Singularities in FLRW spacetimes

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ABSTRACT

We point out that past-incompleteness of geodesics in FLRW spacetimes does not necessarily imply that these spacetimes start from a singularity. Namely, if a test particle that follows such a trajectory has a non-vanishing velocity, its energy was super-Planckian at some time in the past if it kept following that geodesic. That indicates a breakdown of the particle's description, which is why we should not consider those trajectories for the definition of an initial singularity. When one only considers test particles that do not have this breakdown of their trajectory, it turns out that the only singular FLRW spacetimes are the ones that have a scale parameter that vanishes at some initial time.

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1. Introduction

Hubble's law, the observed abundance of elements, the cosmic background radiation and the large scale structure formation in the universe are strong evidence that the universe expanded from an initial very high dense state to how we observe it now. However, what happened exactly during this hot density state is still an open problem. One of the questions that needs to be answered is whether there was a singularity at the beginning of spacetime. Such a singularity is in accordance with the very general theorems of Hawking and Penrose [1,2] defined as a non-spacelike geodesic that is incomplete in the past. One uses this definition because test particles move on these trajectories and thus have only traveled for a finite proper time.

The flatness, horizon and magnetic monopole problem can be solved with a period of exponential expansion in the very early universe [3,4]. To avoid a singularity before that period, it was suggested that one can have past-eternal inflation in which the universe starts from an almost static universe and flows towards a period of exponential expansion. This way the universe would not have a beginning. One of the characteristics of inflationary models is that the Hubble parameter H is positive. In [5] it was shown that when the average Hubble parameter along a geodesic H_{av} is positive, the geodesic is past-incomplete such that we would have a singularity. This is also applicable to models of eternal inflation

in which the average Hubble parameter along geodesics does not go to zero sufficiently fast (i.e. such that we do not have that H_{av} is zero). In [6], a model of eternal inflation was given with all non-spacelike geodesics complete, but in [7] these kind of models were shown to be quantum mechanically unstable. Hence, this would imply that also models of eternal inflation start from a singularity.

In [8] it was pointed out that in De Sitter space the test particles that follow those past-incomplete trajectories and have a non-vanishing velocity, will have an energy that becomes arbitrarily large when going back in the past. This can be generalized to general Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime and means that the energy of such a test particle can become super-Planckian at some initial time such that their description breaks down. This is the reason one should not consider those trajectories when defining a singularity. When one only considers the trajectories of test particles that do not have a breakdown of the description of their trajectory, one finds that the only FLRW spacetimes that start from a singularity are the ones with a scale factor that vanishes at some initial time. This implies that models of eternal inflation or bouncing models are singularity free provided one requires sub-Planckian test particles at all times.

In this paper we first consider the past-(in)completeness of geodesics in spacetimes with an FLRW metric. We review the general singularity theorems of [1,2] applied to these models and we review the more general (in the context of cosmology) argument of [5]. After that we consider how the energies of test particles change in time. We adopt units in which the velocity of light $c = 1$.

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2. Past-(in)completeness of geodesics in FLRW spacetimes

Consider a universe with an FLRW metric which describes a spatially homogeneous, isotropic spacetime:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - \kappa r^2} + r^2 \left(d\theta^2 + \sin^2(\theta) d\varphi^2 \right) \right], \qquad (1)$$

where κ is the curvature of spacelike three-surfaces and the scale factor $a(t)$ is normalized such that $a(t_1) = 1$ for some time t_1. This metric is a good description of our universe, since from experiments as WMAP and Planck, it follows that our universe is spatially homogeneous and isotropic when averaged over large scales. Geodesics $\gamma(\tau)$, where τ is an affine parameter, satisfy

$$\frac{d\gamma^0}{d\tau} = \frac{\sqrt{|\vec{V}(t_1)|^2 - \epsilon a^2}}{a}, \qquad (2)$$

where $|\vec{V}|^2 = g_{ij} \dot{\gamma}^i \dot{\gamma}^j$ and ϵ is the normalization of the geodesic: $\epsilon = 0$ for null geodesics and $\epsilon = -1$ for timelike geodesics. We thus have a past-incomplete geodesic when

$$\int d\tau = \int_{t_0}^{t} \frac{a}{\sqrt{|\vec{V}(t_1)|^2 - \epsilon a^2}} dt \qquad (3)$$

for an initial velocity $|\vec{V}(t_1)|$ is finite. Here t_0 is $-\infty$ if $a(t) > 0$ for all t, otherwise $t_0 \in \mathbb{R}$ is taken such that $a(t_0) = 0$. Notice that when $a(t_0) = 0$ for some time t_0, all non-spacelike geodesics are past-incomplete. When $t_0 = -\infty$ and the integral (3) is converging, we cannot immediately conclude that geodesics are past-incomplete. It is possible that we only consider a part of the actual spacetime. An example is given by $\kappa = 0$, and the Hubble parameter $H = \dot{a}/a$ satisfying $\dot{H}/H^2 = 0$, in which case $a(t) = e^{Ht}$ with H constant. If the whole manifold would be covered by these coordinates, it would result in past-incomplete geodesics. However, this model only describes one half, known as the Poincaré patch, of the larger De Sitter space; the whole space is described by choosing $\kappa = 1$, $a(t) = \cosh(Ht)/H$ which yields complete geodesics. See also [9] and [10]. When the integral (3) is diverging one can conclude that geodesics in that specific coordinate patch are past-complete. Of course, one can also assume that a certain model with $t_0 = -\infty$ covers the whole spacetime. Then the past-(in)completeness of a geodesic is determined by the integral (3).

From (3) we see that in spacetimes with $a(t) > A \in \mathbb{R}_{>0}$ all non-spacelike geodesics are past-complete. Hence for a spacetime to have a non-spacelike geodesic that is past-incomplete, $a(t)$ needs to become arbitrarily small.

There are a few theorems that prove that a spacetime contains a (past-)incomplete geodesic. Hawking and Penrose, [1,2], proved theorems that state that when

$$R_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu \geq 0 \qquad (4)$$

for all geodesics γ and the spacetime obeys a few other conditions such as containing a trapped surface, there is a non-spacelike geodesic that is incomplete. Condition (4) for the metric (1) yields

$$\left(\frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} + 2 \frac{\kappa}{a^2} \right) \epsilon - 2 \left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2} \right] (\dot{\gamma}^0)^2 \geq 0. \qquad (5)$$
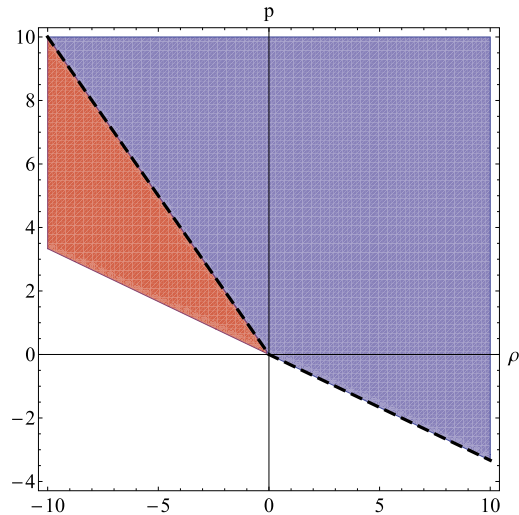

Fig. 1. Illustration of condition (8). For $\kappa < 0$ one needs (ρ, p) in the shaded area above the dashed line to apply the Hawking–Penrose singularity theorems. For $\kappa \geq 0$, we have less restrictions, the shaded area below the dashed line is also included, but it is impossible for an FLRW spacetime with non-negative spatial curvature to be in that area.

Using Eq. (2) one finds that condition (5) becomes

$$\kappa \geq 0 : \ddot{a} \leq 0;$$

$$\kappa < 0 : \begin{cases} \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2} & \leq 0; \\ \ddot{a} & \leq 0. \end{cases} \qquad (6)$$

In particular for all κ we need that $\ddot{a} \leq 0$ at all time, or that the spacetime is non-accelerating. Notice that when $\ddot{a} \leq 0$, a will always be zero at some time t_0 (this might be in the future), unless a is a positive constant ($H = 0$) in which case we do not have past-incomplete geodesics. Hence, when we want to use these theorems to say something about an initial singularity in an FLRW spacetime, we need a metric that has a scale parameter a that becomes zero at some time in the past. Describing the matter content of the universe by a perfect fluid

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}, \qquad (7)$$

where p is the pressure, ρ the energy density and $U^\mu = (1, 0, 0, 0)$, the condition (6) translates via the Friedmann equations to

$$\kappa \geq 0 : \rho + 3p \geq 0;$$

$$\kappa < 0 : \begin{cases} \rho + p & \geq 0; \\ \rho + 3p & \geq 0. \end{cases} \qquad (8)$$

Although it seems that we have less restrictions when $\kappa \geq 0$, it is impossible that $\rho + p < 0$ and $\rho + 3p \geq 0$ for non-negative spatial curvature. In Fig. 1 one finds an illustration of condition (8).

Another theorem that proves that a geodesic is past-incomplete was published in [5] and is also applicable to spacetimes that have $a(t) > 0$ for all t. It says that when the average Hubble parameter $H = \dot{a}/a$ along a non-spacelike geodesic, H_{av}, satisfies $H_{av} > 0$, the geodesic must be past-incomplete. For the metric (1), the argument is as follows. Consider a non-spacelike geodesic $\gamma(\tau)$ between an initial point $\gamma(\tau_i)$ and a final point $\gamma(\tau_f)$. We can in-

tegrate H along the geodesic, using Eq. (2):

$$
\begin{aligned}
\int_{\tau_i}^{\tau_f} H d\tau &= \int_{t_i}^{t_f} \frac{\dot{a}}{\sqrt{|\vec{V}(t_1)|^2 - \epsilon a^2}} dt \\
&= \int_{a(t_i)}^{a(t_f)} \frac{da}{\sqrt{|\vec{V}(t_1)|^2 - \epsilon a^2}} \\
&= \begin{cases} \frac{1}{|\vec{V}(t_1)|} [a(t_f) - a(t_i)], & \epsilon = 0 \\ \log\left(\frac{a(t_f) + \sqrt{|\vec{V}(t_1)|^2 + a(t_f)^2}}{a(t_i) + \sqrt{|\vec{V}(t_1)|^2 + a(t_i)^2}} \right) & \epsilon = -1 \end{cases} \\
&\leq \begin{cases} \frac{a(t_f)}{|\vec{V}(t_1)|}, & \epsilon = 0 \\ \log\left(\frac{a(t_f) + \sqrt{|\vec{V}(t_1)|^2 + a(t_f)^2}}{|\vec{V}(t_1)|} \right) & \epsilon = -1. \end{cases}
\end{aligned} \tag{9}
$$

Notice that for the second equality sign, one should break up the integration domain into parts where $a = a(t)$ is injective, but that one will end up with the same result. Hence, this integral as function of the initial affine parameter τ_i is restricted by some fixed final τ_f. This means that when

$$
H_{av} = \frac{1}{\tau_f - \tau_i} \int_{\tau_i}^{\tau_f} H d\tau > 0 \tag{10}
$$

τ_i has to be some finite value such that the geodesic is past-incomplete. Notice that it is still possible to construct an FLRW spacetime that has $H > 0$ at all times and complete geodesics. For this we need that H_{av} must become zero when $\tau_i \to -\infty$. Examples are for instance given by spacetimes with $H > 0$ and $a \to a_0 > 0$ for $t \to -\infty$ (in this case we will have that $H \to 0$ as $t \to -\infty$).

3. Energy of test particles

As stated before, the definition of a singularity is based on the trajectories of massive test particles and massless particles. For cosmological spacetimes with an FLRW metric, we would like to study the energies of test particles over time. We will generalize the argument given in [8] for De Sitter space to a general FLRW spacetime.

Using Eq. (2) we find that for massive test particles

$$
|\vec{V}|^2 = g_{ij} \dot{\gamma}^i \dot{\gamma}^j = \epsilon + \left(\dot{\gamma}^0 \right)^2 = \frac{|\vec{V}(t_1)|^2}{a^2}. \tag{11}
$$

We already saw that in order for a spacetime to have a past-incomplete non-spacelike geodesic, the scale parameter a needs to become arbitrarily small. With Eq. (11) this then implies that when the particle has a velocity $|\vec{V}(t_1)|$ at time t_1, the velocity and hence the energy $E^2 = m^2 \left(1 + \frac{|\vec{V}(t_1)|^2}{a^2}\right)$ of a test particle with mass m become arbitrarily large when moving back to the past.

The statement above for massive test particles carries over to photons. In this case the angular frequency as observed by a co-moving observer is

$$
\omega = \dot{\gamma}^0 = \frac{\omega(t_1)}{a}. \tag{12}
$$

Thus also the energy of photons $E = \hbar \omega$ will become arbitrarily large when moving back to the past.

In [8] it was noted that one cannot have particles with arbitrarily high energies because if such a particle has a nonvanishing

interaction cross section with any particle with a non-zero physical number density, then the particle will interact with an infinite number of them, breaking the Cosmological principle. However, the particle's energy cannot become arbitrarily high because it will reach the Planck energy $E_P = \sqrt{\frac{\hbar}{G}} \approx 1.22 \cdot 10^{19}$ GeV at some time t. With this energy, the particle's Compton wavelength is approximately equal to its Schwarzschild radius such that it will form a black hole. Therefore, the description of the particle's trajectory will break down. Scattering processes involving vacuum fluctuations may cause the test particle's energy to never reach the Planck energy. If these processes are significant the particle's trajectory is not a geodesic anymore. Near the Planck energy scattering processes are dominated by processes that involve the exchange of a graviton [11]. To estimate this effect we consider photon–photon scattering with the exchange of a graviton. We model the loss of energy of the photon when going back in time as

$$
\frac{d}{dt} E = (-H - \sigma n) E, \tag{13}
$$

where n is the number density of virtual photons and σ is the cross section of the scattering process. The particle gains energy from the expansion of the universe because $-H$ is positive (when going back in time) and it looses energy from the scattering with virtual photons. We estimate the density of virtual photons as one per Hubble volume:

$$
n = \frac{1}{V_H} = -\frac{3H^3}{4\pi}. \tag{14}
$$

The differential cross section for photon–photon scattering with the exchange of a graviton for unpolarized photons is [12]

$$
\frac{d\sigma}{d\Omega} = \frac{\kappa^4}{8\pi^2} \frac{k^2}{\sin^2(\theta)} \left[1 + \cos^{16}\left(\frac{1}{2}\theta\right) + \sin^{16}\left(\frac{1}{2}\theta\right) \right] \tag{15}
$$

where $\kappa = \sqrt{16\pi G}$, k is the momentum of the photon and θ is the scattering angle. Since we are primarily interested in large momentum exchange, we neglect small angle scatterings when calculating the total cross section of this process:

$$
\begin{aligned}
\sigma &= \int \frac{d\sigma}{d\Omega} d\Omega \\
&= \frac{\kappa^4}{\pi} \frac{k^2}{4} \int_{-1+\xi}^{1-\xi} \frac{1 + \frac{1}{256}(1+x)^8 + \frac{1}{256}(1-x)^8}{1-x^2} dx \\
&= \frac{\kappa^4}{\pi} \frac{k^2}{2} \int_{\xi}^{1} \frac{1 + \frac{1}{256}(2-y)^8 + \frac{1}{256}y^8}{y(2-y)} dy \\
&= \frac{\kappa^4}{\pi} \frac{k^2}{4} \left[2\log\frac{1}{\xi} - \frac{363}{140} + \log(4) + \mathcal{O}(\xi) \right],
\end{aligned} \tag{16}
$$

where we have the relation $\sin(\theta/2) = \sqrt{\xi/2}$. Taking only angles $.26\pi < \theta < .74\pi$ into account for the scattering, we have that $2\log\frac{1}{\xi} - \frac{363}{140} + \log(4) \approx 1$. With Eqs. (13), (14) and (16) we find that the energy of the test photon does not increase when

$$
H \sim \sigma n = 48 G^2 E^2 H^3, \tag{17}
$$

where $E = k$ is the photon energy. Using the Hubble parameter of cosmic inflation which typically is about $-\hbar H \approx 10^{13}$ GeV, we find from (17) that the scattering process becomes significant when

$$
\left(\frac{E}{E_P} \right)^2 \sim \frac{E_P^2}{48\hbar^2 H^2} \approx 10^{10}. \tag{18}
$$

Hence, processes involving gravitons will not cause the particle's energy to stay smaller than the Planck energy and a black hole will form. This implies that the description of the particle's trajectory (as a geodesic) breaks down, either because of interaction processes or by the formation of a black hole. The latter definitely happens when the initial energy is near the Planck energy.

Up to now, the maximum energy of a single particle that has been measured is of the order of 10^{20} eV [13] which is eight orders of magnitude smaller than the Planck scale. These particles were all cosmic ray particles, so their probable origin is a supernova, an active galactic nucleus, a quasar or a gamma ray-burst. Even when using this energy as an upper bound for the energy of test particles, we have that the description of the trajectories of non-comoving test particles breaks down at times that are certainly later than the Planck era, the period where we have to take quantum gravitational effects into account. In [8] the arbitrarily high energies of test particles were used to argue that these particles should be forbidden in De Sitter space. This can be done by using a different time arrow in the two patches of De Sitter space that one has in the flat slicing. That way the two coordinate patches become non-communicating and describe eternally inflating space-times. We will not look into these kind of constructions for general FLRW spacetimes but we want to use the arbitrarily high energies of test particles to give a consistent definition of a singularity. When the particle's description breaks down before it reaches the beginning of its trajectory, it is not very useful to use that particle as an indication for an initial singularity. That is the reason why we suggest to define a singularity in spacetimes with an FLRW metric that has a parameter a that becomes arbitrarily small, as a time-like geodesic with $|\vec{V}(t_1)| = 0$ that is past-incomplete. For such trajectories, we have that $dt = d\tau$ which means that a spacetime has no initial singularity when $a(t) > 0$ for all $t \in \mathbb{R}$. Hence, an FLRW spacetime starts from a singularity precisely when $a(t_0) = 0$ at some initial finite time t_0.

4. Conclusion

We pointed out that spacetimes with an FLRW metric such that $a(t) > 0$ for all $t \in \mathbb{R}$ have no initial singularity. This was done by first observing that in models that have $a(t) > A \in \mathbb{R}_{>0}$ all non-spacelike geodesics are past-complete. When a becomes arbitrarily small, it is possible that the spacetime contains a past-incomplete geodesic. With the usual definition of a singularity, this means that the spacetime has an initial singularity. However, that definition is based on a test particle that has that geodesic as trajectory. We pointed out that when this particle has an initial velocity, its energy will become super-Planckian at some time in the past if it

kept following that geodesic. This means that the particle stops being a test particle and it does not matter that its trajectory is past-incomplete. For a model in which the scale factor becomes arbitrarily small, we should define an initial singularity as a trajectory of a comoving particle that is past-incomplete. This implies that the only FLRW spacetimes with an initial singularity are the ones such that $a(t_0) = 0$ at some initial time t_0. Hence, bouncing spacetimes and past-eternal inflationary models do not start from a singularity. One can use similar arguments to show that the only FLRW spacetimes that have a singularity in the future are the ones that have a scale factor such that $a(t)$ vanishes at some time in the future. It would be interesting to examine if similar results hold for universes that are obtained by perturbating an FLRW spacetime.

Acknowledgements

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