NON-CLASSICAL BEHAVIOR OF MOVING RELATIVISTIC UNSTABLE PARTICLES*

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We study the survival probability of moving relativistic unstable particles with definite momentum $\vec{p} \neq 0$. The amplitude of the survival probability of these particles is calculated using its integral representation. We found decay curves of such particles for the quantum mechanical models considered. These model studies show that late time deviations of the survival probability of these particles from the exponential form of the decay law, that is the transition times region between exponential and non-exponential form of the survival probability, should occur much earlier than it follows from the classical standard approach resolving itself into replacing time t by t/γ (where γ is the relativistic Lorentz factor) in the formula for the survival probability and that the survival probabilities should tend to zero as $t \to \infty$ much slower than one would expect using classical time dilation relation. Here, we show also that for some physically admissible models of unstable states, the computed decay curves of the moving particles have a fluctuating form at relatively short times including times of the order of the lifetime.

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1. Introduction

Physicists studying the decay processes are often confronted with the problem of how to predict the form of the decay law for a particle moving in respect to the rest reference frame of the observer knowing the decay law of this particle decaying in its rest frame. From the standard text book considerations, one finds that if the decay law of an unstable particle in rest has the exponential form of $\mathcal{P}_0(t) = \exp\,[-\frac{\varGamma_0\, t}{\hbar}]$, then the decay law of the moving particle with momentum $p \neq 0$ is $\mathcal{P}_p(t) = \exp\,[-\frac{\varGamma_0\, t}{\hbar\,\gamma}]$, where

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t denotes time, Γ_0 is the decay rate (time t and Γ_0 are measured in the rest reference frame of the particle), and γ is the relativistic Lorentz factor, $\gamma \equiv 1/\sqrt{1-\beta^2}$, $\beta = v/c$, v is the velocity of the particle. This equality is the classical physics relation. It is almost common belief that this equality is valid also for any t in the case of quantum decay processes and does not depend on the model of the unstable particles considered. For the proper interpretation of many accelerator experiments with high-energy unstable particles as well as of results of observations of astrophysical processes in which a huge numbers of elementary particles (including unstable one) are produced, we should be sure that this belief is supported by theoretical analysis of quantum models of decay processes. The problem seems to be extremely important because from some theoretical studies, it follows that in the case of quantum decay processes, this relation is valid to a sufficient accuracy only for not more than a few lifetimes $\tau_0 = \hbar/\Gamma_0$ [1–4]. What is more, it appears that this relation may not apply in the case of the famous result of the GSI experiment, where an oscillating decay rate of the ionized isotopes ^{140}Pr and ^{142}Pm moving with relativistic velocity ($\gamma \simeq 1.43$) was observed [5, 6]. So we can see that the problem requires a deeper analysis. In this paper, the basis of such an analysis will be the formalism developed in [1, 2] where within the quantum field theory, the formula for the survival amplitude of moving particles was derived. We will follow the method used in [4] and analyze numerically properties of the survival probability for a model of the unstable particle based on the Breit–Wigner mass distribution considered therein and as well as the other different ones. Here, we show that the relativistic treatment of the problem within the Stefanovich–Shirokov theory [1, 2] yields decay curves tending to zero as $t \to \infty$ much slower than one would expect using classical time dilation relation which confirms and generalizes some conclusions drawn in [4]. We show also that for some physically admissible models of unstable states, decay curves of the moving particles computed using the above-mentioned approach have analogous fluctuating form as the decay curve measured in the GSI experiment, and that in the model considered, these fluctuations begin from times much shorter than the lifetime. Our results show that conclusions relating to the quantum decay processes of moving particles based on the use of the classical physics time dilation relation need not be universally valid.

One of the aims of this paper is to analyze numerically properties of the survival probability in a wide range of times t from very short $t \ll \tau_0$ through $t \sim \tau_0$ until $t \gg \tau_0$ of moving unstable particles derived in [1, 2], and to present results of calculations of decay curves of such particles for the model considered in [4] but for the more realistic parameters of this model, and to confront them with results obtained for another more realistic model.

Another intention is to demonstrate that when considering the relativistic quantum unstable system, the only rational assumption seems to be the assumption that the momentum $\vec{p}$ of such a system is constant. The paper is organized as follows: Section 2 contains preliminaries and the main steps of the derivation of all relations necessary for the numerical studies, which results are presented in Section 3. Consequences of the assumption that the momentum $\vec{p}$ of the moving freely quantum unstable system is constant are analyzed in Section 4. Section 5 contains a discussion and conclusions.

2. Decay law of moving relativistic particles

Let us begin our considerations from the following assumptions: Suppose that in a laboratory, a large number $\mathcal{N}_0$ of unstable particles was created at the instant of time t_0 and then, their decay process is observed there. Suppose also that all these unstable particles do not move or are moving very slowly in relation to the rest frame of the reference of the observer $\mathcal{O}$, and that the observer counts at instants $t_1 < t_2 < \ldots t_n < \ldots$, (where $t_1 > t_0$), how many particles $\mathcal{N}(t)$ survived up to these instants of time. All collected results of these observations can be approximated by a function of time $\mathcal{P}_0(t) \simeq \mathcal{N}(t)/\mathcal{N}_0$ forming a decay curve. If $\mathcal{N}_0$ is large, then $\mathcal{P}_0(t)$ can be considered as the survival probability of the unstable particle. The standard procedure is to confront results of the experiment with theoretical predictions. Within the quantum theory, when one intends to analyze the survival probability $\mathcal{P}_0(t)$ of the unstable state or particle, say ϕ, in the rest system, one starts from the calculation of the probability amplitude $a_0(t)$. This amplitude defines the survival probability $\mathcal{P}_0(t) = |a_0(t)|^2$ we are looking for. There is $a_0(t) \equiv \langle\phi|\phi(t)\rangle$ and $|\phi(t)\rangle = \exp\left[-\frac{i}{\hbar}tH\right]|\phi\rangle$, where H is the total, self-adjoint Hamiltonian of the system considered. Studying the properties of the amplitude $a_0(t)$ it is convenient to use the integral representation of $a_0(t)$ as the Fourier transform of the energy or, equivalently, mass distribution function, $\omega(m)$ (see, *e.g.* [7–12]), with $\omega(m) \geq 0$ and $\omega(m) = 0$ for $m < \mu_0$ (μ_0 is the lower bound of the spectrum of H). It appears that the general form of the decay law $\mathcal{P}_0(t)$ of the particle in its rest reference frame practically does not depend on the form of the all physically acceptable $\omega(m)$ (see, *e.g.* [9–11, 13–16]): There is, $a_0(t) = a_{\text{exp}}(t) + a_{\text{lt}}(t)$, starting from times slightly longer than the extremely short times [14–16]. Here, $a_{\text{exp}}(t) = N\,\exp\left[-i\frac{t}{\hbar}\left(E_0 - \frac{i}{2}\,\mathit{\Gamma}_0\right)\right]$ ($E_0 = m_0\,c^2$ is the energy of the system in the unstable state $|\phi\rangle$ measured at the canonical decay times when $\mathcal{P}_0(t)$ has the exponential form, N is the normalization constant). The component $a_{\text{lt}}(t)$ exhibits inverse power-law behavior at the late time region. The late time region denotes times $t > T$, where T is the cross-over time and it can be found by solving the following equation, $|a_{\text{exp}}(t)|^2 = |a_{\text{lt}}(t)|^2$. There is $|a_{\text{exp}}(t)| \gg |a_{\text{lt}}(t)|$ for $t < T$ and $|a_{\text{exp}}(t)| \ll |a_{\text{lt}}(t)|$ for $t > T$.

We came to the place where a flux of moving relativistic unstable particles investigated by an observer in his laboratory should be considered. According to the fundamental principles of the classical physics and quantum theory (including relativistic quantum field theory), the energy and momentum of the moving particle have to be conserved. There is no an analogous conservation law for the velocity $\vec{v}$. These conservation laws are one of the basic and model-independent tools of the study of reactions between the colliding or decaying particles. So it seems to be reasonable to assume, as it was done in [1, 2, 17], that momentum $\vec{p}$ of the moving unstable particles measured in the rest frame of the observer is constant (see also a discussion in [18]). The question is: what is the picture seen by the observer in such a case and what is the relation between this picture and the picture seen by this observer in the case of non-moving unstable particles? In other words, we should compare the decay law $\mathcal{P}_0(t)$ with the decay law $\mathcal{P}_p(t)$ of the moving relativistic unstable particle with the definite, constant momentum $\vec{p} = \text{const}$. It is important to remember that the decay law $\mathcal{P}_p(t)$ does not describe the quantum decay process of the moving particle in its rest frame but describes the decay process of this particle seen by the observer in his rest laboratory frame. One meets such a picture in numerous experiments in the field of high-energy physics or when detecting cosmic rays: Detectors of a finite volume are stationary in the frame of reference of the observer $\mathcal{O}$ and stable or unstable particles together with their decay products passing through the detector are recorded. According to the broadly shared opinion reproduced in many textbooks, one expects that it should be

$$\mathcal{P}_p(t) = \mathcal{P}_0(t/\gamma) \tag{1}$$

in the considered case. This relation is a simple extension of the standard time dilation formula to quantum decay processes. The question is: how does the time dilation formula being the classical physics formula work in the case of quantum decay processes? From the results reported in [1, 2, 4] and obtained there for the model defined by Breit–Wigner mass (energy) distribution function $\omega(m) = \omega_{\rm BW}(m)$, it follows that relation (1) works in this model only within a limited range of times: for no more than a few lifetimes. What is more, it has been shown in [4] that for times longer than few lifetimes, the difference between the correctly obtained survival probability $\mathcal{P}_p(t)$ and $\mathcal{P}_0(t/\gamma)$ is significant.

Now let us follow [1, 2] and calculate survival probabilities $\mathcal{P}_0(t)$ and $\mathcal{P}_p(t)$. Hamiltonian H and the momentum operator $\boldsymbol{P}$ have common eigenvectors $|m;p\rangle$. Momentum $\vec{p}$ is the eigenvalue of the momentum operator $\boldsymbol{P}$. There is in $\hbar = c = 1$ units

$$\boldsymbol{P}|m;p\rangle = \vec{p}\,|m;p\rangle\,, \tag{2}$$

and

$$H|m;p\rangle = E'(m,p)\,|m;p\rangle\,. \tag{3}$$

In the coordinate system of the unstable quantum state at rest, when $\vec{p} = 0$, we have $|m;0\rangle = |m;p=0\rangle$

$$H|m;0\rangle = m\,|m;0\rangle\,, \qquad m \in \sigma_{\mathrm{c}}(H)\,, \tag{4}$$

where $m \equiv E'(m,0)$ and $\sigma_{\mathrm{c}}(H)$ is the continuous part of the spectrum of the Hamiltonian H. Operators H and $\boldsymbol{P}$ act in the state space $\mathcal{H}$. Eigenvectors $|m;p\rangle$ are normalized as follows

$$\langle p;m|m';p\rangle = \delta\left(m - m'\right)\,. \tag{5}$$

Now, we can model the moving unstable particle ϕ with constant momentum, $\vec{p}$, as the following wave-packet $|\phi_p\rangle$

$$|\phi_p\rangle = \int_{\mu_0}^{\infty} \varsigma(m)\,|m;p\rangle\,\mathrm{d}m\,, \tag{6}$$

where expansion coefficients $\varsigma(m)$ are functions of the mass parameter m, that is of the rest mass m, which is Lorentz invariant and, therefore, the scalar functions $\varsigma(m)$ of m are also Lorentz invariant. (Here, μ_0 is the lower bound of the spectrum $\sigma_{\mathrm{c}}(H)$ of H). We require the state $|\phi_p\rangle$ to be normalized: So it has to be $\int_{\mu_0}^{\infty} |\varsigma(m)|^2\,\mathrm{d}m = 1$.

By means of relation (6) we can define the state vector $|\phi\rangle \stackrel{\mathrm{def}}{=} |\phi_0\rangle \equiv |\phi_{p=0}\rangle \in \mathcal{H}$ describing an unstable state in rest as follows:

$$|\phi_0\rangle = |\phi\rangle = \int_{\mu_0}^{\infty} \varsigma(m)|m;0\rangle\,\mathrm{d}m\,. \tag{7}$$

This expansion and (4) allow one to find the amplitude $a_0(t)$ and to write

$$a_0(t) \equiv \int_{\mu_0}^{\infty} \omega(m)\,e^{-\,i\,m\,t}\,\mathrm{d}m\,, \tag{8}$$

where $\omega(m) \equiv |\varsigma(m)|^2 > 0$.

We need also the probability amplitude $a_p(t) = \langle\phi_p|\phi_p(t)\rangle$, which defines the survival probability $\mathcal{P}_p(t) = |a_p(t)|^2$. There is $|\phi_p(t)\rangle \stackrel{\mathrm{def}}{=} \exp\left[-itH\right]|\phi_p\rangle$ in $\hbar = c = 1$ units. We have the vector $|\phi_p\rangle$ (see (6)) but we still need eigenvalues $E'(m,p)$ solving Eq. (3). Vectors $|\phi\rangle, |\phi_p\rangle$ are elements of the same state space $\mathcal{H}$ connected with the coordinate rest system of the observer $\mathcal{O}$: We are looking for the decay law of the moving particle measured by the observer $\mathcal{O}$. If to assume for simplicity that $\boldsymbol{P} = (P_1, 0, 0)$ and that $\vec{v} = (v_1, 0, 0) \equiv (v, 0, 0)$, then there is $\vec{p} = (p, 0, 0)$ for the eigenvalues $\vec{p}$ of the momentum operator $\boldsymbol{P}$. Let $\Lambda_{p,m}$ be the Lorentz transformation from the reference frame $\mathcal{O}$, where the momentum of the unstable particle considered is zero, $\vec{p} = 0$, into the frame $\mathcal{O}'$, where the momentum of this particle is $\vec{p} \equiv (p, 0, 0) \neq 0$ and $p \geq 0$, or, equivalently, where its velocity equals $\vec{v} = \vec{v}_{p,m} \equiv \frac{\vec{p}}{m\,\gamma_m}$ (where m is the rest mass and $\gamma_m \equiv \sqrt{p^2 + (m)^2}/m$). In this case, the corresponding 4-vectors are: $\wp = (E/c, 0, 0, 0) \equiv (m, 0, 0, 0) \in \mathcal{O}$ within the considered system of units, and $\wp' = (E'/c, p, 0, 0) \equiv (E', p, 0, 0) = \Lambda_{p,m}\ \wp \in \mathcal{O}'$. There is $\wp' \cdot \wp' \equiv (\Lambda_{p,m}\ \wp) \cdot (\Lambda_{p,m}\ \wp) = \wp \cdot \wp$ in Minkowski space, which is an effect of the Lorentz invariance. (Here, the dot denotes the scalar product in Minkowski space). Hence, in our case: $\wp' \cdot \wp' \equiv (E')^2 - p^2 = m^2$ because $\wp \cdot \wp \equiv m^2$ and thus $(E')^2 \equiv (E'(m,p))^2 = p^2 + m^2$.

Another way to find $E'(m,p)$ is to use the unitary representation, $U(\Lambda_{p,m})$ of the transformation $\Lambda_{p,m}$, which acts in the Hilbert space $\mathcal{H}$ of states $|\phi\rangle \equiv |\phi; 0\rangle, |\phi_p\rangle \in \mathcal{H}$: One can show that the vector $U(\Lambda_{p,m})|m; 0\rangle$ is the common eigenvector for operators H and $\boldsymbol{P}$, that is

$$|m; p\rangle \equiv U(\Lambda_{p,m})|m; 0\rangle$$

(see, *e.g.* [19]). Indeed, taking into account that operators H and $\boldsymbol{P}$ form a 4-vector $P_\nu = (P_0, \boldsymbol{P}) \equiv (P_0, P_1, 0, 0)$ and $P_0 \equiv H$, we have

$$U^{-1}(\Lambda_{p,m}) P_\nu U(\Lambda_{p,m}) = \Lambda_{p,m;\,\nu\lambda}\, P_\lambda\,,$$

where $\nu, \lambda = 0, 1, 2, 3$ (see, *e.g.*, [19], Chap. 4). From this general transformation rule, it follows that

$$\begin{aligned} U^{-1}(\Lambda_{p,m}) P_0 U(\Lambda_{p,m}) &= \gamma_m\, (P_0 + v_m\, P_1) \\ &\equiv \gamma_m (H + v_m\, P_1)\,. \end{aligned} \tag{9}$$

Based on this relation, one can show that vectors $U(\Lambda_{p,m})|m; 0\rangle$ are eigenvectors for the Hamiltonian H. There is

$$\begin{aligned} H\, U(\Lambda_{p,m})|m; 0\rangle &= U(\Lambda_{p,m})\, U^{-1}(\Lambda_{p,m})\, H\, U(\Lambda_{p,m})|m; 0\rangle \\ &= \gamma_m\, U(\Lambda_{p,m})\, (H + v_m\, P_1)\, |m; 0\rangle\,. \end{aligned} \tag{10}$$

The Lorentz factor γ_m corresponds to the rest mass m being the eigenvalue of the vector $|m; 0\rangle$. There are $\gamma_m \neq \gamma_{m'}$ and $v_m \neq v_{m'}$ for $m \neq m'$. From (2), it follows that $P_1\,|m;0\rangle = 0$ for $p = 0$, which means that using (4), relation (10) can be rewritten as follows:

$$H\,U(\Lambda_{p,m})|m;0\rangle = m\gamma_m\,U(\Lambda_{p,m})|m;0\rangle\,. \tag{11}$$

Taking into account the form of the γ_m forced by the condition $p = \text{const}$, one concludes that, in fact, the eigenvalue found, $m\gamma_m$, equals $m\gamma_m \equiv \sqrt{p^2+m^2}$. This is exactly the same result as that at the conclusion following from the Lorentz invariance mentioned earlier: $E'(m,p) = \sqrt{p^2+m^2}$, which shows that the above considerations are self-consistent.

Similarly, one can show that vectors $U(\Lambda_{p,m})|m;0\rangle$ are the eigenvectors of the momentum operator $\boldsymbol{P}$ for the eigenvalue $m\gamma_m\,v_m \equiv p$, that is that $U(\Lambda_{p,m})|m;0\rangle \equiv |m;p\rangle$ which was to show.

Thus, finally, we come to desired result

$$H|m;p\rangle = \sqrt{p^2+m^2}\,|m;p\rangle \tag{12}$$

which replaces Eq. (3).

Now, using (12) and equation (6), we obtain the final, required relation for the amplitude $a_p(t)$

$$a_p(t) \;=\; \int_{\mu_0}^{\infty} \omega(m)\; e^{-\,i\sqrt{p^2+m^2}\;t}\,\mathrm{d}m\,. \tag{13}$$

The above derivation of the expression for $a_p(t)$ is similar to that of [4]. It is based on [19] and it is reproduced here for the convenience of readers. This is a shortened and slightly changed, simplified version of the considerations presented in [1] and mainly in [2], and more explanations and more details can be found therein and in [20, 21], where this formula was derived using the quantum field theory approach.

3. Results of numerical studies

According to the literature a reasonable simplified representation of the density of the mass distribution is to choose the Breit–Wigner form $\omega_{\mathrm{BW}}(m)$ for $\omega(m)$, which under rather general condition approximates sufficiently well many real systems [1, 9, 13]

$$\omega_{\mathrm{BW}}(m) \stackrel{\mathrm{def}}{=} \frac{N}{2\pi}\Theta(m-\mu_0)\frac{\Gamma_0}{(m-m_0)^2+\left(\frac{\Gamma_0}{2}\right)^2}\,, \tag{14}$$

where N is a normalization constant and $\Theta(m)$ is the unit step function, m_0 is the rest mass of the particle, and Γ_0 is the decay rate of the particle in the rest. Inserting $\omega(m) \equiv \omega_{\rm BW}(m)$ into (8) and into (13), one can find decay curves (survival probabilities) $\mathcal{P}_0(t)$ and $\mathcal{P}_p(t)$. Results of numerical calculations are presented in Figs. 1 and 2, where calculations were performed for $\mu_0 = 0$, $E_0/\varGamma_0 \equiv m_0/\varGamma_0 = 1000$ and $cp/\varGamma_0 \equiv p/\varGamma_0 = 1000$. Values of these parameters correspond to $\gamma = \sqrt{2}$, which is very close to γ from the experiment performed by the GSI team [5, 6] and this is why such values were chosen in our considerations. Similar calculations were performed in [4] but for different and less realistic values of the ratio $m_0/\varGamma_0$: For $m_0/\varGamma_0 = 10,\ 25$ and 100 and different $p/\varGamma_0$. According to the literature, for laboratory systems, a typical value of the ratio $m_0/\varGamma_0$ is $m_0/\varGamma_0 \geq O(10^3–10^6)$ (see *e.g.* [22]), therefore, the choice $m_0/\varGamma_0 = 1000$ seems to be reasonable and more realistic than those used in [4].

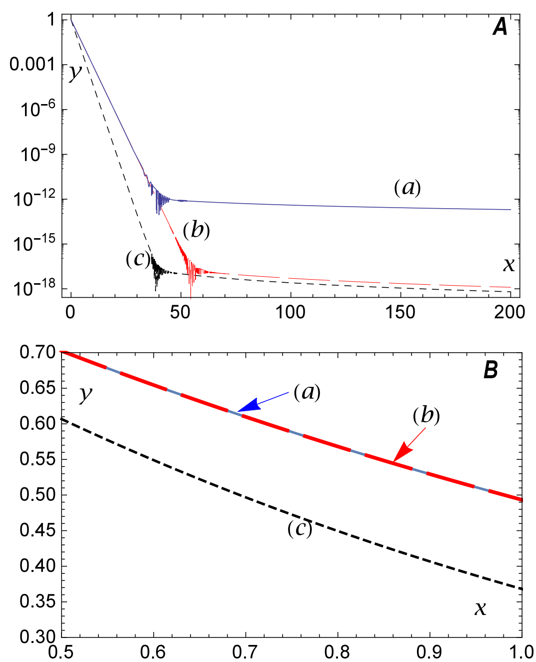

Fig. 1. Decay curves obtained for $\omega_{\rm BW}(m)$ given by Eq. (14). Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — survival probabilities. Panel A: the logarithmic scales, (a) the decay curve $\mathcal{P}_p(t)$, (b) the decay curve $\mathcal{P}_0(t/\gamma)$, (c) the decay curve $\mathcal{P}_0(t)$; Panel B: (a) $\mathcal{P}_p(t)$, (b) $\mathcal{P}_0(t/\gamma)$, (c) $\mathcal{P}_0(t)$.

Results presented in Figs. 1 and 2 show that in the case of $\omega(m)$ having the Breit–Wigner form, the survival probabilities $\mathcal{P}_0(t/\gamma)$ and $\mathcal{P}_p(t)$ overlap for not too long times when $\mathcal{P}_p(t)$ has the canonical, that is the exponential form. This observation confirms conclusions drawn in [1, 2, 4]. On the other

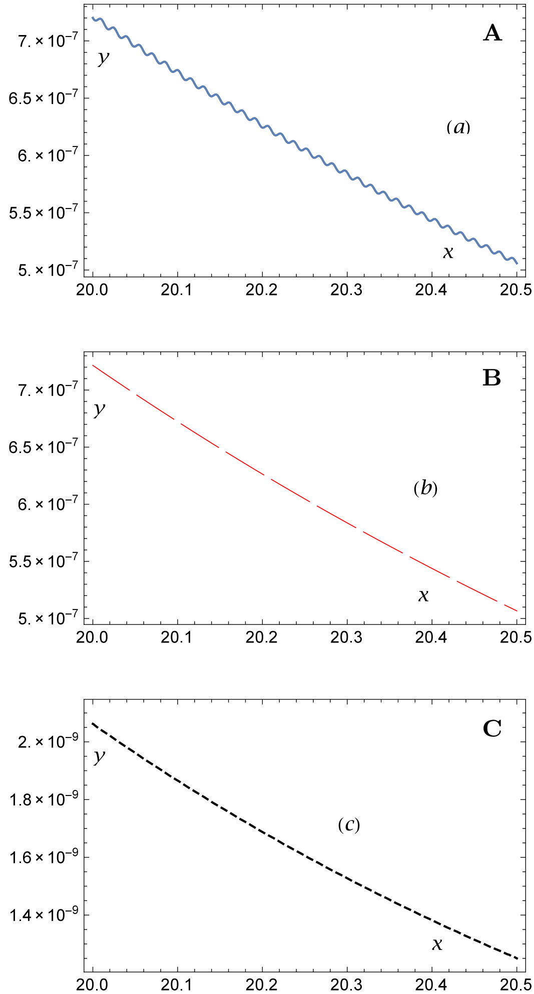

Fig. 2. Decay curves obtained for $\omega_{\text{BW}}(m)$ given by Eq. (14). Axes: $x = t/\tau_0$ and y — survival probabilities: (a) the decay curve $\mathcal{P}_p(t)$, (b) the decay curve $\mathcal{P}_0(t/\gamma)$, (c) the decay curve $\mathcal{P}_0(t)$.

hand, results presented in panel A of Fig. 1 and Fig. 2 show that in the case of moving relativistic unstable particles, the transition times region, when the canonical form of the survival probability $\mathcal{P}_p(t)$ transforms into inverse power-like form of t, begins much earlier than in the case of this particle observed in its rest coordinate system and described by $\mathcal{P}_0(t)$. This observation agrees with results obtained in [4].

To be sure that the above conclusions are valid not only in the approximate case $\omega_{\rm BW}(m)$ of the density of the mass distribution $\omega(m)$, we should consider a more general form of $\omega(m)$. The most general condition for $\omega(m)$ following from (8) is that $\omega(m) \in\ L_1(-\infty,\infty)$. So, if to assume that $\omega(m) \in\ L_1(-\infty,\infty)$ and additionally that $\omega(m) = 0$ for $m < \mu_0$, $\omega(\mu_0 = 0)$ and $\omega(m) \geq 0$ for $m > \mu_0$, that is that

$$\omega(m) = \Theta(m-\mu_0)\,(m-\mu_0)^{\kappa+l}\;\varrho(m)\,, \tag{15}$$

(where $0 \leq \kappa < 1, l = 0,1,2,\ldots;$), and $\varrho(\mu_0) \stackrel{\rm def}{=} \varrho_0 > 0$, $\varrho(m) \geq 0$ for $m > \mu_0$ and $\varrho^{(k)}(m) = \frac{d}{dm}\,\varrho(m)$, $(k = 0,1,\ldots,n)$ exist and they are continuous in $[\mu_0,\infty)$, and limits $\lim_{m\to\mu_0+}\ \varrho^{(k)}(m) \stackrel{\rm def}{=} \varrho_0^{(k)}$ exist, and

$$\lim_{m\to\infty}\,(m-\mu_0)^{\kappa+l}\;\varrho^{(k)}(m) = 0$$

for all above-mentioned k, then one finds for $l = 0$ that in the rest system (see [14, 15])

$$\begin{aligned} a_0(t) \quad \underset{t\to\infty}{\sim} \quad & (-1)\,e^{-i\mu_0 t}\,\Big[\Big(-\frac{i}{t}\Big)^{\kappa+1}\;\Gamma(\kappa+1)\;\varrho_0 \\ & + \kappa\,\Big(-\frac{i}{t}\Big)^{\kappa+2}\;\Gamma(\kappa+2)\;\varrho_0^{(1)}\;+\;\ldots\Big] = a_{\rm lt}(t)\,, \end{aligned} \tag{16}$$

where $\Gamma(x)$ is Euler's gamma function. Hence, one finds that, *e.g.* for $\kappa = 1/2$, the leading term of $a_{\rm lt}(t)$ has the following form:

$$a_{\rm lt}(t)\;\simeq\;(-1)\,e^{-i\mu_0 t}\,\frac{\sqrt{\pi}}{2}\,\Big[\Big(-\frac{i}{t}\Big)^{3/2}\;\varrho_0\;+\ldots\Big]. \tag{17}$$

From an analysis of general properties of the mass (energy) distribution functions $\omega(m)$ of real unstable systems, it follows that they have properties similar to the scattering amplitude, *i.e.*, they can be decomposed into a threshold factor, a pole-function, with a simple pole (often modeled by $\omega_{\rm BW}(m)$) and a smooth form factor $f(m)$ [9, 13]. This means that $\varrho(m)$ in (15) should have the following form $\varrho(m) = \omega_{\rm BW}(m)\,f(m)$, where $f(m) \to 0$ as $m \to \infty$. Guided by this observation, we follow [13] and assume that

$$\omega(m) = N\,\sqrt{m-\mu_0}\;\frac{\sqrt{\Gamma_0}}{(m-m_0)^2 + (\Gamma_0/2)^2}\;e^{-\eta\,\frac{m}{m_0-\mu_0}} \tag{18}$$

with $\eta > 0$. The asymptotic form of the survival amplitude $a_0(t)$ for such a $\omega(m)$ is given by relation (17). Hence, one finds that at late times $t \to \infty$

there is $\mathcal{P}_0(t) \sim 1/t^3$ in the case considered. Decay curves corresponding to $\omega(m)$ defined by (18) were find numerically for the case of the particle decaying in the rest system (the survival probability $\mathcal{P}_0(t)$) as well as for the moving particle (the non-decay probability $\mathcal{P}_p(t)$). Results are presented in Figs. 3 and 4. In order to compare them with the results obtained for $\omega_{\rm BW}(m)$, calculations were performed for the same ratios as in that case: $m_0/\Gamma_0 = p/\Gamma_0 = 1000$, and $\mu_0 = 0$. The ratio $\eta\Gamma_0/(m_0 - \mu_0) \equiv \eta\Gamma_0/m_0$ was chosen to be $\eta\Gamma_0/m_0 = 0.01$ (Fig. 3) and $\eta\Gamma_0/m_0 = 0.006$ (Fig. 4).

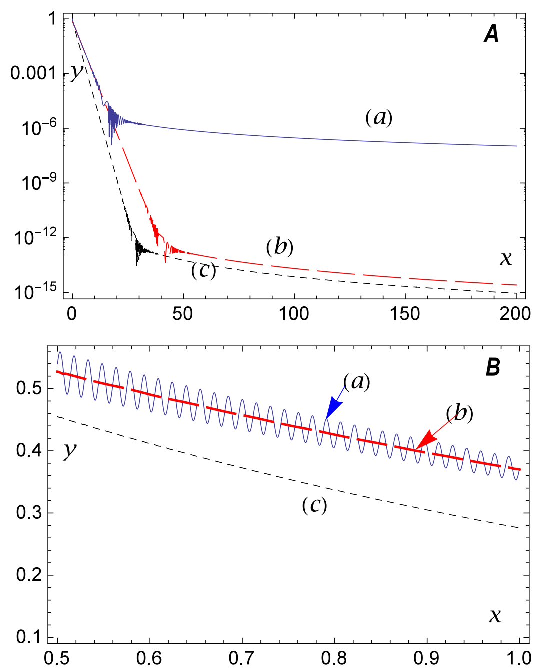

Fig. 3. Decay curves obtained for $\omega(m)$ given by Eq. (18). Axes: $x = t/\tau_0$ and y — survival probabilities; Panel A: the logarithmic scales, (a) the decay curve $\mathcal{P}_p(t)$, (b) the decay curve $\mathcal{P}_0(t/\gamma)$, (c) the decay curve $\mathcal{P}_0(t)$; Panel B: (a) $\mathcal{P}_p(t)$, (b) $\mathcal{P}_0(t/\gamma)$, (c) $\mathcal{P}_0(t)$. The case $\eta\Gamma_0/m_0 = 0.01$.

From Figs. 3, 4, it is seen that in the case of $\omega(m) \neq \omega_{\rm BW}(m)$, *e.g.* when $\omega(m)$ has the form given by Eq. (18), the survival probabilities $\mathcal{P}_0$ and $\mathcal{P}_0(t/\gamma)$ have an analogous form as the corresponding probabilities obtained for $\omega(m) = \omega_{\rm BW}(m)$ both for relatively short times $t \sim \tau_0$ and for long times $t \gg \tau_0$. On the other hand, in the case of the survival probabilities $\mathcal{P}_p(t)$, the difference between decay curves calculated for the density $\omega(m)$ given by formula (18) and for $\omega(m) = \omega_{\rm BW}(m)$ is significant: The decay curves $\mathcal{P}_p$ calculated for $\omega(m)$ defined in (18) have an oscillating form at

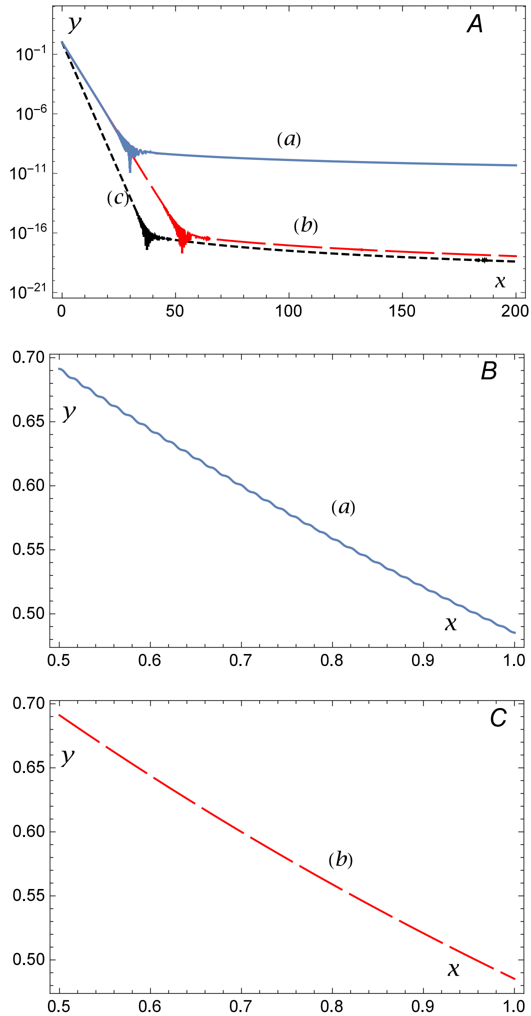

Fig. 4. Decay curves obtained for $\omega(m)$ given by Eq. (18). Axes: $x = t/\tau_0$ and y — survival probabilities. Panel A: the logarithmic scales, (a) the decay curve $\mathcal{P}_p(t)$, (b) the decay curve $\mathcal{P}_0(t/\gamma)$, (c) the decay curve $\mathcal{P}_0(t)$; Panel B: $\mathcal{P}_p(t)$; Panel C: $\mathcal{P}_0(t/\gamma)$. The case $\eta\varGamma_0/m_0 = 0.006$.

times $t \sim \tau_0$ and shorter while those obtained for $\omega_{\rm BW}(m)$ do not have. This is rather unexpected result but it shows that in the case of moving relativistic particles quantum decay processes may have non-classical form even at times shorter than the lifetime.

4. Analysis of masses and velocities of unstable states

It was assumed in Sections 2 and 3 that the momentum $\vec{p}$ of the relativistic unstable particle moving like a free particle is conserved. Using this assumption, one usually concludes that in such a case, the velocity of the particle has to be conserved and constant in time. Such a conclusion is true in the case of the classical particles: In the case of a classical object moving like a free particle, the conservation of the momentum means that the velocity of this object is constant in time. The question is whether such a conclusion is true in the case of moving quantum unstable objects or not. In order to solve this problem, we should analyze relativistic formula for the momentum $\vec{p}$, which within the assumed system of units has the following form: $\vec{p} = m\,\gamma(\vec{v}\,)\,\vec{v}$. In this relation, m is the rest mass of the moving quantum or classical objects and $\vec{v}$ is the velocity of these objects. From the point of view of the quantum theory, the problem is that the state vector $|\phi_p\rangle$ of the form of (6) corresponding to such a quantum object cannot be an eigenvector of the Hamiltonian H (including the case of $\vec{p} = 0$), otherwise, it would be that $\mathcal{P}_p(t) = |\langle\phi_p|\phi_p(t)\rangle|^2 = |\langle\phi_p|\exp\left[-itH\right]\phi_p\rangle|^2 \equiv 1$ for all times t. The fact that the vector $|\phi\rangle$ describing the unstable quantum object is not the eigenvector for H means that the mass (energy) of this object is not defined. Simply, the mass cannot take the exact constant value in this state $|\phi_p\rangle$. In such a case, quantum objects are characterized by the mass (energy) distribution density $\omega(m)$ and the average mass

$$\langle m\rangle = \int_{\mu_0}^{\infty} m\,\omega(m)\,\mathrm{d}m\,,$$

or by the instantaneous mass (energy) $m_\phi(t)$ (see, *e.g.* [23, 24]) but not by the exact value of the mass.

Let us analyze the properties of the instantaneous mass. The instantaneous mass $m_\phi(t)$ (energy) can be found using the exact effective Hamiltonian $h_\phi(t)$ governing the time evolution in the subspace of states spanned by the vector $|\phi\rangle \neq 0$

$$h_\phi(t) = \frac{i}{a_0(t)}\,\frac{\partial a_0(t)}{\partial t}\,, \tag{19}$$

$$\equiv \frac{\langle\phi|H|\phi(t)\rangle}{\langle\phi|\phi(t)\rangle}\,, \tag{20}$$

which results from the Schrödinger equation when one looks for the exact evolution equation for the mentioned subspace of states (for details, see [14, 16, 23–26]), where the system of units $\hbar = c = 1$ is used. It is assumed that the vector $|\phi\rangle$ is not an eigenvector of H: There does not exist any number λ such that $H|\phi\rangle = \lambda|\phi\rangle$.

Within the assumed system of units, the instantaneous mass (energy) of the unstable quantum system in the rest reference frame is the real part of $h_\phi(t)$

$$m_\phi(t) = \Re\left[h_\phi(t)\right], \tag{21}$$

and $\varGamma_\phi(t) = -2\Im\left[h_\phi(t)\right]$ is the instantaneous decay rate.

Using relation (20), one can find some general properties of $h_\phi(t)$ and $m_\phi(t)$. Indeed, if to rewrite the numerator of the right-hand side of (20) as follows

$$\langle\phi|H|\phi(t)\rangle \equiv \langle\phi|H|\phi\rangle\, a_0(t) + \langle\phi|H|\phi(t)\rangle_\perp, \tag{22}$$

where $|\phi(t)\rangle_\perp = Q|\phi(t)\rangle$, $Q = \mathbb{I} - P$ is the projector onto the subspace of decay products, $P = |\phi\rangle\langle\phi|$ and $\langle\phi|\phi(t)\rangle_\perp = 0$, then one can see that there is a permanent contribution of decay products described by $|\phi(t)\rangle_\perp$ to the instantaneous mass (energy) of the unstable state considered. The intensity of this contribution depends on time t. Using (20) and (22), one finds that

$$h_\phi(t) \;=\; \langle\phi|H|\phi\rangle + \frac{\langle\phi|H|\phi(t)\rangle_\perp}{a_0(t)} \tag{23}$$

$$\stackrel{\mathrm{def}}{=} \; \langle\phi|H|\phi\rangle + V_\phi(t)\,. \tag{24}$$

From this relation, one can see that $h_\phi(0) = \langle\phi|H|\phi\rangle$ and $V_\phi(0) = 0$ if the matrix elements $\langle\phi|H|\phi\rangle$ exist. It is because $|\phi(t=0)\rangle_\perp = 0$ and $a_0(t=0) = 1$.

Now, let us assume that $\langle\phi|H|\phi\rangle$ exists and $i\frac{\partial a_0(t)}{\partial t} \equiv \langle\phi|H|\phi;t\rangle$ is a continuous function of time t for $0 \le t < \infty$. If these assumptions are satisfied then $h_\phi(t)$ is a continuous function of time t for $0 \le t < \infty$ and $h_\phi(0) = \langle\phi|H|\phi\rangle$ exists. Now, if to assume that for $0 \le t_1 \neq t_2$ there is $h_\phi(0) = h_\phi(t_1) = h_\phi(t_2) = \text{const}$, then from the continuity of $h_\phi(t)$ immediately follows that there should be $h_\phi(t) = h_\phi(0) \equiv \langle\phi|H|\phi\rangle = \text{const}$ for any $t \ge 0$. Unfortunately, such an observation contradicts implications of (23), (24): From relations (23), (24), one concludes that it is possible if and only if

$$V_\phi(t>0) = 0 \tag{25}$$

for every t such that $0 < t < \infty$. There is $|a_0(t)| > 0$ for $t < \infty$, therefore

$$V_\phi(t>0) = 0 \;\;\Leftrightarrow\;\; \langle\phi|H|\phi(t>0)\rangle_\perp = 0\,, \tag{26}$$

for every $t > 0$ and $t < \infty$. Relation (26) can take place if and only if

$$(\langle\phi|H)^+ \equiv H|\phi\rangle \;\;\perp\;\; |\phi(t>0)\rangle_\perp \quad \text{for all} \quad t>0\,. \tag{27}$$

This last condition leads to the conclusion that

$$\left\{|V_\phi(t>0)| = 0 \quad \text{for every} \quad t>0\right\} \;\;\Leftrightarrow\;\; H|\phi\rangle = \lambda|\phi\rangle\,. \tag{28}$$

This observation means that

$$h_\phi(t) = \text{const} \tag{29}$$

if and only if there is no any decay of the state $|\phi\rangle$ considered (if there is no any transitions between $\mathcal{H}_\| = P\mathcal{H}$ and $\mathcal{H}_\perp$). So, in the case of unstable systems, $h_\phi(t > 0) \neq \text{const}$, which means that the instantaneous mass (energy) $m_\phi(t) \equiv \Re\,[h_\phi(t)]$, and the instantaneous decay rate $\mathit{\Gamma}_\phi(t)$ cannot be constant in time: $m_\phi(t) \neq \text{const}$ and $\mathit{\Gamma}_\phi(t) \neq \text{const}$. Results of numerical calculations presented in Figs. 5–7 (or those one can find in [23, 24]) confirm this conclusion. In Figs. 5–7, the function

$$\kappa(t) = \frac{m_\phi(t) - \mu_0}{m_0 - \mu_0} \tag{30}$$

is presented, which illustrates a typical form of time-varying $m_\phi(t)$. (All calculations were performed for $(m_0 - \mu_0)/\mathit{\Gamma}_0 = 1000$.)

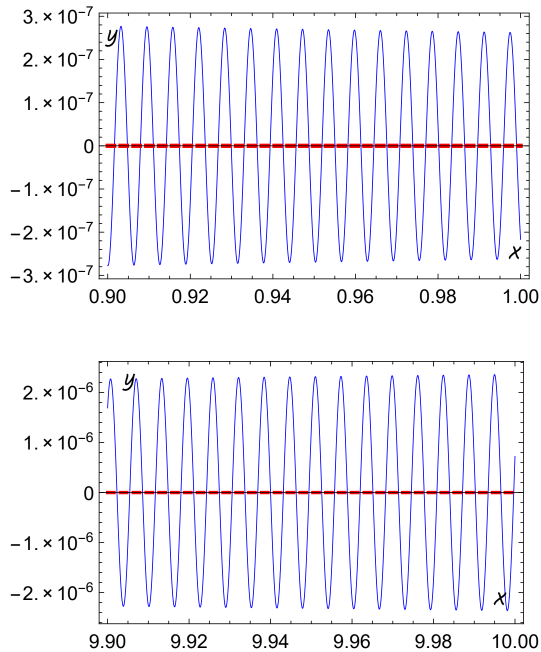

Fig. 5. The instantaneous mass $m_\phi(t)$ as a function of time obtained for $\omega_{\mathrm{BW}}(m)$. Axes: $y = \kappa(t) - 1$, where $\kappa(t)$ is defined by (30); $x = t/\tau_\phi$: Time is measured in lifetimes. The horizontal dashed line represents the value of $m_\phi(t) = m_0$.

As it is seen from Figs. 5, 6, 7, the amplitude of variations of $m_\phi(t)$ needs not be large at relatively short times: It is almost negligible small but these variations always exist (see Figs. 5–7 and results presented in [27]).

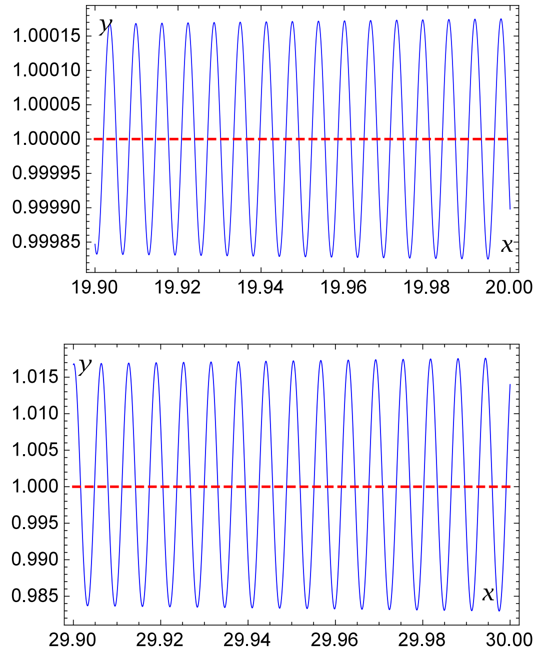

Fig. 6. The instantaneous mass $m_\phi(t)$ as a function of time obtained for $\omega_{\rm BW}(m)$. Axes: $y = \kappa(t)$, where $\kappa(t)$ is defined by (30); $x = t/\tau_\phi$: Time is measured in lifetimes. The horizontal dashed line represents the value of $m_\phi(t) = m_0$.

When the time increases, the amplitude of these variations grows and reaches maximal values for times $t \sim T$. Now, if this particle is a moving relativistic particle, then within the assumed system of units its momentum equals $\vec{p} = m_\phi \, \gamma(\vec{v}\,) \, \vec{v}$, where m_ϕ is the rest mass of the particle ϕ, $\vec{v}$ is the velocity. The total momentum (and energy) of the objects moving like a free particle both quantum and classical must be conserved. Thus, it has to be $\vec{p}(t_1) = \vec{p}(t_2)$, that is $m_\phi(t_1)\,\gamma(\vec{v}\,)\,\vec{v} = m_\phi(t_2)\,\gamma(\vec{v}\,)\,\vec{v}$ for any $t_1 \neq t_2$. It is possible only if changes of $m_\phi(t)$ are compensated by suitable changes of $\gamma(\vec{v}\,)\,\vec{v}$, that is by corresponding changes in the velocity $\vec{v}$. (A similar mechanism was described in [23, 24], where its consequences were analyzed for times of the order of the cross-over time T.) So the principle of conservation of the momentum forces compensation of changes in the instantaneous mass $m_\phi(t)$ through appropriate changes in the velocity of the moving unstable system. (It is a pirouette-like effect.) This is why the assumption $\vec{v} =$ const when considering moving quantum unstable objects leads to the result $P_{\vec{v}}(t) = P_0(\gamma t)$, *i.e.*, to the result never observed in experiments [21]. Thus, the assumption $\vec{p} =$ const mentioned seems to be the only acceptable choice in the case of moving quantum unstable systems (see also a discussion in [18]).

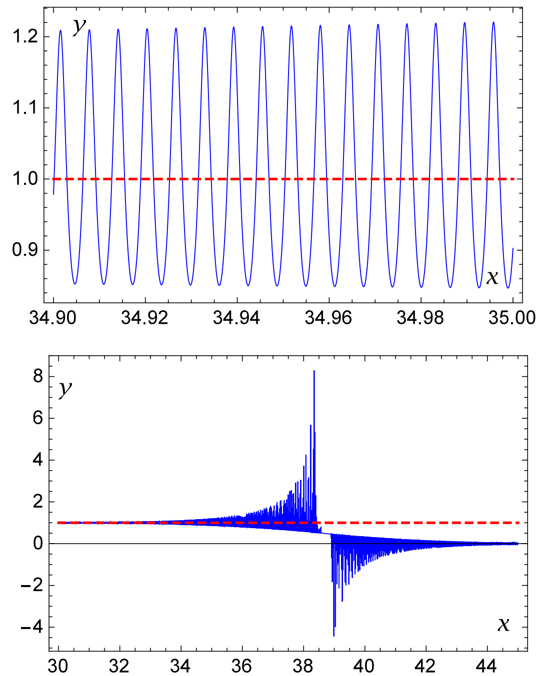

Fig. 7. The same as in Fig. 6 for longer times.

Let us analyze now implications of the observation that the velocity $\vec{v}$ of the quantum unstable system moving like a free particle cannot be constant in time and it has to vary in time $\vec{v} \equiv \vec{v}(t) \neq$ const. This property has an effect that $\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \neq 0$. Now, let us denote by $\mathcal{O}'$ the reference frame which moves together with the moving quantum unstable system considered and in which this system is in rest. This reference frame moves relative to $\mathcal{O}$ with the velocity $\vec{v} = \vec{v}(t) \neq$ const measured in $\mathcal{O}$. The observation that $\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \neq 0$ means that the rest reference frame $\mathcal{O}'$ of the quantum unstable system moving like a free particle cannot be the inertial one.

5. Discussion and conclusions

Let us begin from a general remark: In any case, we should remember that relation (1) is the classical physics relation and that the quantum decay processes are analyzed in this paper. The relativistic time-dilation relation in its form known from classical physics does not need to manifest itself in quantum processes in the same way as in classical physics processes. It is also important to be aware that, as it was shown in [28], the Quantum Field Theory models of the decay processes can be also described within the formalism used in Sec. 2.

All results presented in Figs. 1–4 show decay curves seen by the observer $\mathcal{O}$ in his rest reference frame (curves $\mathcal{P}_0(t/\gamma)$ correspond to the situation when the classical dilation relation (1) is assumed to be true in the case of quantum decay processes). The time (the horizontal axes) in all these figures is the time measured by the observer in his rest system. These results show that the Stefanovich–Shirokov theory [1, 2] predicts such a form of the survival probability $\mathcal{P}_p(t)$ that the expected relation (1) holds to a very good approximation only for times $t \sim \tau_0$ and only for $\omega(m) = \omega_{\rm BW}(m)$. The visible difference between $\mathcal{P}_p(t)$ and $\mathcal{P}_0(t/\gamma)$ takes place at times $t \gg \tau_0$ but this needs not mean that this theory is wrong: To this day, there have been no published reports on experiments analyzing the form of the decay law of moving relativistic unstable particles at times $t \gg \tau_0$ or $t \sim T$ and $t > T$.

Analyzing the results presented in Figs. 1–4, we can conclude that properties of the survival probability of the moving unstable particle, $\mathcal{P}_p(t) = |a_p(t)|^2$, where $a_p(t)$ is calculated using Eq. (13) (*i.e.* the formula derived in [1, 2]), are much more sensitive to the form of $\omega(m)$ than properties of $\mathcal{P}_0(t)$. It is a general observation. Another general conclusion following from these results is that starting from times t from the transition time region, $t > T$, the decay process of moving particles is much slower than one would expect assuming the standard dilation relation (1).

From Figs. 3 and 4, it follows that in the case of moving relativistic unstable particles, the standard relation (1) does not apply in the case of the density $\omega(m)$ of the form of (18) and leads to the wrong conclusions for such densities. Results presented in these figures show also that a conclusion drawn in [1, 2, 4] on the basis of studies of the model defined by the Breit–Wigner density $\omega_{\rm BW}(m)$ that relation (1) is valid for not more than few lifetimes and is true only for the density $\omega_{\rm BW}(m)$, and need not be true for densities $\omega(m)$ having a more general form. Similar limitations concern the result presented in [3], where it is stressed that the approximations used to derive the final result may work only for times no longer than a few lifetimes. What is more, a detailed analysis shows that the final result presented therein was obtained using the non-relativistic limit of $\sqrt{m^2+p^2}$. There was used the following approximation: $\sqrt{m^2+p^2} \simeq m + \frac{p^2}{2m} + \ldots$ (see [3], formula (20) and then (30a), (30b)). So, in general, relation (1) can be considered as sufficiently accurate approximation only for not too long times t if at these times $\mathcal{P}_p(t)$ has the same exponential form as the decay laws obtained within classical physics considerations. If quantum effects force $\mathcal{P}_p(t)$ behave non-classically at these times then relation (1) which is the classical physics relation is not applicable.

In general, as it follows from the results obtained within the considered theory and presented in Figs. 3 and 4 contrary to the standard expectations based on the classical physics time dilation relation of the special relativity, some quantum effects should be registered earlier by the observer $\mathcal{O}$ studying the behavior of moving unstable particles in relation to his rest reference frame than the same effects observed by $\mathcal{O}$ in the case of the particles decaying in the common rest reference frame for the particle and the observer $\mathcal{O}$: The transition times region, that is the time region when contributions from the exponential and late time non-exponential parts of the amplitude $a_p(t)$ or $a_0(t)$ are comparable, which manifest itself as sharp and frequent oscillations of the survival probability, takes place earlier for $\mathcal{P}_p(t)$ (Fig. 2, the curve (a) and Figs. 3 and 4, panel A, curves (a)) than for $\mathcal{P}_0(t)$ (Fig. 2, the curve (c) and Figs. 3 and 4, panel A, curves (c)). The same observation concerns results presented in panels B of Figs. 3 and 4.

These properties that is the form of the decay curves presented in panel A of Figs. 1, 3 and 4 can be easily explained analyzing the equivalent expression of formula (13) for $a_p(t)$

$$a_p(t) \;\equiv\; \int_{\mu_0}^{\infty} \omega(m)\, e^{-\,i\,m\,\gamma_m\,t}\, \mathrm{d}m = a_{<p}(t) + a_{>p}(t)\,, \tag{31}$$

where γ_m can be equivalently written as $\gamma_m \equiv \sqrt{1+\frac{p^2}{m^2}}$ and within the used system of units

$$a_{<p}(t) \;\equiv\; \int_{\mu_0}^{p} \omega(m)\, e^{-\,i\,m\,\gamma_m\,t}\, \mathrm{d}m\,, \tag{32}$$

$$a_{>p}(t) \;\equiv\; \int_{p}^{\infty} \omega(m)\, e^{-\,i\,m\,\gamma_m\,t}\, \mathrm{d}m\,. \tag{33}$$

It is easy to see that for $m < p$, there is $\gamma_m > \sqrt{2}$ and γ_m becomes very large for $m \ll p$, which means that $a_{<p}(t)$ reaches values proper for times t of the order of the crossover time T much earlier comparing with $a_0(t)$ given by formula (8). Therefore, the visible oscillations of decay curves of moving particles can begin earlier than in the case of the particles decaying in the rest system. On the other hand, for $m > p$, one has $\gamma_m < \sqrt{2}$ and for $m \gg p$, we observe that and $\gamma_m \simeq 1$ which shows that contribution of $a_{>p}(t)$ into $a_p(t)$ is almost the same as in the case the of $a_0(t)$. This explains why at very late time the decay curves of moving unstable particles presented in panels A of Figs. 1, 3 and 4 have the same form as in the case of particles decaying in the rest system. The final form of the decay curve $\mathcal{P}_p(t)$ of the moving unstable particle with a definite momentum depends on the balance of contributions to $a_p(t)$ coming from amplitudes $a_{<p}(t)$ and $a_{>p}(t)$ and on the interference between them

$$\mathcal{P}_p(t) = |a_p(t)|^2 \equiv |a_{<p}(t) + a_{>p}(t)|^2 \,. \tag{34}$$

The balance between contributions of $|a_{<p}(t)|^2$ and $|a_{>p}(t)|^2$ into $\mathcal{P}_p(t)$ depends on the form and properties of $\omega(m)$.

In all figures, the time is measured in lifetimes τ_0. So, fluctuations of $\mathcal{P}_p(t)$ calculated for the density $\omega(m) = \omega_{\rm BW}(m)$ and presented in Fig. 2 (the decay curve (a)) are rather unmeasurable: They take place at $t \sim 20\tau_0$. On the other hand, similar fluctuations appearing in the case of $\omega(m)$ given by Eq. (18) and presented in Fig. 3 (panel B, the decay curve (a)) and Fig. 4 (panel B) take place at times $t \leq \tau_0$ and longer. This means that the probability that they can be registered in some cases is very high.

Results of Sec. 4 explain the growing with time differences between $\mathcal{P}_p(t)$ and $\mathcal{P}_0(t/\gamma)$. Note that $\mathcal{P}_0(t/\gamma)$ corresponds to the classical physics expectations. The cause of these differences is a pure quantum effect: fluctuations in time of the instantaneous mass $m_\phi(t)$ of the unstable quantum system. Simply for relatively long times, fluctuations of this instantaneous mass $m_\phi(t)$ become significant and grow with time t. Hence, variations of $\vec{v}(t)$ have to be larger and larger. This means that deviations from the classical physics predictions become also large and grow with the increasing fluctuations of $m_\phi(t)$.

Let us have a look again at Figs. 3 and 4. A more detailed analysis of panels B in these figures indicates striking similarity of the decay curves $\mathcal{P}_p(t)$ presented there by solid lines (curves $\mathcal{P}_0(t/\gamma)$ are represented there by long-dashed lines) to the results presented in Figs. 3–5 in [5] known as the "GSI anomaly". This suggests that the nature of GSI anomaly is probably purely quantum-mechanical. (Readers can meet a several theoretical proposals that attempt to explain the GSI anomaly: There are authors using the interference of two mass eigenstates (see, *e.g.* [29]); Some authors use neutrino oscillations [30]; In [31], time is used as a dynamical variable and the time representation is used; In [32], a truncated Breit–Wigner mass distribution with an energy-dependent decay with $\mathit{\Gamma}$ such that $\omega(m) = 0$ for $m < \mathit{\Lambda}_1$ and $\omega(m) = 0$ for $m > \mathit{\Lambda}_2 > \mathit{\Lambda}_1$ is applied, and so on.)

Let us make one more observation. Note that from properties of the relativistic expression $\mathcal{P}_p(t)$, it follows that within the considered theory the number of unstable particles which are able to survive up to times t longer than the transition time T is much greater than one would expect performing suitable estimations using $\mathcal{P}_0(t/\gamma)$ (see results presented in panels A of Figs. 1, 3 and 4) and that the decay process at times $t > T$ is significantly slower than it results from the properties of $\mathcal{P}_0(t/\gamma)$. These properties seem to be important when one analyzes some accelerator experiments with unstable particles of high energies or results of observations of some astrophysical and cosmological process: In many astrophysical processes, an extremely huge numbers of unstable particles are created and they all are moving with relativistic velocities. These numbers are so huge that many of them may survive up to times $t \sim T$ or even to much longer times $t \gg T$. So, taking into account the results presented in panels A of the above-mentioned figures, one can conclude that at asymptotically late times $t > T$, much more unstable particles may be found undecayed than an observer from Earth expects considering the classical relation (1).

All the above conclusions following from the results presented in Figs. 1–3 are the consequence of the form of the amplitude $a_p(t)$ derived in [1, 2] and briefly described in Sec. 2. The question is if this amplitude reflects correctly real properties of the moving unstable quantum objects (particles) and thus if the possible effects predicted using this $a_p(t)$ and described in this section can occur: Only the suitable experiments can decide about this. The problem is that all known tests of relation (1) were performed for times $t \sim \tau_0$ (where τ_0 is the lifetime) (see, *e.g.* [33, 34]). In the light of the results presented in this paper and of the above discussion, one concludes that the problem of the fundamental importance is to examine how the relativistic dilation really works in quantum decay processes of moving relativistic particles for very long times: From times longer than a few lifetimes up to times longer than the cross-over time T. Only this kind of an experiment can decide how time dilation being classical physics relation is manifested in the quantum decay processes of relativistic particles.

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