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Stringy scaling of n-point Regge string scattering amplitudes

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Abstract: We discover a *stringy scaling* behavior for a class of n-point Regge string scattering amplitudes (RSSA). The number of independent kinematics variables is found to be reduced by dim$\mathcal{M}$.

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Contents

1 Introduction

Recent development of string scattering amplitudes (SSA) has shown that a class of 4-point SSA form representations of the SL$(K+3, C)$ group [1, 2]. These are SSA with three tachyons and one arbitrary string states

$$\left|r_n^T, r_m^P, r_l^L\right\rangle = \prod_{n>0} \left(\alpha_{-n}^T\right)^{r_n^T} \prod_{m>0} \left(\alpha_{-m}^P\right)^{r_m^P} \prod_{l>0} \left(\alpha_{-l}^L\right)^{r_l^L} |0, k\rangle \tag{1.1}$$

where $e^P = \frac{1}{M}(E, \mathrm{k}, 0) = \frac{k_2}{M_2}$ is the momentum polarization, $e^L = \frac{1}{M}(\mathrm{k}, E, 0)$ is the longitudinal polarization and $e^T = (0, 0, 1)$ is the transverse polarization on the $(2+1)$-dimensional scattering plane. Note that SSA of three tachyons and one arbitrary string states with polarizations orthogonal to the scattering plane vanish. In addition to the mass level $M^2 = 2(N-1)$ with

$$N = \sum_{\substack{n,m,l>0 \\ \{r_j^X \neq 0\}}} \left(n r_n^T + m r_m^P + l r_l^L \right), \tag{1.2}$$

another important index K was identified for the state in eq. (1.1) [3]

$$K = \sum_{\substack{n,m,l>0 \\ \{r_j^X \neq 0\}}} (n + m + l) \tag{1.3}$$

where $X = (T, P, L)$ and one has put $r_n^T = r_m^P = r_l^L = 1$ in eq. (1.2) in the definition of K in eq. (1.3). Intuitively, K counts the number of variety of the α_{-j}^X oscillators in eq. (1.1).

The representation bases of the above subclass of 4-point SSA was soon extended to all 4-point SSA with arbitrary four string states, and eventually to all n-point SSA with arbitrary n string states [4, 5]. It is thus important to study whether other known interesting characteristics of the 4-point SSA can be similarly extended to the n-point SSA [6–8]. One such characteristics of the 4-point SSA is the existence of infinite linear relations and their associated *constant ratios*, independent of the scattering angle ϕ, among hard SSA (HSSA) at each fixed mass level of the open bosonic string spectrum. These infinite linear relations and their associated constant ratios were first conjectured by Gross [9, 10] and later explicitly calculated by the method of decoupling of zero-norm states in [11–15].

Indeed, in one of the authors' previous publications [16], we discovered a general *stringy scaling* behavior for all n-point HSSA to all string loop orders. For the simplest case of $n = 4$, the stringy scaling behavior reduces to the infinite linear relations and the constant ratios of HSSA at each mass level mentioned above. For this case, the ratios are independent of 1 scattering angle ϕ and thus the number of independent kinematics variable reduced from 1 to 0 with $\dim\mathcal{M} = 1 - 0 = 1$. In general higher n-point HSSA, the stringy scaling behavior implies that the number of independent kinematics variables of the ratios reduced by $\dim\mathcal{M}$ [16]. See the definition of $\dim\mathcal{M}$ in eq. (2.20) and eq. (2.25). As a result, the linear relations and their associated constant ratios of 4-point HSSA persist only in the parameter spaces $\mathcal{M}$ for the cases of higher n-point HSSA [16]. See the example of constant ratios calculated among 6-point HSSA in eq. (2.24).

In this paper, we will extend our calculation of stringy scaling behavior of HSSA to the case of Regge SSA (RSSA). We will demonstrate a stringy scaling behavior for a class of n-point ($n \geq 5$) RSSA with $m = q = 0$ (see eq. (3.4)), and the number of independent kinematics variables is again found to be reduced by $\dim\mathcal{M}$. However, in contrast to the calculation of complete 4-point RSSA ($m \neq 0$, $q \neq 0$), see the difference between eq. (3.3) and eq. (4.24).

This paper is organized as following. In section 2, we review and give a detailed calculation of the stringy scaling behavior of HSSA [16]. Section 3 and 4 are the main

JHEP09(2023)111

parts of this paper and we extend the calculation of HSSA to the stringy scaling of RSSA. We will derive a stringy scaling behavior for a class of n-point RSSA with arbitrary n in section 4. A brief conclusion was given in section 5. In appendix A, we review examples of ratios among HSSA. In appendix B, we give a detailed saddle-poing calculation of n-point HSSA which was briefly discussed in [16].

2 The hard stringy scaling

A brief report on stringy scaling of n-point hard string scattering amplitudes (HSSA) was recently given in [16]. In this section, we will first give a detailed calculation of hard string scaling behavior. This can also be served as a preparartion for the calculation of stringy scaling of Regge string scattering amplitudes (RSSA) to be discussed in section 3 and section 4.

2.1 Stringy scaling of 4-point HSSA

The first stringy scaling was conjectured by Gross in 1988 [9] which claimed that all 4-point HSSA ($E \to \infty$, fixed ϕ) at each fixed mass level share the same functional form. That is, all HSSA at each fixed mass level are proportional to each other with *constant* ratios independent of the scattering angle ϕ.

To show this remarkable behavior, the starting point is to apply the 4-point l-loop stringy on-shell Ward identities [11, 12]

$$\langle V_1 \chi V_3 V_4 \rangle_{\text{l-loop}} = 0 \tag{2.1}$$

in the hard scattering limit. In eq. (2.1) V_j above can be any string vertex and the second vertex χ is the vertex of a zero-norm state (ZNS). In the hard scattering limit, the kinematic set up is shown in figure 1, and the components of polarization orthogonal to the scattering plane are subleading order in energy. On the other hand, it can be shown that at each fixed mass level $M^2 = 2(N-1)$ only states of the following form [14, 15] (in the hard scattering limit $e^P \simeq e^L$)

$$|N, 2m, q\rangle = \left(\alpha_{-1}^T\right)^{N-2m-2q} \left(\alpha_{-1}^L\right)^{2m} \left(\alpha_{-2}^L\right)^q |0; k\rangle \tag{2.2}$$

are leading order in energy.

There are two types of physical ZNS in the old covariant first quantized open bosonic string spectrum: [17]

$$\text{Type I}: L_{-1}|y\rangle, \qquad \text{where } L_1|y\rangle = L_2|y\rangle = 0,\ L_0|y\rangle = 0; \tag{2.3}$$

$$\text{Type II}: \left(L_{-2} + \frac{3}{2}L_{-1}^2\right)|\tilde{y}\rangle, \text{ where } L_1|\tilde{y}\rangle = L_2|\tilde{y}\rangle = 0,\ (L_0+1)|\tilde{y}\rangle = 0.\ (D = 26 \textbf{ only}). \tag{2.4}$$

(1) We first consider χ to be the type I hard ZNS (HZNS) calculated from Type I ZNS

$$\begin{aligned} L_{-1}|N-1,2m-1,q\rangle &= (M\alpha_{-1}^L + \alpha_{-2}^L\alpha_1^L + \underbrace{\alpha_{-2}^T\alpha_1^T + \alpha_{-3}\cdot\alpha_2 + \cdots}_{\text{irrelevant}})|N-1,2m-1,q\rangle \\ &\simeq M|N,2m,q\rangle + (2m-1)|N,2m-2,q+1\rangle \end{aligned} \tag{2.5}$$

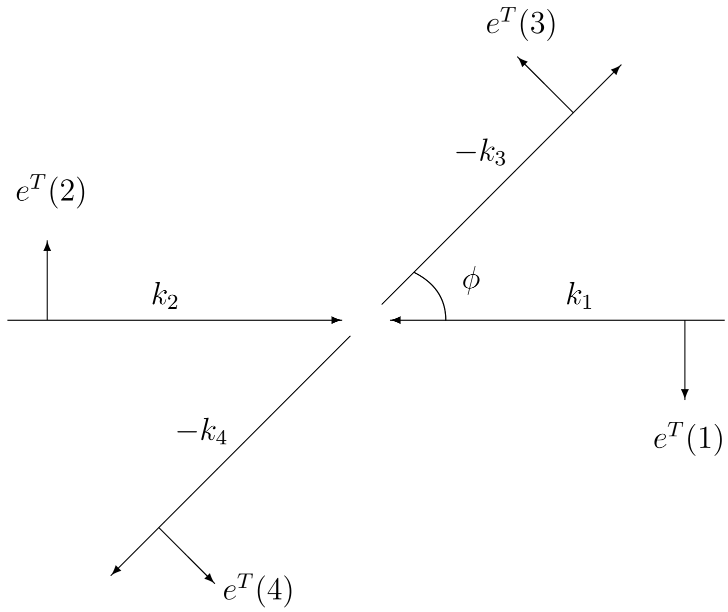

Figure 1. Kinematic variables in the center of mass frame.

where many terms are omitted because they are not of the form of eq. (2.2). This implies the following relation among 4-point amplitudes of three arbitrary string states and one high energy string state of eq. (2.2) (we keep only tensor indice of the state in eq. (2.2) in $\mathcal{T}^{(N,2m,q)}$)

$$\mathcal{T}^{(N,2m,q)} = -\frac{2m-1}{M}\mathcal{T}^{(N,2m-2,q+1)}. \tag{2.6}$$

Using this relation repeatedly, we get

$$\mathcal{T}^{(N,2m,q)} = \frac{(2m-1)!!}{(-M)^m}\mathcal{T}^{(N,0,m+q)}. \tag{2.7}$$

(2) Next, we consider another class of HZNS calculated from type II ZNS

$$\begin{aligned} L_{-2}|N-2,0,q\rangle &= \left(\frac{1}{2}\alpha_{-1}^T\alpha_{-1}^T + M\alpha_{-2}^L + \underbrace{\alpha_{-3}\cdot\alpha_1 + \cdots}_{\text{irrelevant}}\right)|N-2,0,q\rangle \\ &\simeq \frac{1}{2}|N,0,q\rangle + M|N,0,q+1\rangle. \end{aligned} \tag{2.8}$$

Again, irrelevant terms are omitted here. From this we deduce that

$$\mathcal{T}^{(N,0,q+1)} = -\frac{1}{2M}\mathcal{T}^{(N,0,q)}, \tag{2.9}$$

which leads to

$$\mathcal{T}^{(N,0,q)} = \frac{1}{(-2M)^q}\mathcal{T}^{(N,0,0)}. \tag{2.10}$$

In conclusion, the decoupling of ZNS in eq. (2.7) and eq. (2.10) leads to constant ratios among 4-point HSSA [11, 12, 14, 15]

$$\frac{\mathcal{T}^{(N,2m,q)}}{\mathcal{T}^{(N,0,0)}} = \frac{(2m)!}{m!}\left(\frac{-1}{2M}\right)^{2m+q}. \quad (\textbf{independent of } \phi\ \textbf{!}) \tag{2.11}$$

In eq. (2.11) $\mathcal{T}^{(N,2m,q)}$ is the 4-point HSSA of any string vertex V_j with $j = 1, 3, 4$ and V_2 is the high energy state in eq. (2.2); while $\mathcal{T}^{(N,0,0)}$ is the 4-point HSSA of any string vertex V_j with $j = 1, 3, 4$, and V_2 is the leading Regge trajectory string state at mass level N. Note that we have omitted the tensor indice of V_j with $j = 1, 3, 4$ and keep only those of V_2 in $\mathcal{T}^{(N,2m,q)}$.

2.2 Stringy scaling of higher point ($n \geq 5$) HSSA

It is tempted to extend the stringy scaling behavior of 4-point SSA derived in the previous subsection to the higher point SSA. The n-point stringy on-shell Ward identities can be written as

$$\langle V_1 \chi V_3 \cdots V_n \rangle_{\text{1-loop}} = 0 \tag{2.12}$$

where χ again is the vertex of a ZNS. We begin the discussion with a simple kinematics regime on the scattering plane.

2.2.1 On the scattering plane

In the hard scattering limit on the scattering plane, the space part of momenta k_j ($j = 3, 4, \cdots, n$) form a closed 1-chain with $(n-2)$ sides due to momentum conservation. It turned out that all the 4-point calculation in the previous subsection persist and one ends up with eq. (2.11) again [16]. However, while for $n = 4$ the *ratios* are independent of 1 scattering angle ϕ, for $n = 5$, the ratios are independent of 3 kinematics variables (2 angles and 1 fixed ratio of two infinite energies) or, for simplicity, 3 scattering "angles". For $n = 6$, there are 5 scattering "angles" etc.

2.2.2 Out of the scattering plane

The general high energy states at each fixed mass level $M^2 = 2(N-1)$ can be written as [16]

$$|\{p_i\}, 2m, q\rangle = \left(\alpha_{-1}^{T_1}\right)^{N+p_1}\left(\alpha_{-1}^{T_2}\right)^{p_2}\cdots\left(\alpha_{-1}^{T_r}\right)^{p_r}\left(\alpha_{-1}^{L}\right)^{2m}\left(\alpha_{-2}^{L}\right)^{q}|0;k\rangle \tag{2.13}$$

where $\sum_{i=1}^{r} p_i = -2(m+q)$ with $r \leq 24$. In eq. (2.13), T_j is the jth transverse direction orthogonal to k_2. For higher dimensional scattering space, one generalizes the transverse polarization $e^T = (0, 0, 1)$ to $e^{\hat{T}} = (0, 0, \vec{\omega})$ where

$$\omega_i = \cos\theta_i \prod_{\sigma=1}^{i-1} \sin\theta_\sigma \text{with } i = 1, \cdots, r,\ \ \theta_r = 0 \tag{2.14}$$

are the solid angles in the transverse space spanned by 24 transverse directions e^{T_i}. Note that $\alpha_{-1}^{\hat{T}} = \alpha_{-1} \cdot e^{\hat{T}}$ etc. With $\left(\alpha_{-1}^{T_i}\right) = \left(\alpha_{-1}^{\hat{T}}\right)\omega_i$, we easily obtain

$$\left(\alpha_{-1}^{T_1}\right)^{N+p_1} \left(\alpha_{-1}^{T_2}\right)^{p_2} \cdots \left(\alpha_{-1}^{T_r}\right)^{p_r} \left(\alpha_{-1}^{L}\right)^{2m} \left(\alpha_{-2}^{L}\right)^{q} |0;k\rangle$$
$$= \left(\omega_1^N \prod_{i=1}^{r} \omega_i^{p_i}\right) \left(\alpha_{-1}^{\hat{T}}\right)^{N-2m-2q} \left(\alpha_{-1}^{L}\right)^{2m} \left(\alpha_{-2}^{L}\right)^{q} |0;k\rangle\,, \tag{2.15}$$

which leads to the ratios of n-point HSSA [16]

$$\frac{\mathcal{T}^{(\{p_i\},2m,q)}}{\mathcal{T}^{(\{0_i\},0,0)}} = \frac{(2m)!}{m!}\left(\frac{-1}{2M}\right)^{2m+q} \prod_{i=1}^{r} \omega_i^{p_i} \tag{2.16}$$

where $\mathcal{T}^{(\{0_i\},0,0)}$ is the HSSA of leading Regge trajectory state at mass level $M^2 = 2(N-1)$. It is important to note that the number of kinematics variables dependence in the ratios of eq. (2.16) reduced. This stringy scaling behavior of n-point ($n \geq 5$) HSSA is the generalization of that of 4-point HSSA in eq. (2.11). Since the result of ZNS calculation in eq. (2.16) is based on the stringy Ward identity in eq. (2.12), The ratios calculated in eq. (2.16) are valid to all string loop orders.

2.3 Degree of stringy scaling

We see in the previous section that for the simple case with $n = 4$ and $r = 1$, one has two variables, s and t (or E, ϕ). The ratios of all HSSA are independent of the scattering angle ϕ and we will call the degree of the scaling $\dim\mathcal{M} = 1$. The dependence of the number of kinematics variable reduced from 1 to 0 and we have $1 - 0 = \dim\mathcal{M} = 1$. (see the definition of $\mathcal{M}$ below)

For the general n-point HSSA with $r \leq 24$, $d = r+2$, we have k_j vector with $j = 1, \cdots, n$ and $k_j \in R^{d-1,1}$. The number of kinematics variables is $n(d-1) - \frac{d(d+1)}{2}$. Indeed, as $p = E \to \infty$, that implies $q_j \to \infty$ in the hard limit, we define the 26-dimensional momenta in the CM frame to be

$$\begin{aligned}
k_1 &= (E, -E, 0^r)\,,\\
k_2 &= (E, +E, 0^r)\,,\\
&\;\vdots\\
k_j &= \left(-q_j, -q_j\Omega_1^j, -q_j\Omega_2^j, \cdots, -q_j\Omega_r^j, -q_j\Omega_{r+1}^j\right)
\end{aligned} \tag{2.17}$$

where $j = 3, 4, \cdots, n$, and

$$\Omega_i^j = \cos\phi_i^j \prod_{\sigma=1}^{i-1} \sin\phi_\sigma^j \text{ with } \phi_{j-1}^j = 0,\;\; \phi_{i>r}^j = 0 \text{ and } r \leq \min\{n-3, 24\} \tag{2.18}$$

are the solid angles in the $(j-2)$-dimensional spherical space with $\sum_{i=1}^{j-2}\left(\Omega_i^j\right)^2 = 1$. In eq. (2.17), 0^r denotes the r-dimensional null vector. The condition $\phi_{j-1}^j = 0$ in eq. (2.18) was chosen to fix the frame by using the rotational symmetry.

The independent kinematics variables can be chosen to be some φ_i^j and some fixed ratios of infinite q_j. For the kinematics parameter space $\mathcal{M}$ defined by [16]

$$\omega_j \,(\text{kinematics parameters with } E\to\infty) = \text{fixed constant} \quad (j=2,\cdots,r), \tag{2.19}$$

we can count the dimension of $\mathcal{M}$ to be [16]

$$\dim\mathcal{M} = n\,(d-1) - \frac{d\,(d+1)}{2} - 1 - (r-1) = \frac{(r+1)\,(2n-r-6)}{2} \tag{2.20}$$

where $r = d-2$ is the number of transverse directions e^{T_i}. In sum, the ratios among n-point $HSSA$ with $r \leq 24$ are constants and independent of the scattering "angles" in the kinematic regime $\mathcal{M}$.

2.3.1 Examples

(1). For $n=5$ and $r=2$, $d=r+2=4$ and one has $n\,(d-1) - \frac{d(d+1)}{2} = 5$ parameters (r_1 is the ratio of two infinite energies)

$$E, \phi_2^3, \phi_2^4, \phi_3^4, r_1. \tag{2.21}$$

In the hard scattering limit $E\to\infty$, for $\theta_1 = fixed$ we get $\dim\mathcal{M} = 3$.

(2). For $n=6$ and $r=3$, the ratios of 6-point HSSA depends only on 2 variables θ_1 and θ_2 instead of 8 "angles" and $\dim\mathcal{M}=6$. For this case, $\mathcal{M}$ is defined by

$$\theta_j\,(8 \text{ kinematics parameters}) = \text{fixed constant}, \;\; j=1,2, \tag{2.22}$$

and the ratios [16]

$$\frac{\mathcal{T}^{(\{p_1,p_2,p_3\},2m,q)}}{\mathcal{T}^{(\{0,0,0\},0,0)}} = \frac{(2m)!}{m!}\left(\frac{-1}{2M}\right)^{2m+q} (\cos\theta_1)^{p_1}\,(\sin\theta_1\cos\theta_2)^{p_2}\,(\sin\theta_1\sin\theta_2)^{p_3} \tag{2.23}$$

are independent of kinematics parameters in the space $\mathcal{M}$. For example, for say $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{6}$, we get the ratios among 6-point HSSA

$$\frac{\mathcal{T}^{(\{p_1,p_2,p_3\},2m,q)}}{\mathcal{T}^{(\{0,0,0\},0,0)}} = \left(-\frac{1}{M}\right)^{2m+q} (2m-1)!! \left(\frac{1}{2}\right)^{p_2+p_3} \left(\sqrt{3}\right)^{p_3}. \tag{2.24}$$

These ratios for higher point HSSA are one example of generalization of previous ratios calculated in eq. (2.11) for the case 4-point HSSA.

2.3.2 General cases

In general, in the hard scattering limit, the number of scattering "angles" dependence on ratios of n-point $HSSA$ with $r \leq 24$ reduces by $\dim\mathcal{M}$. For a given (n,r), we can calculate some examples of $\dim\mathcal{M}$ [16]

$\dim\mathcal{M}$	$r=1$	$r=2$	$r=3$	$r=4$
$n=4$	1			
$n=5$	3	3		
$n=6$	5	6	6	
$n=7$	7	9	10	10

(2.25)

Note that for the $n=4$ and $r=1$ case, one obtains the previous 4-point case in eq. (2.11).

3 Stringy scaling of Regge string scattering amplitudes

Another important high-energy regime of 4-point SSA is the fixed momentum transfer regime which contains complementary information of the theory. That is in the kinematic regime

$$s \to \infty, \qquad \sqrt{-t} = \text{fixed}, \quad \left(\text{but } \sqrt{-t} \neq \infty\right). \tag{3.1}$$

In this regime, the number of high-energy SSA is much more numerous than that of the fixed angle regime. One of the reason is that in contrast to the identification $e^P \simeq e^L$ in the hard scattering limit, e^P *does not* approach to e^L in the Regge scattering limit. For example, at mass level $M^2 = 4$ of open bosonic string, there are only 4 HSSA while there are 22 RSSA [31, 32]. On the other hand, in the Regge regime both the saddle-point method and the method of decoupling of zero-norm states adopted in the calculation of fixed angle regime do not apply.

The complete leading order high-energy open string states in the Regge regime at each fixed mass level $N = \sum_{n,m,l>0} np_n + mq_m + lr_l$ are

$$|v_n, q_m, r_l\rangle = \prod_{n>0} (\alpha^T_{-n})^{v_n} \prod_{m>0} (\alpha^P_{-m})^{q_m} \prod_{l>0} (\alpha^L_{-l})^{r_l} |0, k\rangle. \tag{3.2}$$

It turned out that the 4-pont RSSA of three tachyons and states in eq. (3.2) are NOT proportional to each other, and the ratios are t-dependent functions. However, it was shown that for the RSSA $A^{(N,2m,q)}$ with $v_1 = N - m - q$, $r_1 = 2m$ and $r_2 = q$ and all others 0 in eq. (3.2), one can extract the ratios of hard string scatterings in eq. (2.11) from $A^{(N,2m,q)}$ [33–35]

$$\lim_{\tilde{t}' \to \infty} \frac{A^{(N,2m,q)}}{A^{(N,0,0)}} = \frac{T^{(N,2m,q)}}{T^{(N,0,0)}} = \frac{(2m)!}{m!} \left(\frac{-1}{2M}\right)^{2m+q} \tag{3.3}$$

where $\tilde{t}' = t + M_2^2 - M_3^2$. It is thus reasonable to expect that for the n-point ($n \geq 5$) RSSA with $n-1$ tachyons and some subset of the high-energy states in eq. (2.13), the RSSA show similar stringy scaling behavior as in eq. (2.16) of HSSA.

In this paper, we will consider a class of n-point ($n \geq 5$) RSSA with $n-1$ tachyons and one high-energy state at mass level N

$$|\{p_i\}, 0, 0\rangle = \left(\alpha^{T_1}_{-1}\right)^{N+p_1} \left(\alpha^{T_2}_{-1}\right)^{p_2} \cdots \left(\alpha^{T_r}_{-1}\right)^{p_r} |0; k\rangle\,, \tag{3.4}$$

which is obtained by setting $m = q = 0$ in eq. (2.13). We will show that these RSSA show stringy scaling behavior for arbitrary n similar to that we obtained for the HSSA in eq. (2.16).

There are many different Regge regimes for the n-point ($n \geq 5$) RSSA. To specify the Regge regime, we first discuss the system of kinematics variables we will use. The standard kinematics variables commonly adopted for the n-point scatterings can be defined as following. One first defines the $(n-3)$ s variables

$$s_{12} = -(k_1 + k_2)^2, s_{123} = -(k_1 + k_2 + k_3)^2, \cdots, s_{1,\cdots,n-2} = -(k_1 + \cdots + k_{n-2})^2, \tag{3.5}$$

and then defines the $\frac{(n-2)(n-3)}{2}$ t variables

$$\begin{aligned} t_{23} &= -\left(k_2+k_3\right)^2, t_{24} = -\left(k_2+k_4\right)^2, \cdots, t_{2,n-1} = -\left(k_2+k_{n-1}\right)^2, \\ t_{34} &= -\left(k_3+k_4\right)^2, \cdots, t_{3,n-1} = -\left(k_3+k_{n-1}\right)^2, \\ &\vdots \\ t_{n-2,n-1} &= -\left(k_{n-2}+k_{n-1}\right)^2, \end{aligned} \tag{3.6}$$

which amount to $\frac{n(n-3)}{2}$ independent kinematics variables.

For our purpose in the calculation of this paper, we will adopt another system of independent kinematics variables. We use the notation $k_{ij} \equiv k_i \cdot k_j$ to define the following $\frac{n(n-3)}{2}$ independent kinematics variables

$$\begin{aligned} &k_{12}, k_{13}, k_{14}, \cdots, k_{1,n-2}, \\ &k_{23}, k_{24}, k_{25}, \cdots, k_{2,n-1}, \\ &k_{34}, k_{35}, \cdots, k_{3,n-1}, \\ &\vdots \\ &k_{n-3,n-2}, k_{n-3,n-1}, \\ &k_{n-2,n-1}. \end{aligned} \tag{3.7}$$

For later use, we also define

$$k_{1,\cdots,i-1,i} = k_{1,\cdots,i-1} + \sum_{j=1}^{i-1} k_{ji}, \tag{3.8}$$

which means, for example,

$$k_{123} = k_{12}+k_{13}+k_{23}, k_{1234} = k_{123}+k_{14}+k_{24}+k_{34}, k_{12345} = k_{1234}+k_{15}+k_{25}+k_{35}+k_{45}. \tag{3.9}$$

3.1 The 5-point and 6-point Regge stringy scaling

Let's begin with the calculation of 5-point RSSA with $r = 2$ in eq. (3.4). The kinematics are

$$\begin{aligned} k_1 &= \left(\sqrt{p^2+M_1^2}, -p, 0, 0\right), \\ k_2 &= \left(\sqrt{p^2+M_2^2}, p, 0, 0\right), \\ k_3 &= \left(-\sqrt{q_3^2+M_3^2}, -q_3\cos\phi_1^3, -q_3\sin\phi_1^3, 0\right), \\ k_4 &= \left(-\sqrt{q_4^2+M_4^2}, -q_4\cos\phi_1^4, -q_4\sin\phi_1^4\cos\phi_2^4, -q_4\sin\phi_1^4\sin\phi_2^4\right), \\ k_5 &= \left(-\sqrt{q_5^2+M_5^2}, -q_5\cos\phi_1^5, -q_5\sin\phi_1^5\cos\phi_2^5, -q_5\sin\phi_1^5\sin\phi_2^5\right). \end{aligned} \tag{3.10}$$

JHEP09(2023)111

During the calculation, we will keep record of the notations used for each step so that eventually we can generalize the calculation to the case of n-point RSSA. The amplitude of state

$$\left(\alpha_{-1}^{T_1}\right)^{N+p_1}\left(\alpha_{-1}^{T_2}\right)^{p_2}|0,k\rangle\,,p_1+p_2=0 \tag{3.11}$$

and 4 tachyon states can be written as

$$\begin{aligned} A^{\{p_1,p_2\},0,0} = & \int_0^1 dx_3 \int_0^{x_3} dx_2 \times x_2^{k_{12}} x_3^{k_{13}} \left(x_3-x_2\right)^{k_{23}} \left(1-x_2\right)^{k_{24}} \left(1-x_3\right)^{k_{34}} \\ & \times \left[\frac{k_3^{T_1}}{x_3-x_2}+\frac{k_4^{T_1}}{1-x_2}\right]^{N+p_1} \left[\underbrace{\frac{k_3^{T_2}}{x_3-x_2}}_{0}+\frac{k_4^{T_2}}{1-x_2}\right]^{p_2}. \end{aligned} \tag{3.12}$$

One can easily find that $k_3^{T_2}=0$. After doing the change of variables

$$x_2=z_2z_3, x_3=z_3, \tag{3.13}$$

we can rewrite the above 5-point amplitude as following

$$\begin{aligned} A^{\{p_1,p_2\},0,0} = & \int_0^1 dz_3 \int_0^1 dz_2 z_2^{k_{12}} z_3^{k_{123}+1} \left(1-z_2\right)^{k_{23}} \left(1-z_2z_3\right)^{k_{24}} \left(1-z_3\right)^{k_{34}} \\ & \times \left[\frac{k_3^{T_1}}{z_3-z_2z_3}+\frac{k_4^{T_1}}{1-z_2z_3}\right]^{N+p_1} \left[\frac{k_4^{T_2}}{1-z_2z_3}\right]^{p_2} \end{aligned} \tag{3.14}$$

where we have defined $k_{123}=k_{12}+k_{23}+k_{13}$. Next, let's perform the binomial expansion on the bracket to obtain

$$\begin{aligned} A^{\{p_1,p_2\},0,0} = & \sum_{J_1^1+J_2^1=N+p_1} \frac{(N+p_1)!}{J_1^1!J_2^1!} \left(k_3^{T_1}\right)^{J_1^1} \left(k_4^{T_1}\right)^{J_2^1} \left(k_4^{T_2}\right)^{p_2} \\ & \times \int_0^1 dz_3 \int_0^1 dz_2 z_2^{k_{12}} z_3^{k_{123}+1-J_1^1} \left(1-z_2\right)^{k_{23}-J_1^1} \left(1-z_2z_3\right)^{k_{24}-J_2^1-p_2} \left(1-z_3\right)^{k_{34}}. \end{aligned} \tag{3.15}$$

For the next step, we expand the crossing term $\left(1-z_2z_3\right)^{k_{24}-J_2^1-p_2}$ to obtain

$$\begin{aligned} A^{\{p_1,p_2\},0,0} = & \sum_{J_1^1+J_2^1=N+p_1} \frac{(N+p_1)!}{J_1^1!J_2^1!} \left(k_3^{T_1}\right)^{J_1^1} \left(k_4^{T_1}\right)^{J_2^1} \left(k_4^{T_2}\right)^{p_2} \\ & \times \sum_{m_{23}} \frac{\left(-k_{24}+p_2+J_2^1\right)_{m_{23}}}{m_{23}!} \int_0^1 dz_2 z_2^{k_{12}+m_{23}} \left(1-z_2\right)^{k_{23}-J_1^1} \int_0^1 dz_3 z_3^{k_{123}+1-J_1^1+m_{23}} \left(1-z_3\right)^{k_{34}} \end{aligned} \tag{3.16}$$

where the subscripts of m_{23} keep record of the subscripts z_2z_3 in $\left(1-z_2z_3\right)^{k_{24}-J_2^1-p_2}$. After the integration, the amplitude can be written as

$$\begin{aligned} A^{\{p_1,p_2\},0,0} = & \sum_{J_1^1+J_2^1=N+p_1} \frac{(N+p_1)!}{J_1^1!J_2^1!} \left(k_3^{T_1}\right)^{J_1^1} \left(k_4^{T_1}\right)^{J_2^1} \left(k_4^{T_2}\right)^{p_2} \\ & \times \sum_{m_{23}} \frac{\left(-k_{24}+p_2+J_2^1\right)_{m_{23}}}{m_{23}!} \frac{\Gamma\left(k_{12}+1+m_{23}\right)\Gamma\left(k_{23}+1-J_1^1\right)}{\Gamma\left(k_{12}+k_{23}+2+m_{23}-J_1^1\right)} \\ & \times \frac{\Gamma\left(k_{123}+2+m_{23}-J_1^1\right)\Gamma\left(k_{34}+1\right)}{\Gamma\left(k_{123}+k_{34}+3+m_{23}-J_1^1\right)}. \end{aligned} \tag{3.17}$$

JHEP09(2023)111

Now we choose to work on the Regge regime defined by

$$k_{123} \sim s, k_{34} \sim s, k_{123} + k_{34} \sim t \tag{3.18}$$

where $s \to \infty$ and $t =$ fixed. (we will use these notations to define a Regge regime for the rest of the paper) In this Regge regime, the amplitude can be approximated as

$$\begin{aligned} A^{\{p_1,p_2\},0,0} \sim & \sum_{J_1^1+J_2^1=N+p_1} \frac{(N+p_1)!}{J_1^1! J_2^1!} \left(k_3^{T_1}\right)^{J_1^1} \left(k_4^{T_1}\right)^{J_2^1} \left(k_4^{T_2}\right)^{p_2} \\ & \times \sum_{m_{23}} \frac{\left(-k_{24}+p_2+J_2^1\right)_{m_{23}}}{m_{23}!} \frac{\Gamma\left(k_{12}+1+m_{23}\right) \Gamma\left(k_{23}+1-J_1^1\right)}{\Gamma\left(k_{12}+k_{23}+2+m_{23}-J_1^1\right)} \\ & \times \frac{\left(k_{123}\right)^{m_{23}-J_1^1} \Gamma\left(k_{123}+2\right) \Gamma\left(k_{34}+1\right)}{\left(k_{123}+k_{34}+3\right)_{m_{23}-J_1^1} \Gamma\left(k_{123}+k_{34}+3\right)}. \end{aligned} \tag{3.19}$$

The leading power of k_{123} occurs when $J_1^1 = 0$ which means $J_2^1 = N + p_1$. Since $p_1 + p_2 = 0$, the leading term of the RSSA is

$$\begin{aligned} A^{\{p_1,p_2\},0,0} \sim \left(k_4^{T_1}\right)^{N+p_1} \left(k_4^{T_2}\right)^{p_2} & \sum_{m_{23}} \frac{\left(-k_{24}+N\right)_{m_{23}}}{m_{23}!} \frac{\Gamma\left(k_{12}+1+m_{23}\right) \Gamma\left(k_{23}+1\right)}{\Gamma\left(k_{12}+k_{23}+2+m_{23}\right)} \\ \times & \frac{\left(k_{123}\right)^{m_{23}} \Gamma\left(k_{123}+2\right) \Gamma\left(k_{34}+1\right)}{\left(k_{123}+k_{34}+3\right)_{m_{23}} \Gamma\left(k_{123}+k_{34}+3\right)}. \end{aligned} \tag{3.20}$$

The ratio of $A^{\{p_1,p_2\},0,0}$ and $A^{\{0,0\},0,0}$ can be easily calculated to be

$$\begin{aligned} \frac{A^{\{p_1,p_2\},0,0}}{A^{\{0,0\},0,0}} &= \frac{\left(k_4^{T_1}\right)^{N+p_1} \left(k_4^{T_2}\right)^{p_2}}{\left(k_4^{T_1}\right)^{N}} = \left(k_4^{T_1}\right)^{p_1} \left(k_4^{T_2}\right)^{p_2} \\ &= \left(-q_4 \sin\phi_1^4 \cos\phi_2^4\right)^{p_1} \left(-q_4 \sin\phi_1^4 \sin\phi_2^4\right)^{p_2} \\ &= \left(\cos\theta_1\right)^{p_1} \left(\sin\theta_1\right)^{p_2} = \left(\omega_1\right)^{p_1} \left(\omega_2\right)^{p_2}, \end{aligned} \tag{3.21}$$

which is the same as eq. (2.16) with $m = q = 0$ and $r = 2$.

Let's now calculate the 6-point RSSA with $r = 3$ in eq. (3.4). The kinematics are

$$\begin{aligned} k_1 &= \left(\sqrt{p^2+M_1^2}, -p, 0, 0, 0\right), \\ k_2 &= \left(\sqrt{p^2+M_2^2}, p, 0, 0, 0\right), \\ k_3 &= \left(-\sqrt{q_3^2+M_3^2}, -q_3 \cos\phi_1^3, -q_3 \sin\phi_1^3, 0, 0\right), \\ k_4 &= \left(-\sqrt{q_4^2+M_4^2}, -q_4 \cos\phi_1^4, -q_4 \sin\phi_1^4 \cos\phi_2^4, -q_4 \sin\phi_1^4 \sin\phi_2^4, 0\right), \\ k_5 &= \left(-\sqrt{q_5^2+M_5^2}, -q_5 \cos\phi_1^5, -q_5 \sin\phi_1^5 \cos\phi_2^5, -q_5 \sin\phi_1^5 \sin\phi_2^5 \cos\phi_3^5, -q_5 \sin\phi_1^5 \sin\phi_2^5 \sin\phi_3^5\right), \\ k_6 &= \left(-\sqrt{q_5^2+M_5^2}, -q_6 \cos\phi_1^6, -q_6 \sin\phi_1^6 \cos\phi_2^6, -q_6 \sin\phi_1^6 \sin\phi_2^6 \cos\phi_3^6, -q_6 \sin\phi_1^6 \sin\phi_2^6 \sin\phi_3^6\right). \end{aligned} \tag{3.22}$$

JHEP09(2023)111

The amplitude of state

$$\left(\alpha_{-1}^{T_1}\right)^{N+p_1}\left(\alpha_{-1}^{T_2}\right)^{p_2}\left(\alpha_{-1}^{T_3}\right)^{p_3}|0,k\rangle\,, p_1+p_2+p_3=0 \tag{3.23}$$

and 5 tachyon states is

$$\begin{aligned}
&A^{\{p_1,p_2,p_3\},0,0}\\
&\quad=\int_0^1 dx_4\int_0^{x_4}dx_3\int_0^{x_3}dx_2\\
&\quad\times x_2^{k_{12}}x_3^{k_{13}}x_4^{k_{14}}\left(x_3-x_2\right)^{k_{23}}\left(x_4-x_2\right)^{k_{24}}\left(1-x_2\right)^{k_{25}}\left(x_4-x_3\right)^{k_{34}}\left(1-x_3\right)^{k_{35}}\left(1-x_4\right)^{k_{45}}\\
&\quad\times\left[\frac{k_3^{T_1}}{x_3-x_2}+\frac{k_4^{T_1}}{x_4-x_2}+\frac{k_5^{T_1}}{\underbrace{1}_{x_5}-x_2}\right]^{N+p_1}\left[\frac{k_3^{T_2}}{\underbrace{x_3-x_2}_{=0}}+\frac{k_4^{T_2}}{x_4-x_2}+\frac{k_5^{T_2}}{1-x_2}\right]^{p_2}\\
&\quad\times\left[\frac{k_3^{T_3}}{\underbrace{x_3-x_2}_{=0}}+\frac{k_4^{T_3}}{\underbrace{x_4-x_2}_{=0}}+\frac{k_5^{T_3}}{1-x_2}\right]^{p_3}.
\end{aligned}\tag{3.24}$$

Since $k_3^{T_2}=k_3^{T_3}=k_4^{T_3}=0$, we can rewrite the amplitude as

$$\begin{aligned}
&A^{\{p_1,p_2,p_3\},0,0}\\
&\quad=\int_0^1 dx_4\int_0^{x_4}dx_3\int_0^{x_3}dx_2\; x_2^{k_{12}}x_3^{k_{13}}x_4^{k_{14}}\left(x_3-x_2\right)^{k_{23}}\left(x_4-x_2\right)^{k_{24}}\left(1-x_2\right)\\
&\quad\times^{k_{25}}\left(x_4-x_3\right)^{k_{34}}\left(1-x_3\right)^{k_{35}}\left(1-x_4\right)^{k_{45}}\\
&\quad\times\left[\frac{k_3^{T_1}}{x_3-x_2}+\frac{k_4^{T_1}}{x_4-x_2}+\frac{k_5^{T_1}}{1-x_2}\right]^{N+p_1}\left[\frac{k_4^{T_2}}{x_4-x_2}+\frac{k_5^{T_2}}{1-x_2}\right]^{p_2}\left[\frac{k_5^{T_3}}{1-x_2}\right]^{p_3}.
\end{aligned}\tag{3.25}$$

We can do the following change of variables

$$x_i=z_i\cdots z_{n-2}, \tag{3.26}$$

or

$$x_2=z_2z_3z_4, x_3=z_3z_4, x_4=z_4 \tag{3.27}$$

to obtain

$$\begin{aligned}
&A^{\{p_1,p_2,p_3\},0,0}\\
&\quad=\int_0^1 dz_4\int_0^1 dz_3\int_0^1 dz_2\times z_2^{k_{12}}z_3^{k_{123}+1}z_4^{k_{1234}+2}\left(1-z_2\right)^{k_{23}}\left(1-z_3\right)^{k_{34}}\left(1-z_4\right)^{k_{45}}\\
&\quad\times\left(1-z_2z_3\right)^{k_{24}}\left(1-z_2z_3z_4\right)^{k_{25}}\left(1-z_3z_4\right)^{k_{35}}\\
&\quad\times\left[\frac{k_3^{T_1}}{z_3z_4-z_2z_3z_4}+\frac{k_4^{T_1}}{z_4-z_2z_3z_4}+\frac{k_5^{T_1}}{1-z_2z_3z_4}\right]^{N+p_1}\\
&\quad\times\left[\frac{k_4^{T_2}}{z_4-z_2z_3z_4}+\frac{k_5^{T_2}}{1-z_2z_3z_4}\right]^{p_2}\left[\frac{k_5^{T_3}}{1-z_2z_3z_4}\right]^{p_3}
\end{aligned}\tag{3.28}$$

JHEP09(2023)111

where we have defined

$$k_{123} = k_{12} + k_{13} + k_{23}, k_{1234} = k_{12} + k_{13} + k_{14} + k_{23} + k_{24} + k_{34}. \tag{3.29}$$

Next, let's perform the binomial expansion on the brackets to obtain

$$\begin{aligned}
&A^{\{p_1,p_2,p_3\},0,0} \\
&= \int_0^1 dz_4 \int_0^1 dz_3 \int_0^1 dz_2 \times z_2^{k_{12}} z_3^{k_{123}+1} z_4^{k_{1234}+2} (1-z_2)^{k_{23}} (1-z_3)^{k_{34}} (1-z_4)^{k_{45}} \\
&\times (1-z_2z_3)^{k_{24}} (1-z_2z_3z_4)^{k_{25}} (1-z_3z_4)^{k_{35}} \\
&\times \sum_{J_1^1+J_2^1+J_3^1=N+p_1}^{N+p_1} \frac{(N+p_1)!}{J_1^1!J_2^1!J_3^1!} \left(\frac{k_3^{T_1}}{z_3z_4 - z_2z_3z_4}\right)^{J_1^1} \left(\frac{k_4^{T_1}}{z_4 - z_2z_3z_4}\right)^{J_2^1} \left(\frac{k_5^{T_1}}{1 - z_2z_3z_4}\right)^{J_3^1} \\
&\times \sum_{J_1^2+J_2^2=p_2}^{p_2} \frac{p_2!}{J_1^2!J_2^2!} \left(\frac{k_4^{T_2}}{z_4 - z_2z_3z_4}\right)^{J_1^2} \left(\frac{k_5^{T_2}}{1 - z_2z_3z_4}\right)^{J_2^2} \left[\frac{k_5^{T_3}}{1 - z_2z_3z_4}\right]^{p_3}
\end{aligned} \tag{3.30}$$

where J_1^1, J_2^1, J_3^1, J_1^2, J_2^2 are non-negative integers with $J_1^1 + J_2^1 = N + p_1$ and $J_1^2 + J_2^2 = p_2$. We then rearrange the above equation

$$\begin{aligned}
&A^{\{p_1,p_2,p_3\},0,0} \\
&= \sum_{J_1^1+J_2^1+J_3^1=N+p_1}^{N+p_1} \frac{(N+p_1)!}{J_1^1!J_2^1!J_3^1!} \left(k_3^{T_1}\right)^{J_1^1} \left(k_4^{T_1}\right)^{J_2^1} \left(k_5^{T_1}\right)^{J_3^1} \\
&\times \sum_{J_1^2+J_2^2=p_2}^{p_2} \frac{p_2!}{J_1^2!J_2^2!} \left(k_4^{T_2}\right)^{J_1^2} \left(k_5^{T_2}\right)^{J_2^2} \left(k_5^{T_3}\right)^{p_3} \\
&\times \int_0^1 dz_4 \int_0^1 dz_3 \int_0^1 dz_2 \times z_2^{k_{12}} z_3^{k_{123}+1-J_1^1} z_4^{k_{1234}+2-J_1^1-\left(J_2^1+J_1^2\right)} \\
&\times (1-z_2)^{k_{23}-J_1^1} (1-z_3)^{k_{34}} (1-z_4)^{k_{45}} (1-z_2z_3)^{k_{24}-\left(J_2^1+J_1^2\right)} \\
&\times (1-z_2z_3z_4)^{k_{25}-\left(J_3^1+J_2^2+p_3\right)} (1-z_3z_4)^{k_{35}},
\end{aligned} \tag{3.31}$$

and expand the crossing terms to obain

$$\begin{aligned}
&A^{\{p_1,p_2,p_3\},0,0} \\
&= \sum_{J_1^1+J_2^1+J_3^1=N+p_1}^{N+p_1} \frac{(N+p_1)!}{J_1^1!J_2^1!J_3^1!} \left(k_3^{T_1}\right)^{J_1^1} \left(k_4^{T_1}\right)^{J_2^1} \left(k_5^{T_1}\right)^{J_3^1} \\
&\times \sum_{J_1^2+J_2^2=p_2}^{p_2} \frac{p_2!}{J_1^2!J_2^2!} \left(k_4^{T_2}\right)^{J_1^2} \left(k_5^{T_2}\right)^{J_2^2} \left(k_5^{T_3}\right)^{p_3} \\
&\times \int_0^1 dz_4 \int_0^1 dz_3 \int_0^1 dz_2 \times z_2^{k_{12}} z_3^{k_{123}+1-J_1^1} z_4^{k_{1234}+2-J_1^1-\left(J_2^1+J_1^2\right)} \\
&\times (1-z_2)^{k_{23}-J_1^1} (1-z_3)^{k_{34}} (1-z_4)^{k_{45}} \\
&\times \sum_{m_{23}=0} \frac{\left[-k_{24}+\left(J_2^1+J_1^2\right)\right]_{m_{23}}}{m_{23}!} (z_2z_3)^{m_{23}} \sum_{m_{24}=0} \frac{\left[-k_{25}+\left(J_3^1+J_2^2+p_3\right)\right]_{m_{24}}}{m_{24}!} (z_2z_3z_4)^{m_{24}} \\
&\times \sum_{m_{34}=0} \frac{\left[-k_{35}\right]_{m_{34}}}{m_{34}!} (z_3z_4)^{m_{34}}
\end{aligned} \tag{3.32}$$

JHEP09(2023)111

where, for example, the subscripts of m_{24} keep record of the first and the last subscripts of $(z_2z_3z_4)$ etc. We rearrange the above equation again

$$
\begin{aligned}
&A^{\{p_1,p_2,p_3\},0,0} \\
&= \sum_{J_1^1+J_2^1+J_3^1=N+p_1}^{N+p_1} \frac{(N+p_1)!}{J_1^1!J_2^1!J_3^1!} \left(k_3^{T_1}\right)^{J_1^1} \left(k_4^{T_1}\right)^{J_2^1} \left(k_5^{T_1}\right)^{J_3^1} \sum_{J_1^2+J_2^2=p_2}^{N+p_1} \left(k_4^{T_2}\right)^{J_1^2} \left(k_5^{T_2}\right)^{J_2^2} \left(k_5^{T_3}\right)^{p_3} \\
&\quad\times \sum_{m_{23}=0} \frac{\left[-k_{24}+\left(J_2^1+J_1^2\right)\right]_{m_{23}}}{m_{23}!} \sum_{m_{24}=0} \frac{\left[-k_{25}+\left(J_3^1+J_2^2+p_3\right)\right]_{m_{24}}}{m_{24}!} \sum_{m_{34}=0} \frac{[-k_{35}]_{m_{34}}}{m_{34}!} \\
&\quad\times \int_0^1 dz_2 z_2^{k_{12}+m_{23}+m_{24}} (1-z_2)^{k_{23}-J_1^1} \\
&\quad\times \int_0^1 dz_3 z_3^{k_{123}+1-J_1^1+m_{23}+m_{24}+m_{34}} (1-z_3)^{k_{34}} \\
&\quad\times \int_0^1 dz_4 z_4^{k_{1234}+2-J_1^1-\left(J_2^1+J_1^2\right)+m_{24}+m_{34}} (1-z_4)^{k_{45}} ,
\end{aligned}
\tag{3.33}
$$

and perform the integration to obtain

$$
\begin{aligned}
&A^{\{p_1,p_2,p_3\},0,0} \\
&= \sum_{J_1^1+J_2^1+J_3^1=N+p_1}^{N+p_1} \frac{(N+p_1)!}{J_1^1!J_2^1!J_3^1!} \left(k_3^{T_1}\right)^{J_1^1} \left(k_4^{T_1}\right)^{J_2^1} \left(k_5^{T_1}\right)^{J_3^1} \\
&\quad\times \sum_{J_1^2+J_2^2=p_2}^{p_2} \frac{p_2!}{J_1^2!J_2^2!} \left(k_4^{T_2}\right)^{J_1^2} \left(k_5^{T_2}\right)^{J_2^2} \left(k_5^{T_3}\right)^{p_3} \\
&\quad\times \sum_{m_{23}=0} \frac{\left[-k_{24}+\left(J_2^1+J_1^2\right)\right]_{m_{23}}}{m_{23}!} \sum_{m_{24}=0} \frac{\left[-k_{25}+\left(J_3^1+J_2^2+p_3\right)\right]_{m_{24}}}{m_{24}!} \sum_{m_{34}=0} \frac{[-k_{35}]_{m_{34}}}{m_{34}!} \\
&\quad\times \frac{\Gamma\left(k_{12}+1+m_{23}+m_{24}\right)\Gamma\left(k_{23}+1-J_1^1\right)}{\Gamma\left(k_{12}+k_{23}+2-J_1^1+m_{23}+m_{24}\right)} \\
&\quad\times \frac{\Gamma\left(k_{123}+2-J_1^1+m_{23}+m_{24}+m_{34}\right)\Gamma\left(k_{34}+1\right)}{\Gamma\left(k_{123}+k_{34}+3-J_1^1+m_{23}+m_{24}+m_{34}\right)} \\
&\quad\times \frac{\Gamma\left(k_{1234}+3-\left(J_1^1+J_2^1+J_1^2\right)+m_{24}+m_{34}\right)\Gamma\left(k_{45}+1\right)}{\Gamma\left(k_{1234}+k_{23}+4-\left(J_1^1+J_2^1+J_1^2\right)+m_{24}+m_{34}\right)} .
\end{aligned}
\tag{3.34}
$$

Now we choose to work on the Regge regime defined by

$$
k_{1234} \sim s, k_{1234}+k_{23} \sim t. \tag{3.35}
$$

In this Regge regime, the amplitude can be approximated as

$$\begin{aligned}
&A^{\{p_1,p_2,p_3\},0,0} \\
&\quad\sim \sum_{J_1^1+J_2^1+J_3^1=N+p_1}^{N+p_1} \frac{(N+p_1)!}{J_1^1!J_2^1!J_3^1!}\left(k_3^{T_1}\right)^{J_1^1}\left(k_4^{T_1}\right)^{J_2^1}\left(k_5^{T_1}\right)^{J_3^1} \\
&\qquad\times \sum_{J_1^2+J_2^2=p_2}^{p_2} \frac{p_2!}{J_1^2!J_2^2!}\left(k_4^{T_2}\right)^{J_1^2}\left(k_5^{T_2}\right)^{J_2^2}\left(k_5^{T_3}\right)^{p_3} \\
&\qquad\times \sum_{m_{23}=0} \frac{\left[-k_{24}+\left(J_2^1+J_1^2\right)\right]_{m_{23}}}{m_{23}!} \sum_{m_{24}=0} \frac{\left[-k_{25}+\left(J_3^1+J_2^2+p_3\right)\right]_{m_{24}}}{m_{24}!} \sum_{m_{34}=0} \frac{[-k_{35}]_{m_{34}}}{m_{34}!} \\
&\qquad\times \frac{\Gamma\left(k_{12}+1+m_{23}+m_{24}\right)\Gamma\left(k_{23}+1-J_1^1\right)}{\Gamma\left(k_{12}+k_{23}+2-J_1^1+m_{23}+m_{24}\right)} \\
&\qquad\times \frac{\Gamma\left(k_{123}+2-J_1^1+m_{23}+m_{24}+m_{34}\right)\Gamma\left(k_{34}+1\right)}{\Gamma\left(k_{123}+k_{34}+3-J_1^1+m_{23}+m_{24}+m_{34}\right)} \\
&\qquad\times \frac{\left(k_{1234}\right)^{-\left(J_1^1+J_2^1+J_1^2\right)+m_{24}+m_{34}}}{\left(k_{1234}+k_{23}+4\right)_{-\left(J_1^1+J_2^1+J_1^2\right)+m_{24}+m_{34}}} \frac{\Gamma\left(k_{1234}+3\right)\Gamma\left(k_{45}+1\right)}{\Gamma\left(k_{1234}+k_{23}+4\right)}. \qquad (3.36)
\end{aligned}$$

We can now take $J_1^1 = J_2^1 = J_1^2 = 0$ to extract the leading order term in k_{1234}. This implies $J_3^1 = N + p_1$ and $J_2^2 = p_2$ which give

$$\begin{aligned}
A^{\{p_1,p_2,p_3\},0,0} \sim{}& \left(k_5^{T_1}\right)^{N+p_1}\left(k_5^{T_2}\right)^{p_2}\left(k_5^{T_3}\right)^{p_3} \\
&\times \sum_{m_{23}=0} \frac{[-k_{24}]_{m_{23}}}{m_{23}!} \sum_{m_{24}=0} \frac{[-k_{25}+N]_{m_{24}}}{m_{24}!} \sum_{m_{34}=0} \frac{[-k_{35}]_{m_{34}}}{m_{34}!} \\
&\times \frac{\Gamma\left(k_{12}+1+m_{23}+m_{24}\right)\Gamma\left(k_{23}+1\right)}{\Gamma\left(k_{12}+k_{23}+2+m_{23}+m_{24}\right)} \\
&\times \frac{\Gamma\left(k_{123}+2+m_{23}+m_{24}+m_{34}\right)\Gamma\left(k_{34}+1\right)}{\Gamma\left(k_{123}+k_{34}+3+m_{23}+m_{24}+m_{34}\right)} \\
&\times \frac{\left(k_{1234}\right)^{m_{24}+m_{34}}}{\left(k_{1234}+k_{23}+4\right)_{m_{24}+m_{34}}} \frac{\Gamma\left(k_{1234}+3\right)\Gamma\left(k_{45}+1\right)}{\Gamma\left(k_{1234}+k_{23}+4\right)}. \qquad (3.37)
\end{aligned}$$

Finally, the ratios of the 6-point RSSA can be easily calculated to be

$$\begin{aligned}
\frac{A^{\{p_1,p_2,p_3\},0,0}}{A^{\{0,0,0\},0,0}} &= \left(k_5^{T_1}\right)^{p_1}\left(k_5^{T_2}\right)^{p_2}\left(k_5^{T_3}\right)^{p_3} \\
&= \left(\cos\phi_2^5\right)^{p_1}\left(\sin\phi_2^5\cos\phi_3^5\right)^{p_2}\left(\sin\phi_2^5\sin\phi_3^5\right)^{p_3} \\
&= \left(\cos\theta_1\right)^{p_1}\left(\sin\theta_1\cos\theta_2\right)^{p_2}\left(\sin\theta_1\sin\theta_2\right)^{p_3} \\
&= \left(\omega_1\right)^{p_1}\left(\omega_2\right)^{p_2}\left(\omega_3\right)^{p_3}, \qquad (3.38)
\end{aligned}$$

which is the same as eq. (2.16) with $m = q = 0$ and $r = 3$.

3.2 The 7-point Regge stringy scaling

In this section we calculate the 7-point RSSA with $r = 4$ in eq. (3.4). The kinematics are

$$
\begin{aligned}
k_1 &= \left(\sqrt{p^2+M_1^2}, -p, 0, 0, 0, 0\right),\\
k_2 &= \left(\sqrt{p^2+M_2^2}, p, 0, 0, 0, 0\right),\\
k_3 &= \left(-\sqrt{q_3^2+M_3^2}, -q_3\cos\phi_1^3, -q_3\sin\phi_1^3, 0, 0, 0\right),\\
k_4 &= \left(-\sqrt{q_4^2+M_4^2}, -q_4\cos\phi_1^4, -q_4\sin\phi_1^4\cos\phi_2^4, -q_4\sin\phi_1^4\sin\phi_2^4, 0, 0\right),\\
k_5 &= \left(-\sqrt{q_5^2+M_5^2}, -q_5\cos\phi_1^5, -q_5\sin\phi_1^5\cos\phi_2^5, -q_5\sin\phi_1^5\sin\phi_2^5\cos\phi_3^5,\right.\\
&\qquad \left. -\, q_5\sin\phi_1^5\sin\phi_2^5\sin\phi_3^5, 0\right),\\
k_6 &= \begin{pmatrix} -\sqrt{q_5^2+M_5^2}, -q_6\cos\phi_1^6, -q_6\sin\phi_1^6\cos\phi_2^6, -q_6\sin\phi_1^6\sin\phi_2^6\cos\phi_3^6, \\ -q_6\sin\phi_1^6\sin\phi_2^6\sin\phi_3^6\cos\phi_4^6, -q_6\sin\phi_1^6\sin\phi_2^6\sin\phi_3^6\sin\phi_4^6 \end{pmatrix},\\
k_7 &= \begin{pmatrix} -\sqrt{q_5^2+M_5^2}, -q_7\cos\phi_1^7, -q_7\sin\phi_1^7\cos\phi_2^7, -q_7\sin\phi_1^7\sin\phi_2^7\cos\phi_3^7, \\ -q_7\sin\phi_1^7\sin\phi_2^7\sin\phi_3^7\cos\phi_4^7, -q_7\sin\phi_1^7\sin\phi_2^7\sin\phi_3^7\sin\phi_4^7 \end{pmatrix}.
\end{aligned}
\tag{3.39}
$$

The tensor state we are going to consider is

$$
\left(\alpha_{-1}^{T_1}\right)^{N+p_1}\left(\alpha_{-1}^{T_2}\right)^{p_2}\left(\alpha_{-1}^{T_3}\right)^{p_3}\left(\alpha_{-1}^{T_4}\right)^{p_4}|0,k\rangle, p_1+p_2+p_3+p_4=0. \tag{3.40}
$$

We will use the notation defined in eq. (3.7), so we have the following $\frac{7(7-3)}{2} = 14$ independent kinematics variables

$$
k_{12}, k_{13}, k_{14}, k_{15}, k_{23}, k_{24}, k_{25}, k_{26}, k_{34}, k_{35}, k_{36}, k_{45}, k_{46}, k_{56}. \tag{3.41}
$$

The RSSA of one tensor state and 6 tachyon states is

$$
\begin{aligned}
&A^{\{p_1,p_2,p_3,p_4\},0,0}\\
&\quad = \int_0^1 dx_5 \int_0^{x_5} dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2\\
&\qquad \cdot x_2^{k_{12}} x_3^{k_{13}} x_4^{k_{14}} x_5^{k_{15}} (x_3-x_2)^{k_{23}} (x_4-x_2)^{k_{24}} (x_5-x_2)^{k_{25}} (1-x_2)^{k_{26}}\\
&\qquad \times (x_4-x_3)^{k_{34}} (x_5-x_3)^{k_{35}} (1-x_3)^{k_{36}} (x_5-x_4)^{k_{45}} (1-x_4)^{k_{46}} (1-x_5)^{k_{56}}\\
&\qquad \times \left[\frac{k_3^{T_1}}{x_3-x_2} + \frac{k_4^{T_1}}{x_4-x_2} + \frac{k_5^{T_1}}{x_5-x_2} + \frac{k_6^{T_1}}{1-x_2}\right]^{N+p_1}\\
&\qquad \times \left[\frac{k_4^{T_2}}{x_4-x_2} + \frac{k_5^{T_2}}{x_5-x_2} + \frac{k_6^{T_2}}{1-x_2}\right]^{p_2}\\
&\qquad \times \left[\frac{k_5^{T_3}}{x_5-x_2} + \frac{k_6^{T_3}}{1-x_2}\right]^{p_3} \left[\frac{k_6^{T_4}}{1-x_2}\right]^{p_4}.
\end{aligned}
\tag{3.42}
$$

JHEP09(2023)111

Note that $k_3^{T_2}, k_3^{T_3}, k_3^{T_4}, k_4^{T_3}, k_4^{T_4}, k_5^{T_4}$ are all zeros. Let us make the following change of variables

$$x_2 = z_2 z_3 z_4 z_5, x_3 = z_3 z_4 z_5, x_4 = z_4 z_5, x_5 = z_5, x_6 = z_6 = 1 \tag{3.43}$$

to obtain

$$\begin{aligned}
&A^{\{p_1,p_2,p_3,p_4\},0,0} \\
&\quad = \int_0^1 dz_5 \int_0^1 dz_4 \int_0^1 dz_3 \int_0^1 dz_2 \cdot \left(z_3 z_4^2 z_5^3\right) (z_2 z_3 z_4 z_5)^{k_{12}} (z_3 z_4 z_5)^{k_{13}} (z_4 z_5)^{k_{14}} z_5^{k_{15}} \\
&\quad \times (z_3 z_4 z_5 - z_2 z_3 z_4 z_5)^{k_{23}} (z_4 z_5 - z_2 z_3 z_4 z_5)^{k_{24}} (z_5 - z_2 z_3 z_4 z_5)^{k_{25}} (1 - z_2 z_3 z_4 z_5)^{k_{26}} \\
&\quad \times (z_4 z_5 - z_3 z_4 z_5)^{k_{34}} (z_5 - z_3 z_4 z_5)^{k_{35}} (1 - z_3 z_4 z_5)^{k_{36}} (z_5 - z_4 z_5)^{k_{45}} (1 - z_4 z_5)^{k_{46}} (1 - z_5)^{k_{56}} \\
&\quad \times \left[\frac{k_3^{T_1}}{z_3 z_4 z_5 - z_2 z_3 z_4 z_5} + \frac{k_4^{T_1}}{z_4 z_5 - z_2 z_3 z_4 z_5} + \frac{k_5^{T_1}}{z_5 - z_2 z_3 z_4 z_5} + \frac{k_6^{T_1}}{1 - z_2 z_3 z_4 z_5}\right]^{N+p_1} \\
&\quad \times \left[\frac{k_4^{T_2}}{z_4 z_5 - z_2 z_3 z_4 z_5} + \frac{k_5^{T_2}}{z_5 - z_2 z_3 z_4 z_5} + \frac{k_6^{T_2}}{1 - z_2 z_3 z_4 z_5}\right]^{p_2} \\
&\quad \times \left[\frac{k_5^{T_3}}{z_5 - z_2 z_3 z_4 z_5} + \frac{k_6^{T_3}}{1 - z_2 z_3 z_4 z_5}\right]^{p_3} \left[\frac{k_6^{T_4}}{1 - z_2 z_3 z_4 z_5}\right]^{p_4}.
\end{aligned} \tag{3.44}$$

We use the definition in eq. (3.8) to obtain

$$k_{123} = k_{12} + k_{13} + k_{23}, k_{1234} = k_{123} + k_{14} + k_{24} + k_{34}, k_{12345} = k_{1234} + k_{15} + k_{25} + k_{35} + k_{45}. \tag{3.45}$$

After some calculation, we get

$$\begin{aligned}
&A^{\{p_1,p_2,p_3,p_4\},0,0} \\
&\quad = \int_0^1 dz_5 \int_0^1 dz_4 \int_0^1 dz_3 \int_0^1 dz_2 \cdot z_2^{k_{12}} z_3^{k_{123}+1} z_4^{k_{1234}+2} z_5^{k_{12345}+3} \\
&\quad \times (1 - z_2)^{k_{23}} (1 - z_2 z_3)^{k_{24}} (1 - z_2 z_3 z_4)^{k_{25}} (1 - z_2 z_3 z_4 z_5)^{k_{26}} \\
&\quad \times (1 - z_3)^{k_{34}} (1 - z_3 z_4)^{k_{35}} (1 - z_3 z_4 z_5)^{k_{36}} \\
&\quad \times (1 - z_4)^{k_{45}} (1 - z_4 z_5)^{k_{46}} \\
&\quad \times (1 - z_5)^{k_{56}} \\
&\quad \times \left[\frac{k_3^{T_1}}{z_3 z_4 z_5 - z_2 z_3 z_4 z_5} + \frac{k_4^{T_1}}{z_4 z_5 - z_2 z_3 z_4 z_5} + \frac{k_5^{T_1}}{z_5 - z_2 z_3 z_4 z_5} + \frac{k_6^{T_1}}{1 - z_2 z_3 z_4 z_5}\right]^{N+p_1} \\
&\quad \times \left[\frac{k_4^{T_2}}{z_4 z_5 - z_2 z_3 z_4 z_5} + \frac{k_5^{T_2}}{z_5 - z_2 z_3 z_4 z_5} + \frac{k_6^{T_2}}{1 - z_2 z_3 z_4 z_5}\right]^{p_2} \\
&\quad \times \left[\frac{k_5^{T_3}}{z_5 - z_2 z_3 z_4 z_5} + \frac{k_6^{T_3}}{1 - z_2 z_3 z_4 z_5}\right]^{p_3} \left[\frac{k_6^{T_4}}{1 - z_2 z_3 z_4 z_5}\right]^{p_4}.
\end{aligned} \tag{3.46}$$

JHEP09(2023)111

The next step is to expand the brackets to get

$$
\begin{aligned}
&A^{\{p_1,p_2,p_3,p_4\},0,0}\\
&\quad=\int_0^1 dz_5\int_0^1 dz_4\int_0^1 dz_3\int_0^1 dz_2\cdot z_2^{k_{12}}z_3^{k_{123}+1}z_4^{k_{1234}+2}z_5^{k_{12345}+3}\\
&\qquad\times(1-z_2)^{k_{23}}(1-z_2z_3)^{k_{24}}(1-z_2z_3z_4)^{k_{25}}(1-z_2z_3z_4z_5)^{k_{26}}\\
&\qquad\times(1-z_3)^{k_{34}}(1-z_3z_4)^{k_{35}}(1-z_3z_4z_5)^{k_{36}}\\
&\qquad\times(1-z_4)^{k_{45}}(1-z_4z_5)^{k_{46}}\\
&\qquad\times(1-z_5)^{k_{56}}\\
&\qquad\times\sum_{J_1^1+J_2^1+J_3^1+J_4^1=N+p_1}^{N+p_1}\left[\frac{(N+p_1)!}{J_1^1!J_2^1!J_3^1!J_4^1!}\left(\frac{k_3^{T_1}}{z_3z_4z_5-z_2z_3z_4z_5}\right)^{J_1^1}\left(\frac{k_4^{T_1}}{z_4z_5-z_2z_3z_4z_5}\right)^{J_2^1}\right.\\
&\qquad\qquad\left.\left(\frac{k_5^{T_1}}{z_5-z_2z_3z_4z_5}\right)^{J_3^1}\left(\frac{k_6^{T_1}}{1-z_2z_3z_4z_5}\right)^{J_4^1}\right]\\
&\qquad\times\sum_{J_1^2+J_2^2+J_3^2=p_2}^{p_2}\frac{P_2!}{J_1^2!J_2^2!J_3^2!}\left(\frac{k_4^{T_2}}{z_4z_5-z_2z_3z_4z_5}\right)^{J_1^2}\left(\frac{k_5^{T_2}}{z_5-z_2z_3z_4z_5}\right)^{J_2^2}\left(\frac{k_6^{T_2}}{1-z_2z_3z_4z_5}\right)^{J_3^2}\\
&\qquad\times\sum_{J_1^3+J_2^3=p_3}^{p_2}\frac{P_3!}{J_1^3!J_2^3!}\left(\frac{k_5^{T_3}}{z_5-z_2z_3z_4z_5}\right)^{J_1^3}\left(\frac{k_6^{T_3}}{1-z_2z_3z_4z_5}\right)^{J_2^3}\\
&\qquad\times\left(\frac{k_6^{T_4}}{1-z_2z_3z_4z_5}\right)^{p_4}
\end{aligned}\tag{3.47}
$$

where $J_1^1, J_2^1, J_3^1, J_4^1, J_1^2, J_2^2, J_3^2, J_1^3, J_2^3$ are non-negative integers with $J_1^1+J_2^1+J_3^1+J_4^1=N+p_1$, $J_1^2+J_2^2+J_3^2=p_2$ and $J_1^3+J_2^3=p_3$. Let us rearrange the above equation as

$$
\begin{aligned}
A^{\{p_1,p_2,p_3,p_4\},0,0}=&\sum_{J_1^1+J_2^1+J_3^1+J_4^1=N+p_1}^{N+p_1}\frac{(N+p_1)!}{J_1^1!J_2^1!J_3^1!J_4^1!}\left(k_3^{T_1}\right)^{J_1^1}\left(k_4^{T_1}\right)^{J_2^1}\left(k_5^{T_1}\right)^{J_3^1}\left(k_6^{T_1}\right)^{J_4^1}\\
&\times\sum_{J_1^2+J_2^2+J_3^2=p_2}^{p_2}\frac{P_2!}{J_1^2!J_2^2!J_3^2!}\left(k_4^{T_2}\right)^{J_1^2}\left(k_5^{T_2}\right)^{J_2^2}\left(k_6^{T_2}\right)^{J_3^2}\\
&\times\sum_{J_1^3+J_2^3=p_3}^{p_2}\frac{P_3!}{J_1^3!J_2^3!}\left(k_5^{T_3}\right)^{J_1^3}\left(k_6^{T_3}\right)^{J_2^3}\left(k_6^{T_4}\right)^{p_4}.\\
&\times\int_0^1 dz_5\int_0^1 dz_4\int_0^1 dz_3\int_0^1 dz_2\cdot z_2^{k_{12}}z_3^{k_{123}+1}z_4^{k_{1234}+2}z_5^{k_{12345}+3}\\
&\times(1-z_2)^{k_{23}}(1-z_2z_3)^{k_{24}}(1-z_2z_3z_4)^{k_{25}}(1-z_2z_3z_4z_5)^{k_{26}}\\
&\times(1-z_3)^{k_{34}}(1-z_3z_4)^{k_{35}}(1-z_3z_4z_5)^{k_{36}}\\
&\times(1-z_4)^{k_{45}}(1-z_4z_5)^{k_{46}}\\
&\times(1-z_5)^{k_{56}}\\
&\times(z_3z_4z_5-z_2z_3z_4z_5)^{-\left(J_1^1\right)}(z_4z_5-z_2z_3z_4z_5)^{-\left(J_2^1+J_1^2\right)}\\
&\times(z_5-z_2z_3z_4z_5)^{-\left(J_3^1+J_2^2+J_1^3\right)}(1-z_2z_3z_4z_5)^{-\left(J_4^1+J_3^2+J_2^3+p_4\right)},
\end{aligned}\tag{3.48}
$$

which means

$$
\begin{aligned}
&A^{\{p_1,p_2,p_3,p_4\},0,0} \\
&= \sum_{J_1^1+J_2^1+J_3^1+J_4^1=N+p_1}^{N+p_1} \frac{(N+p_1)!}{J_1^1!J_2^1!J_3^1!J_4^1!} \left(k_3^{T_1}\right)^{J_1^1} \left(k_4^{T_1}\right)^{J_2^1} \left(k_5^{T_1}\right)^{J_3^1} \left(k_6^{T_1}\right)^{J_4^1} \\
&\times \sum_{J_1^2+J_2^2+J_3^2=p_2}^{p_2} \frac{P_2!}{J_1^2!J_2^2!J_3^2!} \left(k_4^{T_2}\right)^{J_1^2} \left(k_5^{T_2}\right)^{J_2^2} \left(k_6^{T_2}\right)^{J_3^2} \\
&\times \sum_{J_1^3+J_2^3=p_3}^{p_2} \frac{P_3!}{J_1^3!J_2^3!} \left(k_5^{T_3}\right)^{J_1^3} \left(k_6^{T_3}\right)^{J_2^3} \left(k_6^{T_4}\right)^{p_4} . \\
&\times \int_0^1 dz_5 \int_0^1 dz_4 \int_0^1 dz_3 \int_0^1 dz_2 \cdot z_2^{k_{12}} z_3^{k_{123}+1-\left(J_1^1\right)} z_4^{k_{1234}+2-\left(J_1^1+J_2^1+J_1^2\right)} \\
&\times z_5^{k_{12345}+3-\left(J_1^1+J_2^1+J_1^2+J_3^1+J_2^2+J_1^3\right)} \\
&\times (1-z_2)^{k_{23}-J_1^1} (1-z_2z_3)^{k_{24}-\left(J_2^1+J_1^2\right)} (1-z_2z_3z_4)^{k_{25}-\left(J_3^1+J_2^2+J_1^3\right)} \\
&\times (1-z_2z_3z_4z_5)^{k_{26}-\left(J_4^1+J_3^2+J_2^3+p_4\right)} \\
&\times (1-z_3)^{k_{34}} (1-z_3z_4)^{k_{35}} (1-z_3z_4z_5)^{k_{36}} \\
&\times (1-z_4)^{k_{45}} (1-z_4z_5)^{k_{46}} \\
&\times (1-z_5)^{k_{56}} .
\end{aligned}
\tag{3.49}
$$

Then we expand the crossing terms to get

$$
\begin{aligned}
&A^{\{p_1,p_2,p_3,p_4\},0,0} \\
&= \sum_{J_1^1+J_2^1+J_3^1+J_4^1=N+p_1}^{N+p_1} \frac{(N+p_1)!}{J_1^1!J_2^1!J_3^1!J_4^1!} \left(k_3^{T_1}\right)^{J_1^1} \left(k_4^{T_1}\right)^{J_2^1} \left(k_5^{T_1}\right)^{J_3^1} \left(k_6^{T_1}\right)^{J_4^1} \\
&\times \sum_{J_1^2+J_2^2+J_3^2=p_2}^{p_2} \frac{P_2!}{J_1^2!J_2^2!J_3^2!} \left(k_4^{T_2}\right)^{J_1^2} \left(k_5^{T_2}\right)^{J_2^2} \left(k_6^{T_2}\right)^{J_3^2} \\
&\times \sum_{J_1^3+J_2^3=p_3}^{p_2} \frac{P_3!}{J_1^3!J_2^3!} \left(k_5^{T_3}\right)^{J_1^3} \left(k_6^{T_3}\right)^{J_2^3} \left(k_6^{T_4}\right)^{p_4} . \\
&\times \int_0^1 dz_5 \int_0^1 dz_4 \int_0^1 dz_3 \int_0^1 dz_2 \cdot z_2^{k_{12}} z_3^{k_{123}+1-\left(J_1^1\right)} z_4^{k_{1234}+2-\left(J_1^1+J_2^1+J_1^2\right)} \\
&\times z_5^{k_{12345}+3-\left(J_1^1+J_2^1+J_1^2+J_3^1+J_2^2+J_1^3\right)} \\
&\times (1-z_2)^{k_{23}-J_1^1} (1-z_3)^{k_{34}} (1-z_4)^{k_{45}} (1-z_5)^{k_{56}} \\
&\times \sum_{m_{23}} \frac{\left[-k_{24}+\left(J_2^1+J_1^2\right)\right]_{m_{23}}}{m_{23}!} (z_2z_3)^{m_{23}} \sum_{m_{24}} \frac{\left[-k_{25}+\left(J_3^1+J_2^2+J_1^3\right)\right]_{m_{24}}}{m_{24}!} (z_2z_3z_4)^{m_{24}} \\
&\times \sum_{m_{25}} \frac{\left[-k_{26}+\left(J_4^1+J_3^2+J_2^3+p_4\right)\right]_{m_{25}}}{m_{25}!} (z_2z_3z_4z_5)^{m_{25}} \\
&\times \sum_{m_{34}} \frac{[-k_{35}]_{m_{34}}}{m_{34}!} (z_3z_4)^{m_{34}} \sum_{m_{35}} \frac{[-k_{36}]_{m_{34}}}{m_{35}!} (z_3z_4z_5)^{m_{35}} \sum_{m_{45}} \frac{[-k_{46}]_{m_{34}}}{m_{45}!} (z_4z_5)^{m_{45}} ,
\end{aligned}
\tag{3.50}
$$

JHEP09(2023)111

which gives

$$
\begin{aligned}
&A^{\{p_1,p_2,p_3,p_4\},0,0}\\
&\quad=\sum_{J_1^1+J_2^1+J_3^1+J_4^1=N+p_1}^{N+p_1}\frac{(N+p_1)!}{J_1^1!J_2^1!J_3^1!J_4^1!}\left(k_3^{T_1}\right)^{J_1^1}\left(k_4^{T_1}\right)^{J_2^1}\left(k_5^{T_1}\right)^{J_3^1}\left(k_6^{T_1}\right)^{J_4^1}\\
&\qquad\times\sum_{J_1^2+J_2^2+J_3^2=p_2}^{p_2}\frac{P_2!}{J_1^2!J_2^2!J_3^2!}\left(k_4^{T_2}\right)^{J_1^2}\left(k_5^{T_2}\right)^{J_2^2}\left(k_6^{T_2}\right)^{J_3^2}\\
&\qquad\times\sum_{J_1^3+J_2^3=p_3}^{p_2}\frac{P_3!}{J_1^3!J_2^3!}\left(k_5^{T_3}\right)^{J_1^3}\left(k_6^{T_3}\right)^{J_2^3}\left(k_6^{T_4}\right)^{p_4}\\
&\qquad\times\sum_{m_{23}}\frac{\left[-k_{24}+\left(J_2^1+J_1^2\right)\right]_{m_{23}}}{m_{23}!}\sum_{m_{24}}\frac{\left[-k_{25}+\left(J_3^1+J_2^2+J_1^3\right)\right]_{m_{24}}}{m_{24}!}\\
&\qquad\times\sum_{m_{25}}\frac{\left[-k_{26}+\left(J_4^1+J_3^2+J_2^3+p_4\right)\right]_{m_{25}}}{m_{25}!}\\
&\qquad\times\sum_{m_{34}}\frac{[-k_{35}]_{m_{34}}}{m_{34}!}\sum_{m_{35}}\frac{[-k_{36}]_{m_{35}}}{m_{35}!}\sum_{m_{45}}\frac{[-k_{46}]_{m_{45}}}{m_{45}!}\\
&\qquad\times\int_0^1 dz_2\cdot z_2^{k_{12}+m_{23}+m_{24}+m_{25}}\left(1-z_2\right)^{k_{23}-J_1^1}\\
&\qquad\times\int_0^1 dz_3\cdot z_3^{k_{123}+1-\left(J_1^1\right)+m_{23}+m_{24}+m_{25}+m_{34}+m_{35}}\left(1-z_3\right)^{k_{34}}\\
&\qquad\times\int_0^1 dz_4\cdot z_4^{k_{1234}+2-\left(J_1^1+J_2^1+J_1^2\right)+m_{24}+m_{25}+m_{34}+m_{35}+m_{45}}\left(1-z_4\right)^{k_{45}}\\
&\qquad\times\int_0^1 dz_5 z_5^{k_{12345}+3-\left(J_1^1+J_2^1+J_1^2+J_3^1+J_2^2+J_1^3\right)+m_{25}+m_{35}+m_{45}}\left(1-z_5\right)^{k_{56}}.
\end{aligned}
\tag{3.51}
$$

After integration, we obtain

$$
\begin{aligned}
&A^{\{p_1,p_2,p_3,p_4\},0,0}\\
&=\sum_{J_1^1+J_2^1+J_3^1+J_4^1=N+p_1}^{N+p_1}\frac{(N+p_1)!}{J_1^1!J_2^1!J_3^1!J_4^1!}\left(k_3^{T_1}\right)^{J_1^1}\left(k_4^{T_1}\right)^{J_2^1}\left(k_5^{T_1}\right)^{J_3^1}\left(k_6^{T_1}\right)^{J_4^1}\\
&\quad\times\sum_{J_1^2+J_2^2+J_3^2=p_2}^{p_2}\frac{P_2!}{J_1^2!J_2^2!J_3^2!}\left(k_4^{T_2}\right)^{J_1^2}\left(k_5^{T_2}\right)^{J_2^2}\left(k_6^{T_2}\right)^{J_3^2}\\
&\quad\times\sum_{J_1^3+J_2^3=p_3}^{p_2}\frac{P_3!}{J_1^3!J_2^3!}\left(k_5^{T_3}\right)^{J_1^3}\left(k_6^{T_3}\right)^{J_2^3}\left(k_6^{T_4}\right)^{p_4}\\
&\quad\times\sum_{m_{23}}\frac{\left[-k_{24}+\left(J_2^1+J_1^2\right)\right]_{m_{23}}}{m_{23}!}\sum_{m_{24}}\frac{\left[-k_{25}+\left(J_3^1+J_2^2+J_1^3\right)\right]_{m_{24}}}{m_{24}!}\\
&\quad\times\sum_{m_{25}}\frac{\left[-k_{26}+\left(J_4^1+J_3^2+J_2^3+p_4\right)\right]_{m_{25}}}{m_{25}!}\\
&\quad\times\sum_{m_{34}}\frac{[-k_{35}]_{m_{34}}}{m_{34}!}\sum_{m_{35}}\frac{[-k_{36}]_{m_{35}}}{m_{35}!}\sum_{m_{45}}\frac{[-k_{46}]_{m_{45}}}{m_{45}!}
\end{aligned}
$$

JHEP09(2023)111

$$\times \frac{\Gamma\left(k_{12}+1+m_{23}+m_{24}+m_{25}\right)\Gamma\left(k_{23}+1-J_1^1\right)}{\Gamma\left(k_{12}+k_{23}+2+m_{23}+m_{24}+m_{25}\right)}$$
$$\times \frac{\Gamma\left(k_{123}+2-J_1^1+m_{23}+m_{24}+m_{25}+m_{34}+m_{35}\right)\Gamma\left(k_{34}+1\right)}{\Gamma\left(k_{123}+k_{34}+3-J_1^1+m_{23}+m_{24}+m_{25}+m_{34}+m_{35}\right)}$$
$$\times \frac{\Gamma\left(k_{1234}+3-\left(J_1^1+J_2^1+J_1^2\right)+m_{24}+m_{25}+m_{34}+m_{35}+m_{45}\right)\Gamma\left(k_{45}+1\right)}{\Gamma\left(k_{1234}+k_{45}+4-\left(J_1^1+J_2^1+J_1^2\right)+m_{24}+m_{25}+m_{34}+m_{35}+m_{45}\right)}$$
$$\times \frac{\Gamma\left(k_{12345}+4-\left(J_1^1+J_2^1+J_1^2+J_3^1+J_2^2+J_1^3\right)+m_{25}+m_{35}+m_{45}\right)\Gamma\left(k_{56}+1\right)}{\Gamma\left(k_{12345}+k_{56}+5-\left(J_1^1+J_2^1+J_1^2+J_3^1+J_2^2+J_1^3\right)+m_{25}+m_{35}+m_{45}\right)}. \tag{3.52}$$

Now we choose to work on the Regge regime defined by

$$k_{12345} \sim s, k_{12345}+k_{56} \sim t. \tag{3.53}$$

In this regime, the RSSA can be approximated as

$$A^{\{p_1,p_2,p_3,p_4\},0,0}$$
$$\sim \sum_{J_1^1+J_2^1+J_3^1+J_4^1=N+p_1}^{N+p_1} \frac{(N+p_1)!}{J_1^1!J_2^1!J_3^1!J_4^1!}\left(k_3^{T_1}\right)^{J_1^1}\left(k_4^{T_1}\right)^{J_2^1}\left(k_5^{T_1}\right)^{J_3^1}\left(k_6^{T_1}\right)^{J_4^1}$$
$$\times \sum_{J_1^2+J_2^2+J_3^2=p_2}^{p_2} \frac{P_2!}{J_1^2!J_2^2!J_3^2!}\left(k_4^{T_2}\right)^{J_1^2}\left(k_5^{T_2}\right)^{J_2^2}\left(k_6^{T_2}\right)^{J_3^2}$$
$$\times \sum_{J_1^3+J_2^3=p_3}^{p_2} \frac{P_3!}{J_1^3!J_2^3!}\left(k_5^{T_3}\right)^{J_1^3}\left(k_6^{T_3}\right)^{J_2^3}\left(k_6^{T_4}\right)^{p_4}$$
$$\times \sum_{m_{23}} \frac{\left[-k_{24}+\left(J_2^1+J_1^2\right)\right]_{m_{23}}}{m_{23}!} \sum_{m_{24}} \frac{\left[-k_{25}+\left(J_3^1+J_2^2+J_1^3\right)\right]_{m_{24}}}{m_{24}!}$$
$$\times \sum_{m_{25}} \frac{\left[-k_{26}+\left(J_4^1+J_3^2+J_2^3+p_4\right)\right]_{m_{25}}}{m_{25}!}$$
$$\times \sum_{m_{34}} \frac{\left[-k_{35}\right]_{m_{34}}}{m_{34}!} \sum_{m_{35}} \frac{\left[-k_{36}\right]_{m_{35}}}{m_{35}!} \sum_{m_{45}} \frac{\left[-k_{46}\right]_{m_{45}}}{m_{45}!}$$
$$\times \frac{\Gamma\left(k_{12}+1+m_{23}+m_{24}+m_{25}\right)\Gamma\left(k_{23}+1-J_1^1\right)}{\Gamma\left(k_{12}+k_{23}+2-J_1^1+m_{23}+m_{24}+m_{25}\right)}$$
$$\times \frac{\Gamma\left(k_{123}+2-J_1^1+m_{23}+m_{24}+m_{25}+m_{34}+m_{35}\right)\Gamma\left(k_{34}+1\right)}{\Gamma\left(k_{123}+k_{34}+3-J_1^1+m_{23}+m_{24}+m_{25}+m_{34}+m_{35}\right)}$$
$$\times \frac{\Gamma\left(k_{1234}+3-\left(J_1^1+J_2^1+J_1^2\right)+m_{24}+m_{25}+m_{34}+m_{35}+m_{45}\right)\Gamma\left(k_{45}+1\right)}{\Gamma\left(k_{1234}+k_{45}+4-\left(J_1^1+J_2^1+J_1^2\right)+m_{24}+m_{25}+m_{34}+m_{35}+m_{45}\right)}$$
$$\times \frac{\left(k_{12345}\right)^{-\left(J_1^1+J_2^1+J_1^2+J_3^1+J_2^2+J_1^3\right)+m_{25}+m_{35}+m_{45}}}{\left(k_{12345}+k_{56}+5\right)_{-\left(J_1^1+J_2^1+J_1^2+J_3^1+J_2^2+J_1^3\right)+m_{25}+m_{35}+m_{45}}} \frac{\Gamma\left(k_{12345}+4\right)\Gamma\left(k_{56}+1\right)}{\Gamma\left(k_{12345}+k_{56}+5\right)}. \tag{3.54}$$

To get the leading order in k_{12345}, we take

$$J_1^1 = J_2^1 = J_3^1 = J_1^2 = J_2^2 = J_1^3 = 0,$$

which implies

$$J_4^1 = N + p_1, J_3^2 = p_2, J_2^3 = p_3. \tag{3.55}$$

With $p_1 + p_2 + p_3 + p_4 = 0$, the leading term is

$$\begin{aligned}
&A^{\{p_1,p_2,p_3,p_4\},0,0} \\
&\sim \left(k_6^{T_1}\right)^{N+p_1} \left(k_6^{T_2}\right)^{p_2} \left(k_6^{T_3}\right)^{p_3} \left(k_6^{T_4}\right)^{p_4} \\
&\quad \times \sum_{m_{23}} \frac{[-k_{24}]_{m_{23}}}{m_{23}!} \sum_{m_{24}} \frac{[-k_{25}]_{m_{24}}}{m_{24}!} \sum_{m_{25}} \frac{[-k_{26}+N]_{m_{25}}}{m_{25}!} \\
&\quad \times \sum_{m_{34}} \frac{[-k_{35}]_{m_{34}}}{m_{34}!} \sum_{m_{35}} \frac{[-k_{36}]_{m_{35}}}{m_{35}!} \sum_{m_{45}} \frac{[-k_{46}]_{m_{45}}}{m_{45}!} \\
&\quad \times \frac{\Gamma(k_{12}+1+m_{23}+m_{24}+m_{25})\,\Gamma(k_{23}+1)}{\Gamma(k_{12}+k_{23}+2+m_{23}+m_{24}+m_{25})} \\
&\quad \times \frac{\Gamma(k_{123}+2+m_{23}+m_{24}+m_{25}+m_{34}+m_{35})\,\Gamma(k_{34}+1)}{\Gamma(k_{123}+k_{34}+3+m_{23}+m_{24}+m_{25}+m_{34}+m_{35})} \\
&\quad \times \frac{\Gamma(k_{1234}+3+m_{24}+m_{25}+m_{34}+m_{35}+m_{45})\,\Gamma(k_{45}+1)}{\Gamma(k_{1234}+k_{45}+4+m_{24}+m_{25}+m_{34}+m_{35}+m_{45})} \\
&\quad \times \frac{(k_{12345})^{+m_{25}+m_{35}+m_{45}}}{(k_{12345}+k_{56}+5)_{m_{25}+m_{35}+m_{45}}} \frac{\Gamma(k_{12345}+4)\,\Gamma(k_{56}+1)}{\Gamma(k_{12345}+k_{56}+5)}.
\end{aligned} \tag{3.56}$$

So the ratios of the 7-point RSSA is

$$\begin{aligned}
\frac{A^{\{p_1,p_2,p_3,p_4\},0,0}}{A^{\{0,0,0,0\},0,0}} &= \left(k_6^{T_1}\right)^{p_1} \left(k_6^{T_2}\right)^{p_2} \left(k_6^{T_3}\right)^{p_3} \left(k_6^{T_4}\right)^{p_4} \\
&= \left(\cos\phi_2^6\right)^{p_1} \left(\sin\phi_2^6 \cos\phi_3^6\right)^{p_2} \left(\sin\phi_2^6 \sin\phi_3^6 \cos\phi_4^6\right)^{p_3} \left(\sin\phi_2^6 \sin\phi_3^6 \sin\phi_4^6\right)^{p_4} \\
&= (\cos\theta_1)^{p_1} (\sin\theta_1 \cos\theta_2)^{p_2} (\sin\theta_1 \sin\theta_2 \cos\theta_3)^{p_3} (\sin\theta_1 \sin\theta_2 \sin\theta_3)^{p_4} \\
&= (\omega_1)^{p_1} (\omega_2)^{p_2} (\omega_3)^{p_3} (\omega_4)^{p_4},
\end{aligned} \tag{3.57}$$

which is the same as eq. (2.16) with $m = q = 0$ and $r = 4$.

4 The n-point Regge stringy scaling

In this section, we generalize the previous calculations to the case of n-point RSSA. We first define the 26-dimensional momenta in the CM frame to be

$$\begin{aligned}
k_1 &= \left(\sqrt{p^2+M_1^2}, -p, 0^r\right), \\
k_2 &= \left(\sqrt{p^2+M_2^2}, p, 0^r\right), \\
&\vdots \\
k_j &= \left(-\sqrt{q_j^2+M_j^2}, -q_j\Omega_1^j, -q_j\Omega_2^j, \cdots, -q_j\Omega_r^j, -q_j\Omega_{r+1}^j\right)
\end{aligned} \tag{4.1}$$

where $j = 3, 4, \cdots, n$, and

$$\Omega_i^j = \cos\phi_i^j \prod_{\sigma=1}^{i-1} \sin\phi_\sigma^j \text{ with } \phi_{j-1}^j = 0,\ \phi_{i>r}^j = 0 \text{ and } r \leq \min\{n-3, 24\} \tag{4.2}$$

are the solid angles in the $(j-2)$-dimensional spherical space with $\sum_{i=1}^{j-2}\left(\Omega_i^j\right)^2=1$. In eq. (4.1), 0^r denotes the r-dimensional null vector. The amplitude of one tensor state

$$\left(\alpha_{-1}^{T_1}\right)^{N+p_1}\left(\alpha_{-1}^{T_2}\right)^{p_2}\cdots\left(\alpha_{-1}^{T_r}\right)^{p_r}|0,k\rangle\,, p_1+p_2+\cdots+p_r=0 \tag{4.3}$$

and $n-1$ tachyon states is

$$\begin{aligned}&A^{\{p_1,p_2,\cdots,p_r\},0,0}\\&=\int_0^1 dx_{n-2}\int_0^{x_{n-2}}dx_{n-3}\cdots\int_0^{x_4}dx_3\int_0^{x_3}dx_2\\&\quad\times\prod_{0\le i<j\le n-1}(x_j-x_i)^{k_{ij}}\prod_{\sigma=1}^{r}\left[\sum_{j=\sigma+2}^{n-1}\left(\frac{k_j^{T_\sigma}}{x_j-x_2}\right)\right]^{\mathcal{P}_\sigma}\end{aligned} \tag{4.4}$$

where we have defined

$$\mathcal{P}_1=N+p_1,\mathcal{P}_{\sigma\neq 1}=p_\sigma. \tag{4.5}$$

Now, let's explicitly write down the second product part of eq. (4.4) as

$$\begin{aligned}&A^{\{p_1,p_2,\cdots,p_r\},0,0}\\&=\int_0^1 dx_{n-2}\int_0^{x_{n-2}}dx_{n-3}\cdots\int_0^{x_4}dx_3\int_0^{x_3}dx_2\times\prod_{0\le i<j\le n-1}(x_j-x_i)^{k_{ij}}\\&\quad\times\left[\frac{k_3^{T_1}}{x_3-x_2}+\frac{k_4^{T_1}}{x_4-x_2}+\frac{k_5^{T_1}}{x_5-x_2}\cdots+\frac{k_{n-1}^{T_1}}{1-x_2}\right]^{\mathcal{P}_1}\\&\quad\times\left[\frac{k_4^{T_2}}{x_4-x_2}+\frac{k_5^{T_2}}{x_5-x_2}+\cdots+\frac{k_{n-1}^{T_2}}{1-x_2}\right]^{\mathcal{P}_2}\\&\quad\vdots\\&\quad\times\left[\frac{k_{r+2}^{T_r}}{x_{r+2}-x_2}+\cdots+\frac{k_{n-1}^{T_r}}{1-x_2}\right]^{\mathcal{P}_r}.\end{aligned} \tag{4.6}$$

For convience, from now on we add trivial terms with $\mathcal{P}_\sigma=0$ $(r+1\le\sigma\le n-3)$ to the amplitude and obtain

$$\begin{aligned}&A^{\{p_1,p_2,\cdots,p_r\},0,0}\\&=\int_0^1 dx_{n-2}\int_0^{x_{n-2}}dx_{n-3}\cdots\int_0^{x_4}dx_3\int_0^{x_3}dx_2\times\prod_{0\le i<j\le n-1}(x_j-x_i)^{k_{ij}}\\&\quad\times\left[\frac{k_3^{T_1}}{x_3-x_2}+\frac{k_4^{T_1}}{x_4-x_2}+\frac{k_5^{T_1}}{x_5-x_2}+\frac{k_5^{T_1}}{x_6-x_2}+\cdots+\frac{k_{n-1}^{T_1}}{1-x_2}\right]^{\mathcal{P}_1}\\&\quad\times\left[\frac{k_4^{T_2}}{x_4-x_2}+\frac{k_5^{T_2}}{x_5-x_2}+\frac{k_6^{T_2}}{x_6-x_2}+\cdots+\frac{k_{n-1}^{T_2}}{1-x_2}\right]^{\mathcal{P}_2}\\&\quad\vdots\end{aligned}$$

$$
\times\left[\frac{k_{r+2}^{T_r}}{x_{r+2}-x_2}+\frac{k_{r+3}^{T_r}}{x_{r+3}-x_2}+\cdots+\frac{k_{n-1}^{T_r}}{1-x_2}\right]^{\mathcal{P}_r}\left[\frac{k_{r+3}^{T_{r+1}}}{x_{r+3}-x_2}+\cdots+\frac{k_{n-1}^{T_{r+1}}}{1-x_2}\right]^{\mathcal{P}_{r+1}}
$$

$$
\vdots
$$

$$
\times\left[\frac{k_{n-1}^{T_{n-3}}}{1-x_2}\right]^{\mathcal{P}_{n-3}}. \tag{4.7}
$$

Now we can expand the brackets

$$
\begin{aligned}
&A^{\{p_1,p_2,\cdots,p_r\},0,0}\\
&=\int_0^1 dx_{n-2}\int_0^{x_{n-2}}dx_{n-3}\cdots\int_0^{x_4}dx_3\int_0^{x_3}dx_2\times\prod_{0\le i<j\le n-1}(x_j-x_i)^{k_{ij}}\\
&\times\sum_{J_1^1+J_2^1+\cdots+J_{n-4}^1+J_{n-3}^1\equiv\mathcal{P}_1}\frac{\mathcal{P}_1!}{J_1^1!J_2^1!\cdots J_{n-4}^1!J_{n-3}^1!}\left(\frac{k_3^{T_1}}{x_3-x_2}\right)^{J_1^1}\left(\frac{k_4^{T_1}}{x_4-x_2}\right)^{J_2^1}\cdots\left(\frac{k_{n-1}^{T_1}}{1-x_2}\right)^{J_{n-3}^1}\\
&\times\sum_{J_1^2+J_2^2+\cdots+J_{n-5}^2+J_{n-4}^2=\mathcal{P}_2}\frac{\mathcal{P}_2!}{J_1^2!J_2^2!\cdots J_{n-5}^2!J_{n-4}^2!}\left(\frac{k_4^{T_2}}{x_4-x_2}\right)^{J_1^2}\cdots\left(\frac{k_{n-1}^{T_2}}{1-x_2}\right)^{J_{n-4}^2}\\
&\vdots\\
&\times\sum_{J_1^{n-3}=\mathcal{P}_{n-3}=0}\frac{\mathcal{P}_{n-3}!}{J_1^{n-3}!}\left(\frac{k_{n-1}^{T_{n-3}}}{1-x_2}\right)^{J_1^{n-3}}
\end{aligned}\tag{4.8}
$$

where all J are non-negative integers. (Note that all $J_j^{\sigma\ge r+1}=0$ due to $\mathcal{P}_{\sigma\ge r+1}=0$.)

After some rearrangements, we can derive

$$
\begin{aligned}
&A^{\{p_1,p_2,\cdots,p_{n-3}\},0,0}\\
&=\sum_{J_1^1+J_2^1+\cdots+J_{n-4}^1+J_{n-3}^1\equiv\mathcal{P}_1}\frac{\mathcal{P}_1!}{J_1^1!J_2^1!\cdots J_{n-4}^1!J_{n-3}^1!}\left(k_3^{T_1}\right)^{J_1^1}\left(k_4^{T_1}\right)^{J_2^1}\cdots\left(k_{n-1}^{T_1}\right)^{J_{n-3}^1}\\
&\times\sum_{J_1^2+J_2^2+\cdots+J_{n-5}^2+J_{n-4}^2=\mathcal{P}_2}\frac{\mathcal{P}_2!}{J_1^2!J_2^2!\cdots J_{n-5}^2!J_{n-4}^2!}\left(k_4^{T_2}\right)^{J_1^2}\left(k_5^{T_2}\right)^{J_2^2}\cdots\left(k_{n-1}^{T_2}\right)^{J_{n-4}^2}\\
&\vdots\\
&\times\sum_{J_1^{n-3}=\mathcal{P}_{n-3}=0}\frac{\mathcal{P}_{n-3}!}{J_1^{n-3}!}\left(k_{n-1}^{T_{n-3}}\right)^{J_1^{n-3}}\\
&\times\int_0^1 dx_{n-2}\int_0^{x_{n-2}}dx_{n-3}\cdots\int_0^{x_4}dx_3\int_0^{x_3}dx_2\times\prod_{0\le i<j\le n-1}(x_j-x_i)^{k_{ij}}\\
&\times\left(\frac{1}{x_3-x_2}\right)^{J_1^1}\left(\frac{1}{x_4-x_2}\right)^{J_2^1}\cdots\left(\frac{1}{1-x_2}\right)^{J_{n-3}^1}\left(\frac{1}{x_4-x_2}\right)^{J_1^2}\cdots\left(\frac{1}{1-x_2}\right)^{J_{n-4}^2}\\
&\vdots\\
&\times\left(\frac{1}{1-x_2}\right)^{J_1^{n-3}}.
\end{aligned}\tag{4.9}
$$

JHEP09(2023)111

We can collect terms with the same sum of subscripts and superscripts to get

$$
\begin{aligned}
&A^{\{p_1,p_2,\cdots,p_{n-3}\},0,0} \\
&\quad=\prod_{\sigma=1}^{n-3}\left[\sum_{\sum_{j=1}^{n-2-\sigma}J_j^\sigma=\mathcal{P}_\sigma}\left(\mathcal{P}_\sigma!\prod_{j=1}^{n-2-\sigma}\frac{\left(k_{j+\sigma+1}^{T_\sigma}\right)^{J_j^\sigma}}{J_j^\sigma!}\right)\right] \\
&\quad\quad\times\int_0^1 dx_{n-2}\int_0^{x_{n-2}}dx_{n-3}\cdots\int_0^{x_4}dx_3\int_0^{x_3}dx_2\times\prod_{0\le i<j\le n-1}(x_j-x_i)^{k_{ij}-\delta_{i2}\left(J_{j-2}^1+\cdots+J_1^{j-2}\right)}.
\end{aligned}
\tag{4.10}
$$

To perform the integral of the last line in eq. (4.10), let's do the following change of variables

$$
x_i=\prod_{k=i}^{n-2}z_k,\ \text{ or } x_2=z_2z_3\cdots z_{n-2}, x_3=z_3z_4\cdots z_{n-2},\cdots,x_{n-2}=z_{n-2},x_{n-1}=z_{n-1}=1
\tag{4.11}
$$

to make all the integral intervals from 0 to 1. The integral becomes

$$
\begin{aligned}
&\int_0^1 dx_{n-2}\int_0^{x_{n-2}}dx_{n-3}\cdots\int_0^{x_4}dx_3\int_0^{x_3}dx_2\times\prod_{0\le i<j\le n-1}(x_j-x_i)^{k_{ij}-\delta_{i2}\left(J_{j-2}^1+\cdots+J_1^{j-2}\right)} \\
&=\int_0^1 dz_{n-2}\cdots\int_0^1 dz_3\int_0^1 dz_2\prod_{i=1}^{n-4}(z_{i+2})^i\prod_{0\le i<j\le n-1}\left(\prod_{k=j}^{n-2}z_k-\prod_{k=i}^{n-2}z_k\right)^{k_{ij}-\delta_{i2}\left(J_{j-2}^1+\cdots+J_1^{j-2}\right)} \\
&=\int_0^1 dz_{n-2}\cdots\int_0^1 dz_2\times\prod_{i=1}^{n-4}(z_{i+2})^i\prod_{0\le i<j\le n-1}\left[\prod_{k=j}^{n-2}z_k\left(1-\prod_{k=i}^{j-1}z_k\right)\right]^{k_{ij}-\delta_{i2}\left(J_{j-2}^1+\cdots+J_1^{j-2}\right)}.
\end{aligned}
\tag{4.12}
$$

Now, the amplitude can be explicitly written as

$$
\begin{aligned}
&A^{\{p_1,p_2,\cdots,p_{n-3}\},0,0} \\
&=\prod_{\sigma=1}^{n-3}\left[\sum_{\sum_{j=1}^{n-2-\sigma}J_j^\sigma=\mathcal{P}_\sigma}\left(\mathcal{P}_\sigma!\prod_{j=1}^{n-2-\sigma}\frac{\left(k_{j+\sigma+1}^{T_\sigma}\right)^{J_j^\sigma}}{J_j^\sigma!}\right)\right] \\
&\times\int_0^1 dz_{n-2}\int_0^1 dz_{n-3}\cdots\int_0^1 dz_3\int_0^1 dz_2 \\
&\times z_2^{k_{12}}z_3^{k_{123}+1-J_1^1}z_4^{k_{1234}+2-J_1^1-\left(J_1^2+J_2^1\right)}\cdots z_{n-2}^{k_{1,\cdots,n-2}+(n-4)-J_1^1-\left(J_1^2+J_2^1\right)-\cdots-\left(J_1^{n-4}+\cdots+J_{n-4}^1\right)} \\
&\times(1-z_2)^{k_{23}-J_1^1}(1-z_2z_3)^{k_{24}-\left(J_1^2+J_2^1\right)}\cdots(1-z_2z_3z_4\cdots z_{n-2})^{k_{2,n-1}-\left(J_1^{n-3}+\cdots+J_{n-3}^1\right)} \\
&\times(1-z_3)^{k_{34}}(1-z_3z_4)^{k_{35}}\cdots(1-z_3z_4\cdots z_{n-2})^{k_{3,n-1}} \\
&\vdots \\
&\times(1-z_{n-3})^{k_{n-3,n-2}}(1-z_{n-3}z_{n-2})^{k_{n-3,n-1}} \\
&\times(1-z_{n-2})^{k_{n-2,n-1}}.
\end{aligned}
\tag{4.13}
$$

JHEP09(2023)111

Let's rearrange the above equation to get a more symmetric form in the following

$$
\begin{aligned}
&A^{\{p_1,p_2,\cdots,p_{n-3}\},0,0} \\
&\quad= \prod_{\sigma=1}^{n-3}\left[\sum_{\sum_{j=1}^{n-2-\sigma} J_j^\sigma=\mathcal{P}_\sigma}\left(\mathcal{P}_\sigma! \prod_{j=1}^{n-2-\sigma}\frac{\left(k_{j+\sigma+1}^{T_\sigma}\right)^{J_j^\sigma}}{J_j^\sigma!}\right)\right] \\
&\quad\times \int_0^1 dz_{n-2}\int_0^1 dz_{n-3}\cdots\int_0^1 dz_3\int_0^1 dz_2 \cdot \\
&\quad\times z_2^{k_{12}} z_3^{k_{123}+1-J_1^1} z_4^{k_{1234}+2-J_1^1-\left(J_1^2+J_2^1\right)}\cdots z_{n-2}^{k_{1,\cdots,n-2}+(n-4)-J_1^1-\left(J_1^2+J_2^1\right)-\cdots-\left(J_1^{n-4}+\cdots+J_{n-4}^1\right)} \\
&\quad\times (1-z_2)^{k_{23}-J_1^1}(1-z_3)^{k_{34}}(1-z_4)^{k_{45}}\cdots(1-z_{n-2})^{k_{n-2,n-1}} \\
&\quad\times (1-z_2z_3)^{k_{24}-\left(J_1^2+J_1^2\right)}(1-z_2z_3z_4)^{k_{25}-\left(J_1^3+J_2^2+J_3^1\right)}\cdots \\
&\quad\times (1-z_2z_3z_4\cdots z_{n-2})^{k_{2,n-1}-\left(J_1^{n-3}+\cdots+J_{n-3}^1\right)} \\
&\quad\times (1-z_3z_4)^{k_{35}}\cdots(1-z_3z_4\cdots z_{n-2})^{k_{3,n-1}} \\
&\quad\vdots \\
&\quad\times (1-z_{n-3}z_{n-2})^{k_{n-3,n-1}} .
\end{aligned}
\tag{4.14}
$$

Then we expand the crossing terms to get

$$
\begin{aligned}
&A^{\{p_1,p_2,\cdots,p_{n-3}\},0,0} \\
&\quad= \prod_{\sigma=1}^{n-3}\left[\sum_{\sum_{j=1}^{n-2-\sigma} J_j^\sigma=\mathcal{P}_\sigma}\left(\mathcal{P}_\sigma! \prod_{j=1}^{n-2-\sigma}\frac{\left(k_{j+\sigma+1}^{T_\sigma}\right)^{J_j^\sigma}}{J_j^\sigma!}\right)\right] \\
&\quad\times \int_0^1 dz_{n-2}\int_0^1 dz_{n-3}\cdots\int_0^1 dz_3\int_0^1 dz_2 \\
&\quad\times z_2^{k_{12}} z_3^{k_{123}+1-J_1^1} z_4^{k_{1234}+2-J_1^1-\left(J_1^2+J_2^1\right)}\cdots z_{n-2}^{k_{1,\cdots,n-2}+(n-4)-J_1^1-\left(J_1^2+J_2^1\right)-\cdots-\left(J_1^{n-4}+\cdots+J_{n-4}^1\right)} \\
&\quad\times (1-z_2)^{k_{23}-J_1^1}(1-z_3)^{k_{34}}(1-z_4)^{k_{45}}\cdots(1-z_{n-2})^{k_{n-2,n-1}} \\
&\quad\times \sum_{m_{23}}\frac{\left[-k_{24}+\left(J_1^2+J_2^1\right)\right]_{m_{23}}}{m_{23}!}(z_2z_3)^{m_{23}}\cdots \\
&\quad\times \sum_{m_{2,n-2}}\frac{\left[-k_{2,n-1}+\left(J_1^{n-3}+\cdots+J_{n-3}^1\right)\right]_{m_{2,n-2}}}{m_{2,n-2}!}(z_2z_3z_4\cdots z_{n-2})^{m_{2,n-2}} \\
&\quad\times \sum_{m_{34}}\frac{(-k_{35})_{m_{34}}}{m_{34}!}(z_3z_4)^{m_{34}}\cdots\sum_{m_{3,n-2}}\frac{(-k_{3,n-1})_{m_{3,n-2}}}{m_{3,n-2}!}(z_3z_4\cdots z_{n-2})^{m_{3,n-2}} \\
&\quad\vdots \\
&\quad\times \sum_{m_{n-3,n-2}}\frac{(-k_{n-3,n-1})_{m_{3,n-2}}}{m_{n-3,n-2}!}(z_{n-3}z_{n-2})^{m_{n-3,n-2}}
\end{aligned}
\tag{4.15}
$$

where the subscripts of m_{ij} keep record of the first and the last subscripts of $(z_i z_{i+1} \cdots z_{j-1} z_j)$ etc. The amplitude becomes

$$
\begin{aligned}
&A^{\{p_1,p_2,\cdots,p_{n-3}\},0,0} \\
&=\prod_{\sigma=1}^{n-3}\left[\sum_{\sum_{j=1}^{n-2-\sigma} J_j^{\sigma}=\mathcal{P}_\sigma}\left(\mathcal{P}_\sigma!\prod_{j=1}^{n-2-\sigma}\frac{\left(k_{j+\sigma+1}^{T_\sigma}\right)^{J_j^\sigma}}{J_j^\sigma!}\right)\right] \\
&\times\sum_{m_{23}}\frac{\left[-k_{24}+\left(J_1^2+J_2^1\right)\right]_{m_{23}}}{m_{23}!}\cdots\sum_{m_{2,n-2}}\frac{\left[-k_{2,n-1}+\left(J_1^{n-3}+\cdots+J_{n-3}^1\right)\right]_{m_{2,n-2}}}{m_{2,n-2}!} \\
&\times\sum_{m_{34}}\frac{(-k_{35})_{m_{34}}}{m_{34}!}\cdots\sum_{m_{3,n-2}}\frac{(-k_{3,n-1})_{m_{3,n-2}}}{m_{3,n-2}!} \\
&\vdots \\
&\times\sum_{m_{n-3,n-2}}\frac{(-k_{n-3,n-1})_{m_{3,n-2}}}{m_{n-3,n-2}!} \\
&\times\int_0^1 dz_{n-2}\int_0^1 dz_{n-3}\cdots\int_0^1 dz_3\int_0^1 dz_2\cdot \\
&\times z_2^{k_{12}+\sum_{i\leq 2\leq j} m_{ij}}\, z_3^{k_{123}+1-\sum_{i+j\leq 2} J_j^i+\sum_{i\leq 3\leq j} m_{ij}}\cdots z_{n-2}^{k_{1,\cdots,n-2}+(n-4)-\sum_{i+j\leq n-3} J_j^i+\sum_{i\leq n-2\leq j} m_{ij}} \\
&\times(1-z_2)^{k_{23}-J_1^1}(1-z_3)^{k_{34}}\cdots(1-z_{n-2})^{k_{n-2,n-1}}\,.
\end{aligned}
\tag{4.16}
$$

After integration, we can write it as

$$
\begin{aligned}
&A^{\{p_1,p_2,\cdots,p_{n-3}\},0,0} \\
&\quad=\prod_{\sigma=1}^{n-3}\left[\sum_{\sum_{j=1}^{n-2-\sigma} J_j^{\sigma}=\mathcal{P}_\sigma}\left(\mathcal{P}_\sigma!\prod_{j=1}^{n-2-\sigma}\frac{\left(k_{j+\sigma+1}^{T_\sigma}\right)^{J_j^\sigma}}{J_j^\sigma!}\right)\right] \\
&\quad\times\sum_{m_{23}}\frac{\left[-k_{24}+\left(J_1^2+J_2^1\right)\right]_{m_{23}}}{m_{23}!}\sum_{m_{24}}\frac{\left[-k_{25}+\left(J_1^3+J_2^2+J_3^1\right)\right]_{m_{24}}}{m_{24}!} \\
&\quad\cdots\sum_{m_{2,n-2}}\frac{\left[-k_{2,n-1}+\left(J_1^{n-3}+\cdots+J_{n-3}^1\right)\right]_{m_{2,n-2}}}{m_{2,n-2}!} \\
&\quad\times\sum_{m_{34}}\frac{(-k_{35})_{m_{34}}}{m_{34}!}\cdots\sum_{m_{3,n-2}}\frac{(-k_{3,n-1})_{m_{3,n-2}}}{m_{3,n-2}!} \\
&\quad\vdots \\
&\quad\times\sum_{m_{n-3,n-2}}\frac{(-k_{n-3,n-1})_{m_{3,n-2}}}{m_{n-3,n-2}!} \\
&\quad\times\frac{\Gamma\left(k_{12}+1+\sum_{i\leq 2\leq j} m_{ij}\right)\Gamma\left(k_{23}+1-J_1^1\right)}{\Gamma\left(k_{12}+k_{23}+2-J_1^1+\sum_{i\leq 2\leq j} m_{ij}\right)}
\end{aligned}
$$

$$\times \frac{\Gamma\left(k_{123}+2-\sum_{i+j\leq 2}J_j^i+\sum_{i\leq 3\leq j}m_{ij}\right)\Gamma\left(k_{34}+1\right)}{\Gamma\left(k_{123}+k_{34}+3-\sum_{i+j\leq 2}J_j^i+\sum_{i\leq 3\leq j}m_{ij}\right)}$$

$$\vdots$$

$$\times \frac{\Gamma\left(k_{1,\cdots,n-2}+(n-3)-\sum_{i+j\leq n-3}J_j^i+\sum_{i\leq n-2\leq j}m_{ij}\right)\Gamma\left(k_{n-2,n-1}+1\right)}{\Gamma\left(k_{1,\cdots,n-2}+k_{n-2,n-1}+(n-2)-\sum_{i+j\leq n-3}J_j^i+\sum_{i\leq n-2\leq j}m_{ij}\right)}. \tag{4.17}$$

Now we choose to work on the Regge regime defined by

$$k_{1,\cdots,n-2}\sim s, k_{1,\cdots,n-2}+k_{n-2,n-1}\sim t. \tag{4.18}$$

In this regime, the RSSA can be approximated as

$$A^{\{p_1,p_2,\cdots,p_{n-3}\},0,0}$$

$$\sim\prod_{\sigma=1}^{n-3}\left[\sum_{\sum_{j=1}^{n-2-\sigma}J_j^\sigma=\mathcal{P}_\sigma}\left(\mathcal{P}_\sigma!\prod_{j=1}^{n-2-\sigma}\frac{\left(k_{j+\sigma+1}^{T_\sigma}\right)^{J_j^\sigma}}{J_j^\sigma!}\right)\right]$$

$$\times\sum_{m_{23}}\frac{\left[-k_{24}+\left(J_1^2+J_2^1\right)\right]_{m_{23}}}{m_{23}!}\sum_{m_{24}}\frac{\left[-k_{25}+\left(J_1^3+J_2^2+J_3^1\right)\right]_{m_{24}}}{m_{24}!}$$

$$\cdots\sum_{m_{2,n-2}}\frac{\left[-k_{2,n-1}+\left(J_1^{n-3}+\cdots+J_{n-3}^1\right)\right]_{m_{2,n-2}}}{m_{2,n-2}!}$$

$$\times\sum_{m_{34}}\frac{\left(-k_{35}\right)_{m_{34}}}{m_{34}!}\cdots\sum_{m_{3,n-2}}\frac{\left(-k_{3,n-1}\right)_{m_{3,n-2}}}{m_{3,n-2}!}$$

$$\vdots$$

$$\times\sum_{m_{n-3,n-2}}\frac{\left(-k_{n-3,n-1}\right)_{m_{3,n-2}}}{m_{n-3,n-2}!}$$

$$\times\frac{\Gamma\left(k_{12}+1+\sum_{i\leq 2\leq j}m_{ij}\right)\Gamma\left(k_{23}+1-J_1^1\right)}{\Gamma\left(k_{12}+k_{23}+2-J_1^1+\sum_{i\leq 2\leq j}m_{ij}\right)}$$

$$\times\frac{\Gamma\left(k_{123}+2-\sum_{i+j\leq 2}J_j^i+\sum_{i\leq 3\leq j}m_{ij}\right)\Gamma\left(k_{34}+1\right)}{\Gamma\left(k_{123}+k_{34}+3-\sum_{i+j\leq 2}J_j^i+\sum_{i\leq 3\leq j}m_{ij}\right)}$$

$$\vdots$$

$$\times\frac{\left(k_{1,\cdots,n-2}\right)^{-\sum_{i+j\leq n-3}J_j^i+\sum_{i\leq n-2\leq j}m_{ij}}\Gamma\left(k_{1,\cdots,n-2}+(n-3)\right)\Gamma\left(k_{n-2,n-1}+1\right)}{\left(k_{1,\cdots,n-2}+k_{n-2,n-1}+(n-2)\right)_{-\sum_{i+j\leq n-3}J_j^i+\sum_{i\leq n-2\leq j}m_{ij}}\Gamma\left(k_{1,\cdots,n-2}+k_{n-2,n-1}+(n-2)\right)}. \tag{4.19}$$

To get the leading order in $k_{1,\cdots,n-2}\tilde{}\ s$, we take

$$J_j^i=0,\ \text{(for all } i+j\leq n-3) \tag{4.20}$$

or

$$\begin{aligned}
&J_1^1 = J_2^1 = \cdots = J_{n-4}^1 = 0,\\
&J_1^2 = \cdots = J_{n-5}^2 = 0,\\
&J_1^r = \cdots = J_{n-r-3}^r = 0
\end{aligned} \tag{4.21}$$

which imply

$$J_{n-3}^1 = N + p_1, J_{n-4}^2 = p_2, \cdots, J_{n-r-2}^r = p_r. \tag{4.22}$$

Finally, the leading order term of the amplitude is

$$\begin{aligned}
&A^{\{p_1,p_2,\cdots,p_r\},0,0}\\
&\sim \prod_{\sigma=1}^{r}\left[\left(k_{n-1}^{T_\sigma}\right)^{\mathcal{P}_\sigma}\right]\\
&\quad\times \sum_{m_{23}} \frac{[-k_{24}]_{m_{23}}}{m_{23}!} \sum_{m_{24}} \frac{[-k_{25}]_{m_{24}}}{m_{24}!} \cdots \sum_{m_{24}} \frac{[-k_{2,n-1}]_{m_{2,n-2}}}{m_{2,n-2}!}\\
&\quad\times \sum_{m_{34}} \frac{(-k_{35})_{m_{34}}}{m_{34}!} \cdots \sum_{m_{3,n-2}} \frac{(-k_{3,n-1})_{m_{3,n-2}}}{m_{3,n-2}!}\\
&\quad\times \sum_{m_{n-3,n-2}} \frac{(-k_{n-3,n-1})_{m_{3,n-2}}}{m_{n-3,n-2}!}\\
&\quad\times \frac{\Gamma\left(k_{12}+1+\sum_{i\le 2\le j} m_{ij}\right)\Gamma\left(k_{23}+1\right)}{\Gamma\left(k_{12}+k_{23}+2+\sum_{i\le 2\le j} m_{ij}\right)}\\
&\quad\times \frac{\Gamma\left(k_{123}+2+\sum_{i\le 3\le j} m_{ij}\right)\Gamma\left(k_{34}+1\right)}{\Gamma\left(k_{123}+k_{34}+3-\sum_{i+j\le 2} J_j^i+\sum_{i\le 3\le j} m_{ij}\right)}\\
&\quad\vdots\\
&\quad\times \frac{(k_{1,\cdots,n-2})^{\sum_{i\le n-2\le j} m_{ij}}\,\Gamma\left(k_{1,\cdots,n-2}+(n-3)\right)\Gamma\left(k_{n-2,n-1}+1\right)}{(k_{1,\cdots,n-2}+k_{n-2,n-1}+(n-2))_{\sum_{i\le n-2\le j} m_{ij}}\,\Gamma\left(k_{1,\cdots,n-2}+k_{n-2,n-1}+(n-2)\right)}\\
&= \prod_{\sigma=1}^{r}\left[\left(k_{n-1}^{T_\sigma}\right)^{\mathcal{P}_\sigma}\right] \times (\text{factors independent of } J_q^r\text{'s }).
\end{aligned} \tag{4.23}$$

The ratios of the amplitudes are

$$\begin{aligned}
\frac{A^{\{p_1,p_2,\cdots,p_r\},0,0}}{A^{\{0,0,\cdots,0\},0,0}} &= \left(k_{n-1}^{T_1}\right)^{p_1}\left(k_{n-1}^{T_2}\right)^{p_2}\cdots\left(k_{n-1}^{T_r}\right)^{p_r},\\
&= \left(\Omega_2^{n-1}\right)^{p_1}\left(\Omega_3^{n-1}\right)^{p_2}\cdots\left(\Omega_{r+1}^{n-1}\right)^{p_r},\\
&= (\omega_1)^{p_1}(\omega_2)^{p_2}\cdots(\omega_r)^{p_r}\\
&= \frac{\mathcal{T}^{(\{p_i\},0,0)}}{\mathcal{T}^{(\{0_i\},0,0)}}
\end{aligned} \tag{4.24}$$

which is the same as eq. (2.16) with $m = q = 0$.

Note the difference between the 4-point calculation in eq. (3.3) and the n-point calculation in eq. (4.24). In eq. (4.24), as we have calculated only a subclass of RSSA with $m = q = 0$, we do not need to take limit as we did for eq. (3.3). We expect that the ratios calculated in eq. (2.16) can also be **extracted** from the n-point RSSA with $n - 1$ tachyons and the state in eq. (2.13) should one can explicitly calculate the amplitudes.

5 Conclusion

In this paper, we first give a review with detailed calculations of ratios among HSSA at each fixed mass level to demonstrate the stringy scaling behavior in the hard scattering limit. We then extend the calculations and discover a similar stringy scaling behavior for a class of n-point RSSA. The number of independent kinematics variables of these RSSA is found to be reduced by $\dim\mathcal{M}$, similar to those of the HSSA.

These stringy scaling behaviors are reminiscent of deep inelastic scattering of electron and proton where the two structure functions $W_1(Q^2, \nu)$ and $W_2(Q^2, \nu)$ scale, and become not functions of 2 kinematics variables Q^2 and ν independently but only of their ratio Q^2/ν. Thus the number of independent kinematics variables reduces from 2 to 1. Indeed, it is now well-known that the structure functions scale as [36]

$$MW_1(Q^2, \nu) \to F_1(x), \quad \nu W_2(Q^2, \nu) \to F_2(x) \tag{5.1}$$

where x is the Bjorken variable and M is the proton mass. Moreover, due to the spin-$\frac{1}{2}$ assumption of quark, Callan and Gross derived the following relation [37]

$$2xF_1(x) = F_2(x). \tag{5.2}$$

Both of these scaling behaviors, the reduction of the number of kinematics variables in eq. (5.1) and the number of structure functions in eq. (5.2) in the hard scattering limit of quark-parton model in QCD seem to persist in some way in the HSSA and some RSSA of string theory. We believe that, comparing to hard QCD scaling, high energy stringy scaling in general has not been well studied yet in the literature [38]. More new phenomena of stringy scaling remain to be uncovered.

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A Examples of 4-point stringy scaling

A.1 Bosonic open string

Since the ratios of the amplitudes in eq. (2.11) are independent of the choices of V_1, V_3 and V_4, we choose them to be tachyons and V_2 to be eq. (2.2). On the other hand, since the

JHEP09(2023)111

ratios are independent of the loop order, we choose to calculate HSSA of $l = 0$ loop. An explicit amplitude calculation for $M^2 = 4$, 6 and 8 gives [11, 12, 14, 15]

$$\mathcal{T}_{TTT} : \mathcal{T}_{(LLT)} : \mathcal{T}_{(LT)} : \mathcal{T}_{[LT]} = 8 : 1 : -1 : -1, \tag{A.1}$$

$$\mathcal{T}_{(TTTT)} : \mathcal{T}_{(TTLL)} : \mathcal{T}_{(LLLL)} : \mathcal{T}_{TT,L} : \mathcal{T}_{(TTL)} : \mathcal{T}_{(LLL)} : \mathcal{T}_{(LL)}$$
$$= 16 : \frac{4}{3} : \frac{1}{3} : -\frac{2\sqrt{6}}{3} : -\frac{4\sqrt{6}}{9} : -\frac{\sqrt{6}}{9} : \frac{2}{3} \tag{A.2}$$

and

$$\mathcal{T}_{(TTTTT)} : \mathcal{T}_{(TTTL)} : \mathcal{T}_{(TTTLL)} : \mathcal{T}_{(TLLL)} : \mathcal{T}_{(TLLLL)} : \mathcal{T}_{(TLL)} : \mathcal{T}_{T,LL} : \mathcal{T}_{TLL,L} : \mathcal{T}_{TTT,L}$$
$$= 32 : \sqrt{2} : 2 : \frac{3\sqrt{2}}{16} : \frac{3}{8} : \frac{1}{3} : \frac{2}{3} : \frac{\sqrt{2}}{16} : 3\sqrt{2}, \tag{A.3}$$

respectively. These are all remarkably consistent with eq. (2.11) of ZNS calculation [18, 19].

It is important to note that for subleading order amplitudes, they are in general *not* proportional to each other. For $M^2 = 4$, for example, one gets 6 subleading order amplitudes and 4 linear relations (on-shell Ward identities) in the ZNS calculation. An explicit subleading order amplitude calculation gives [11, 12]

$$\mathcal{T}^2_{LLL} \sim -4E^8 \sin\phi \cos\phi,$$
$$\mathcal{T}^2_{LTT} \sim -8E^8 \sin^2\phi \cos\phi, \tag{A.4}$$

which show that the proportional coefficients do depend on the scattering angle ϕ.

A.2 Bosonic closed string and D-particle

For closed string scatterings [20, 21], one can use the KLT formula [22], which expresses the relation between tree amplitudes of closed and two channels of open string ($\alpha'_{\text{closed}} = 4\alpha'_{\text{open}} = 2$), to obtain the closed string ratios which are the tensor product of two open string ratios in eq. (2.11). On the other hand, it is interesting to find that the ratios of hard closed string D-particle scatterings are again given by the tensor product of two open string ratios [23]

$$\frac{T_{SD}^{\left(N;2m,2m';q,q'\right)}}{T_{SD}^{(N;0,0;0,0)}} = \left(-\frac{1}{M_2}\right)^{2(m+m')+q+q'} \left(\frac{1}{2}\right)^{m+m'+q+q'} (2m-1)!!(2m'-1)!!, \tag{A.5}$$

which came as a surprise since there is no physical picture for open string D-particle tree scattering amplitudes and thus no factorization for closed string D-particle scatterings into two channels of open string D-particle scatterings, and hence no KLT-like formula there. However, these ratios are consistent with the decoupling of high energy ZNS calculation.

A.3 Stringy scaling of superstring

It turned out to be nontrivial to extend the linear relations and their associated constant ratios of the HSSA of bosonic string to the case of $10D$ open superstring. First of all, in

JHEP09(2023)111

addition to the NS-sector, there are massive fermionic states in the R-sector whose vertex operators are still unknown except the leading Regge trajectory states in the spectrum [24]. So the only known complete vertex operators so far are those for the mass level $M^2 = 2$ [25] which contains no off leading massive Regge trajectory fermionic string states.

Secondly, in the NS-sector of $M^2 = 2$ it was surprised to note that [27] there exists no "inter-particle gauge transformation" induced by bosonic ZNS for the two positive-norm physical propagating states, the symmetric spin three and the anti-symmetric spin two states. However, the 4-point HSSA among these two positive-norm states are still related and are indeed again proportional to each others. Presumably, this is due to the massive spacetime SUSY and the existence of spacetime massive fermion string scattering amplitudes of the R-sector of the theory [26].

Thirdly, it was noted that for the HSSA of the NS sector of superstring, there existed leading order HSSA with polarizations orthogonal to the scattering plane [27]. This was due to the "worldsheet fermion exchange" [28] in the correlation functions and was argued to be related to the HSSA of massive spacetime fermion of R-sector of the theory [26].

The first calculation of the 4-point superstringy scaling was performed for the NS-sector of $10D$ open superstring theory. There are four classes of HSSA of superstring which are all proportional to each other [28]

$$|N, 2m, q\rangle \otimes \left|b^P_{-\frac{3}{2}}\right\rangle = \left(-\frac{1}{2M_2}\right)^{q+m} \frac{(2m-1)!!}{(-M_2)^m} |N, 0, 0\rangle \otimes \left|b^P_{-\frac{3}{2}}\right\rangle, \quad \text{(A.6)}$$

$$|N+1, 2m+1, q\rangle \otimes \left|b^P_{-\frac{1}{2}}\right\rangle = \left(-\frac{1}{2M_2}\right)^{q+m} \frac{(2m+1)!!}{(-M_2)^{m+1}} |N, 0, 0\rangle \otimes \left|b^P_{-\frac{3}{2}}\right\rangle, \quad \text{(A.7)}$$

$$|N+1, 2m, q\rangle \otimes \left|b^T_{-\frac{1}{2}}\right\rangle = \left(-\frac{1}{2M_2}\right)^{q+m} \frac{(2m-1)!!}{(-M_2)^{m-1}} |N, 0, 0\rangle \otimes \left|b^P_{-\frac{3}{2}}\right\rangle, \quad \text{(A.8)}$$

$$|N-1, 2m, q-1\rangle \otimes \left|b^T_{-\frac{1}{2}} b^P_{-\frac{1}{2}} b^P_{-\frac{3}{2}}\right\rangle = \left(-\frac{1}{2M_2}\right)^{q+m} \frac{(2m-1)!!}{(-M_2)^m} |N, 0, 0\rangle \otimes \left|b^P_{-\frac{3}{2}}\right\rangle. \quad \text{(A.9)}$$

Note that, in order to simplify the notation, we have only shown the second state of the four point functions to represent the scattering amplitudes on both sides of each equation above. Eqs. (A.6) to (A.9) are thus the SUSY generalization of eq. (2.11) for the bosonic string.

Moreover, a recent calculation showed that [26] among $2^4 \times 2^4 = 256$ 4-point polarized fermion SSA (PFSSA) in the R-sector of $M^2 = 2$ states, only 16 of them are of leading order in energy and all of them share the same functional form in the hard scattering limit. On the other hand, the ratios of the *complete* 4-point HSSA in the NS sector of mass level $M^2 = 2$ which include HSSA with polarizations orthogonal to the sacttering plane are [27]

$$\left\langle b^T_{\frac{-1}{2}}, \alpha^T_{-1} b^T_{\frac{-1}{2}} \right\rangle : \left\langle b^T_{\frac{-1}{2}}, \left(2b^L_{\frac{-1}{2}} \alpha^L_{-1} - M b^L_{\frac{-3}{2}}\right) \right\rangle : \left\langle b^{T_i}_{\frac{-1}{2}}, \alpha^T_{-1} b^{T_j}_{\frac{-1}{2}} \right\rangle : \left\langle b^{T_k}_{\frac{-1}{2}}, b^L_{\frac{-1}{2}} b^T_{\frac{-1}{2}} b^{T_l}_{\frac{-1}{2}} \right\rangle \quad \text{(A.10)}$$

$$= -2k_3^T E^2 : -2\left(\frac{2}{M^2} + 1\right) k_3^T E^2 : \delta_{ij} 2k_3^T E^2 : \delta_{lk} \frac{-2k_3^T E^2}{M}$$

$$= 1 : 2 : -\delta_{ij} : \frac{\delta_{lk}}{\sqrt{2}}. \quad (i, j, k, l = 3, 4, 5, \ldots, 9) \quad \text{(A.11)}$$

where we have, for simplicity, omitted the last two tachyon vertices in the notation of each HSSA in eq. (A.10). In sum, in the NS sector one gets $1+1+7+7=16$ HSSA in eq. (A.10). This result agrees with those of 16 hard massive PFSSA in the R-sector calculated recently [26].

A.4 Field theory

On the other hand, in field theory, as an example, the leading order process of the elastic scattering of a spin-$\frac{1}{2}$ particle by a spin-0 particle such as $e^-\pi^+ \longrightarrow e^-\pi^+$, the non-vanishing amplitudes were shown to be [29]

$$\mathcal{T}\,(e_R^-\pi^+ \longrightarrow e_R^-\pi^+) = \mathcal{T}\,(e_L^-\pi^+ \longrightarrow e_L^-\pi^+) \sim \cos\frac{\phi}{2}, \tag{A.12}$$

$$\mathcal{T}\,(e_R^-\pi^+ \longrightarrow e_L^-\pi^+) = \mathcal{T}\,(e_L^-\pi^+ \longrightarrow e_R^-\pi^+) \sim \sin\frac{\phi}{2}, \tag{A.13}$$

which are *not* proportional to each other. In QED, as another example, for the leading order process of $e^-e^+ \longrightarrow \mu^-\mu^+$, there are 4 non-vanishing among 16 hard polarized amplitudes [30]

$$\mathcal{T}\,(e_R^-e_L^+ \longrightarrow \mu_R^-\mu_L^+) = \mathcal{T}\,(e_L^-e_R^+ \longrightarrow \mu_L^-\mu_R^+) \sim (1+\cos\theta) = 2\ \cos^2\frac{\phi}{2}, \tag{A.14}$$

$$\mathcal{T}\,(e_R^-e_L^+ \longrightarrow \mu_L^-\mu_R^+) = \mathcal{T}\,(e_L^-e_R^+ \longrightarrow \mu_R^-\mu_L^+) \sim (1-\cos\theta) = 2\ \sin^2\frac{\phi}{2}, \tag{A.15}$$

and they are *not* all proportional to each other.

B Saddle point calculation

In this appendix, to justify the ZNS calculation in eq. (2.11) and eq. (2.16), we use the saddle point calculation to explicitly calculate the HSSA. Since the ratios are independent of the choices of V_j ($J=1,3,4\cdots,n$), we choose them to be tachyons and V_2 to be the high energy state in eq. (2.2). On the other hand, since the ratios are independent of the loop order, we choose to calculate $l=0$ loop. We begin with the 4-point case [14, 15].

B.1 The four point calculation

The $t-u$ channel contribution to the stringy amplitude at tree level is (after SL$(2,R)$ fixing)

$$\mathcal{T}^{(N,2m,q)} = \int_1^\infty dx x^{(1,2)}(1-x)^{(2,3)}\left[\frac{e^T\cdot k_1}{x} - \frac{e^T\cdot k_3}{1-x}\right]^{N-2m-2q} \cdot\left[\frac{e^P\cdot k_1}{x} - \frac{e^P\cdot k_3}{1-x}\right]^{2m}\left[-\frac{e^P\cdot k_1}{x^2} - \frac{e^P\cdot k_3}{(1-x)^2}\right]^{q} \tag{B.1}$$

where $(1,2)=k_1\cdot k_2$ etc.

In order to apply the saddle-point method, we rewrite the amplitude above into the following form

$$\mathcal{T}^{(N,2m,q)}(K) = \int_1^\infty dx\ u(x)e^{-Kf(x)}, \tag{B.2}$$

where

$$K \equiv -(1,2) \to \frac{s}{2} \to 2E^2, \tag{B.3}$$

$$\tau \equiv -\frac{(2,3)}{(1,2)} \to -\frac{t}{s} \to \sin^2\frac{\phi}{2}, \tag{B.4}$$

$$f(x) \equiv \ln x - \tau \ln(1-x), \tag{B.5}$$

$$u(x) \equiv \left[\frac{(1,2)}{M}\right]^{2m+q} (1-x)^{-N+2m+2q} \underbrace{(f')^{2m}} (f'')^q (-e^T \cdot k_3)^{N-2m-2q}. \tag{B.6}$$

The saddle-point for the integration of moduli, $x = x_0$, is defined by

$$f'(x_0) = 0, \tag{B.7}$$

and we have

$$x_0 = \frac{1}{1-\tau} = \sec^2\frac{\phi}{2}, \qquad 1 - x_0 = -\frac{\tau}{1-\tau}, \qquad f''(x_0) = (1-\tau)^3\tau^{-1}. \tag{B.8}$$

Due to the factor $(f')^{2m}$ in eq. (B.6), it is easy to see that [14, 15]

$$u(x_0) = u'(x_0) = \ldots = u^{(2m-1)}(x_0) = 0, \tag{B.9}$$

and

$$u^{(2m)}(x_0) = \left[\frac{(1,2)}{M}\right]^{2m+q} (1-x_0)^{-N+2m+2q} (2m)! (f_0'')^{2m+q} (-e^T \cdot k_3)^{N-2m-2q}. \tag{B.10}$$

With these inputs, one can easily evaluate the Gaussian integral associated with the four-point amplitudes [14, 15]

$$\begin{aligned} &\int_1^\infty dx\ u(x) e^{-Kf(x)} \\ &= \sqrt{\frac{2\pi}{Kf_0''}} e^{-Kf_0} \left[\frac{u_0^{(2m)}}{2^m\ m!\ (f_0'')^m\ K^m} + O\left(\frac{1}{K^{m+1}}\right)\right] \\ &= \sqrt{\frac{2\pi}{Kf_0''}} e^{-Kf_0} \left[(-1)^{N-q} \frac{2^{N-2m-q}(2m)!}{m!\ M^{2m+q}}\ \tau^{-\frac{N}{2}} (1-\tau)^{\frac{3N}{2}} E^N + O(E^{N-2})\right]. \end{aligned} \tag{B.11}$$

This result shows explicitly that with one tensor and three tachyons, the energy and angle dependence for the four-point HSS amplitudes only depend on the level N [14, 15]

$$\begin{aligned} \lim_{E\to\infty} \frac{\mathcal{T}^{(N,2m,q)}}{\mathcal{T}^{(N,0,0)}} &= \frac{(-1)^q (2m)!}{m!(2M)^{2m+q}} \\ &= \left(-\frac{2m-1}{M}\right) \cdots \left(-\frac{3}{M}\right)\left(-\frac{1}{M}\right)\left(-\frac{1}{2M}\right)^{m+q}, \end{aligned} \tag{B.12}$$

which is remarkably consistent with calculation of decoupling of high energy ZNS obtained in eq. (2.11).

B.2 The n-point HSSA with $r = 1$

To illustrate the n-point HSSA calculation, we begin with n-point HSSA with $r = 1$. We want to calculate n-point HSSA with $(n-1)$ tachyons and 1 high energy state in eq. (2.2). With the change of variables $z_i = \frac{x_i}{x_{i+1}}$ or $x_i = z_i \cdots z_{n-2}$, the HSSA can be written as

$$\begin{aligned}
\mathcal{T}^{(\{p_i\},m,q)} &= \int_0^1 dx_{n-2} \cdots \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 u e^{-Kf} \\
&= \int_0^1 dz_{n-2} \cdots \int_0^1 dz_3 \int_0^1 dz_2 \begin{vmatrix} z_3 \cdots z_{n-2} & z_2 z_4 \cdots z_{n-2} & \cdots & z_2 \cdots z_{n-3} \\ 0 & z_4 \cdots z_{n-2} & \cdots & \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \end{vmatrix} u e^{-Kf} \\
&= \left(\prod_{i=3}^{n-2} \int_0^1 dz_i \; z_i^{i-2-N} \right) \int_0^1 dz_2 u e^{-Kf}
\end{aligned} \tag{B.13}$$

where

$$f(x_i) = -\sum_{i<j} \frac{k_i \cdot k_j}{K} \ln(x_j - x_i) = -\sum_{i<j} \frac{k_i \cdot k_j}{K} \ln(z_j \cdots z_{n-2} - z_i \cdots z_{n-2})$$
$$= -\sum_{i<j} \frac{k_i \cdot k_j}{K} \left[\ln(z_j \cdots z_{n-2}) + \ln(1 - z_i \cdots z_{j-1}) \right], \; K = -k_1 \cdot k_2, \tag{B.14}$$

$$u(x_i) = \left(k^T\right)^{N-2m-q} \underbrace{\left(k^L\right)^{2m}} \left(k'^L\right)^q \cdot \left(k'^L = \frac{\partial k^L}{\partial x_2} \right) \tag{B.15}$$

In eq. (B.15), we have defined

$$k = \sum_{i \neq 2,n} \frac{k_i}{x_i - x_2} = \sum_{i \neq 2,n} \frac{k_i}{z_i \cdots z_{n-2} - z_2 \cdots z_{n-2}}, \tag{B.16}$$

and $k_\perp = |k_\perp| \sum_{i=1}^r e^{T_i} \omega_i = |k_\perp| e^{\hat{T}}$.

The saddle points $(\tilde{z}_2, \cdots, \tilde{z}_{n-2})$ are the solution of

$$\frac{\partial f}{\partial z_2} = 0, \; \cdots, \; \frac{\partial f}{\partial z_{n-2}} = 0. \tag{B.17}$$

Note that eq. (B.17) implies

$$\tilde{k}^L = \frac{\tilde{k} \cdot k_2}{M} = \frac{k_{12}}{M} \frac{\partial f}{\partial x_2}\bigg|_{z_i = \tilde{z}_i} = \frac{k_{12}}{M} \frac{\partial z_j}{\partial x_2} \frac{\partial f}{\partial z_j}\bigg|_{z_i = \tilde{z}_i} = 0 \;, \; \left|\tilde{k}\right| = \left|\tilde{k}_\perp\right|. \tag{B.18}$$

We also define

$$f_2 \equiv \frac{\partial f}{\partial z_2}, \; f_{22} \equiv \frac{\partial^2 f}{\partial z_2^2}, \; \tilde{f} = f(\tilde{z}_2, \cdots, \tilde{z}_{n-2}), \; \tilde{f}_{22} = \frac{\partial^2 f}{\partial z_2^2}\bigg|_{(\tilde{z}_2, \cdots, \tilde{z}_{n-2})}. \tag{B.19}$$

JHEP09(2023)111

In view of the factor $\left(k^L\right)^{2m}$ in eq. (B.15) and eq. (B.18), all up to $(2m)$-order differentiations of u function in eq. (B.15) at the saddle point vanish except [16]

$$\begin{aligned}\left.\frac{\partial^{2m}u}{\partial z_2^{2m}}\right|_{(\tilde{z}_2,\cdots,\tilde{z}_{n-2})} &= \left(\frac{k_{12}}{M}\right)^{2m+q}\left(-\sum_{i\neq 2,n}\frac{k_i^T}{\tilde{x}_i-\tilde{x}_2}\right)^{N-2m-2q}(2m)!\left(\tilde{f}_{22}\right)^{q+2m}\\ &= \left(\frac{k_{12}}{M}\right)^{2m+q}\left(\tilde{k}^T\right)^{N-2m-2q}(2m)!\left(\tilde{f}_{22}\right)^{q+2m}. \end{aligned} \tag{B.20}$$

Finally, with the saddle point, we can calculate the HSSA to be [16]

$$\mathcal{T}^{(N,2m,2q)} = \left(\prod_{i=3}^{n-2}\int_0^1 dz_i\, z_i^{i-2-N}\right)\int_0^1 dz_2\left(\frac{\partial^{2m}\tilde{u}}{\partial z_2^{2m}}\frac{(z_2-\tilde{z}_2)^{2m}}{(2m)!}\right)e^{-Kf} \tag{B.21}$$

$$\simeq \frac{1}{(2m)!}\frac{\partial^{2m}\tilde{u}}{\partial z_2^{2m}}\left(\prod_{i=3}^{n-2}\tilde{z}_i^{i-2-N}\right)\int_0^1 dz_2\,(z_2-\tilde{z}_2)^{2m}\,e^{-Kf(z_2)} \tag{B.22}$$

$$\simeq \frac{1}{(2m)!}\frac{\partial^{2m}\tilde{u}}{\partial z_2^{2m}}\left(\prod_{i=3}^{n-2}\tilde{z}_i^{i-2-N}\right)\int_0^\infty dz_2\,(z_2-\tilde{z}_2)^{2m}\,e^{-Kf(z_2)} \tag{B.23}$$

$$= \frac{2\sqrt{\pi}}{m!}\left(\prod_{i=3}^{n-2}\tilde{z}_i^{i-2-N}\right)\frac{e^{-K\tilde{f}}}{\left|\tilde{k}\right|^{2m+1}}\left.\frac{\partial^{2m}u}{\partial z_2^{2m}}\right|_{z_i=\tilde{z}_i} \tag{B.24}$$

$$= 2\sqrt{\pi}e^{-K\tilde{f}}\left|\tilde{k}\right|^{N-1}\left(\prod_{i=3}^{n-2}\tilde{z}_i^{i-2-N}\right)\frac{(2m)!}{m!}\left(\frac{-1}{2M}\right)^{2m+q}\left(\frac{2K\tilde{f}_{22}}{\left(\sum_{i\neq 2,n}\frac{k_i^T}{\tilde{x}_i-\tilde{x}_2}\right)^2}\right)^{m+q} \tag{B.25}$$

where $f(z_2) = f(z_2,\tilde{z}_3,\cdots,\tilde{z}_{n-2})$. The ratios of n-point $HSSA$ with $r=1$ is

$$\begin{aligned}\frac{\mathcal{T}^{(N,m,q)}}{\mathcal{T}^{(N,0,0)}} &= \frac{(2m)!}{m!}\left(\frac{-1}{2M}\right)^{2m+q}\left(\frac{2K\tilde{f}_{22}}{\left(\sum_{i\neq 2,n}\frac{k_i^T}{\tilde{x}_i-\tilde{x}_2}\right)^2}\right)^{m+q} && \text{(B.26)}\\ &= \frac{(2m)!}{m!}\left(\frac{-1}{2M}\right)^{2m+q} && \text{(B.27)}\end{aligned}$$

where the second equality followed from the calculation of decoupling of ZNS in eq. (2.11).

This suggests the identity

$$\frac{2K\tilde{f}_{22}}{\left(\sum_{i\neq 2,n}\frac{k_i^T}{\tilde{x}_i-\tilde{x}_2}\right)^2} = 1. \tag{B.28}$$

For the case of $n=4$, one can easily solve the saddle point $\tilde{z}_2 = \sec^2\frac{\phi}{2}$ to verify the identity. We have also proved the identity for $n=5$ by using maple numerically. Similar proof can be done by maple for the case of $n=6$.

B.3 The n-point HSSA with $r = 2$

Now we calculate the case of n-point HSSA with $r = 2$. We want to calculate n-point HSSA with $(n-1)$ tachyons and 1 high energy state

$$\left(\alpha_{-1}^{T_1}\right)^{N+p_1}\left(\alpha_{-1}^{T_2}\right)^{p_2}\left(\alpha_{-1}^{L}\right)^{2m}\left(\alpha_{-2}^{L}\right)^{q}|0;k\rangle\,,\quad p_1+p_2=-2(m+q). \tag{B.29}$$

The ratios of n-point HSSA with $r = 2$ can be similarly calculated to be

$$\begin{aligned}\frac{\mathcal{T}^{(p_1,p_2,m,q)}}{\mathcal{T}^{(N,0,0,0)}} &= \frac{(2m)!}{m!}\left(\frac{-1}{2M}\right)^{2m+q}\frac{\left(2K\tilde{f}_{22}\right)^{m+q}}{\left(\sum_{i\neq 2,n}\frac{k_i^{T_1}}{\tilde{x}_i-\tilde{x}_2}\right)^{2m+2q+p_2}\left(\sum_{i\neq 2,n}\frac{k_i^{T_2}}{\tilde{x}_i-\tilde{x}_2}\right)^{-p_2}}\\ &= \frac{(2m)!}{m!}\left(\frac{-1}{2M}\right)^{2m+q}\frac{\left(\frac{\sum_{i\neq 2,n}\frac{k_i^{T_2}}{\tilde{x}_i-\tilde{x}_2}}{\sum_{i\neq 2,n}\frac{k_i^{T_1}}{\tilde{x}_i-\tilde{x}_2}}\right)^{p_2}}{\left(\frac{\sum_{i\neq 2,n}\frac{k_i^{T_1}}{\tilde{x}_i-\tilde{x}_2}}{\sqrt{2K\tilde{f}_{22}}}\right)^{2m+2q}}.\end{aligned} \tag{B.30}$$

On the other hand, the decoupling of ZNS calculated in eq. (2.16) gives

$$\frac{\mathcal{T}^{(p_1,p_2,m,q)}}{\mathcal{T}^{(N,0,0,0)}} = \frac{(2m)!}{m!}\left(\frac{-1}{2M}\right)^{2m+q}\omega_1^{p_1}\omega_2^{p_2} = \frac{(2m)!}{m!}\left(\frac{-1}{2M}\right)^{2m+q}\frac{(\tan\theta_1)^{p_2}}{(\cos\theta_1)^{2m+2q}}. \tag{B.31}$$

Eq. (B.30) and eq. (B.31) can be identified for any p_2, m and q if

$$\left(\sum_{i\neq 2,n}\frac{k_i^{T_1}}{\tilde{x}_i-\tilde{x}_2}\right) = \sqrt{2K\tilde{f}_{22}}\cos\theta_1,\quad \left(\sum_{i\neq 2,n}\frac{k_i^{T_2}}{\tilde{x}_i-\tilde{x}_2}\right) = \sqrt{2K\tilde{f}_{22}}\sin\theta_1, \tag{B.32}$$

which implies the identity

$$\left(\sum_{i\neq 2,n}\frac{k_i^{T_1}}{\tilde{x}_i-\tilde{x}_2}\right)^2 + \left(\sum_{i\neq 2,n}\frac{k_i^{T_2}}{\tilde{x}_i-\tilde{x}_2}\right)^2 = 2K\tilde{f}_{22}. \tag{B.33}$$

It is not surprising that eq. (B.33) is a generalization of eq. (B.28) to two transverse directions T_1 and T_2.

B.4 The n-point HSSA with $r \leq 24$

It is now easy to generalize eq. (B.33) to any r (number of T_i) with $r \leq 24$

$$\left(\sum_{i\neq 2,n}\frac{k_i^{T_1}}{\tilde{x}_i-\tilde{x}_2}\right)^2 + \left(\sum_{i\neq 2,n}\frac{k_i^{T_2}}{\tilde{x}_i-\tilde{x}_2}\right)^2 + \cdots + \left(\sum_{i\neq 2,n}\frac{k_i^{T_r}}{\tilde{x}_i-\tilde{x}_2}\right)^2 = 2K\tilde{f}_{22}. \tag{B.34}$$

By using eq. (B.16) and eq. (B.18), we see that the key identity eq. (B.34) can be written as [16]

$$\tilde{k}^2 + 2M\tilde{k}'^L = 0. \tag{B.35}$$

The ratios in eq. (2.16) are thus proved by the saddle point method.

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JHEP09(2023)111