Eur. Phys. J. C (2023) 83:579
https://doi.org/10.1140/epjc/s10052-023-11727-0

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

A_4 modular flavour model of quark mass hierarchies close to the fixed point $\tau = \omega$

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Received: 6 May 2023 / Accepted: 19 June 2023 / Published online: 7 July 2023

Abstract We investigate the possibility to describe the quark mass hierarchies as well as the CKM quark mixing matrix without fine-tuning in a quark flavour model with modular A_4 symmetry. The quark mass hierarchies are considered in the vicinity of the fixed point $\tau = \omega \equiv \exp(i\,2\pi/3)$ (the left cusp of the fundamental domain of the modular group), τ being the VEV of the modulus. The model involves modular forms of level 3 and weights 6, 4 and 2, and contains eight constants, only two of which, g_u and g_d, can be a source of CP violation in addition to the VEV of the modulus, $\tau = \omega + \epsilon$, $(\epsilon)^* \neq \epsilon$, $|\epsilon| \ll 1$. We find that in the case of real (CP-conserving) g_u and g_d and common τ (ϵ) in the down-type and up-type quark sectors, the down-type quark mass hierarchies can be reproduced without fine tuning with $|\epsilon| \cong 0.03$, all other constants being of the same order in magnitude, and correspond approximately to $1 : |\epsilon| : |\epsilon|^2$. The up-type quark mass hierarchies can be achieved with the same $|\epsilon| \cong 0.03$ but allowing $g_u \sim \mathcal{O}(10)$ and correspond to $1 : |\epsilon|/|g_u| : |\epsilon|^2/|g_u|^2$. In this setting the reproduction of the value of the CKM element $|V_{\rm cb}|$ is problematic. A much more severe problem is the correct description of the CP violation in the quark sector since it arises as a higher order correction in ϵ with $|\epsilon| \ll 1$. We show that this problem is somewhat alleviated in the case of complex g_u and g_d although the rephasing invariant $J_{\rm CP}$ is larger by a factor of ~ 1.8 than the correct value. A correct no-fine-tuned description of the quark mass hierarchies, the quark mixing and CP violation is possible with all constants being of the same order in magnitude and complex g_u and g_d, if one allows different values of ϵ in the down-type and up-type quark sectors, or in a modification of the considered model which involves modular forms of level 3 and weights 8, 6 and 4.

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1 Introduction

In spite of the remarkable success of the standard model (SM), the flavour problem of quarks and leptons is still a challenging issue. In order to solve the flavour problem, a remarkable step was made in Ref. [1], where the idea of using modular invariance as a flavour symmetry was put forward. This new original approach based on modular invariance opened up a new promising direction in the studies of the flavour physics and correspondingly in flavour model building.

The main feature of the approach proposed in Ref. [1] is that the elements of the Yukawa coupling and fermion mass matrices in the Lagrangian of the theory are modular forms of a certain level N which are functions of a single complex scalar field τ—the modulus—and have specific transformation properties under the action of the modular group. In addition, both the couplings and the matter fields (supermultiplets) are assumed to transform in representations of an inhomogeneous (homogeneous) finite modular group $\Gamma_N^{(\prime)}$. For $N \leq 5$, the finite modular groups Γ_N are isomorphic to the permutation groups S_3, A_4, S_4 and A_5 (see, e.g., [2]), while the groups Γ'_N are isomorphic to the double covers of the indicated permutation groups, $S'_3 \equiv S_3$, $A'_4 \equiv T'$, S'_4 and A'_5. These discrete groups are widely used in flavour model building. The theory is assumed to possess the modular symmetry described by the finite modular group $\Gamma_N^{(\prime)}$, which plays the role of a flavour symmetry. In the simplest class of such models, the vacuum expectation value (VEV) of modulus τ

is the only source of flavour symmetry breaking, such that no flavons are needed.

Another appealing feature of the proposed framework is that the VEV of τ can also be the only source of breaking of the CP symmetry [3]. When the flavour symmetry is broken, the elements of the Yukawa coupling and fermion mass matrices get fixed, and a certain flavour structure arises. As a consequence of the modular symmetry, in the lepton sector, for example, the charged-lepton and neutrino masses, neutrino mixing and the leptonic phases of CP violation (CPV) are simultaneously determined in terms of a limited number of coupling constant parameters. This together with the fact that they are also functions of a single complex VEV—that of the modulus τ—leads to experimentally testable correlations between, e.g., the neutrino mass and mixing observables. Models of flavour based on modular invariance have then an increased predictive power.

The modular symmetry approach to the flavour problem has been widely implemented so far primarily in theories with global (rigid) supersymmetry. Within the SUSY framework, modular invariance is assumed to be a feature of the Kähler potential and the superpotential of the theory.[1] Bottom-up modular invariance approaches to the lepton flavour problem have been exploited first using the groups $\Gamma_3 \simeq A_4$ [1,5], $\Gamma_2 \simeq S_3$ [6], $\Gamma_4 \simeq S_4$ [7]. After the first studies, the interest in the approach grew significantly and models based on the groups $\Gamma_4 \simeq S_4$ [8–13], $\Gamma_3 \simeq A_4$ [14–31,33–42], $\Gamma_5 \simeq A_5$ [13,43,44], $\Gamma_2 \simeq S_3$ [45,46] and $\Gamma_7 \simeq PSL(2, \mathbb{Z}_7)$ [47] have been constructed and extensively studied. Similarly, attempts have been made to construct viable models of quark flavour [48] and of quark-lepton unification [32,49–61]. The formalism of the interplay of modular and generalised CP (gCP) symmetries has been developed and first applications made in [3]. It was explored further in [62–65], as was the possibility of coexistence of multiple moduli [66–69], considered first phenomenologically in [8,15]. Such bottom-up analyses are expected to eventually connect with top-down results [70–101] based on ultraviolet-complete theories. The problem of modulus stabilisation was also addressed in [102–105].

While the aforementioned finite quotients Γ_N of the modular group have been widely used in the literature to construct modular-invariant models of flavour from the bottom-up perspective, top-down constructions typically lead to their double covers Γ'_N (see, e.g., [73,75,76,106]). The formalism of such double covers has been developed first in Refs. [107,108] and [109,110] for the cases of $\Gamma'_3 \simeq T'$, $\Gamma'_4 \simeq S'_4$ and $\Gamma'_5 \simeq A'_5$, respectively, where viable lepton flavour models have also been constructed. Subsequently these groups have been used for flavour model building, e.g., in Refs. [111–115].

In almost all phenomenologically viable flavour models based on modular invariance constructed so far the hierarchy of the charged-lepton and quark masses is obtained by fine-tuning some of the constant parameters present in the models.[2] Perhaps, the only notable exceptions are Refs. [116–118], in which modular weights are used as Froggatt-Nielsen charges [119], and additional scalar fields of non-zero modular weights play the role of flavons. The recent work in Ref. [120] has proposed the formalism that allows to construct models in which the fermion (e.g. charged-lepton and quark) mass hierarchies follow solely from the properties of the modular forms, thus avoiding the fine-tuning without the need to introduce extra fields. Indeed, authors have succeeded to reproduce the charged lepton mass hierarchy without fine-tuning keeping the observed lepton mixing angles. On the other hand, it is still challenging to reproduce quark masses and the Cabibbo, Kobayashi, Maskawa (CKM) quark mixing matrix in quark flavour models with modular symmetry.

It was noticed in [8] and further exploited in [15,33,43,62] that for the three fixed points of the VEV of τ in the modular group fundamental domain, $\tau_{\text{sym}} = i$, $\tau_{\text{sym}} = \omega \equiv \exp(i\,2\pi/3) = -1/2 + i\sqrt{3}/2$ (the 'left cusp'), and $\tau_{\text{sym}} = i\infty$, the theories based on the Γ_N invariance have respectively $\mathbb{Z}_2^S$, $\mathbb{Z}_3^{ST}$, and $\mathbb{Z}_N^T$ residual symmetries. In the case of the double cover groups Γ'_N, the $\mathbb{Z}_2^S$ residual symmetry is replaced by the $\mathbb{Z}_4^S$ and there is an additional $\mathbb{Z}_2^R$ symmetry that is unbroken for any value of τ (see [108] for further details).

The fermion mass matrices are strongly constrained in the points of residual symmetries [8,15,33,43,62,120–122]. This suggests that fine-tuning could be avoided in the vicinity of these points if the charged-lepton and quark mass hierarchies follow from the properties of the modular forms present in the corresponding fermion mass matrices rather than being determined by the values of the accompanying constants also present in the matrices. Relatively small deviations of the modulus VEV from the symmetric point might also be needed to ensure the breaking of the CP symmetry [3].

In this work, we study the possibility of obtaining the quark mass hierarchies as well as the CKM matrix without fine-tuning along the lines proposed in Ref. [120] in a "minimal" model with A_4 modular quark flavour symmetry. Since A_4 symmetry is rather simple, it can be used to clearly understand the problems facing the construction of no-fine-tuned modular invariant flavour models of quark mass hierarchies and CKM mixing. After introducing the necessary tools in Sect. 2, we present the A_4 modular invariant model in Sect. 3.

[1] Possible non-minimal additions to the Kähler potential, compatible with the modular symmetry, may jeopardise the predictive power of the approach [4]. This problem is the subject of ongoing research.

[2] By fine-tuning we refer to either (i) unjustified hierarchies between parameters which are introduced in the model on an equal footing and/or (ii) high sensitivity of observables to model parameters.

In Sect. 3, we describe how one can naturally generate hierarchical mass patterns in the vicinity of symmetric points, and then, investigate the flavour structure of the quark mass matrices. In Sect. 4, we discuss the possibility of reproducing the observed quark masses and CKM quark mixing parameters without fine-tuning in the vicinity of $\tau = \omega$, i.e., without strong dependence of the results on the constants present in the model. In Sect. 5, the problem of reproducing correctly the CP violation in the quark sector is discussed. In Sect. 6 we consider alternative quark flavour models, in which it is possible to describe correctly the quark observables without fine-tuning of the constant parameters present in the quark mass matrices. We summarize our results in Sect. 7. In Appendix A, the decomposition of tensor products are presented. In Appendix B, the relevant modular forms with higher weights are listed. In Appendix C, the modular forms are presented at close to $\tau = \omega$. In Appendix D, the measure of goodness of numerical fitting is presented.

2 Modular symmetry of flavours and residual symmetries

We start by briefly reviewing the modular invariance approach to flavour. In this supersymmetric (SUSY) framework, one introduces a chiral superfield, the modulus τ, transforming non-trivially under the modular group $\Gamma \equiv SL(2, \mathbb{Z})$. The group Γ is generated by the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

obeying $S^2 = R$, $(ST)^3 = R^2 = \mathbb{1}$, and $RT = TR$. The elements γ of the modular group act on τ via fractional linear transformations,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \quad \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \tag{2}$$

while matter superfields transform as "weighted" multiplets [1,106,123],

$$\psi_i \to (c\tau + d)^{-k} \rho_{ij}(\gamma)\, \psi_j, \tag{3}$$

where $k \in \mathbb{Z}$ is the so-called modular weight[3] and $\rho(\gamma)$ is a unitary representation of Γ.

In using modular symmetry as a flavour symmetry, an integer level $N \geq 2$ is fixed and one assumes that $\rho(\gamma) = \mathbb{1}$ for elements γ of the principal congruence subgroup

$$\Gamma(N) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod } N) \right\}. \tag{4}$$

[3] While we restrict ourselves to integer k, it is also possible for weights to be fractional [76,124–126].

Hence, ρ is effectively a representation of the (homogeneous) finite modular group $\Gamma'_N \equiv \Gamma \big/ \Gamma(N) \simeq SL(2, \mathbb{Z}_N)$. For $N \leq 5$, this group admits the presentation

$$\Gamma'_N = \left\langle S,\ T,\ R \mid S^2 = R,\ (ST)^3 = \mathbb{1},\ R^2 = \mathbb{1},\ RT = TR,\ T^N = \mathbb{1} \right\rangle. \tag{5}$$

The modulus τ acquires a VEV which is restricted to the upper half-plane and plays the role of a spurion, parameterising the breaking of modular invariance. Additional flavon fields are not required, and we do not consider them here. Since τ does not transform under the R generator, a $\mathbb{Z}_2^R$ symmetry is preserved in such scenarios [108]. If also matter fields transform trivially under R, one may identify the matrices γ and $-\gamma$, thereby restricting oneself to the inhomogeneous modular group $\overline{\Gamma} \equiv PSL(2, \mathbb{Z}) \equiv SL(2, \mathbb{Z}) \,/\, \mathbb{Z}_2^R$. In such a case, ρ is effectively a representation of a smaller (inhomogeneous) finite modular group $\Gamma_N \equiv \Gamma \big/ \langle \Gamma(N) \cup \mathbb{Z}_2^R \rangle$. For $N \leq 5$, this group admits the presentation

$$\Gamma_N = \left\langle S,\ T \mid S^2 = \mathbb{1},\ (ST)^3 = \mathbb{1}, T^N = \mathbb{1} \right\rangle. \tag{6}$$

In general, however, R-odd fields may be present in the theory and Γ and Γ'_N are then the relevant symmetry groups.

Finally, to understand how modular symmetry may constrain the Yukawa couplings and mass structures of a model in a predictive way, we turn to the Lagrangian—which for an $\mathcal{N} = 1$ global supersymmetric theory is given by

$$\mathcal{L} = \int d^2\theta\, d^2\bar{\theta}\, K(\tau, \bar{\tau}, \psi_I, \bar{\psi}_I) + \left[\int d^2\theta\, W(\tau, \psi_I) + \text{h.c.} \right]. \tag{7}$$

Here K and W are the Kähler potential and the superpotential, respectively. The superpotential W can be expanded in powers of matter superfields ψ_I,

$$W(\tau, \psi_I) = \sum \left(Y_{I_1 \ldots I_n}(\tau)\, \psi_{I_1} \ldots \psi_{I_n} \right)_{\mathbf{1}}, \tag{8}$$

where one has summed over all possible field combinations and independent singlets of the finite modular group. By requiring the invariance of the superpotential under modular transformations, one finds that the field couplings $Y_{I_1 \ldots I_n}(\tau)$ have to be modular forms of level N. These are severely constrained holomorphic functions of τ, which under modular transformations obey

$$Y_{I_1 \ldots I_n}(\tau) \xrightarrow{\gamma} Y_{I_1 \ldots I_n}(\gamma\tau) = (c\tau + d)^k \rho_Y(\gamma)\, Y_{I_1 \ldots I_n}(\tau). \tag{9}$$

Modular forms carry weights $k = k_{I_1} + \ldots + k_{I_n}$ and furnish unitary irreducible representations ρ_Y of the finite modular group such that $\rho_Y \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_n} \supset \mathbf{1}$. Non-trivial modular forms of a given level exist only for $k \in \mathbb{N}$, span finite-dimensional linear spaces $\mathcal{M}_k(\Gamma(N))$, and can be arranged into multiplets of $\Gamma_N^{(\prime)}$.

The breakdown of modular symmetry is parameterised by the VEV of the modulus and there is no value of τ which preserves the full symmetry. Nevertheless, at certain so-called symmetric points $\tau = \tau_{\text{sym}}$ the modular group is only partially broken, with the unbroken generators giving rise to residual symmetries. In addition, as we have noticed, the R generator is unbroken for any value of τ, so that a $\mathbb{Z}_2^R$ symmetry is always preserved. There are only three inequivalent symmetric points, namely [8]:

- $\tau_{\text{sym}} = i\infty$, invariant under T, preserving $\mathbb{Z}_N^T \times \mathbb{Z}_2^R$;
- $\tau_{\text{sym}} = i$, invariant under S, preserving $\mathbb{Z}_4^S$ (recall that $S^2 = R$);
- $\tau_{\text{sym}} = \omega \equiv \exp(2\pi i/3)$, 'the left cusp', invariant under ST, preserving $\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R$.

3 Quark mass hierarchies in A_4 modular invariant models

3.1 Fermion mass hierarchy without fine-tuning close to $\tau = \omega$

In theories where modular invariance is broken only by the VEV of modulus, the flavour structure of mass matrices in the limit of unbroken supersymmetry is determined by the value of τ and by the couplings in the superpotential. At a symmetric point $\tau = \tau_{\text{sym}}$, flavour textures can be severely constrained by the residual symmetry group, which may enforce the presence of multiple zero entries in the mass matrices. As τ moves away from its symmetric value, these entries will generically become non-zero. The magnitudes of such (residual-)symmetry-breaking entries will be controlled by the size of the departure ϵ from τ_{sym} and by the field transformation properties under the residual symmetry group (which may depend on the modular weights). We present below a more detailed discussion of this approach to the fermion (charged lepton and quark) mass hierarchies following [120].

Consider a modular-invariant bilinear

$$\psi_i^c \, M(\tau)_{ij} \, \psi_j, \tag{10}$$

where the superfields ψ and ψ^c transform under the modular group as[4]

$$\begin{aligned} \psi &\xrightarrow{\gamma} (c\tau + d)^{-k} \rho(\gamma)\, \psi, \\ \psi^c &\xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho^c(\gamma)\, \psi^c, \end{aligned} \tag{11}$$

[4] Note that in the case of a Dirac bilinear ψ and ψ^c are independent fields, so in general $k^c \neq k$ and $\rho^c \neq \rho, \rho^*$.

so that each $M(\tau)_{ij}$ is a modular form of level N and weight $K \equiv k + k^c$. Modular invariance requires $M(\tau)$ to transform as

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger. \tag{12}$$

Taking τ to be close to the symmetric point, and setting γ to the residual symmetry generator, one can use this transformation rule to constrain the form of the mass matrix $M(\tau)$.

Let us discuss the case, where τ is in the vicinity of $\tau_{\text{sym}} = \omega$. We consider the basis where the product ST is represented by a diagonal matrix. In this ST-diagonal basis, we define

$$\tilde{\rho}_i^{(c)} \equiv \omega^{k^{(c)}} \rho_i^{(c)}, \tag{13}$$

which are representations under the residual symmetry group. By setting $\gamma = ST$ in Eq. (12), one finds

$$M_{ij}(ST\tau) = [-\omega(\tau + 1)]^K \left(\tilde{\rho}_i^c \tilde{\rho}_j\right)^* M_{ij}(\tau). \tag{14}$$

It is now convenient to treat the M_{ij} as functions of [120]

$$u \equiv \frac{\tau - \omega}{\tau - \omega^2}, \tag{15}$$

so that, in this context, $|u|$ denotes the deviation of τ from the symmetric point. Note that the entries $M_{ij}(u)$ depend analytically on u and that $u \xrightarrow{ST} \omega^2 u$. Thus, in terms of u, Eq. (14) reads

$$\begin{aligned} M_{ij}(\omega^2 u) &= \left(\frac{1 - \omega^2 u}{1 - u}\right)^K (\tilde{\rho}_i^c \tilde{\rho}_j)^* M_{ij}(u) \\ &\Rightarrow \quad \tilde{M}_{ij}(\omega^2 u) = (\tilde{\rho}_i^c \tilde{\rho}_j)^* \tilde{M}_{ij}(u), \end{aligned} \tag{16}$$

where $\tilde{M}_{ij}(u) \equiv (1 - u)^{-K} M_{ij}(u)$. Expanding both sides in powers of u, one obtains

$$\omega^{2n} \tilde{M}_{ij}^{(n)}(0) = (\tilde{\rho}_i^c \tilde{\rho}_j)^* \tilde{M}_{ij}^{(n)}(0), \tag{17}$$

where $\tilde{M}_{ij}^{(n)}$ denotes the n-th derivative of $\tilde{M}_{ij}$ with respect to u.

It follows that for $\tau \simeq \omega$ the mass matrix entry $M_{ij} \sim \tilde{M}_{ij}$ is only allowed to be $\mathcal{O}(1)$ when $\tilde{\rho}_i^c \tilde{\rho}_j = 1$. More generally, if $\tilde{\rho}_i^c \tilde{\rho}_j = \omega^\ell$ with $\ell = 0, 1, 2$, then the entry $M_{ij} \sim \tilde{M}_{ij}$ is expected to be $\mathcal{O}(|u|^\ell)$ in the vicinity of $\tau = \omega$. The factors $\tilde{\rho}_i^{(c)}$ depend on the weights $k^{(c)}$, see Eq. (13). Thus, the leading terms of the components of the mass matrix is any of $\mathcal{O}(1)$, $\mathcal{O}(|u|)$ and $\mathcal{O}(|u|^2)$ in the vicinity of $\tau = \omega$. This result allows to obtain fermion mass hierarchies without fine-tuning [120].

3.2 Flavour structure of the A_4 modular invariant quark model

3.2.1 The anatomy of the minimal model of quarks

We present next a simple model of quark mass matrices with modular A_4 flavour symmetry, which we consider in the

Table 1 Assignments of A_4 representations and weights for relevant chiral super-fields

	$Q=(Q_1,Q_2,Q_3)$	$(d^c,s^c,b^c),(u^c,c^c,t^c)$	H_u	H_d
$SU(2)$	2	1	2	2
A_4	3	$(1,\ 1'',1')$	1	1
k	2	$(4,\ 2,\ 0)$	0	0

vicinity of the fixed point $\tau=\omega$. We assign the A_4 representation and the weights for the relevant chiral superfields of quarks as

- The three left-handed doublets $Q=(Q_1,Q_2,Q_3)$ form a A_4 triplet with weight 2.
- The right-handed singlets (d^c,s^c,b^c) and (u^c,c^c,t^c) are A_4 singlets $(1,\ 1'',1')$ with weight $(4,\ 2,0)$, respectively.
- The Higgs fields coupled to down-type and up-type quark sectors $H_d,\ H_u$ are A_4 singlets with weight 0.

These are summarized in Table 1.

The model with the indicated assignments of quark fields was considered in Refs. [32,33]. One of the important attractive features of the model is that it has minimal number of constant parameters that allows to generate non-zero masses for all down-type and up-type quarks and possibly to obtain the quark mass hierarchies without fine-tuning of the constants present in the quark mass matrices.

In [32] the model was analysed in the vicinity of the symmetric point $\tau=i$. The quark mass hierarchies were obtained by severe fine-tuning of the constants present in the down-type and up-type quark mass matrices. Actually, it is impossible to avoid the fine-tuning in the description of quark mass hierarchies in the vicinity of $\tau=i$ [120].

The same model was analysed in [33], in particular, in the vicinity of the symmetric point $\tau=\omega$. According to the general results presented in [120], which was published six months after [33], in this case it should be possible, in principle, to describe the quark mass hierarchies without fine tuning of the constants in the quark mass matrices. However, the authors of [33] did not investigate this possibility. The statistical analysis of the model performed numerically in [33] showed that the model cannot describe correctly the observables in the quark sector, namely, the quark masses, the CKM mixing angles and the quark CP violation. The quark observables whose description was problematic were not identified in [33].

Working in the vicinity of the symmetric point $\tau=\omega$ we first revisit the model considered in [32,33]. Our primary goal is to understand whether the model can describe the quark mass hierarchies without fine tuning and which of the quark observables cannot be reproduced correctly by the model. After identifying the problems of the model, we can consider alternative quark flavour models in which these problems might not arise.

Let us write the superpotential terms giving rise to quark mass matrices by using modular forms with weights 2, 4 and 6 as follows:

$$\begin{aligned}W_d&=\Big[\alpha_d(\mathbf{Y_3^{(6)}}Q)_1d_1^c+\alpha'_d(\mathbf{Y_{3'}^{(6)}}Q)_1d_1^c\\&\quad+\beta_d(\mathbf{Y_3^{(4)}}Q)_{1'}s_{1'}^c+\gamma_d(\mathbf{Y_3^{(2)}}Q)_{1''}b_{1'}^c\Big]H_d,\\W_u&=\Big[\alpha_u(\mathbf{Y_3^{(6)}}Q)_1u_1^c+\alpha'_u(\mathbf{Y_{3'}^{(6)}}Q)_1u_1^c\\&\quad+\beta_u(\mathbf{Y_3^{(4)}}Q)_{1'}c_{1'}^c+\gamma_u(\mathbf{Y_3^{(2)}}Q)_{1''}b_{1'}^t\Big]H_u.\end{aligned}\tag{18}$$

where the subscripts of $1,1',1''$ denote the A_4 representations. The parameters $\alpha_q,\alpha'_q,\beta_q$ and γ_q $(q=d,u)$ are, in general, arbitrary complex constants.

We take the modular invariant kinetic terms simply by

$$\sum_I\frac{|\partial_\mu\psi^{(I)}|^2}{\langle-i\tau+i\bar\tau\rangle^{k_I}}\,,\tag{19}$$

where $\psi^{(I)}$ denotes a chiral superfield with weight k_I, and $\bar\tau$ is the anti-holomorphic modulus. After taking VEV of modulus, one can set $\bar\tau=\tau^*$. It is important to address the transformation needed to get the kinetic terms of matter superfields in the canonical form because the terms in Eq. (19) are not canonical. Therefore, we normalize the superfields as:

$$\psi^{(I)}\to\sqrt{(2\mathrm{Im}\tau_q)^{k_I}}\,\psi^{(I)}.\tag{20}$$

The canonical form is obtained by an overall normalization, which shifts our parameters such as

$$\begin{aligned}\alpha_q&\to\hat\alpha_q=\alpha_q\sqrt{(2\mathrm{Im}\tau_q)^6}=\alpha_q(\sqrt3+2\mathrm{Im}\,\epsilon)^3,\\\alpha'_q&\to\hat\alpha'_q=\alpha'_q\sqrt{(2\mathrm{Im}\tau_q)^6}=\alpha'_q(\sqrt3+2\mathrm{Im}\,\epsilon)^3,\\\beta_q&\to\hat\beta_q=\beta_q\sqrt{(2\mathrm{Im}\tau_q)^4}=\beta_q(\sqrt3+2\mathrm{Im}\,\epsilon)^2,\\\gamma_q&\to\hat\gamma_q=\gamma_q\sqrt{(2\mathrm{Im}\tau_q)^2}=\gamma_q(\sqrt3+2\mathrm{Im}\,\epsilon),\end{aligned}\tag{21}$$

where $\tau=\omega+\epsilon$ $(|\epsilon|\ll1)$. We have:

$$\frac{\hat\alpha'_q}{\hat\alpha_q}=\frac{\alpha'_q}{\alpha_q},\quad\frac{\hat\beta_q}{\hat\alpha_q}\simeq\frac{1}{\sqrt3}\frac{\beta_q}{\alpha_q},\qquad\frac{\hat\gamma_q}{\hat\alpha_q}\simeq\frac13\frac{\gamma_q}{\alpha_q}.\tag{22}$$

By using the tensor product decomposition rules given in Appendix A, we obtain the following expressions for the mass matrices M_d and M_u of down-type and up-type quarks:[5]

$$M_d=v_d\begin{pmatrix}\hat\alpha_d&0&0\\0&\hat\beta_d&0\\0&0&\hat\gamma_d\end{pmatrix}\begin{pmatrix}\tilde Y_1^{(6)}&\tilde Y_3^{(6)}&\tilde Y_2^{(6)}\\Y_2^{(4)}&Y_1^{(4)}&Y_3^{(4)}\\Y_3^{(2)}&Y_2^{(2)}&Y_1^{(2)}\end{pmatrix},$$

[5] We note that the mass matrices are written in RL convention.

$$M_u = v_u \begin{pmatrix} \hat{\alpha}_u & 0 & 0 \\ 0 & \hat{\beta}_u & 0 \\ 0 & 0 & \hat{\gamma}_u \end{pmatrix} \begin{pmatrix} \tilde{Y}_1^{(6)} & \tilde{Y}_3^{(6)} & \tilde{Y}_2^{(6)} \\ Y_2^{(4)} & Y_1^{(4)} & Y_3^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}, \tag{23}$$

where $v_{d(u)}$ denotes VEV of the neutral component of $H_{d(u)}$,

$$\tilde{Y}_i^{(6)} = Y_i^{(6)} + g_q\, Y_i^{'(6)}, \quad g_q \equiv \hat{\alpha}'_q/\hat{\alpha}_q = \alpha'_q/\alpha_q \quad (i = 1,\, 2,\, 3, \quad q = d,\, u), \tag{24}$$

and $Y_i^{(2)}$, $Y_i^{(4)}$ and $Y_i^{(6)}$ $(i = 1, 2, 3)$, are the components of the weight 2, 4 and 6 modular forms furnishing triplet representations of A_4. Explicit expressions for the the modular forms of interest are presented in Appendix B.

We note that the CKM quark mixing matrix $\mathrm{U_{CKM}}$ is given by the product of the unitary matrices $\mathrm{U_{uL}}$ and $\mathrm{U_{dL}}$, which diagonalise respectively $M_u^\dagger M_u$ and $M_d^\dagger M_d$: $\mathrm{U_{CKM}} = \mathrm{U_{uL}^\dagger U_{dL}}$. It is clear from the expressions of M_u and M_d in Eq. (23) that only the absolute values squared of the constants $\alpha_q(\hat{\alpha}_q)$, $\beta_q(\hat{\beta}_q)$ and $\gamma_q(\hat{\gamma}_q)$, $q = d, u$, enter into the expressions for $M_u^\dagger M_u$ and $M_d^\dagger M_d$. Thus, these constants cannot be a source of CP violation and without loss of generality can be taken to be real. In contrast, the constants g_q, $q = d, u$, if complex, may cause violation of the CP symmetry and therefore we will consider them, in general, as complex parameters.

3.2.2 Quark mass matrices at $\tau = \omega$

Consider the quark mass matrices in Eq. (23) at the fixed point $\tau = \omega$. In the symmetric basis of S and T generators given in Appendix A (Eq. (62)), in which the mass matrices in Eq. (23) are obtained, the modular forms $\mathbf{Y_3^{(2)}}$, $\mathbf{Y_3^{(4)}}$ and $\mathbf{Y_{3'}^{(6)}}$ take simple forms at $\tau = \omega$ as shown in Appendix B. Correspondingly, at $\tau = \omega$ the quark mass matrices can written as:

$$M_q = \begin{pmatrix} -g_q\,\hat{\alpha}_q & 0 & 0 \\ 0 & \hat{\beta}_q & 0 \\ 0 & 0 & \hat{\gamma}_q \end{pmatrix} \begin{pmatrix} 1 & -2\omega^2 & -2\omega \\ -\frac{1}{2}\omega & 1 & \omega^2 \\ -\frac{1}{2}\omega^2 & \omega & 1 \end{pmatrix}_{RL} \quad q = d, u. \tag{25}$$

It is easily checked that the 1st, 2nd and 3rd rows of M_q are proportional to each other. That is, the mass matrix of Eq. (25) is of rank one.

It proves convenient to analyse the quark mass matrices M_q in the diagonal basis of the ST generator for the A_4 triplet, in which the flavour structure of M_q becomes explicit. The ST-transformation of the A_4 triplet of the left-handed quarks Q with weight $k = 2$ is

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \xrightarrow{ST} (-\omega - 1)^{-2} \rho(ST) \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \omega^2 \times \frac{1}{3} \begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2 & -\omega & 2\omega^2 \\ 2 & 2\omega & -\omega^2 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}, \tag{26}$$

where the representations of S and T for the triplet are given explicitly in Appendix A. The ST-eigenstate Q^{ST} is obtained with the help of a unitary transformation. We use the unitary matrix V_{ST},

$$V_{\mathrm{ST}} = \frac{1}{3} \begin{pmatrix} -2\omega^2 & -2\omega & 1 \\ 2\omega^2 & -\omega & 2 \\ -\omega^2 & 2\omega & 2 \end{pmatrix}. \tag{27}$$

which leads to the diagonal basis of the ST generator of interest:

$$V_{\mathrm{ST}}\; \omega^2 ST\; V_{\mathrm{ST}}^\dagger = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{28}$$

Then, the ST-eigenstate is $Q^{ST} = V_{\mathrm{ST}}\, Q$.

The right-handed quarks q_i^c $(q = d, u)$, which are singlets $(1, 1'', 1')$ with weights $(4, 2, 0)$, are eigenstates of ST:

$$\begin{pmatrix} q_1^c \\ q_2^c \\ q_3^c \end{pmatrix} \xrightarrow{ST} \begin{pmatrix} (-\omega - 1)^{-4} & 0 & 0 \\ 0 & (-\omega - 1)^{-2}\,\omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \begin{pmatrix} q_1^c \\ q_2^c \\ q_3^c \end{pmatrix} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \begin{pmatrix} q_1^c \\ q_2^c \\ q_3^c \end{pmatrix}, \tag{29}$$

where we have used ST charges of $(1, 1'', 1')$ which read $(1,\, \omega^2,\, \omega)$ (see Eq. (65)). Finally, we have

$$\begin{pmatrix} u^c \\ c^c \\ t^c \end{pmatrix} \xrightarrow{ST} \omega \begin{pmatrix} u^c \\ c^c \\ t^c \end{pmatrix}, \qquad \begin{pmatrix} d^c \\ s^c \\ b^c \end{pmatrix} \xrightarrow{ST} \omega \begin{pmatrix} d^c \\ s^c \\ b^c \end{pmatrix}, \tag{30}$$

It can be shown using, in particular, the preceding results that the Dirac mass matrix in the ST diagonal basis, $\mathcal{M}_q$, is related to the mass matrix in the initial S and T symmetric basis as follows:

$$\mathcal{M}_q = M_q V_{ST}^\dagger, \quad \mathrm{q = d, u}. \tag{31}$$

Thus, we get:

$$\mathcal{M}_q^{(0)} = M_q V_{\rm ST}^\dagger = v_q \begin{pmatrix} 0 & 0 & \frac{27}{8}\,\hat\alpha_q\, g_q\, \omega \\ 0 & 0 & \frac{9}{4}\,\hat\beta_q\, \omega^2 \\ 0 & 0 & \frac{3}{2}\,\hat\gamma_q \end{pmatrix}, \quad \text{q} = \text{d, u}, \tag{32}$$

which is a rank one matrix. We have two massless down-type quarks and two massless up-type quarks at the fixed point $\tau = \omega$. The matrix $\mathcal{M}_q^\dagger \mathcal{M}_q$ is transformed as:

$$\mathcal{M}_q^{(0)\,2} \equiv V_{\rm ST} M_q^\dagger M_q V_{\rm ST}^\dagger = v_q^2\,\frac{9}{4} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{81}{16}\,|g_q|^2\,\hat\alpha_q^2 + \frac{9}{4}\hat\beta_q^2 + \hat\gamma_q^2 \end{pmatrix} \tag{33}$$

We see that only the third generation of quarks get non-zero masses.

3.2.3 Quark mass matrices in the vicinity of $\tau = \omega$

The quark mass matrices in Eq. (32) are corrected due to the small deviation of τ from the fixed point of $\tau = \omega$. By the Taylor expansion of the modular forms in the vicinity of $\tau = \omega$ as seen in Appendix C, we estimate the off-diagonal elements of $\mathcal{M}_q^2$ in Eq. (33). In the ST diagonal basis, the correction is parametrised by a relatively small variable ϵ, where

$$\tau = \omega + \epsilon. \tag{34}$$

The parameter ϵ describing the deviation of τ from ω is related to the "deviation" parameter u introduced in [120] (see Eq. (15)): $\epsilon = i\,\sqrt{3}u/(1-u) \simeq i\,\sqrt{3}u$, $|u| \ll 1$. Up to 2nd order approximation in ϵ, the quark mass matrix $\mathcal{M}_q$ is given by:

$$\mathcal{M}_q^{(2)} = v_q \begin{pmatrix} \hat\alpha_q \omega Y_1^3 & 0 & 0 \\ 0 & \hat\beta_q \omega^2 Y_1^2 & 0 \\ 0 & 0 & \hat\gamma_q Y_1 \end{pmatrix} \times \begin{pmatrix} (-3+\frac{3}{4}g_q)\epsilon_1^2 - \frac{3}{2}[3\epsilon_1 + 2\epsilon_1^2(\frac{5}{2}+\frac{3}{2}k_2)](1+\frac{g_q}{2}) & g_q\,\frac{9}{2}[\frac{3}{4}+\frac{3}{2}\epsilon_1+\epsilon_1^2(\frac{5}{4}+\frac{3}{2}k_2)] \\ -\frac{3}{2}\epsilon_1^2 & \frac{3}{2}[\epsilon_1+\epsilon_1^2(1+k_2)] & \frac{9}{4}+3\epsilon_1+\epsilon_1^2(\frac{3}{2}+3k_2) \\ \frac{1}{3}\epsilon_1^2 & -[\epsilon_1+\epsilon_1^2(\frac{1}{3}+k_2)] & \frac{3}{2}+\epsilon_1+\epsilon_1^2(\frac{1}{6}+k_2) \end{pmatrix}, \tag{35}$$

where ϵ_1, ϵ_2, k_2 and k_3 are given in Appendix C:

$$\frac{Y_2}{Y_1} \simeq \omega\,(1+\epsilon_1+k_2\epsilon_1^2)\,, \qquad \frac{Y_3}{Y_1} \simeq -\frac{1}{2}\omega^2\,(1+\epsilon_2+k_3\epsilon_2^2), \tag{36}$$

with

$$\epsilon_1 \simeq 2.235\, i\,\epsilon,\, \epsilon_2 = 2\epsilon_1,\, k_2 = 0.592,\, k_3 = 0.546. \tag{37}$$

Using Eq. (35) we obtain the elements of $(\mathcal{M}_q^{(2)})^\dagger \mathcal{M}_q^{(2)} \equiv \mathcal{M}_q^2$ in leading order in ϵ_1:

$$\begin{aligned}
\mathcal{M}_q^2[1,1] &= v_q^2\,|\epsilon_1|^4 \left[\frac{1}{9}|\hat\gamma_q|^2 + \frac{9}{16}(4|\hat\beta_q|^2 + |\hat\alpha_q|^2|g_q-4|^2)\right],\\
\mathcal{M}_q^2[2,2] &= v_q^2\,|\epsilon_1|^2 \left[|\hat\gamma_q|^2 + \frac{9}{4}|\hat\beta_q|^2 + \frac{81}{16}|\hat\alpha_q|^2|g_q+2|^2\right],\\
\mathcal{M}_q^2[3,3] &= v_q^2 \left[\frac{9}{4}|\hat\gamma_q|^2 + \frac{81}{16}|\hat\beta_q|^2 + \frac{729}{64}|\hat\alpha_q|^2|g_q|^2\right],\\
\mathcal{M}_q^2[1,2] &= -\,v_q^2\,|\epsilon_1|^2\epsilon_1^* \left[\frac{1}{3}|\hat\gamma_q|^2 + \frac{9}{4}|\hat\beta_q|^2 \right.\\
&\quad \left. -\frac{27}{16}|\hat\alpha_q|^2(2+g_q)(4-g_q^*)\right],\\
\mathcal{M}_q^2[1,3] &= v_q^2\,(\epsilon_1^*)^2 \left[\frac{1}{2}|\hat\gamma_q|^2 - \frac{27}{8}|\hat\beta_q|^2 - \frac{81}{32}|\hat\alpha_q|^2 g_q(4-g_q^*)\right],\\
\mathcal{M}_q^2[2,3] &= -\,v_q^2\,\epsilon_1^* \left[\frac{3}{2}|\hat\gamma_q|^2 - \frac{9}{8}|\hat\beta_q|^2 + \frac{243}{32}|\hat\alpha_q|^2 g_q(2+g_q^*)\right],\\
\mathcal{M}_q^2[2,1] &= \mathcal{M}_q^2[1,2]^*, \qquad \mathcal{M}_q^2[3,1] = \mathcal{M}_q^2[1,3]^*,\\
\mathcal{M}_q^2[3,2] &= \mathcal{M}_q^2[2,3]^*,
\end{aligned} \tag{38}$$

where the factors Y_1^3, Y_1^2 and Y_1 in $\hat\alpha_q Y_1^3$, $\hat\beta_q Y_1^2$ and $\hat\gamma_q Y_1$ are absorbed in $\hat\alpha_q$, $\hat\beta_q$ and $\hat\gamma_q$, respectively. We note that in the case of $|\epsilon| \ll 1$ of interest the factor Y_1 is close to 1.[6] The flavour structure of $(\mathcal{M}_q^{(2)})^\dagger \mathcal{M}_q^{(2)}$ is given in terms of powers of ϵ as:

$$\mathcal{M}_q^2 \equiv (\mathcal{M}_q^{(2)})^\dagger \mathcal{M}_q^{(2)} \sim v_q^2 \begin{pmatrix} \mathcal{O}|(\epsilon|^4) & \mathcal{O}(|\epsilon|^2\epsilon^*) & \mathcal{O}(\epsilon^{2*}) \\ \mathcal{O}(|\epsilon|^2\epsilon) & \mathcal{O}(|\epsilon|^2) & \mathcal{O}(\epsilon^*) \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) & \mathcal{O}(1) \end{pmatrix}. \tag{39}$$

We can obtain the mass eigenvalues m_{q1}, m_{q2} and m_{q3} approximately as follows. The determinant of $\mathcal{M}_q^2$ is given as

$$|\det[\mathcal{M}_q^2]| = m_{q1}^2 m_{q2}^2 m_{q3}^2 \simeq 729\, v_q^6\, \hat\alpha_q^2 \hat\beta_q^2 \hat\gamma_q^2 |\epsilon_1|^6, \tag{40}$$

which is independent of g_q. We also have

[6] To give a more precise idea of the magnitude of $Y_1(\tau)$ we give its value at $\epsilon = 0.0199 + i\,0.02055$, which is one of the values of ϵ relevant for our analysis (see further): $Y_1 \simeq 0.95516 - i\,0.00557$.

$$m_{q3}^2 \simeq \mathcal{M}_q^2\,[3,3] = v_q^2\,\frac{9}{64}(81\hat{\alpha}_q^2|g_q|^2 + 36\hat{\beta}_q^2 + 16\hat{\gamma}_q^2), \tag{41}$$

and

$$m_{q2}^2 m_{q3}^2 \simeq v_q^4\,\frac{81}{64}|\epsilon_1|^2(81\hat{\alpha}_q^2\hat{\beta}_q^2|1+g_q|^2 + 36\hat{\alpha}_q^2\hat{\gamma}_q^2 + 16\hat{\beta}_q^2\hat{\gamma}_q^2). \tag{42}$$

It is easy to find that in the case of $|g_q| \sim 1$ and $\hat{\alpha}_q \sim \hat{\beta}_a \sim \hat{\gamma}_q$ the mass ratios satisfy:

$$m_{q3} : m_{q2} : m_{q1} \simeq 1 : |\epsilon_1| : |\epsilon_1|^2 \simeq 1 : |\epsilon| : |\epsilon|^2. \tag{43}$$

On the other hand, if $|g_q| \gg 1$ and $\hat{\alpha}_q \sim \hat{\beta}_a \sim \hat{\gamma}_q$, we have

$$m_{q3} \simeq \frac{27}{8}\,v_q\,\hat{\alpha}_q|g_q|, \quad m_{q2} \simeq 3\,v_q\,\hat{\beta}_q|\epsilon_1|,$$
$$m_{q1} = \frac{8}{3}\,v_q\,\hat{\gamma}_q|\epsilon_1|^2\frac{1}{|g_q|}, \quad \epsilon_1 \simeq 2.235\,i\,\epsilon. \tag{44}$$

Then, the quark mass ratios are approximately given by

$$m_{q3} : m_{q2} : m_{q1} \simeq \frac{27}{8}|g_q| : 3|\epsilon_1| : \frac{8}{3}\frac{1}{|g_q|}|\epsilon_1|^2$$
$$= 1 : \frac{8}{9}\frac{|\epsilon_1|}{|g_q|} : \frac{64}{81}\left(\frac{|\epsilon_1|}{|g_q|}\right)^2$$
$$\simeq 1 : \frac{|\epsilon_1|}{|g_q|} : \left(\frac{|\epsilon_1|}{|g_q|}\right)^2. \tag{45}$$

Namely, in the case of $|g_q| \gg 1$ and $\hat{\alpha}_q \sim \hat{\beta}_a \sim \hat{\gamma}_q$ the quark mass hierarchies are given effectively in terms of $|\epsilon_1/g_q| \sim |\epsilon/g_q|$. Indeed, we have succeeded in explaining both down-type and up-type quark mass hierarchies numerically for $|g_u| \sim 15$ and $|g_d| \sim 1$.

Thus, we see the scaling of quark masses with ϵ. The CKM elements are also scale roughly with ϵ. However, they depend also on the constants and phases of both mass matrices because both contribute to the CKM elements.

4 Describing the quark masses and CKM mixing

In this section, we discuss the possibility of reproducing the observed quark masses and CKM quark mixing parameters without fine-tuning in the vicinity of $\tau = \omega$, i.e., without strong dependence of the results on the constants present in the model.

4.1 Quark mass hierarchies with common τ in $\mathcal{M}_d$ and $\mathcal{M}_u$

We investigate first whether it is possible to describe the down-type and up-type mass hierarchies in terms of powers of the small parameter $\epsilon \equiv \tau - \omega$ avoiding fine-tuning of the constants present in the model. Correspondingly, we suppose that the constants α_q, α'_q, β_q and γ_q in Eq. (18) are real and are of the same order, i.e., $g_q \equiv \alpha'_q/\alpha_q \simeq \beta_q/\alpha_q \simeq \gamma_q/\alpha_q \sim \mathcal{O}(1)$, so that their influence on the strong quark mass hierarchies of interest is insignificant [120]. The reality of the constants can be ensured by imposing the condition of exact gCP symmetry in the considered model [3]. The gCP symmetry will be broken by the complex value of $\epsilon = \tau - \omega \neq 0$.[7] It can be broken also by some (or all) constants being complex.

In the modular invariance approach to the flavour problem the modulus τ obtains a VEV, which breaks the modular flavour symmetry, at some high scale. Thus, the quark mass matrices, and correspondingly the quark masses, mixing angles and CP violating phase, are derived theoretically in the model at this high scale. The values of these observables at the high scale are obtained from the values measured at the electroweak scale by the use of the renormalization group (RG) equations. In the framework of the minimal SUSY scenario the RG running effects depend, in particular, on the chosen high scale and $\tan\beta$. In the analysis which follows we use the GUT scale of 2×10^{16} GeV and $\tan\beta = 5$ as reference values. The numerical values of the quark Yukawa couplings at the GUT scale for $\tan\beta = 5$ are given by [127,128]:

$$\frac{y_d}{y_b} = 9.21 \times 10^{-4}\ (1 \pm 0.111),$$
$$\frac{y_s}{y_b} = 1.82 \times 10^{-2}\ (1 \pm 0.055),$$
$$\frac{y_u}{y_t} = 5.39 \times 10^{-6}\ (1 \pm 0.311),$$
$$\frac{y_c}{y_t} = 2.80 \times 10^{-3}\ (1 \pm 0.043). \tag{46}$$

The quark masses are obtained from the relation $m_q = y_q v_H$ with $v_H = 174$ GeV. The choice of relatively small value of $\tan\beta$ allows us to avoid relatively large $\tan\beta$-enhanced threshold corrections in the RG running of the Yukawa couplings. We set these corrections to zero.

Assuming that both the ratios of the down-type and up-type quark masses, m_b, m_s, m_d and m_t, m_c, m_u, are determined in the model by the small parameter $|\epsilon|$ (or $|\epsilon_1| = 2.235|\epsilon|$), we have

$$m_b : m_s : m_d \simeq 1 : |\epsilon| : |\epsilon|^2,$$
$$|\epsilon| = 0.02 \sim 0.03\ \ (|\epsilon_1| = 0.045 \sim 0.067), \tag{47}$$
$$m_t : m_c : m_u \simeq 1 : |\epsilon| : |\epsilon|^2,$$
$$|\epsilon| = 0.002 \sim 0.003\ \ (|\epsilon_1| = 0.0045 \sim 0.0067), \tag{48}$$

[7] In order for the gCP symmetry to be broken the value of $\tau = \omega + \epsilon$ should not lie on the border of the fundamental domain of the modular group and $\mathrm{Re}(\tau) \neq 0$ [120].

Table 2 Values of the constant parameters obtained in the fit of the quark mass ratios given in Eq. (46)

ϵ	$\frac{\beta_d}{\alpha_d}$	$\frac{\gamma_d}{\alpha_d}$	g_d	$\frac{\beta_u}{\alpha_u}$	$\frac{\gamma_u}{\alpha_u}$	g_u
$0.03188 + i\,0.02151$	1.69	0.91	−1.94	1.02	0.88	17.83

Table 3 Results on the quark mass ratios compared with those at the GUT scale including 1σ error, given in Eq. (46)

	$\frac{m_s}{m_b} \times 10^2$	$\frac{m_d}{m_b} \times 10^4$	$\frac{m_c}{m_t} \times 10^3$	$\frac{m_u}{m_t} \times 10^6$
Output	1.81	9.38	2.77	5.51
Observed	1.82 ± 0.10	9.21 ± 1.02	2.80 ± 0.12	5.39 ± 1.68

where we have given also the values of $|\epsilon_1|$ suggested by fitting the down-type and up-type quark mass ratios given in Eq. (46). Thus, the required $|\epsilon|$ for the description of the down-type and up-type quark mass hierarchies differ approximately by one order of magnitude. As indicated by Eq. (45), this inconsistency can be "rescued" by relaxing the requirement on the constant $|g_u|$ in the up-quark sector, such as $|g_u| = \mathcal{O}(1) \to \mathcal{O}(10)$ leading to

$$m_t : m_c : m_u \simeq 1 : \frac{|\epsilon|}{|g_u|} : \left(\frac{|\epsilon|}{|g_u|}\right)^2, \tag{49}$$

with $|\epsilon| = 0.02 \sim 0.03$ (corresponding to $|\epsilon_1| = 0.045 \sim 0.067$).

In the considered case we have eight real parameters in the down-type and up-type quark mass matrices, $\alpha_q, \beta_q, \gamma_q, g_q \equiv \alpha'_q/\alpha_q$, $q = d, u$, and one complex parameter $\tau = \omega + \epsilon$. Taking $\alpha_q \sim \beta_q \sim \gamma_q$, $|g_d| = \mathcal{O}(1)$ and $|g_u| = \mathcal{O}(10)$, we can reproduce the observed quark mass values. A sample set of values of these parameters for which the quark mass hierarchies are described correctly is given in Tables 2 and 3.

A quantitative criterion of fine-tuning, i.e., of high sensitivity of observables to model parameters, was proposed by Barbieri and Giudice [135] in a different context, but is applicable also in the case of quark mass hierarchies studied by us. The Barbieri–Giudice measure of fine-tuning [135] in the quark sector, max(BG), corresponds to the largest of quantities $|\partial \ln(\text{mass ratio})/\partial \ln \tilde{\alpha}(')_q|$, $|\partial \ln(\text{mass ratio})/\partial \ln \tilde{\beta}_q|$ and $|\partial \ln(\text{mass ratio})/\partial \ln \tilde{\gamma}_q|$, equivalently, to the largest of $|\partial \ln(\text{mass ratio})/\partial \ln \alpha(')_q|$, $|\partial \ln(\text{mass ratio})/\partial \ln \beta_q|$ and $|\partial \ln(\text{mass ratio})/\partial \ln \gamma_q|$. An observable O is typically considered fine-tuned with respect to some parameter p if BG $\equiv |\partial \ln O/\partial \ln p| \gtrsim 10$ [135]. The criterion is satisfied by the quark mass ratios in the models considered in our work. This can be easily checked using the analytic expressions for the quark masses in terms of the constant parameters, Eqs. (40)–(42) and (44). We should add that, as was shown in [120], when applied to mixing angles the Barbieri-Giudice criterion leads to incorrect results. At present there does not exist a reliable formal no-fine-tuning criterion for the mixing angles and the CP violating phase. So, we and other authors use the simple criterion that the constant parameters present in the quark mass matrices be of the same order of magnitude, the rationale being that these parameters are introduced on equal footing and there is no a priori reason why they should have vastly different values.

4.2 Reproducing the CKM mixing angles

As discussed in the previous sections, the quark mass hierarchies are reproduced due to ϵ, which denotes the deviation from the fixed point $\tau = \omega$, and the help of $|g_u| \sim \mathcal{O}(10)$. Next, we study the CKM mixing angles by taking the values of β_q/α_q,γ_q/α_q and g_d to be of order 1. The present data on the CKM mixing angles are given in Particle Data Group (PDG) edition of Review of Particle Physics [129] as:

$$|V_{us}| = 0.22500 \pm 0.00067$$
$$|V_{cb}| = 0.04182^{\pm 0.00085}_{-0.00074}$$
$$|V_{ub}| = 0.00369 \pm 0.00011. \tag{50}$$

By using these values as input and $\tan\beta = 5$ we obtain the CKM mixing angles at the GUT scale of 2×10^{16} GeV [127,128]:

$$|V_{us}| = 0.2250\,(1 \pm 0.0032)$$
$$|V_{cb}| = 0.0400\,(1 \pm 0.020)$$
$$|V_{ub}| = 0.00353\,(1 \pm 0.036). \tag{51}$$

We give also the data on the CP violation in the quark sector which we will use in our further analyses. The tree-level decays of $B \to D^{(*)}K^{(*)}$ are used as the standard candle of the CP violation. The CP violating phase of latest world average is given in PDG2022 [129] as:

$$\delta_{CP} = 66.2^{\circ}{}^{+3.4^{\circ}}_{-3.6^{\circ}}. \tag{52}$$

Since the phase is almost independent of the evolution of RGE's, we refer to this value in the numerical discussions. The rephasing invariant CP violating measure J_{CP} [130] is also given in [129]:

$$J_{\text{CP}} = 3.08^{+0.15}_{-0.13} \times 10^{-5}. \tag{53}$$

Taking into account the RG effects on the mixing angles for $\tan\beta = 5$, we have at the GUT scale 2×10^{16} GeV:

$$J_{\text{CP}} = 2.80^{+0.14}_{-0.12} \times 10^{-5}. \tag{54}$$

We will discuss the CP violation in the considered model in the next section.

Table 4 Values of the constant parameters obtained in the fit of the quark mass ratios and of CKM mixing angles. See text for details

ϵ	$\frac{\beta_d}{\alpha_d}$	$\frac{\gamma_d}{\alpha_d}$	g_d	$\frac{\beta_u}{\alpha_u}$	$\frac{\gamma_u}{\alpha_u}$	g_u
$0.01779 + i\,0.02926$	3.26	0.43	-1.40	1.05	0.80	-16.1

Table 5 Results of the fit of the quark mass ratios and CKM mixing angles. 'Exp' denotes the respective values at the GUT scale, including 1σ errors, quoted in Eqs. (46) and (51). The value of the $J_{\rm CP}$ factor is calculated as an output using the values of ϵ and the constants given in Table 4. See text for further details

	$\frac{m_s}{m_b} \times 10^2$	$\frac{m_d}{m_b} \times 10^4$	$\frac{m_c}{m_t} \times 10^3$	$\frac{m_u}{m_t} \times 10^6$	$\lvert V_{us}\rvert$	$\lvert V_{cb}\rvert$	$\lvert V_{ub}\rvert$	$J_{\rm CP}$
Fit	1.52	8.62	2.50	5.43	0.2230	0.0786	0.00368	-2.9×10^{-8}
Exp	1.82	9.21	2.80	5.39	0.2250	0.0400	0.00353	2.8×10^{-5}
$1\,\sigma$	± 0.10	± 1.02	± 0.12	± 1.68	± 0.0007	± 0.0008	± 0.00013	$^{+0.14}_{-0.12} \times 10^{-5}$

Table 6 Values of the constant parameters obtained in the fit of the quark mass ratios, CKM mixing angles, the CPV phase $\delta_{\rm CP}$ and the $J_{\rm CP}$ factor with complex ϵ, g_d and g_u

ϵ	$\frac{\beta_d}{\alpha_d}$	$\frac{\gamma_d}{\alpha_d}$	$\lvert g_d\rvert$	arg $[g_d]$	$\frac{\beta_u}{\alpha_u}$	$\frac{\gamma_u}{\alpha_u}$	$\lvert g_u\rvert$	arg $[g_u]$
$0.00048 + i\,0.02670$	2.30	0.39	0.88	161°	1.69	0.49	16.2	205°

We try to reproduce approximately the observed CKM mixing angles with real g_q, $q = d, u$. In our scheme, the CKM mixing angles are given roughly in terms of powers of ϵ_1, as seen in Eq. (38). In order to reproduce the observed ones precisely, the numerical values of the order one ratios of the parameters β_q/α_q,γ_q/α_q as well as of g_d "help" somewhat (no fine-tuning) since both down-type and up-type quark mass matrices contribute to them. We will show those numerical values in Tables.

We scan parameters with the constraint of reproducing the observed values of the quark masses, $|V_{us}|$ (Cabibbo angle), $|V_{ub}|$ and $|V_{cb}|$ including the 3σ uncertainties. A sample set for the fitting and the results are presented in Tables 4 and 5.

As seen in Table 5, the values of the CKM elements $|V_{us}|$ and $|V_{ub}|$ found in the fit are consistent with the observed values. On the other hand, the magnitude of V_{cb} is large, almost twice as large as the observed one. This result can be understood using the results in Eq. (38). For the values of the parameters in Table 4 both down-type quark sector and up-type quark one contribute to $|V_{cb}|$ additively in $\mathcal{O}(\epsilon)$, each contribution being close to the observed one.

4.3 Reproducing CP violation

In Table 5 we give also the value of the $J_{\rm CP}$ factor calculated as an output using the values of ϵ and the constants given in Table 4. The CP violating measure $J_{\rm CP}$ is much smaller than the observed one. As g_q, $q = u, d$, are real, the CP violating phase is generated by Im ϵ_1, which corresponds to Re ϵ. This contribution is strongly suppressed, as discussed in Sect. 5. The CP violating measure $J_{\rm CP}$ is suppressed in the case of real parameters of g_d and g_u due to the extremely small value of the CPV phase $\delta_{\rm CP}$. In order to try to reproduce the observed value of $\delta_{\rm CP}$ and of the $J_{\rm CP}$ factor, we take either of the two parameters (or both) to be complex. Having both g_d and g_u complex is effectively equivalent to not imposing the gCP symmetry requirement at all. As in the preceding subsection, we scan the parameters with the constraint of reproducing the observed values of the quark masses, Cabibbo angle and $|V_{ub}|$ including 3σ uncertainties.

For the case of complex g_d and g_u, a sample set of the results of the fitting is given in Tables 6 and 7. We obtain a value of the CPV phase $\delta_{\rm CP}$ consistent with measured one. On the other hand, the magnitude of V_{cb} is still larger than that at the GUT scale given in Eq. (51). Therefore, $|J_{\rm CP}|$ is also larger approximately by a factor of 1.8 than its value at the GUT scale (see Eq. (54)). It is possible to improve the result for $|V_{cb}|$, e.g., by modification of the model, as is discussed in Sect. 6.

5 The CP violation problem

In the preceding subsections, we have analysed the minimal A_4 quark model. We have seen that the mass hierarchies of both down-type and up-type quarks can be reproduced with a mildly fine-tuned real constant g_u having a value $|g_u|$ of $\mathcal{O}(10)$ and all other constant ratios being $\mathcal{O}(1)$. However, reproducing the CP violation in the quark sector turned out to be highly problematic in the case of real CP conserving constants g_u and g_d. In the present section we analyse the origin of this problem.

Table 7 Results of the fit of the quark mass ratios, CKM mixing angles, δ_{CP} and J_{CP} with complex ϵ, g_d and g_u. 'Exp' denotes the values of the observables at the GUT scale, including 1σ error, quoted in Eqs. (46), (51), (52) and (54) and obtained from the measured ones

	$\frac{m_s}{m_b} \times 10^2$	$\frac{m_d}{m_b} \times 10^4$	$\frac{m_c}{m_t} \times 10^3$	$\frac{m_u}{m_t} \times 10^6$	$\lvert V_{us}\rvert$	$\lvert V_{cb}\rvert$	$\lvert V_{ub}\rvert$	$\lvert J_{CP}\rvert$	δ_{CP}
Fit	1.53	8.88	3.13	2.02	0.2229	0.0777	0.00333	5.2×10^{-5}	67.0°
Exp	1.82	9.21	2.80	5.39	0.2250	0.0400	0.00353	2.8×10^{-5}	66.2°
1σ	± 0.10	± 1.02	± 0.12	± 1.68	± 0.0007	± 0.0008	± 0.00013	${}^{+0.14}_{-0.12} \times 10^{-5}$	${}^{+3.4^\circ}_{-3.6^\circ}$

The mass matrices $\mathcal{M}_q$, $q = d, u$, we have obtained, Eq. (35), are written in the RL basis of the right-handed and left-handed quark fields. They are diagonalised by the bi-unitary transformations: $\mathcal{M}_{\rm q} = {\rm U}_{\rm qR} \mathcal{M}_{\rm q}^{\rm diag} {\rm U}_{\rm qL}^\dagger$, where $\mathcal{M}_{\rm d,u}^{\rm diag}$ are diagonal matrices with the masses of d, s, b and u, c, t quarks, and ${\rm U}_{\rm qR}$ and ${\rm U}_{\rm qL}$ are unitary matrices. The CKM quark mixing matrix ${\rm U}_{\rm CKM}$ is given by ${\rm U}_{\rm CKM} = {\rm U}_{\rm uL}^\dagger {\rm U}_{\rm dL}$. The matrices ${\rm U}_{\rm dL}$ and ${\rm U}_{\rm uL}$ diagonalise $\mathcal{M}_d^\dagger \mathcal{M}_d$ and $\mathcal{M}_u^\dagger \mathcal{M}_u$, respectively: $\mathcal{M}_q^\dagger \mathcal{M}_q = {\rm U}_{\rm qL} (\mathcal{M}_q^{diag})^2 {\rm U}_{\rm qL}^\dagger$. Thus, the possibility to have CP-violating ${\rm U}_{\rm CKM}$, as required by the data, is determined by whether $\mathcal{M}_q^\dagger \mathcal{M}_q$ violate the CP symmetry or not, i.e., by whether $\mathcal{M}_q^\dagger \mathcal{M}_q$ contain complex elements that break the CP symmetry.

We note first that $\mathcal{M}_q^\dagger \mathcal{M}_q$ depend on $|\hat{\alpha}_q \omega Y_1^3|^2$, $|\hat{\beta}_q \omega^2 Y_1^2|^2$ and $|\hat{\gamma}_q Y_1|^2$, and therefore the constants $\hat{\alpha}_q$, $\hat{\beta}_q$, $\hat{\gamma}_q$ and Y_1 cannot be a source of CP violation.

Suppose next that the constants g_q, $q = d, u$, present in the expression Eq. (35) of $\mathcal{M}_q$ are real and that ϵ is complex such that $\tau = \omega + \epsilon$ is CP-violating,[8] i.e., that τ is the only source of breaking of CP symmetry [3].[9] The deviation ϵ of τ from the left cusp plays the role of a small parameter in terms of which the quark mass hierarchies are expressed. We have found that in this case the correct description of the down-quark and up-quark mass hierarchies, which are very different, can be achieved with the help of the real constant g_u having a relatively large absolute value, $|g_u| = \mathcal{O}(10)$. which provides the necessary "enhancement" of the up-quark mass hierarchies, with all the other ratios of constants being of the same order in magnitude and noticeably smaller than g_u. The mass hierarchies thus obtained have the following forms:

$$m_b : m_s : m_d \sim m_b\, (1 : |\epsilon| : |\epsilon|^2), \tag{55}$$

$$m_t : m_c : m_u \sim m_t\, (1 : |\epsilon/g_u| : |\epsilon/g_u|^2). \tag{56}$$

For the discussion of the problem of CP violation, however, the presence of the real constant g_u in Eq. (56) and in the up-quark mass matrix $\mathcal{M}_u$ is not relevant and for simplicity of the presentation we will refer in what follows in this section to both hierarchies in Eqs. (55) and (56) as being of the type $1 : |\epsilon| : |\epsilon|^2$. Then from the point of view of CP violation the quark mass matrices $\mathcal{M}_{d,u}$ in Eq. (35) have the following generic structure:

$$\mathcal{M}_q^{gen} = v_q \begin{pmatrix} i^2\, \epsilon^2 & i\, \epsilon & 1 \\ i^2\, \epsilon^2 & i\, \epsilon & 1 \\ i^2\, \epsilon^2 & i\, \epsilon & 1 \end{pmatrix}, \quad q = d, u, \tag{57}$$

where we have used $\epsilon_1 = i\, 2.235\epsilon$ and kept only the leading order terms in ϵ in the first, second and third columns of $\mathcal{M}_q$. The different real coefficients in the elements of $\mathcal{M}_q$ shown in the second matrix in Eq. (35), including the factors $(2.235)^2$ and 2.235 in the first and second column, as well as the common factors of the 1st, 2nd and 3rd rows of $\mathcal{M}_q$, namely, $\hat{\alpha}_q \omega Y_1^3$, $\hat{\beta}_q \omega^2 Y_1^2$ and $\hat{\gamma}_q Y_1$, which we have not included in the expression Eq. (57) of $\mathcal{M}_q$, are not relevant for the present general discussion of CP violation. It is not difficult to show that $(\mathcal{M}_q^{gen})^\dagger \mathcal{M}_q^{gen}$ of interest can be cast in the form:

$$(\mathcal{M}_q^{gen})^\dagger \mathcal{M}_q^{gen} = v_q^2 \begin{pmatrix} -i\, e^{-i\,\kappa_q} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i\, e^{i\,\kappa_q} \end{pmatrix} \times \begin{pmatrix} |\epsilon_q|^4 & |\epsilon_q|^3 & |\epsilon_q|^2 \\ |\epsilon_q|^3 & |\epsilon_q|^2 & |\epsilon_q| \\ |\epsilon_q|^2 & |\epsilon_q| & 1 \end{pmatrix} \times \begin{pmatrix} i\, e^{i\,\kappa_q} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i\, e^{-i\,\kappa_q} \end{pmatrix}, \quad q = d, u, \tag{58}$$

where we have used $\epsilon_q = |\epsilon_q| e^{i\,\kappa_q}$ taking into account the possibility of two different deviations from $\tau = \omega$ in the down-quark and up-quark sectors.[10] The matrix in Eq. (58) is diagonalised by ${\rm U}_{\rm qL}^{\rm gen} = {\rm P}(\kappa_{\rm q}) {\rm O}_{\rm q}$, where ${\rm P}(\kappa_{\rm q}) = {\rm diag}({\rm e}^{-{\rm i}\,(\kappa_{\rm q} + \pi/2)}, 1, {\rm e}^{{\rm i}\,(\kappa_{\rm q} + \pi/2)})$ and ${\rm O}_{\rm q}$ is a real orthogonal matrix. The CKM matrix in this schematic analysis of "leading order" CP violation is given by:

$${\rm U}_{\rm CKM}^{\rm gen} = {\rm O}_{\rm u}^{\rm T}\, {\rm P}^*(\kappa_{\rm u}) {\rm P}(\kappa_{\rm d}) {\rm O}_{\rm d}. \tag{59}$$

In the "minimal" case of one and the same deviation of τ from ω we have $\epsilon_{1d} = \epsilon_{1u}$ and, consequently, $\kappa_d = \kappa_u$. It

[8] We recall that $\epsilon_1 \simeq i\, 2.235\epsilon$ in Eq. (76) of Appendix C.

[9] The reality of the constants present in the model can be ensured by imposing the gCP symmetry [3].

[10] We have omitted also the overall factor 3 in the right-hand side of Eq. (58), which is irrelevant for the current discussion.

follows from Eq. (59) that in this case $\mathrm{U}^{\text{gen}}_{\text{CKM}}$ is real and thus CP conserving. This implies that to leading order in ϵ in the elements of the quark mass matrices $\mathcal{M}_q$, there will be no CP violation in the quark sector in the considered minimal model: the CP violating phase in U_{CKM}, $\delta^{th}_{\text{CP}} = 0, \pi$, while it follows from the data that $\delta_{\text{CP}} \simeq 66.2°$. The CP violation arises at a higher order due to the corrections to the leading terms in the elements of $\mathcal{M}_q$. Since it is possible to describe correctly the quark mass hierarchies only if we have $|\epsilon| \ll 1$, the CPV phase δ^{th}_{CP}, which is generated by the higher order corrections in ϵ in the elements of $\mathcal{M}_q$, is generically much smaller than the measured value of δ_{CP}, i.e., $\delta^{th}_{\text{CP}} \ll 66.2°$, which is incompatible with the data. This conclusion is confirmed by our numerical analysis in Sect. 4.2.

Thus, we arrive at the conclusion that in the considered minimal quark flavour model with A_4 modular symmetry, supplemented by the gCP symmetry, and one modulus τ having a VEV in the vicinity of the left cusp, $\tau = \omega + \epsilon$, the description of the quark mass hierarchies in terms of ϵ, which has the generic structure $1 : |\epsilon| : |\epsilon|^2$ and thus implies $|\epsilon| \ll 1$, is incompatible with the description of CP violation in the quark sector. On the basis of the general results presented in [120] we suppose that the problem of incompatibility between the "no-fine-tuned" description of the quark mass hierarchies in the vicinity of the left cusp $\tau = \omega + \epsilon$ with $|\epsilon| \ll 1$, and the description of CP violation in the quark sector, will be present in any quark flavour model based on the finite modular groups S_3, $A^{(\prime)}$, $S_4^{(\prime)}$ and $A_5^{(\prime)}$ and gCP symmetry.

As a modification of the studied minimal modular A_4 model we can considered phenomenologically the possibility of having two different moduli in down-type and up-type quark sectors, τ_q, $q = d, u$, acquiring VEVs in the vicinity of the left cusp, $\tau_q = \omega + \epsilon_q$, $q = d, u$, with $\epsilon_d \neq \epsilon_u$.[11] The down-type and up-type quark mass hierarchies in this case are given respectively by $1 : |\epsilon_d| : |\epsilon_d|^2$ and $1 : |\epsilon_u| : |\epsilon_u|^2$, $|\epsilon_{d,u}| \ll 1$. Since $\epsilon_d \neq \epsilon_u$, we have also $\kappa_d \neq \kappa_u$ and thus $\mathrm{P}^*(\kappa_\mathrm{u})\mathrm{P}(\kappa_\mathrm{d}) = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\,(\kappa_\mathrm{u}-\kappa_\mathrm{d})}, 1, \mathrm{e}^{-\mathrm{i}\,(\kappa_\mathrm{u}-\kappa_\mathrm{d})}) \neq \mathbb{1}$ in Eq. (59), where $\mathbb{1}$ is the unit matrix. The factor $\mathrm{P}^*(\kappa_\mathrm{u})\mathrm{P}(\kappa_\mathrm{d})$ may, in principle, be a source of the requisite CP violation provided $\kappa_u - \kappa_d$ is sufficiently large. We analyse this possibility in the next section.

6 Beyond the minimal A_4 quark model

We have seen that although in the minimal A_4 quark model it is possible to reproduce the quark mass hierarchies with the real constant g_u having a value $|g_u|$ of $\mathcal{O}(10)$ and all other constant ratios being $\mathcal{O}(1)$, the model fails to describe correctly the CP violation in the quark sector. In the present section we consider two A_4 extensions of the minimal model which we hope can provide correct description of the quark mass ratios, the CKM mixing angles and of the CP violation in the quark sector without severe fine-tuning of the constants present in the quark mass matrices.

6.1 CKM mixing angle and CPV phase with two moduli τ_d and τ_u

In the previous subsections, we have analysed the case of a common modulus τ in the mass matrices $\mathcal{M}_d$ and $\mathcal{M}_u$. We have seen that the mass hierarchies of both down-type and up-type quarks can be reproduced with the real constant g_u having a value $|g_u|$ of $\mathcal{O}(10)$ and all other constant ratios being $\mathcal{O}(1)$. As suggested in Sect. 5, we consider next phenomenologically the possibility of having two different moduli in down-type and up-type quark sectors, τ_q, $q = d, u$, hoping that in this case one can get a correct description also of the quark CP violation.[12] The down-type and up-type quark mass hierarchies in this case are given respectively by $1 : |\epsilon_u| : |\epsilon_u|^2$ and $1 : |\epsilon_d| : |\epsilon_d|^2$, where $|\epsilon_{d,u}| = |\tau_{d,u} - \omega| \ll 1$. The mass hierarchies of interest given in Eqs. (47) and (48) can be easily reproduced now without fine-tuning with $|g_u|$ and all other constant ratios being $\mathcal{O}(1)$. However, performing a numerical analysis we find that in the case of real constants g_u and g_d the CPV phase δ_{CP} thus generated is too small to be compatible with the measured value. Thus, the problem of correct description of the CP violation in the quark sector persist also in the case of having two different moduli in down-type and up-type quark sectors, as long as the gCP symmetry constraint is imposed.

We have analysed numerically the phenomenological two moduli "model" without imposing the gCP symmetry constraint, with the constants g_u and g_d having complex CP violating values. We present a sample set of results of the fitting with complex g_d and g_u in Tables 8 and 9.

We see that the magnitude of V_{cb} is almost consistent with the observed one at the GUT scale. In this case, the up-type quark sector contribution to $|V_{cb}|$ is of $\mathcal{O}(|\epsilon_u|)$, which is much smaller than the down-type quark sector one that is of $\mathcal{O}(|\epsilon_d|)$. Although we did not search for the χ^2 minimum, we show the magnitude of the measure of goodness of the fitting $N\sigma$, which is defined in Appendix D, as a reference value. In this numerical result, we obtain $N\sigma = 6.8$. The fit does not look so good. It is a consequence of the extremely

[11] Models based on simplectic modular groups contain naturally more than one modulus [131]. Constructing a self-consistent quark flavour model with two moduli is beyond the scope of the present study.

[12] Constructing a self-consistent model in which the down-type and up-type quark mass matrices depend on different moduli is beyond the scope of the present study. This can possibly be done by employing, e.g., $A_4 \times A_4$ modular symmetry.

Table 8 Values of the constant parameters obtained in the fit of the quark mass ratios, CKM mixing angles and of the CPV phase $\delta_{\rm CP}$ in the case of two moduli $\tau_q = \omega + \epsilon_q$ with complex ϵ_q, $q = d, u$, $\epsilon_d \neq \epsilon_u$

ϵ_d	ϵ_u	$\frac{\beta_d}{\alpha_d}$	$\frac{\gamma_d}{\alpha_d}$	$\lvert g_d\rvert$	$\arg[g_d]$	$\frac{\beta_u}{\alpha_u}$	$\frac{\gamma_u}{\alpha_u}$	$\lvert g_u\rvert$	$\arg[g_u]$
$0.02331 + i\,0.02269$	$0.000016 + i\,0.00192$	1.43	0.38	1.10	159°	1.58	1.58	0.895	230°

Table 9 Results of the fits of the quark mass ratios, CKM mixing angles, $J_{\rm CP}$ and $\delta_{\rm CP}$ in the vicinity of two different moduli in the down-type and up-type quark sectors, τ_d and τ_u, $\tau_d \neq \tau_u$. 'Exp' denotes the values of the observables at the GUT scale, including 1σ error, quoted in Eqs. (46), (51), (52) and (54) and obtained from the measured ones

	$\frac{m_s}{m_b} \times 10^2$	$\frac{m_d}{m_b} \times 10^4$	$\frac{m_c}{m_t} \times 10^3$	$\frac{m_u}{m_t} \times 10^6$	$\lvert V_{us}\rvert$	$\lvert V_{cb}\rvert$	$\lvert V_{ub}\rvert$	$\lvert J_{\rm CP}\rvert$	$\delta_{\rm CP}$
Fit	1.52	10.91	2.71	7.66	0.2252	0.0419	0.00351	3.2×10^{-5}	83.8°
Exp	1.82	9.21	2.80	5.39	0.2250	0.0400	0.00353	2.8×10^{-5}	66.2°
$1\,\sigma$	± 0.10	± 1.02	± 0.12	± 1.68	± 0.0007	± 0.0008	± 0.00013	${}^{+0.14}_{-0.12} \times 10^{-5}$	${}^{+3.4°}_{-3.6°}$

high precision of the data which we are fitting using a simple method without making efforts to improve the quality of the fit by varying the value of $\tan\beta$ and/or the threshold effects in the RG running.

6.2 Alternative A_4 model

Finally, we discuss an alternative model. In this model we introduce weight 8 modular forms in addition to the weights 4 and 6 ones in order to get a correct description of the observed three CKM mixing angles and CP violating phase with one modulus τ. The model is obtained formally from the considered one by replacing in Table 1 the weights (4, 2, 0) of the right-handed quarks (d^c, s^c, b^c) and (u^c, c^c, t^c) with weights (6, 4, 2), respectively.

Then, the quark mass matrices are given as follows:

$$M_d = v_d \begin{pmatrix} \hat{\alpha}_d & 0 & 0 \\ 0 & \hat{\beta}_d & 0 \\ 0 & 0 & \hat{\gamma}_d \end{pmatrix} \begin{pmatrix} \tilde{Y}_1^{(8)} & \tilde{Y}_3^{(8)} & \tilde{Y}_2^{(8)} \\ \tilde{Y}_2^{(6)} & \tilde{Y}_1^{(6)} & \tilde{Y}_3^{(6)} \\ Y_3^{(4)} & Y_2^{(4)} & Y_1^{(4)} \end{pmatrix}$$

$$M_u = v_u \begin{pmatrix} \alpha_u & 0 & 0 \\ 0 & \beta_u & 0 \\ 0 & 0 & \gamma_u \end{pmatrix} \begin{pmatrix} \tilde{Y}_1^{(8)} & \tilde{Y}_3^{(8)} & \tilde{Y}_2^{(8)} \\ \tilde{Y}_2^{(6)} & \tilde{Y}_1^{(6)} & \tilde{Y}_3^{(6)} \\ Y_3^{(4)} & Y_2^{(4)} & Y_1^{(4)} \end{pmatrix}, \tag{60}$$

where

$$\begin{aligned} \tilde{Y}_i^{(8)} &= Y_i^{(8)} + f_q Y_i^{'(8)}, \\ \tilde{Y}_i^{(6)} &= Y_i^{(6)} + g_q Y_i^{'(6)}, \\ f_q &\equiv \alpha'_q/\alpha_q, \\ g_q &\equiv \beta'_q/\beta_q \ (i = 1, 2, 3,\ q = d, u). \end{aligned} \tag{61}$$

The additional parameters f_d and f_u of the model play an important role in reproducing the observed CKM parameters. Indeed, we have obtained a good fit of CKM matrix with $|g_d| \simeq |g_u| \simeq |f_d| \simeq 1$, $|f_u| \simeq 30$ and one τ. We show the numerical result with complex ϵ and f_d, while real g_d, g_u and f_u in order to reduce the number of free parameters. The numerical results are presented in Tables 10 and 11. The measure of goodness of fit is considerably improved as $N\sigma = 1.6$ in this case. We can get better fit with $N\sigma < 1$ in the case when g_d, g_u and f_u also complex. The goodness of the fit might be also improved by using a different value of $\tan\beta$ and/or different set of threshold corrections.

7 Summary

We have investigated the possibility to describe the quark mass hierarchies as well as the CKM quark mixing matrix without fine-tuning in a quark flavour model with modular A_4 symmetry. The quark mass hierarchies are considered in the vicinity of the fixed point $\tau = \omega \equiv \exp(i\,2\pi/3)$ (the left cusp of the fundamental domain of the modular group), τ being the VEV of the modulus. In the considered "minimal" A_4 model the three left-handed (LH) quark doublets $Q = (Q_1, Q_2, Q_3)$ are assumed to furnish a triplet irreducible representation of A_4 and to carry weight 2, while the three right-handed (RH) down-type and up-type quarks are supposed to be the A_4 singlets $(\mathbf{1}, \mathbf{1}'', \mathbf{1}')$ carrying weights (4,2,0), respectively. The model involves modular forms of level 3 and weights 6, 4 and 2, and contains eight constants, only two of which, g_d and g_u, can be a source of CP violation in addition to the VEV of the modulus, $\tau = \omega + \epsilon$, $(\epsilon)^* \neq \epsilon$, $|\epsilon| \ll 1$.

We find that in the case of real (CP-conserving) g_d and g_u and common τ (ϵ) for both quark sectors, the down-type quark mass hierarchies can be reproduced without fine tuning with $|\epsilon| \cong 0.03$, all other constants being of the same order in magnitude, and correspond approximately to $1 : |\epsilon| : |\epsilon|^2$. The description of the up-type quark mass hierarchies requires a ten times smaller value of $|\epsilon|$. It can be

Table 10 Values of the constant parameters obtained in the fit of the quark mass ratios, CKM mixing angles, the CPV phase $\delta_{\rm CP}$ and $J_{\rm CP}$ with complex ϵ and f_d, while real g_d, g_u and f_u

ϵ	$\frac{\beta_d}{\alpha_d}$	$\frac{\gamma_d}{\alpha_d}$	g_d	$\lvert f_d\rvert$	$\arg[f_d]$	$\frac{\beta_u}{\alpha_u}$	$\frac{\gamma_u}{\alpha_u}$	g_u	f_u
$0.03612 + i\,0.020133$	1.78	2.01	−1.43	2.63	78.3°	1.08	2.11	0.65	30.3

Table 11 Results of the fit of the quark mass ratios, CKM mixing angles, $\delta_{\rm CP}$ and $J_{\rm CP}$ with complex ϵ and f_d, while real g_d, g_u and f_u. 'Exp' denotes the values of the observables at the GUT scale, including 1σ error, quoted in Eqs. (46), (51), (52) and (54) and obtained from the measured ones

	$\frac{m_s}{m_b}\times 10^2$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_u}{m_t}\times 10^6$	$\lvert V_{us}\rvert$	$\lvert V_{cb}\rvert$	$\lvert V_{ub}\rvert$	$\lvert J_{\rm CP}\rvert$	$\delta_{\rm CP}$
Fit	1.89	9.88	2.84	3.39	0.2250	0.0396	0.00352	2.76×10^{-5}	64.7°
Exp	1.82	9.21	2.80	5.39	0.2250	0.0400	0.00353	2.8×10^{-5}	66.2°
$1\,\sigma$	±0.10	±1.02	±0.12	±1.68	±0.0007	±0.0008	±0.00013	$^{+0.14}_{-0.12}\times 10^{-5}$	$^{+3.4^\circ}_{-3.6^\circ}$

achieved with the same $|\epsilon|\cong 0.03$ allowing the constant g_u to be larger in magnitude than the other constants of the model, $|g_u|\sim\mathcal{O}(10)$, and corresponds to $1:|\epsilon|/|g_u|:|\epsilon|^2/|g_u|^2$. In this setting the description of the CKM element $|V_{\rm cb}|$ is problematic. We have shown that a much more severe problem is the correct description of the CP violation in the quark sector since it arises as a higher order correction in ϵ, which has to be sufficiently small in order to reproduce the quark mass hierarchies. This problem may be generic to modular invariant quark flavour models with one modulus, in which the gCP symmetry is imposed and the quark mass hierarchies are obtained in the vicinity of fixed point $\tau=\omega$. In the considered "minimal" model this problem is somewhat alleviated in the case of complex g_u and g_d. The rephasing invariant $J_{\rm CP}$ has a value which is larger by a factor of ~ 1.8 than the correct value due to a larger than observed value of $|V_{\rm cb}|$. We show also that an essentially correct description of the quark mass hierarchies and the CKM mixing matrix, including the CP violation in the quark sector, is possible in the vicinity of the left cusp with all constants being of the same $\sim\mathcal{O}(1)$ in magnitude and complex g_d and g_u, if there are two different moduli $\tau_d=\omega+\epsilon_d$ and $\tau_u=\omega+\epsilon_u$ in the down-type and up-type quark sectors, with down-type and up-type quark mass hierarchies given by $1:|\epsilon_d|:|\epsilon_d|^2$ and $1:|\epsilon_u|:|\epsilon_u|^2$, respectively. A correct description is also possible in a modification of the considered "minimal" A_4 model which involves level 3 modular forms of weights 8, 6 and 4. Thus, the ten observables of the quark sector can be reproduced quite well. Since these observables have been measured with a very high precision, there is no room for predictions in what concerns the quark masses and mixings. However, the model has predictive power for the flavor phenomena in SMEFT, such as in B meson decays as well as the flavor violation of the charged lepton decays [39].

The results of our study show that describing correctly without severe fine-tuning the quark mass hierarchies, the quark mixing and the CP violation in the quark sector is remarkably challenging within the modular invariance approach to the quark flavour problem.

Acknowledgements The work of S. T. P. was supported in part by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 860881-HIDDeN, by the Italian INFN program on Theoretical Astroparticle Physics and by the World Premier International Research Center Initiative (WPI Initiative, MEXT), Japan. The authors would like to thank Kavli IPMU, University of Tokyo, where part of this study was performed, for the kind hospitality.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and no experimental data.]

Funded by SCOAP^3. SCOAP^3 supports the goals of the International Year of Basic Sciences for Sustainable Development.

Appendix A: Tensor product of A_4 group

We take the generators of A_4 group for the triplet as follows:

$$S=\frac{1}{3}\begin{pmatrix}-1&2&2\\2&-1&2\\2&2&-1\end{pmatrix},\qquad T=\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix},\tag{62}$$

where $\omega = e^{i\frac{2}{3}\pi}$. In this basis, the multiplication rules are:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = (a_1 b_1 + a_2 b_3 + a_3 b_2)_{\mathbf{1}}$$
$$\oplus (a_3 b_3 + a_1 b_2 + a_2 b_1)_{\mathbf{1}'}$$
$$\oplus (a_2 b_2 + a_1 b_3 + a_3 b_1)_{\mathbf{1}''}$$
$$\oplus \frac{1}{3} \begin{pmatrix} 2a_1 b_1 - a_2 b_3 - a_3 b_2 \\ 2a_3 b_3 - a_1 b_2 - a_2 b_1 \\ 2a_2 b_2 - a_1 b_3 - a_3 b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2} \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_1 b_2 - a_2 b_1 \\ a_3 b_1 - a_1 b_3 \end{pmatrix}_{\mathbf{3}} , \tag{63}$$

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}, \quad \mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}'', \quad \mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}', \quad \mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1},$$
$$\mathbf{1}' \otimes \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} = \begin{pmatrix} a_3 \\ a_1 \\ a_2 \end{pmatrix}_{\mathbf{3}} , \quad \mathbf{1}'' \otimes \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} = \begin{pmatrix} a_2 \\ a_3 \\ a_1 \end{pmatrix}_{\mathbf{3}} , \tag{64}$$

where

$$S(\mathbf{1}') = 1, \quad S(\mathbf{1}'') = 1, \quad T(\mathbf{1}') = \omega, \quad T(\mathbf{1}'') = \omega^2. \tag{65}$$

Further details can be found in the reviews [132–134].

Appendix B: Modular forms of A_4

The modular forms of weight 2 transforming as a triplet of A_4 can be written in terms of $\eta(\tau)$ and its derivative [1]:

$$\begin{aligned} Y_1 &= \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right), \\ Y_2 &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \\ Y_3 &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \end{aligned} \tag{66}$$

which satisfy also the constraint [1]:

$$Y_2^2 + 2Y_1 Y_3 = 0 . \tag{67}$$

They have the following q-expansions:

$$\mathbf{Y}_{\mathbf{3}}^{(\mathbf{2})} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \cdots \\ -6q^{1/3}(1 + 7q + 8q^2 + \cdots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \cdots) \end{pmatrix}, \tag{68}$$

where

$$q = \exp(2\pi i\, \tau). \tag{69}$$

The five modular forms of weight 4 are given as:

$$\begin{aligned} &\mathbf{Y}_{\mathbf{1}}^{(4)} = Y_1^2 + 2Y_2 Y_3 , \quad \mathbf{Y}_{\mathbf{1}'}^{(4)} = Y_3^2 + 2Y_1 Y_2 , \\ &\mathbf{Y}_{\mathbf{1}''}^{(4)} = Y_2^2 + 2Y_1 Y_3 = 0 , \\ &\mathbf{Y}_{\mathbf{3}}^{(4)} = \begin{pmatrix} Y_1^{(4)} \\ Y_2^{(4)} \\ Y_3^{(4)} \end{pmatrix} = \begin{pmatrix} Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \\ Y_2^2 - Y_1 Y_3 \end{pmatrix} , \end{aligned} \tag{70}$$

where $\mathbf{Y}_{\mathbf{1}''}^{(4)}$ vanishes due to the constraint of Eq. (67).

There are seven modular forms of weight 6:

$$\begin{aligned} &\mathbf{Y}_{\mathbf{1}}^{(6)} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1 Y_2 Y_3 , \\ &\mathbf{Y}_{\mathbf{3}}^{(6)} \equiv \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = (Y_1^2 + 2Y_2 Y_3) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} , \\ &\mathbf{Y}_{\mathbf{3}'}^{(6)} \equiv \begin{pmatrix} Y_1^{'(6)} \\ Y_2^{'(6)} \\ Y_3^{'(6)} \end{pmatrix} = (Y_3^2 + 2Y_1 Y_2) \begin{pmatrix} Y_3 \\ Y_1 \\ Y_2 \end{pmatrix} . \end{aligned} \tag{71}$$

The weigh 8 modular forms are nine:

$$\begin{aligned} &\mathbf{Y}_{\mathbf{1}}^{(8)} = (Y_1^2 + 2Y_2 Y_3)^2, \quad \mathbf{Y}_{\mathbf{1}'}^{(8)} = (Y_1^2 + 2Y_2 Y_3)(Y_3^2 + 2Y_1 Y_2), \\ &\mathbf{Y}_{\mathbf{1}''}^{(8)} = (Y_3^2 + 2Y_1 Y_2)^2, \\ &\mathbf{Y}_{\mathbf{3}}^{(8)} \equiv \begin{pmatrix} Y_1^{(8)} \\ Y_2^{(8)} \\ Y_3^{(8)} \end{pmatrix} = (Y_1^2 + 2Y_2 Y_3) \begin{pmatrix} Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \\ Y_2^2 - Y_1 Y_3 \end{pmatrix}, \\ &\mathbf{Y}_{\mathbf{3}'}^{(8)} \equiv \begin{pmatrix} Y_1^{'(8)} \\ Y_2^{'(8)} \\ Y_3^{'(8)} \end{pmatrix} = (Y_3^2 + 2Y_1 Y_2) \begin{pmatrix} Y_2^2 - Y_1 Y_3 \\ Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \end{pmatrix} . \end{aligned} \tag{72}$$

At the fixed point $\tau = \omega$ the modular forms take simple forms:

$$\begin{aligned} &\mathbf{Y}_{\mathbf{3}}^{(2)} = Y_0 \begin{pmatrix} 1 \\ \omega \\ -\frac{1}{2}\omega^2 \end{pmatrix} \\ &\mathbf{Y}_{\mathbf{3}}^{(4)} = \frac{3}{2} Y_0^2 \begin{pmatrix} 1 \\ -\frac{1}{2}\omega \\ \omega^2 \end{pmatrix} , \\ &\mathbf{Y}_{\mathbf{1}}^{(4)} = 0 , \quad \mathbf{Y}_{\mathbf{1}'}^{(4)} = \frac{9}{4} Y_0^2\, \omega , \\ &\mathbf{Y}_{\mathbf{3}}^{(6)} = 0 \\ &\mathbf{Y}_{\mathbf{3}'}^{(6)} = \frac{9}{8} Y_0^3 \begin{pmatrix} -1 \\ 2\omega \\ 2\omega^2 \end{pmatrix} \end{aligned}$$

$$\mathbf{Y}_{\mathbf{1}}^{(6)}=\frac{27}{8}Y_0^3,$$
$$\mathbf{Y}_{\mathbf{3}}^{(8)}=0$$
$$\mathbf{Y}_{\mathbf{3}'}^{(8)}=\frac{27}{8}Y_0^4\begin{pmatrix}1\\ \omega\\ -\frac{1}{2}\omega^2\end{pmatrix}$$
$$\mathbf{Y}_{\mathbf{1}}^{(8)}=0,\quad \mathbf{Y}_{\mathbf{1}'}^{(8)}=0$$
$$\mathbf{Y}_{\mathbf{1}''}^{(8)}=\frac{9}{4}\omega Y_0^4. \tag{73}$$

Appendix C: Modular forms at close to $\tau=\omega$

In what follows we present the behavior of modular forms in the vicinity of $\tau=\omega$. We perform Taylor expansion of modular forms $Y_1(\tau)$, $Y_2(\tau)$ and $Y_3(\tau)$ around $\tau=\omega$. We parametrize τ as:

$$\tau=\omega+\epsilon, \tag{74}$$

where $|\epsilon|\ll 1$. Then, the modular forms are expanded in terms of ϵ. In order to get up to 2nd order expansions of ϵ, we parametrize the modular forms as:

$$\frac{Y_2(\tau)}{Y_1(\tau)}\simeq\omega\,(1+\epsilon_1+k_2\epsilon_1^2),\qquad \frac{Y_3(\tau)}{Y_1(\tau)}\simeq-\frac{1}{2}\omega^2\,(1+\epsilon_2+k_3\epsilon_2^2). \tag{75}$$

These parameters are determined numerically in Taylor expansions as:

$$\epsilon_2=2\epsilon_1\simeq 4.47\,i\,\epsilon,\qquad k_2=0.592,\qquad k_3=0.546. \tag{76}$$

The values are obtained by using the first and second derivarives. The constraint $Y_2^2+2Y_1Y_3=0$ in Eq. (67) gives $\epsilon_2=2\epsilon_1$ and $4k_3=1+2k_2$, which are satisfied also numerically.

We expres also the higher weight triplet modular forms $Y_i^{(k)}$, $k=4,6$, in terms of ϵ_1, k_2 and k_3 using $\epsilon_2=2\epsilon_1$. For the weight 4 modular form we get:

$$\frac{Y_1^{(4)}(\tau)}{Y_1^2(\tau)}\simeq\frac{3}{2}[1+\epsilon_1+\frac{2}{3}\epsilon_1^2(1+\frac{1}{2}k_2+2k_3)], \tag{77}$$
$$\frac{Y_2^{(4)}(\tau)}{Y_1^2(\tau)}\simeq\omega\left(-\frac{3}{4}+\epsilon_1^2(1+2k_3-k_2)\right), \tag{78}$$
$$\frac{Y_3^{(4)}(\tau)}{Y_1^2(\tau)}\simeq\omega^2\left(\frac{3}{2}+3\epsilon_1+\epsilon_1^2(1+2k_2+2k_3)\right). \tag{79}$$

In a similar way we get the expressions for the two relevant weight 6 modular forms:

$$\frac{Y_1^{(6)}(\tau)}{Y_1^3(\tau)}\simeq-3\epsilon_1-\epsilon_1^2(2+k_2+4k_3), \tag{80}$$
$$\frac{Y_2^{(6)}(\tau)}{Y_1^3(\tau)}\simeq-\omega\,[3\epsilon_1+\epsilon_1^2(5+k_2+4k_3)], \tag{81}$$
$$\frac{Y_3^{(6)}(\tau)}{Y_1^3(\tau)}\simeq\omega^2\left[\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_1^2(8+k_2+4k_3)\right], \tag{82}$$
$$\frac{Y_1^{'(6)}(\tau)}{Y_1^3(\tau)}\simeq-\left(\frac{9}{8}+\frac{15}{4}\epsilon_1+\epsilon_1^2(\frac{7}{2}+k_2+\frac{11}{2}k_3)\right), \tag{83}$$
$$\frac{Y_2^{'(6)}(\tau)}{Y_1^3(\tau)}\simeq\omega\left(\frac{9}{4}+3\epsilon_1+\epsilon_1^2(1+2k_2+2k_3)\right), \tag{84}$$
$$\frac{Y_3^{'(6)}(\tau)}{Y_1^3(\tau)}\simeq\omega^2\left(\frac{9}{4}+\frac{21}{4}\epsilon_1+\epsilon_1^2(4+\frac{17}{4}k_2+2k_3)\right). \tag{85}$$

Appendix D: A measure of fit

As a measure of goodness of fit, we use the sum of one-dimensional $\Delta\chi^2$ for eight observable quantities $q_j=(m_d/m_b,\ m_s/m_b,\ m_u/m_t,\ m_c/m_t,\ |V_{us}|,\ |V_{cb}|,\ |V_{ub}|,\ \delta_{CP})$. By employing the Gaussian approximation, we difine $N\sigma\equiv\sqrt{\Delta\chi^2}$, where

$$\Delta\chi^2=\sum_j\left(\frac{q_j-q_{j,\text{best fit}}}{\sigma_j}\right)^2. \tag{86}$$

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