 \leq i < j \leq M} \left(\hat{\mathfrak{t}}_{i-1}\hat{\mathfrak{t}}_j/\hat{\mathfrak{t}}_i\hat{\mathfrak{t}}_{j-1};\mathfrak{q}\right)_\infty \left(\mathfrak{q}\,\hat{\mathfrak{t}}_i\hat{\mathfrak{t}}_{j-1}/\hat{\mathfrak{t}}_{i-1}\hat{\mathfrak{t}}_j;\mathfrak{q}\right)_\infty} \\ &\times \prod_{i=1}^{M-1} \hat{\mathfrak{t}}_i^{d_i},\end{aligned} \tag{A.35}$$

where, with $\lambda = \mathrm{par}(\Lambda)$,

$$\mathcal{N}_\Lambda(\mathfrak{q}, \hat{\mathfrak{t}}) = \sum_{\substack{k_1,\ldots,k_M \in \mathbb{Z} \\ k_1+\cdots+k_M=0}} \det_{1 \leq i,j \leq M} \left(\left(\hat{\mathfrak{t}}_i/\hat{\mathfrak{t}}_{i-1}\right)^{(M+m)k_i+\lambda_j-j-\lambda_i+i} \mathfrak{q}^{(\lambda_j-j)k_i+\frac{1}{2}(M+m)k_i^2} \right). \tag{A.36}$$

Appendix B. Some AGT correspondences

Following [1,7,8,11], we summarize some explicit AGT correspondences to identify our conventions in Section 3.2 and to confirm the $U(1)$ factor $Z_{\mathcal{H}}\left(\boldsymbol{m}, \boldsymbol{m}'; \mathfrak{q}\right)$ in (3.9).

B.1. $(N, n) = (2, 1)$ *and Virasoro conformal blocks*

For $(N, n) = (2, 1)$, the $SU(2)$ instanton partition function (2.23) with $a_1 = -a_2 = a$ is computed as

$$\begin{aligned}Z^{\mathbf{0},\mathbf{0}}_{\mathbf{0};\emptyset}(a, \boldsymbol{m}, \boldsymbol{m}'; \mathfrak{q}) = 1 + \mathfrak{q}\Bigg\{&\frac{(a-m_1)(a-m_2)(a+m_1'-\epsilon_1-\epsilon_2)(a+m_2'-\epsilon_1-\epsilon_2)}{2a\,\epsilon_1\,\epsilon_2\,(-2a+\epsilon_1+\epsilon_2)} \\ &- \frac{(a+m_1)(a+m_2)(a-m_1'+\epsilon_1+\epsilon_2)(a-m_2'+\epsilon_1+\epsilon_2)}{2a\,\epsilon_1\,\epsilon_2\,(2a+\epsilon_1+\epsilon_2)}\Bigg\} \\ &+ O\left[\mathfrak{q}^2\right].\end{aligned} \tag{B.1}$$

In [1] (see [60,61] for non-conformal/Whittaker limits) it was found that, by subtracting the $U(1)$ factor (3.9) for $(N, n) = (2, 1)$, the normalized instanton partition function

$$\widehat{Z}^{\mathbf{0},\mathbf{0}}_{\mathbf{0};\emptyset}(a, \boldsymbol{m}, \boldsymbol{m}'; \mathfrak{q}) := (1-\mathfrak{q})^{-\frac{\left[\sum_{I=1}^{2} m_I\right]\left[\epsilon_1+\epsilon_2-\frac{1}{2}\sum_{I=1}^{2} m'_I\right]}{\epsilon_1\epsilon_2}} Z^{\mathbf{0},\mathbf{0}}_{\mathbf{0};\emptyset}(a, \boldsymbol{m}, \boldsymbol{m}'; \mathfrak{q}) \tag{B.2}$$

gives the $c = 1 + 6\,\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}$ Virasoro conformal blocks of 4-point function (3.4) on $\mathbb{P}^1$ by the parameter identifications (3.6) and (3.7):

$$\begin{aligned} \mu^v &= \frac{\epsilon_1+\epsilon_2}{2} + a, \quad & \mu_1 &= \frac{\epsilon_1+\epsilon_2}{2} + \frac{m_1-m_2}{2}, \quad & \mu_2 &= \frac{m_1+m_2}{2}, \\ & & \mu_4 &= \frac{\epsilon_1+\epsilon_2}{2} - \frac{m'_1-m'_2}{2}, \quad & \mu_3 &= \frac{m'_1+m'_2}{2}. \end{aligned} \tag{B.3}$$

Here, note that, in the $n = 1$ cases, the WZW factor $\widehat{\mathfrak{sl}}(n)_N$ in the algebra (3.1) is absent. For example, the Virasoro conformal block at level 1,

$$\frac{\left[\Delta_{\mu^v} - \Delta_{\mu_1} + \Delta_{\mu_2}\right]\left[\Delta_{\mu^v} + \Delta_{\mu_3} - \Delta_{\mu_4}\right]}{2\,\Delta_{\mu^v}}, \qquad \Delta_\mu = \frac{\mu\left[\epsilon_1+\epsilon_2-\mu\right]}{\epsilon_1\epsilon_2}, \tag{B.4}$$

agrees with the coefficient of $\mathfrak{q}$ in (B.2).

B.2. $(N, n) = (3, 1)$ *and* $\mathcal{W}_3$ *conformal blocks*

For $(N, n) = (3, 1)$, the $SU(3)$ instanton partition function (2.23) with $a_3 = -a_1 - a_2$ is computed as

$$\begin{aligned} & Z^{\mathbf{0},\mathbf{0}}_{\mathbf{0};\emptyset}(\boldsymbol{a}, \boldsymbol{m}, \boldsymbol{m}'; \mathfrak{q}) \\ &= 1 + \mathfrak{q}\left[\frac{(a_1-m_1)(a_1-m_2)(a_1-m_3)(-a_1-m'_1+\epsilon_1+\epsilon_2)(-a_1-m'_2+\epsilon_1+\epsilon_2)(-a_1-m'_3+\epsilon_1+\epsilon_2)}{\epsilon_1\epsilon_2(a_1-a_2)(2a_1+a_2)(-2a_1-a_2+\epsilon_1+\epsilon_2)(-a_1+a_2+\epsilon_1+\epsilon_2)}\right. \\ &\quad + \frac{(a_2-m_1)(a_2-m_2)(a_2-m_3)(-a_2-m'_1+\epsilon_1+\epsilon_2)(-a_2-m'_2+\epsilon_1+\epsilon_2)(-a_2-m'_3+\epsilon_1+\epsilon_2)}{\epsilon_1\epsilon_2(a_2-a_1)(a_1+2a_2)(-a_1-2a_2+\epsilon_1+\epsilon_2)(a_1-a_2+\epsilon_1+\epsilon_2)} \\ &\quad \left. - \frac{(a_1+a_2+m_1)(a_1+a_2+m_2)(a_1+a_2+m_3)(a_1+a_2-m'_1+\epsilon_1+\epsilon_2)(a_1+a_2-m'_2+\epsilon_1+\epsilon_2)(a_1+a_2-m'_3+\epsilon_1+\epsilon_2)}{\epsilon_1\epsilon_2(2a_1+a_2)(a_1+2a_2)(2a_1+a_2+\epsilon_1+\epsilon_2)(a_1+2a_2+\epsilon_1+\epsilon_2)}\right] \\ &\quad + O\left[\mathfrak{q}^2\right]. \end{aligned} \tag{B.5}$$

By subtracting the $U(1)$ factor (3.9) for $(N, n) = (3, 1)$, one finds that the normalized instanton partition function

$$\widehat{Z}^{\mathbf{0},\mathbf{0}}_{\mathbf{0};\emptyset}(\boldsymbol{a}, \boldsymbol{m}, \boldsymbol{m}'; \mathfrak{q}) := (1-\mathfrak{q})^{-\frac{\left[\sum_{I=1}^{3} m_I\right]\left[\epsilon_1+\epsilon_2-\frac{1}{3}\sum_{I=1}^{3} m'_I\right]}{\epsilon_1\epsilon_2}} Z^{\mathbf{0},\mathbf{0}}_{\mathbf{0};\emptyset}(\boldsymbol{a}, \boldsymbol{m}, \boldsymbol{m}'; \mathfrak{q}) \tag{B.6}$$

gives the $\mathcal{W}_3$ conformal blocks of 4-point function (3.4) on $\mathbb{P}^1$, with $c = 2 + 24\,\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}$, by the parameter identifications (3.6) and (3.7) [7,8] (see [62–64] for non-conformal/Whittaker limits):

$$\begin{aligned} \mu^v_1 &= \frac{\epsilon_1+\epsilon_2}{2} + \frac{a_1-a_2}{2}, \quad & \mu^v_2 &= \frac{\epsilon_1+\epsilon_2}{2} + \frac{a_1+2a_2}{2}, \\ \mu_{1,1} &= \frac{\epsilon_1+\epsilon_2}{2} + \frac{m_1-m_2}{2}, \quad & \mu_{1,2} &= \frac{\epsilon_1+\epsilon_2}{2} + \frac{m_2-m_3}{2}, \quad & \mu_2 &= \frac{m_1+m_2+m_3}{2}, \\ \mu_{4,1} &= \frac{\epsilon_1+\epsilon_2}{2} - \frac{m'_1-m'_2}{2}, \quad & \mu_{4,2} &= \frac{\epsilon_1+\epsilon_2}{2} - \frac{m'_2-m'_3}{2}, \quad & \mu_3 &= \frac{m'_1+m'_2+m'_3}{2}. \end{aligned} \tag{B.7}$$

For example, the $\mathcal{W}_3$ conformal block at level 1,

$$\begin{aligned}
&\frac{\left[\Delta_{\boldsymbol{\mu}^v} - \Delta_{\boldsymbol{\mu}_1} + \Delta_{0,\mu_2}\right]\left[\Delta_{\boldsymbol{\mu}^v} - \Delta_{\boldsymbol{\mu}_4} + \Delta_{\mu_3,0}\right]}{2\,\Delta_{\boldsymbol{\mu}^v}} \\
&+ \left[-\frac{w_{\boldsymbol{\mu}^v}}{2} - w_{\boldsymbol{\mu}_1} + \frac{w_{0,\mu_2}}{2} + \frac{3\left(\Delta_{\boldsymbol{\mu}^v} - \Delta_{\boldsymbol{\mu}_1}\right) w_{0,\mu_2}}{2\,\Delta_{0,\mu_2}} - \frac{3\left(\Delta_{0,\mu_2} - \Delta_{\boldsymbol{\mu}_1}\right) w_{\boldsymbol{\mu}^v}}{2\,\Delta_{\boldsymbol{\mu}^v}}\right] \\
&\times \left[-\frac{w_{\boldsymbol{\mu}^v}}{2} - w_{\boldsymbol{\mu}_4} + \frac{w_{\mu_3,0}}{2} + \frac{3\left(\Delta_{\boldsymbol{\mu}^v} - \Delta_{\boldsymbol{\mu}_4}\right) w_{\mu_3,0}}{2\,\Delta_{\mu_3,0}} - \frac{3\left(\Delta_{\mu_3,0} - \Delta_{\boldsymbol{\mu}_4}\right) w_{\boldsymbol{\mu}^v}}{2\,\Delta_{\boldsymbol{\mu}^v}}\right] \\
&\times \left[\Delta_{\boldsymbol{\mu}^v}\left[\frac{4\,\epsilon_1\,\epsilon_2\,\Delta_{\boldsymbol{\mu}^v}}{4\,\epsilon_1\,\epsilon_2 + 15\,(\epsilon_1+\epsilon_2)^2} - \frac{3\,(\epsilon_1+\epsilon_2)^2}{4\,\epsilon_1\,\epsilon_2 + 15\,(\epsilon_1+\epsilon_2)^2}\right] - \frac{9\,w_{\boldsymbol{\mu}^v}^2}{2\,\Delta_{\boldsymbol{\mu}^v}}\right]^{-1},
\end{aligned} \tag{B.8}$$

agrees with the coefficient of $\mathfrak{q}$ in (B.6), where

$$\begin{aligned}
\Delta_{\boldsymbol{\mu}} &= \Delta_{\mu_1,\mu_2} = -\frac{2\left[2\,\mu_1^2 + 2\,\mu_1\,\mu_2 + 2\,\mu_2^2 - 3\,(\epsilon_1+\epsilon_2)\,(\mu_1+\mu_2)\right]}{3\,\epsilon_1\,\epsilon_2}, \\
w_{\boldsymbol{\mu}} &= w_{\mu_1,\mu_2} = \frac{1}{\epsilon_1\,\epsilon_2}\left[\frac{2}{3}\,(2\,\mu_1+\mu_2) - (\epsilon_1+\epsilon_2)\right] \\
&\quad\times \left[\frac{2}{3}\,(\mu_1+2\,\mu_2) - (\epsilon_1+\epsilon_2)\right]\left[\frac{2}{3}\,(\mu_1-\mu_2)\right] \\
&\quad\times \left[\frac{-6}{4\,\epsilon_1\,\epsilon_2 + 15\,(\epsilon_1+\epsilon_2)^2}\right]^{\frac{1}{2}}.
\end{aligned} \tag{B.9}$$

B.3. $(N,n) = (2,2)$ *and* $\mathcal{N}=1$ *super-Virasoro conformal blocks*

For $(N,n) = (2,2)$, the $SU(2)$ instanton partition functions (2.23) with $a_1 = -a_2 = a$ are computed as *e.g.*,

$$\begin{aligned}
&Z^{(0,0),(0,0)}_{(0,0);(0)}(a, \boldsymbol{m}, \boldsymbol{m}'; \mathfrak{q}) \\
&= 1 + \mathfrak{q}\left[\frac{(a-m_1)\,(a-m_2)\,(a+m_1'-\epsilon_1-\epsilon_2)\,(a+m_2'-\epsilon_1-\epsilon_2)}{4\,a\,\epsilon_2\,(\epsilon_1-\epsilon_2)\,(-2\,a+\epsilon_1+\epsilon_2)}\right. \\
&+ \frac{(a-m_1)\,(a-m_2)\,(a+m_1'-\epsilon_1-\epsilon_2)\,(a+m_2'-\epsilon_1-\epsilon_2)}{4\,a\,\epsilon_1\,(\epsilon_2-\epsilon_1)\,(-2\,a+\epsilon_1+\epsilon_2)} \\
&+ \frac{(a+m_1)\,(a+m_2)\,(a-m_1'+\epsilon_1+\epsilon_2)\,(a-m_2'+\epsilon_1+\epsilon_2)}{4\,a\,\epsilon_1\,(\epsilon_1-\epsilon_2)\,(2\,a+\epsilon_1+\epsilon_2)} \\
&\left. - \frac{(a+m_1)\,(a+m_2)\,(a-m_1'+\epsilon_1+\epsilon_2)\,(a-m_2'+\epsilon_1+\epsilon_2)}{4\,a\,\epsilon_2\,(\epsilon_1-\epsilon_2)\,(2\,a+\epsilon_1+\epsilon_2)}\right] + O\left[\mathfrak{q}^2\right], \\
&Z^{(0,0),(0,0)}_{(1,1);(1)}(a, \boldsymbol{m}, \boldsymbol{m}'; \mathfrak{q}) = \mathfrak{q}^{\frac{1}{2}}\left[\frac{1}{2\,a\,(-2\,a+\epsilon_1+\epsilon_2)} - \frac{1}{2\,a\,(2\,a+\epsilon_1+\epsilon_2)}\right] + O\left[\mathfrak{q}^{\frac{3}{2}}\right],
\end{aligned} \tag{B.10}$$

for the vanishing first Chern class $c_1 = 0$ in (2.7).

We consider the subtraction of the $U(1)$ factor (3.9) for $(N,n) = (2,2)$ from the instanton partition functions

$$\widehat{Z}^{\boldsymbol{b},\boldsymbol{b}'}_{\boldsymbol{\sigma};(\ell)}(a,\boldsymbol{m},\boldsymbol{m}';\mathfrak{q}) := (1-\mathfrak{q})^{-\frac{\left[\sum_{I=1}^{2} m_I\right]\left[\epsilon_1+\epsilon_2-\frac{1}{2}\sum_{I=1}^{2} m'_I\right]}{2\epsilon_1\epsilon_2}}\, Z^{\boldsymbol{b},\boldsymbol{b}'}_{\boldsymbol{\sigma};(\ell)}(a,\boldsymbol{m},\boldsymbol{m}';\mathfrak{q}). \tag{B.11}$$

In [11,14] (see also [12]), it was shown that the normalized instanton partition functions (B.11) give the $\mathcal{N}=1$ super-Virasoro conformal blocks of 4-point function (3.4) on $\mathbb{P}^1$, with $c=\frac{3}{2}+3\,\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}$, by the parameter identifications (B.3) (see [9,10,13] for non-conformal/Whittaker limits). For example, the instanton partition functions (B.10) correspond to the conformal blocks of four NS primary fields, and actually the conformal block at level 1,

$$\frac{\left[\Delta_{\mu^v}-\Delta_{\mu_1}+\Delta_{\mu_2}\right]\left[\Delta_{\mu^v}+\Delta_{\mu_3}-\Delta_{\mu_4}\right]}{2\,\Delta_{\mu^v}}, \qquad \Delta_\mu=\frac{\mu\,\left(\epsilon_1+\epsilon_2-\mu\right)}{2\,\epsilon_1\,\epsilon_2}, \tag{B.12}$$

agrees with the coefficient of $\mathfrak{q}$ in $\widehat{Z}^{(0,0),(0,0)}_{(0,0);(0)}(a,\boldsymbol{m},\boldsymbol{m}';\mathfrak{q})$, and the conformal blocks

$$\begin{aligned}
&\text{at level } \frac{1}{2}: && \frac{1}{2\,\Delta_{\mu^v}},\\
&\text{at level } \frac{3}{2}: && \frac{\left[1+2\,\Delta_{\mu^v}-2\,\Delta_{\mu_1}+2\,\Delta_{\mu_2}\right]\left[1+2\,\Delta_{\mu^v}+2\,\Delta_{\mu_3}-2\,\Delta_{\mu_4}\right]}{8\,\Delta_{\mu^v}\left[1+2\,\Delta_{\mu^v}\right]}\\
& && +\frac{6\left[\Delta_{\mu_2}-\Delta_{\mu_1}\right]\left[\Delta_{\mu_3}-\Delta_{\mu_4}\right]}{\left[c-\left(9-2c\right)\,\Delta_{\mu^v}+6\,\Delta_{\mu^v}^2\right]\left[1+2\,\Delta_{\mu^v}\right]},
\end{aligned} \tag{B.13}$$

agree with the coefficients of $\mathfrak{q}^{\frac{1}{2}}$ and $\mathfrak{q}^{\frac{3}{2}}$ in $2\epsilon_1\,\epsilon_2\,\widehat{Z}^{(0,0),(0,0)}_{(1,1);(1)}(a,\boldsymbol{m},\boldsymbol{m}';\mathfrak{q})$ [11].

Appendix C. Integrable $\widehat{\mathfrak{sl}}(n)_N$ WZW 4-point conformal blocks for fundamental representations

The integrable $\widehat{\mathfrak{sl}}(n)_N$ WZW conformal blocks of 4-point function on $\mathbb{P}^1$ of primary fields with (anti-)fundamental representations $\square$, $\square$, $\overline{\square}$, and $\overline{\square}$, schematically denoted by

$$\left\langle \overline{\square}(\infty)\,\square(1)\,\square(z)\,\overline{\square}(0)\right\rangle^{\widehat{\mathfrak{sl}}(n)_N}_{\mathbb{P}^1}, \tag{C.1}$$

were obtained in [28] (see also [45]), as solutions to the Knizhnik-Zamolodchikov equation, as

$$\begin{aligned}
\mathcal{F}^{(0)}_1(z) &= z^{-2h_\square}\,(1-z)^{h_\theta-2h_\square}\,{}_2F_1\left(-\frac{1}{n+N},\frac{1}{n+N};\frac{N}{n+N};z\right),\\
\mathcal{F}^{(0)}_2(z) &= \frac{1}{N}\,z^{1-2h_\square}\,(1-z)^{h_\theta-2h_\square}\,{}_2F_1\left(1-\frac{1}{n+N},1+\frac{1}{n+N};1+\frac{N}{n+N};z\right),\\
\mathcal{F}^{(1)}_1(z) &= z^{h_\theta-2h_\square}\,(1-z)^{h_\theta-2h_\square}\,{}_2F_1\left(\frac{n-1}{n+N},\frac{n+1}{n+N};1+\frac{n}{n+N};z\right),\\
\mathcal{F}^{(1)}_2(z) &= -n\,z^{h_\theta-2h_\square}\,(1-z)^{h_\theta-2h_\square}\,{}_2F_1\left(\frac{n-1}{n+N},\frac{n+1}{n+N};\frac{n}{n+N};z\right),
\end{aligned} \tag{C.2}$$

where $h_\square=\frac{n^2-1}{2n(n+N)}$ is the conformal dimension of the four primary fields, and $h_\theta=\frac{n}{n+N}$ is the conformal dimension of the adjoint field with weight $\theta=[N-1,1,0,\ldots,0,1]$. These four solutions correspond to two choices of the representations of states in the internal channel which follow from the fusion of $\square$ and $\overline{\square}$, and $\mathcal{F}_1^{(0)}(z), \mathcal{F}_2^{(0)}(z)$ (resp. $\mathcal{F}_1^{(1)}(z), \mathcal{F}_2^{(1)}(z)$) corresponds to the identity (resp. adjoint) field conformal block of "s-channel". Under a hypergeometric transformation

$$z \quad \to \quad \mathfrak{q} := \frac{z}{z-1}, \tag{C.3}$$

the Gauss hypergeometric function transforms as

$${}_2F_1\left(\alpha, \beta; \gamma; z\right) = (1-\mathfrak{q})^{\alpha}\, {}_2F_1\left(\alpha, \gamma-\beta; \gamma; \mathfrak{q}\right), \tag{C.4}$$

and the $\widehat{\mathfrak{sl}}(n)_N$ WZW 4-point conformal blocks (C.2) are expressed, in the $\mathfrak{q}$-module, as

$$\begin{aligned}
\widehat{\mathcal{F}}_1^{(0)}(\mathfrak{q}) &:= z^{2h_\square}\mathcal{F}_1^{(0)}(z) = (1-\mathfrak{q})^{2h_\square - \frac{n+1}{n+N}}\, {}_2F_1\left(-\frac{1}{n+N}, \frac{N-1}{n+N}; \frac{N}{n+N}; \mathfrak{q}\right), \\
\widehat{\mathcal{F}}_2^{(0)}(\mathfrak{q}) &:= z^{2h_\square}\mathcal{F}_2^{(0)}(z) \\
&= -\frac{\mathfrak{q}}{N}\left(1-\mathfrak{q}\right)^{2h_\square - \frac{n+1}{n+N}}\, {}_2F_1\left(\frac{N-1}{n+N}, 1-\frac{1}{n+N}; 1+\frac{N}{n+N}; \mathfrak{q}\right), \\
\widehat{\mathcal{F}}_1^{(1)}(\mathfrak{q}) &:= \frac{z^{2h_\square}}{n}\mathcal{F}_1^{(1)}(z) \\
&= \frac{(-\mathfrak{q})^{h_\theta}}{n}\left(1-\mathfrak{q}\right)^{2h_\square - \frac{n+1}{n+N}}\, {}_2F_1\left(\frac{n-1}{n+N}, 1-\frac{1}{n+N}; 1+\frac{n}{n+N}; \mathfrak{q}\right), \\
\widehat{\mathcal{F}}_2^{(1)}(\mathfrak{q}) &:= \frac{z^{2h_\square}}{n}\mathcal{F}_2^{(1)}(z) \\
&= -\left(-\mathfrak{q}\right)^{h_\theta}\left(1-\mathfrak{q}\right)^{2h_\square - \frac{n+1}{n+N}}\, {}_2F_1\left(-\frac{1}{n+N}, \frac{n-1}{n+N}; \frac{n}{n+N}; \mathfrak{q}\right).
\end{aligned} \tag{C.5}$$

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