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Chiral symmetry breaking and the quark bilinear condensate in large-N QCD

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We discuss spontaneous chiral symmetry breaking and the quark bilinear condensate in large-N_c quantum chromodynamics (QCD). It is known that the existence of the η' meson is implied in large-N_c QCD, as pointed out by Witten[27] and Veneziano[28]. First, we show that the existence of η' and the Ward–Takahashi identities implies the existence of Nambu–Goldstone bosons from chiral symmetry breaking $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$. Second, we show that a QCD inequality implies a non-zero lower bound on the quark bilinear condensate.

Subject Index B02, B31

1. Introduction

Chiral symmetry breaking [1–3] is one of the most important features in quantum chromodynamics (QCD). QCD with N_f flavors of massless quarks has continuous global symmetry $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$ in the classical Lagrangian. $U(1)_A$ symmetry is explicitly broken by the quantum anomaly [4–8], and the global symmetry at quantum level is $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$. It is shown that $SU(N_f)_V \times U(1)_V$ is unbroken in the vacuum [9]. On the other hand, it is believed that the chiral symmetry is spontaneously broken as $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$ if N_f is below some threshold value. Although chiral symmetry breaking successfully describes hadron physics, we do yet not understand why chiral symmetry breaking occurs. There are numerous pieces of evidence of chiral symmetry breaking in lattice QCD calculations (see, e.g., Ref. [10] and references therein). Also, there are investigations from anomaly matching [11,12], supersymmetric QCD [13–20], and holographic QCD [21–25].

In this paper, we discuss spontaneous chiral symmetry breaking in a QCD-like theory, i.e., $SU(N_c)$ gauge theory with N_f flavors of quarks in the limit of large N_c [26]. First, in Sect. 2, we review the Witten–Veneziano relation [27,28], which claims the existence of a light particle whose mass scales as $1/\sqrt{N_c}$; and this particle can be interpreted as η', the (pseudo-)Nambu–Goldstone (NG) boson from spontaneous breaking of $U(1)_A$ symmetry. Then, in Sect. 3, we will see that consistency with the existence of η' and the Ward–Takahashi identities with quark mass implies the existence of light scalar particles that are associated with axial currents. These particles become massless in the massless quark limit, and they are nothing but NG bosons

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from chiral symmetry breaking. Finally, in Sect. 4, we estimate a lower bound on $\langle \bar{q}q \rangle$ by a QCD inequality [29] in the same way as Kogan et al. [30], and see that $\langle \bar{q}q \rangle$ becomes non-zero.

1.1 *Comparison with the previous literature*

Let us compare our discussion with the previous literature. Proof of chiral symmetry breaking in large-N_c QCD has been identified by Coleman and Witten [31] and Veneziano [32]. Coleman and Witten [31] have shown the existence of massless scalar poles that are associated with axial currents by anomaly matching in a elegant way; however, they have simply assumed that the order parameter is a bifundamental of $SU(N_f)_L \times SU(N_f)_R$ and have not concluded that $\langle \bar{q}q \rangle \neq 0$. Veneziano [32] uses the Ward–Takahashi identities that are equivalent to our Eqs. (10, 11). They claim that $\langle \bar{q}q \rangle \neq 0$; however, they have not realized a subtle issue that will be discussed in Sect. 4.2 and footnote 5. In addition to this point, the lower bound on $\langle q\bar{q} \rangle$ given in Eq. (32) has not been reported in the previous literature.

1.2 *Our assumptions*

Before going to the main part, let us summarize our setup and assumptions in this paper. The Lagrangian of the QCD-like theory is given as

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \sum_i \bar{q}_i (i\not{D} - m) q_i. \tag{1}$$

We take $\theta = 0$ and the quark mass m to be non-negative. We assume $SU(N_f)$ invariance of the quark mass term for analytic convenience. In addition, we make the following two assumptions:

(1) In QCD-like theories with $\theta = 0$[1] and sufficiently small non-negative quark mass m and $1/N_c$, there is no phase transition and we can take a smooth limit of $m \to 0$ and $1/N_c \to 0$.
(2) The topological susceptibility in pure $SU(N_c)$ Yang–Mills theory and QCD-like theories with massive quarks is non-zero positive[2].

Note that assumption 1 leads to the idea that the two limits $1/N_c \to 0$ and $m \to 0$ are commutable when m is real positive.

2. N_f flavors of massless quarks

First, we review the Witten–Veneziano relation [27,28]. We assume that $m = 0$ in this section. Let us define the following two-point correlation function:

$$\chi_t(p^2) \equiv -i \int d^4x e^{ipx} \langle 0| T \frac{g_s^2}{32\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu}(x) \frac{g_s^2}{32\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu}(0) |0\rangle. \tag{2}$$

$\chi_t(0)$ is called the topological susceptibility. Note that $\chi_t(0)$ is also obtained as $d^2V/d\theta^2$, where $V(\theta)$ is the vacuum energy of the θ vacuum [40,41]:

$$V(\theta) \equiv i \log \int \mathcal{D}A_\mu^a \mathcal{D}q \mathcal{D}\bar{q} \exp\left(i \int d^4x \left(\mathcal{L}_{\text{QCD}} + \frac{\theta g_s^2}{32\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu} \right) \right). \tag{3}$$

When we have a massless quark, the θ parameter can be absorbed by chiral rotation of quarks and $V(\theta)$ becomes independent of θ, i.e., $\chi_t(0) = 0$.

[1] In the case of non-zero θ, a non-trivial phase structure has been discussed in a lot of literature [33–38].
[2] Note that the Vafa–Witten theorem [39] guarantees that the topological susceptibility cannot be negative in pure Yang–Mills theory.

Assumption 1 allows us to use $1/N_c$ expansion to evaluate $\chi_t(p^2)$:

$$\chi_t(p^2) = \chi_{t,g}(p^2) + \frac{N_f}{N_c}\chi_{t,q}(p^2) + \mathcal{O}\left(N_c^{-2}\right). \tag{4}$$

$\chi_{t,g}(p^2)$ is the contribution from pure gluonic diagrams, which is the same as pure $SU(N_c)$ Yang–Mills theory. $\chi_{t,g}(0)$ is non-zero because of assumption 2. This term behaves as N_c^0 for large N_c [27,28,42]. $\chi_{t,q}(p^2)$ is the leading contribution from diagrams with a single quark loop. The leading contribution of the quark loop is proportional to the number of quarks N_f and suppressed by $1/N_c$ compared to $\chi_{t,g}(q^2)$. We explicitly use the N_f/N_c factor in Eq. (4) so that $\chi_{t,q}(p^2)$ itself does not have N_f and N_c dependence at the leading term of $1/N_c$ expansion.

$\chi_t(0)$ should become zero once we introduce a massless quark [40,41]. This means that the $\chi_{t,q}(0)$ term should cancel $\chi_{t,g}(0)$; however, one could naively think that this is impossible because of the N_f/N_c factor in front of $\chi_{t,q}$. This puzzle has been solved by Witten [27] and Veneziano [28] by assuming that $\chi_{t,q}(p^2)$ has a pole from the CP-odd scalar particle η' whose mass squared scales as $1/N_c$. Then, $\chi_{t,q}$ can be written as

$$\chi_{t,q}(p^2) = \frac{a_{\eta'}^2}{p^2 - m_{\eta'}^{(0)2}}, \tag{5}$$

where

$$a_{\eta'} \equiv \sqrt{\frac{N_c}{N_f}} \times \left\langle 0 \right| \frac{g_s^2}{32\pi^2} G_{\mu\nu}\tilde{G}_{\mu\nu} \left| \eta' \right\rangle. \tag{6}$$

$a_{\eta'}$ is defined so that $a_{\eta'}$ does not depend on N_c and N_f at the leading order of $1/N_c$ expansion. $\chi_t(0) = 0$ leads to the following mass formula [27,28]:

$$m_{\eta'}^{(0)2} = \frac{N_f}{N_c}\frac{a_{\eta'}^2}{\chi_{t,g}}. \tag{7}$$

Here we attach the superscript to emphasize that this is the η' mass formula for $m = 0$. The chiral anomaly equation is

$$\partial_\mu j_5^\mu = \frac{N_f g_s^2}{16\pi^2} G_{\mu\nu}\tilde{G}^{\mu\nu}. \tag{8}$$

Since $\langle 0|G_{\mu\nu}\tilde{G}^{\mu\nu}|\eta'\rangle$ is non-zero, we obtain

$$\left\langle 0 \right| j_5^\mu \left| \eta' \right\rangle = \sqrt{N_f} f p_\mu, \quad f \equiv \frac{\sqrt{N_f \chi_{t,g}}}{2m_{\eta'}^{(0)}}. \tag{9}$$

This indicates that η' is the NG boson from spontaneous $U(1)_A$ symmetry breaking in the limit of $1/N_c \to 0$. For finite N_c, $U(1)_A$ is explicitly broken by the chiral anomaly and the η' mass is non-zero and scales as $1/\sqrt{N_c}$. In this definition, f scales as $\sqrt{N_c}$ and is independent of N_f at the leading order of $1/N_c$.

Note that, even if η' is identified as the NG boson for $U(1)_A$ symmetry breaking, we do not know which order parameter induces this $U(1)_A$ symmetry breaking at this point. For example, if the order parameter is the 't Hooft determinant $\epsilon_{i_1 \ldots i_{N_f}}\epsilon_{j_1 \ldots j_{N_f}}(\bar{q}_{i_1} q_{j_1}) \cdots (\bar{q}_{i_{N_f}} q_{j_{N_f}})$, $U(1)_A$ is broken but $SU(N_f)_L \times SU(N_f)_R$ is unbroken [43].

3. N_f flavors of quarks with small mass

Next, let us assume that m is non-zero positive. A non-zero quark mass breaks the chiral symmetry explicitly and the remaining global symmetry is $SU(N_f)_V \times U(1)_V$. We can derive the following Ward–Takahashi identities [44–46] (for the derivation, see Appendix A):

$$N_f \chi_t(0) = m\langle \bar{q}q \rangle + m^2 \chi_s(0), \tag{10}$$

$$0 = m\langle \bar{q}q \rangle + m^2 \chi_{\rm adj}(0), \tag{11}$$

where χ_s and $\chi_{\rm adj}$ have pseudo-scalar susceptibility and are defined as

$$\chi_s(p^2) \equiv \frac{1}{N_f} \sum_{i,j} \int d^4 x e^{ipx} \langle 0| T(\bar{q}_i \gamma^5 q_i)(x)(\bar{q}_j \gamma^5 q_j)(0)|0\rangle, \tag{12}$$

$$\chi_{\rm adj}(p^2)\delta_{ab} \equiv \int d^4 x e^{ipx} \langle 0| T(\bar{q}_i \gamma^5 T^a q_i)(x)(\bar{q}_j \gamma^5 T^b q_j)(0)|0\rangle. \tag{13}$$

Here T^a are Hermitian matrices such that $\mathrm{tr}[T^a T^b] = \delta_{ab}$ and $\mathrm{tr} T^a = 0$. We have used the fact that the $SU(N_f)_V \times U(1)_V$ symmetry is unbroken in the vacuum [9] and parameterize $\langle \bar{q}_i q_j \rangle = \langle \bar{q}q \rangle \delta_{ij}$.

Assumption 1 implies the existence of η' for at least sufficiently small m. Because of assumption 2, the cancellation between $\chi_{t,\,g}(0)$ and $\chi_{t,\,q}(0)$ in Eq. (4) should be broken for $m \neq 0$. This means that the η' mass formula should be modified as

$$m_{\eta'}^2 = m_{\eta'}^{(0)2} + \delta m_{\eta'}^2. \tag{14}$$

$\delta m_{\eta'}^2$ is the leading contribution from the non-zero quark mass. $\delta m_{\eta'}^2(m = 0) = 0$ is satisfied and $\delta m_{\eta'}^2$ is an increasing function for quark mass m, at least if m is sufficiently small. The quark mass term is a source of the explicit breaking of $U(1)_A$ symmetry and this effect should remain in the limit of $1/N_c \to 0$. Thus, $\delta m_{\eta'}^2$ scales as N_c^0 with a given quark mass m.

3.1 *Singlet pseudo-scalar susceptibility*

In the current discussion, we have two small parameters; m and $1/N_c$. Since the η' mass becomes small in the limit of small m and $1/N_c$, some amplitude and correlation functions could have singular behavior if there is an η' contribution. Let us discuss χ_s and χ_t with this point in mind.

First let us discuss χ_s. As we have seen, the leading contribution of the quark loop in χ_t is coming from the η' one-particle state. Since $G_{\mu\nu}\tilde{G}^{\mu\nu}$ and $\sum_i \bar{q}_i \gamma^5 q_i$ have the same quantum number, η' gives a dominant contribution to χ_s as

$$\chi_s(k^2) \simeq \frac{1}{N_f} \frac{1}{k^2 - m_{\eta'}^{(0)2} - \delta m_{\eta'}^2} \left| \sum_i \langle 0|\bar{q}_i \gamma^5 q_i|\eta'\rangle \right|^2. \tag{15}$$

For sufficiently small m, $\delta m_{\eta'}^2$ satisfies

$$\delta m_{\eta'}^2 \ll M^2, \tag{16}$$

where M is the typical mass of heavier hadrons. Furthermore, since $\delta m_{\eta'}^2$ scales as N_c^0, we can take sufficiently small $1/N_c$ for a given quark mass m such that

$$m_{\eta'}^{(0)2} \ll \delta m_{\eta'}^2. \tag{17}$$

By using assumption 1, we assume that there is no phase transition between $m_{\eta'}^{(0)2} \ll \delta m_{\eta'}^2$ and $m_{\eta'}^{(0)2} \gg \delta m_{\eta'}^2$ as long as both of $\delta m_{\eta'}^2$ and $m_{\eta'}^{(0)2}$ are sufficiently smaller than M^2.[3] Then, in the

[3] It is known that this assumption is broken in the case of $\theta = \pi$ [33–38]. In this paper, we only focus on the case of $\theta = 0$ and make assumption 1.

case of $m_{\eta'}^{(0)2} \ll \delta m_{\eta'}^2 \ll M^2$, Eq. (15) can be expanded as[4]

$$\chi_s(0) \simeq -\frac{1}{N_f}\frac{1}{\delta m_{\eta'}^2}\left|\sum_i \langle 0|\bar{q}_i\gamma^5 q_i|\eta'\rangle\right|^2 + \frac{1}{N_f}\frac{m_{\eta'}^{(0)2}}{\left(\delta m_{\eta'}^2\right)^2}\left|\sum_i \langle 0|\bar{q}_i\gamma^5 q_i|\eta'\rangle\right|^2. \tag{18}$$

Next, let us discuss $\chi_t(0)$. In the case of $m_{\eta'}^{(0)2} \ll \delta m_{\eta'}^2 \ll M^2$, by using Eqs. (10) and (18), we obtain

$$N_f\chi_t(0) \simeq m\langle\bar{q}q\rangle - \frac{m^2}{N_f}\frac{1}{\delta m_{\eta'}^2}\left|\sum_i \langle 0|\bar{q}_i\gamma^5 q_i|\eta'\rangle\right|^2 + \frac{m^2}{N_f}\frac{m_{\eta'}^{(0)2}}{\left(\delta m_{\eta'}^2\right)^2}\left|\sum_i \langle 0|\bar{q}_i\gamma^5 q_i|\eta'\rangle\right|^2. \tag{19}$$

Let us compare the above equation with Eq. (4). Since $m_{\eta'}^{(0)2}/\delta m_{\eta'}^2$ suppresses $(N_f/N_c)\chi_{t,q}(0)$ in Eq. (4), the dominant contribution in $\chi_t(0)$ is from $\chi_{t,g}(0)$ and $\chi_t(0)$ becomes independent of m. Assumption 1 implies that $\langle\bar{q}q\rangle$ is not singular at small m; i.e., the $m\langle\bar{q}q\rangle$ term depends on m. In order for the RHS of Eq. (19) to be independent of m, the first and second terms should cancel because the third term is suppressed by $m_{\eta'}^{(0)}/\delta m_{\eta'}^2$. Then, we obtain the following two equations:

$$m\langle\bar{q}q\rangle = \frac{m^2}{N_f}\frac{1}{\delta m_{\eta'}^2}\left|\sum_i \langle 0|\bar{q}_i\gamma^5 q_i|\eta'\rangle\right|^2, \tag{20}$$

$$N_f\chi_{t,g} = \frac{m^2}{N_f}\frac{m_{\eta'}^{(0)2}}{\left(\delta m_{\eta'}^2\right)^2}\left|\sum_i \langle 0|\bar{q}_i\gamma^5 q_i|\eta'\rangle\right|^2. \tag{21}$$

These equations correspond to Eqs. (1a) and (1c) in Ref. [32]. We can show that

$$\langle\bar{q}q\rangle = 4f^2\frac{\delta m_{\eta'}^2}{m}, \tag{22}$$

$$\langle 0|\bar{q}_i\gamma_5 q_j|\eta'\rangle = \delta_{ij}\times\frac{2if}{\sqrt{N_f}}\frac{\delta m_{\eta'}^2}{m}, \tag{23}$$

$$\chi_s(k^2) = -\frac{4f^2}{k^2 - m_{\eta'}^{(0)2} - \delta m_{\eta'}^2}\frac{\left(\delta m_{\eta'}^2\right)^2}{m^2}, \tag{24}$$

where f is defined in Eq. (9). Note that these equations should be valid at the leading order of m and $1/N_c$ as long as both of $\delta m_{\eta'}^2$ and $m_{\eta'}^{(0)2}$ are sufficiently smaller than M^2, though we have derived these relations by assuming $m_{\eta'}^{(0)2} \ll \delta m_{\eta'}^2 \ll M^2$.

Equation (22) shows that $\langle\bar{q}q\rangle$ gives a non-zero vacuum expectation value (VEV) for small non-zero m because of $\delta m_{\eta'}^2 \neq 0$. However, it is non-trivial whether $\langle\bar{q}q\rangle \neq 0$ in the limit of $m \to 0$. We discuss this point in Sect. 4.

[4]On the other hand, in the limit of $\delta m_{\eta'}^2 \ll m_{\eta'}^{(0)2} \ll M^2$, we obtain

$$\chi_s(0) \simeq -\frac{1}{N_f}\frac{1}{m_{\eta'}^{(0)2}}\left|\sum_i \langle 0|\bar{q}_i\gamma^5 q_i|\eta'\rangle\right|^2 + \frac{1}{N_f}\frac{\delta m_{\eta'}^2}{m_{\eta'}^{(0)4}}\left|\sum_i \langle 0|\bar{q}_i\gamma^5 q_i|\eta'\rangle\right|^2.$$

Note that this equation cannot be directly derived from Eq. (18) and vice versa. In this sense, we have to be careful about the order of taking limits. See also Ref. [47].

3.2 *Adjoint pseudo-scalar susceptibility*

In $1/N_c$ expansion, the leading contributions to χ_s and χ_{adj} are the same because they come from a similar diagram with connected quark lines. In particular, in comparison with Eq. (24), the behavior in the limit of large $k^2 \gg m_{\eta'}^{(0)2}$, we obtain

$$\chi_{\text{adj}}(k^2) = -\frac{4f^2}{k^2}\frac{\left(\delta m_{\eta'}^2\right)^2}{m^2} \times (1 + \mathcal{O}(N_c^{-1})). \tag{25}$$

Thus, the one-particle state should dominate at the leading order of $1/N_c$ expansion. Therefore, to be consistent with Eqs. (11, 22), $\chi_{\text{adj}}(k^2)$ is given as

$$\chi_{\text{adj}}(k^2) = -\frac{4f^2}{k^2 - \delta m_{\eta'}^2}\frac{\left(\delta m_{\eta'}^2\right)^2}{m^2}, \tag{26}$$

and there should exist CP-odd $SU(N_f)$ adjoint scalar particles π such that

$$\langle 0|\bar{q}\gamma^5 T^a q|\pi^a\rangle = 2if\frac{\delta m_{\eta'}^2}{m}, \qquad m_\pi^2 = \delta m_{\eta'}^2. \tag{27}$$

We can immediately show that

$$\langle 0|\bar{q}\gamma^5\gamma^\mu T^a q|\pi^a\rangle = ifp^\mu. \tag{28}$$

4. Massless quark limit

Finally, let us discuss massless QCD again. Assumption 1 allows us to take a limit of $m \to 0$ for the results in the previous section.

4.1 *Chiral symmetry breaking*

In the massless quark limit, m_π becomes 0 because of Eq. (27) and $\delta m_{\eta'}^2 \to 0$. On the other hand, the matrix element (28) does not depend on the quark mass m and it keeps its non-zero value. Thus, in massless large-N_c QCD, we conclude that there exist massless scalar particles that can be created by the axial current operator. This means chiral symmetry breaking $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$; the π are NG bosons from this symmetry breaking. Now we can see that Eq. (22) is nothing but the Gell-Mann–Oakes–Renner relation [48].

4.2 *Bilinear condensate*

We have shown the chiral symmetry breaking $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$; however, we have not specified the order parameter for this symmetry breaking. As shown in Eq. (22), $\langle\bar{q}q\rangle$ in the massless quark limit depends on the quark mass dependence in $\delta m_{\eta'}^2$. For example, $\langle\bar{q}q\rangle$ becomes zero if $\delta m_{\eta'}^2$ behaves as m^n with $n \geq 2$. The possibility of chiral symmetry breaking with $\langle\bar{q}q\rangle = 0$[5] was pointed out by Refs. [49,50] and, later, Kogan et al. [30] excluded this possibility by utilizing a QCD inequality[6]. One of the QCD inequalities [29,30] yields

$$\langle(\bar{q}_i\gamma^5 q_j)(x)(\bar{q}_j\gamma^5 q_i)(0)\rangle \geq |\langle(\bar{q}_i\gamma^\mu\gamma^5 q_j)(x)(\bar{q}_j\gamma^\nu\gamma^5 q_i)(0)\rangle|, \tag{29}$$

where $i \neq j$. Note that this inequality is exact and should hold for any μ, ν, and any x. For the derivation of this inequality, see Appendix B. For $x \ll m_\pi^{-1}$, the pion contributions on both sides are

[5] This possibility was not realized and $m_\pi^2 \propto m$ was implicitly assumed in Ref. [32].

[6] Note that chiral symmetry breaking without the bilinear condensate can occur with a different matter content; see, e.g., Ref. [51].

$$\langle(\bar{q}_i\gamma^5 q_j)(x)(\bar{q}_j\gamma^5 q_i)(0)\rangle \simeq \tfrac{\langle\bar{q}q\rangle^2}{4\pi^2 f^2 x^2}, \tag{30}$$

$$\langle(\bar{q}_i\gamma^\mu\gamma^5 q_j)(x)(\bar{q}_j\gamma^\nu\gamma^5 q_i)(0)\rangle \simeq \tfrac{f^2}{8\pi^2}\left(\tfrac{g^{\mu\nu}}{x^4} - \tfrac{4x^\mu x^\nu}{x^6}\right). \tag{31}$$

Let us denote M as the mass of the next-to-lightest particle that couples to $\bar{q}_i\gamma^5 q_j$ or $\bar{q}_i\gamma^\mu\gamma^5 q_j$. For $M^{-1} \lesssim x \ll m_\pi^{-1}$, the pion contribution dominates both correlation functions. By comparing the pion contributions for $M^{-1} \lesssim x \ll m_\pi^{-1}$, $\langle\bar{q}q\rangle$ cannot be zero and we obtain a lower bound:

$$\langle\bar{q}q\rangle \gtrsim f^2 M. \tag{32}$$

Here we are sloppy about the $\mathcal{O}(1)$ numerical factor. The QCD Lagrangian with massless quarks has non-anomalous discrete Z_{2N_f} symmetry, which is a subgroup of $U(1)_A$, and non-zero $\langle\bar{q}q\rangle$ means spontaneous breaking of this Z_{2N_f} symmetry. This conclusion is consistent with the anomaly matching discussion in Ref. [52].

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Appendix A. Derivation of the Ward–Takahashi identities

In this appendix, we derive Eqs. (10) and (11). Similar equations were first derived in Refs. [44,45]. The equivalent equations were used in Ref. [32] to show chiral symmetry breaking in large-N_c QCD, and also used in Ref. [46] in the context of the strong CP problem.

For non-zero quark mass m, the chiral anomaly equation is

$$\partial_\mu j^\mu_{5T} - 2im\bar{q}\gamma_5 T q = \mathrm{tr}T \times \frac{g_s^2}{16\pi^2} G_{\mu\nu}\tilde{G}^{\mu\nu}, \tag{A1}$$

where T is an $N_f \times N_f$ matrix and $j^\mu_{5T} = \bar{q}\gamma^5\gamma^\mu T q$. By using Eqs. (2) and (A1), $\chi_t(0)$ can be rewritten as

$$(\mathrm{tr}T)^2\chi_t(0) = \chi_{t,v} + \chi_{t,m}, \tag{A2}$$

where

$$\chi_{t,v} \equiv -\frac{i}{4}\lim_{p\to 0}\int d^4x e^{ipx}\langle 0|\left(\partial_\mu j^\mu_{5T}\right)(x)\left(\partial_\mu j^\mu_{5T} - 4im\sum_i \bar{q}\gamma^5 T q\right)(0)|0\rangle \tag{A3}$$

$$\chi_{t,m} \equiv m^2 \lim_{p\to 0}\int d^4x e^{ipx}\langle 0|\left(\bar{q}\gamma^5 T q\right)(x)\left(\bar{q}\gamma^5 T q\right)(0)|0\rangle. \tag{A4}$$

$\chi_{t,v}$ is simplified by using Eq. (A1) again and taking integration by parts as

$$\begin{aligned}\chi_{t,v} &= -\frac{i}{4}\lim_{p\to 0}\int d^4x e^{ipx}\langle 0|\left(\partial_\mu j^\mu_{5T}\right)(x)\left(\mathrm{tr}T\frac{g_s^2}{32\pi^2}G_{\mu\nu}\tilde{G}^{\mu\nu} - 2im\bar{q}\gamma^5 T q\right)(0)|0\rangle, \\ &= \frac{i}{4}\lim_{p\to 0}\int d^4x e^{ipx}\delta(x^0)\langle 0|[j^0_{5T}(x), (-2im\bar{q}\gamma^5 T\bar{q})(0)]|0\rangle \\ &= m\sum_i\langle 0|\bar{q}T^2 q|0\rangle. \end{aligned}\tag{A5}$$

We obtain the following equation:

$$(\mathrm{tr}T)^2\chi_t(0) = m\langle 0|\bar{q}T^2q|0\rangle + m^2 \lim_{p\to 0}\int d^4x e^{ipx}\langle 0|T(\bar{q}\gamma^5Tq)(x)(\bar{q}\gamma^5Tq)(0)|0\rangle. \quad \text{(A6)}$$

By using χ_s and χ_{adj} defined in Eqs. (12, 13), we obtain the following simple equations:

$$N_f\chi_t(0) = m\langle\bar{q}q\rangle + m^2\chi_s(0), \quad \text{(A7)}$$

$$0 = m\langle\bar{q}q\rangle + m^2\chi_{\mathrm{adj}}(0). \quad \text{(A8)}$$

Here we have used $\langle\bar{q}_iq_j\rangle = \langle\bar{q}q\rangle\delta_{ij}$.

Appendix B. Derivation of the QCD inequality

In this appendix, we derive a QCD inequality (29). This inequality was first derived in Ref. [29]. Kogan et al. [30] pointed out that this inequality can be used to exclude the possibility of $\langle\bar{q}q\rangle = 0$. See also Sect. 11 of Ref. [53].

The inequality (29) can be shown in Euclidean QCD on a lattice. The action is

$$S = S_U + \sum_{x,y}\bar{q}(x)D(x,y)q(y). \quad \text{(B1)}$$

x and y are the lattice sites and S_U is the action for the link (gauge) field U. $D(x, y)$ is given as [54–56]

$$D(x,y) = (4+m_0a)\delta_{x,y} - \frac{1}{2}\sum_{\mu}\Big[(1-\gamma_\mu)U(x,y)\delta_{y,x+\hat{\mu}} + (1+\gamma_\mu)U(x,y)\delta_{y,x-\hat{\mu}}\Big]. \quad \text{(B2)}$$

m_0 is the bare quark mass, a is the lattice spacing, and $\hat{\mu}$ is the unit vectors for four directions. The correlation function of operators that are made from quark fields is

$$\left\langle\prod_i q_{i_1}(x)\bar{q}_{i_2}(y)\right\rangle = \frac{\displaystyle\int\mathcal{D}A\left[\prod_i D^{-1}_{i_1i_2}(x,y)\right](\det D)^{N_f}e^{-S_U}}{\displaystyle\int\mathcal{D}A(\det D)^{N_f}e^{-S_U}}. \quad \text{(B3)}$$

Let us show

$$\langle(\bar{q}_i\gamma^5q_j)(x)(\bar{q}_j\gamma^5q_i)(0)\rangle \geq |\langle(\bar{q}_i\gamma^M\gamma^5q_j)(x)(\bar{q}_j\gamma^N\gamma^5q_i)(0)\rangle| \quad \text{(B4)}$$

for any $M, N = 0, 1, 2, 3$ and $i \neq j$.

It is known that $\det D \geq 0$ is satisfied in QCD-like theories [9]. Then, a sufficient condition to satisfy the above inequality is

$$\mathrm{tr}[D^{-1}(x,y)\gamma^5D^{-1}(y,x)\gamma^5] \geq \left|\mathrm{tr}[D^{-1}(x,y)\gamma^5\gamma^\mu D^{-1}(y,x)\gamma^5\gamma^\nu]\right|. \quad \text{(B5)}$$

for any $\mu, \nu = 0, 1, 2, 3$. We can show $\gamma^5D(x,y)\gamma^5 = D(y,x)^\dagger$; then $\gamma^5D^{-1}(x,y)\gamma^5 = D^{-1}(y, x)^\dagger$. The difference between the LHS and RHS of the above inequality is

$$\begin{aligned}&\mathrm{tr}[D^{-1}(x,y)\gamma^5D^{-1}(y,x)\gamma^5] - \left|\mathrm{tr}[D^{-1}(x,y)\gamma^5\gamma^\mu D^{-1}(y,x)\gamma^5\gamma^\nu]\right| \\ &= \mathrm{tr}[D^{-1}(x,y)(D^{-1}(x,y))^\dagger] - \left|\mathrm{tr}[D^{-1}(x,y)\gamma^\mu(D^{-1}(x,y))^\dagger\gamma^\nu]\right|. \end{aligned} \quad \text{(B6)}$$

To show that this is non-negative, let us evaluate this expression explicitly. The Dirac matrices in 4D Euclidean space are given as

$$\gamma^0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \gamma^i = \begin{pmatrix}0 & i\sigma^i\\ -i\sigma^i & 0\end{pmatrix}, \quad \gamma^5 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \text{(B7)}$$

where σ^i are the Pauli matrices. Let us take the following 4×4 matrix D^{-1}:

$$D^{-1} = \begin{pmatrix} a_{11} & \cdots & a_{14} \\ \vdots & \ddots & \vdots \\ a_{41} & \cdots & a_{44} \end{pmatrix}. \tag{B8}$$

An explicit calculation shows

$$\mathrm{tr}[D^{-1}D^{-1\dagger}] = \sum_{i,j} |a_{ij}|^2, \tag{B9}$$

$$\begin{aligned}\mathrm{tr}[D^{-1}\gamma^0 D^{-1\dagger}\gamma^0] = 2\mathrm{Re}(&a_{11}a_{33}^* + a_{12}a_{34}^* + a_{13}a_{31}^* + a_{14}a_{32}^* + a_{21}a_{43}^* + a_{22}a_{44}^* \\ &+ a_{23}a_{41}^* + a_{24}a_{42}^*)\end{aligned} \tag{B10}$$

$$\begin{aligned}\mathrm{tr}[D^{-1}\gamma^3 D^{-1\dagger}\gamma^0] = 2i\mathrm{Im}(&-a_{11}a_{33}^* - a_{12}a_{34}^* - a_{13}a_{31}^* - a_{14}a_{32}^* + a_{21}a_{43}^* + a_{22}a_{44}^* \\ &+ a_{23}a_{41}^* + a_{24}a_{42}^*).\end{aligned} \tag{B11}$$

By using the Cauchy–Schwartz inequality, we can show

$$\mathrm{tr}[D^{-1}D^{-1\dagger}] \geq \left|\mathrm{tr}[D^{-1}\gamma^0 D^{-1\dagger}\gamma^0]\right|, \tag{B12}$$

$$\mathrm{tr}[D^{-1}D^{-1\dagger}] \geq \left|\mathrm{tr}[D^{-1}\gamma^3 D^{-1\dagger}\gamma^0]\right|. \tag{B13}$$

In a similar way, we can show that Eq. (B6) is non-negative for any μ and ν, and then Eq. (B4) is satisfied.

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