Off-shell Yang-Mills amplitude in the Cachazo-He-Yuan formalism

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Möbius invariance is used to construct gluon tree amplitudes in the Cachazo, He, and Yuan (CHY) formalism. If it is equally effective in steering the construction of off-shell tree amplitudes, then the S-matrix CHY theory can be used to replace the Lagrangian Yang-Mills theory. Unfortunately that is not possible. We find that the CHY formula can indeed be modified to obtain a Möbius-invariant off-shell amplitude M_P, but this modified amplitude lacks local gauge invariance, which can be restored to give the correct Yang-Mills amplitude only by the addition of a complementary amplitude M_Q. Although neither M_P nor M_Q is fully gauge invariant, both are partially gauge invariant in a sense to be explained.

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I. INTRODUCTION

S-matrix theory popular in the 1960s failed to take off because there was no way to incorporate interaction without a Lagrangian. This situation changed in 2014 when Cachazo, He, and Yuan (CHY) [1–5] came up with an S-matrix theory which can reproduce tree-level scattering of gluons, gravitons, and many others, with the additional advantage that double-copy relations appear naturally. These refer to relations that are very difficult to understand in the Lagrangian approach, linking together pairs of amplitudes such as graviton amplitude and the square of Yang-Mills amplitude. See [6–29] for some of the subsequent developments.

n-body CHY amplitudes are given by a complex integral with Möbius invariance, an invariance crucial in steering the construction of these amplitudes. Such construction enables local interaction and local propagation to appear in an S-matrix theory, a very remarkable feat because S-matrix *a priori* knows nothing about a local structure of space-time. This success raises the hope that maybe Möbius invariance is also able to simulate fully local space-time interaction, to reproduce off-shell tree amplitudes and hence loops without a Lagrangian.

In the case of ϕ^3 interaction, this is indeed possible. A simple modification of the scattering function enables all correct scalar Feynman tree diagrams to be reproduced, including those with off-shell external legs [30,31].

In the case of off-shell Yang-Mills kinematics, Möbius invariance forces not only a modification of the scattering function, as in the ϕ^3 case, but also a modification of the Pfaffian. This modified M_P describes an amplitude with a local interaction and local propagation, but unfortunately it is not the correct Yang-Mills amplitude for $n > 3$. The original on-shell M_P is gauge invariant, but the modified off-shell M_P retains only a partial gauge invariance. To restore full local gauge invariance, the hallmark of the Yang-Mills theory, an additional term M_Q must be added, which by itself also has partial but not full gauge invariance.

Unfortunately Möbius invariance is no longer a useful guide to the construction of M_Q when $n \geq 4$. Its appearance is related to the emergence of ghosts in Yang-Mills loops and off-shell Yang-Mills tree amplitudes, so it is unavoidable.

On-shell Yang-Mills amplitude in the CHY formalism is reviewed in Sec. II, to show the power of Möbius invariance, and to see what modification is required to maintain the invariance for off-shell kinematics. The details of such modifications will be discussed in Secs. III and IV. This modification does enable M_P to retain Möbius invariance off shell, but an additional term M_Q is needed to match the Feynman amplitude M_F. In Sec. V, we show how M_Q can be constructed and illustrate the procedure with the explicit construction for $n = 4$. The reason behind the necessary appearance of M_Q can be traced back to local gauge invariance, a topic which is discussed in Sec. VI. Amplitudes for $n \geq 5$ are discussed in Sec. VII, to illustrate how the Feynman amplitude can be simplified by its split into M_P and M_Q and to show how partial gauge invariance

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can be used to check calculations for a larger n. Section VIII provides a conclusion.

II. MÖBIUS-INVARIANT AMPLITUDE

A color-stripped n-gluon scattering amplitude in the natural order $(12\ldots n)$ is given by the CHY formula [2]

$$M_P = \left(-\frac{2g}{2\pi i}\right)^{n-3} \oint_\Gamma \frac{\sigma^2_{(pqr)}}{\sigma_{(12\ldots n)}} \left(\prod_{i=1,i\neq p,q,r}^{n} \frac{d\sigma_i}{f_i}\right) P, \quad (1)$$

where g is the coupling constant henceforth taken to be 1, $\sigma_{(pqr)} = \sigma_{pq}\sigma_{qr}\sigma_{rp}, \sigma_{(12\ldots n)} = \prod_{i=1}^{n} \sigma_{i,i+1}$ with $\sigma_{n+1} \equiv \sigma_1$, and $\sigma_{ij} = \sigma_i - \sigma_j$. The scattering functions f_i are defined by

$$f_i = \sum_{j=1,j\neq i}^{n} \frac{2a_{ij}}{\sigma_{ij}} (1 \le i \le n), \quad (2)$$

with k_i being the outgoing momentum of the ith gluon. The quantity $a_{ij} = a_{ji}$ is a linear function of scalar products of momenta whose explicit form will be discussed later. The reduced Pfaffian $P = \mathrm{Pf}'(\Psi)$ is related to the Pfaffian of a matrix $\Psi^{\lambda\nu}_{\lambda\nu}$ by

$$P = \mathrm{Pf}'(\Psi) = \frac{(-1)^{\lambda+\nu+n+1}}{\sigma_{\lambda\nu}} \mathrm{Pf}(\Psi^{\lambda\nu}_{\lambda\nu}) \qquad (\lambda < \nu), \quad (3)$$

where $\Psi^{\lambda\nu}_{\lambda\nu}$ is obtained from the matrix Ψ with its λth and νth columns and rows removed. The antisymmetric matrix Ψ is made up of three $n \times n$ matrices A, B, C:

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}. \quad (4)$$

The nondiagonal elements of these three submatrices are

$$A_{ij} = \frac{a_{ij}}{\sigma_{ij}}, \qquad B_{ij} = \frac{\epsilon_i \cdot \epsilon_j}{\sigma_{ij}} \coloneqq \frac{b_{ij}}{\sigma_{ij}},$$
$$C_{ij} = \frac{c_{ij}}{\sigma_{ij}}, \qquad -C^T_{ij} = \frac{c_{ji}}{\sigma_{ij}} \quad (1 \le i \neq j \le n), \quad (5)$$

where c_{ij} is a linear function of the scalar products $\epsilon \cdot k$ whose exact form will be decided later and ϵ_i is the polarization of the ith gluon. The diagonal elements of A and B are zero, and that of C is defined by

$$C_{ii} = -\sum_{j=1}^{n} C_{ij}, \quad (6)$$

so that $\sum_j C_{ij} = 0$ for all i. A similar property is true for A if the scattering equations $f_i = 0$ are obeyed. This is the case because the integration contour Γ encloses these zeros anticlockwise.

The factors in Eq. (1) are designed to transform covariantly under the Möbius transformation

$$\sigma_i \to \frac{\alpha\sigma_i + \beta}{\gamma\sigma_i + \delta} \qquad (\alpha\delta - \beta\gamma = 1), \quad (7)$$

in such a way that the total weight of the integrand is zero, thus resulting in a Möbius-invariant integrand. Specifically, under the Möbius transformation, if we let $\lambda_i = 1/(\gamma\sigma_i + \delta)$, then

$$d\sigma_i \to \lambda_i^2 d\sigma_i,$$
$$\sigma_{ij} \to \lambda_i\lambda_j\sigma_{ij},$$
$$\sigma_{(p,q,r)} \to (\lambda_p\lambda_q\lambda_r)^2 \sigma_{(p,q,r)},$$
$$\sigma_{(12\ldots n)} \to \left(\prod_{i=1}^{n} \lambda_i^2\right) \sigma_{(12\ldots n)}. \quad (8)$$

The scattering function transforms covariantly like

$$f_i \to \lambda_i^{-2} f_i, \quad (9)$$

as long as

$$\sum_{j=1,j\neq i}^{n} a_{ij} = 0. \quad (10)$$

Thus the integrand of Eq. (1) is Möbius invariant as long as

$$P \to \left(\prod_{i=1}^{n} \lambda_i^{-2}\right) P \quad (11)$$

whatever p, q, r are.

Using Eq. (8), as well as Eqs. (4)–(6), we see that $P = \mathrm{Pf}'(\Psi)$ in Eq. (3) does transform that way, whatever λ, ν are, provided

$$C_{ii} \to \lambda_i^{-2} C_{ii}, \quad (12)$$

which is the case if

$$\sum_{j=1,j\neq i}^{n} c_{ij} = 0. \quad (13)$$

As long as Eq. (1) is Möbius invariant, the integral M_P can be shown to be independent of the choice of p, q, r, as well as the choice of λ, ν. To be invariant, a_{ij} and c_{ij} must be chosen to satisfy Eqs. (10) and (13).

For on-shell gluons with transverse polarization, $k_i^2 = 0$ and $\epsilon_i \cdot k_i = 0$, momentum conservation guarantees these conditions to be satisfied if

$$a_{ij} = k_i \cdot k_j \coloneqq a'_{ij},$$
$$c_{ij} = \epsilon_i \cdot k_j \coloneqq c'_{ij}, \qquad (14)$$

which is the choice in the CHY theory. For off-shell kinematics with possibly longitudinal and timelike polarizations, $k_i^2 \neq 0$ and $\epsilon_i \cdot k_i \coloneqq d_i \neq 0$, Eq. (14) no longer satisfies Eqs. (10) and (13), so the expression for a_{ij} and c_{ij} must be modified. How this can be done will be discussed in the next two sections.

III. a_{ij} DETERMINED BY THE PROPAGATORS

Let

$$a_{ij} = a'_{ij} + \rho_{ij},$$
$$c_{ij} = c'_{ij} + \eta_{ij}. \qquad (15)$$

The constraints Eqs. (10) and (13) restrict the additional terms to satisfy

$$\sum_{j\neq i,j=1}^{n} \rho_{ij} = k_i^2, \qquad (16)$$

$$\sum_{j\neq i,j=1}^{n} \eta_{ij} = \epsilon_i \cdot k_i \coloneqq d_i. \qquad (17)$$

In this section we will discuss how to obtain $\rho_{ij} = \rho_{ji}$, leaving the determination of η_{ij} to the next section.

Equation (16) alone is not sufficient to determine all ρ_{ij}. Since we want to retain local propagation for off-shell amplitudes, we demand Eq. (1) to yield correct propagators in the Feynman gauge. For the color-stripped amplitude M_P in natural order, this requires $\sum_{i\neq j;i,j\in\mathcal{D}} a_{ij} = (\sum_{i\in\mathcal{D}} k_i)^2 \coloneqq s_{\mathcal{D}}$ for every consecutive set of numbers $\mathcal{D}$. This requirement has a unique solution for ρ given by [30,31]

$$\rho_{i,i\pm1} = +\frac{1}{2}(k_i^2 + k_{i\pm1}^2),$$
$$\rho_{i\mp1,i\pm1} = -\frac{1}{2}k_i^2,$$
$$\rho_{ij} = 0 \quad \text{otherwise}, \qquad (18)$$

where all indices are understood to be mod n.

There is another way to retain Möbius covariance of f_i off shell without modifying $a_{ij} = a'_{ij}$: one can add an extra dimension and use the extra momentum component to simulate k_i^2. However, this does not retain local propagation as the resulting propagators turn out to be incorrect.

IV. c_{ij} DETERMINED BY THE TRIPLE-GLUON VERTEX

There are also many solutions of η_{ij} to satisfy Eq. (17), but unlike ρ_{ij}, which can be fixed by the local propagation requirement, there is no obvious way to settle what η_{ij} should be.

One of the many solutions of Eq. (17) is

$$c_{i,i\pm1} = c'_{i,i\pm1} + \frac{1}{2}d_i,$$
$$c_{ij} = c'_{ij} \quad \text{otherwise}. \qquad (19)$$

We shall adopt this solution throughout because it is the simplest and because it yields the correct $n = 3$ off-shell amplitude.

To see that, recall that the triple-gluon vertex (with a unit coupling constant, and the color factor stripped) depicted in Fig. 1 is

$$\begin{aligned} V &= \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot (k_1 - k_2) + \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot (k_2 - k_3) \\ &\quad + \epsilon_3 \cdot \epsilon_1 \epsilon_2 \cdot (k_3 - k_1) \\ &= b_{12}(c'_{31} - c'_{32}) + b_{23}(c'_{12} - c'_{13}) \\ &\quad + b_{31}(c'_{23} - c'_{21}). \end{aligned} \qquad (20)$$

Using Eq. (19), this becomes

$$\begin{aligned} V &= b_{12}(c_{31} - c_{32}) + b_{23}(c_{12} - c_{13}) + b_{31}(c_{23} - c_{21}) \\ &= 2(-b_{12}c_{32} + b_{23}c_{12} - b_{31}c_{21}), \end{aligned} \qquad (21)$$

which is precisely what Eq. (1) yields when $n = 3$. Therefore, the choice of Eq. (19) enables the triple-gluon vertex to be reproduced correctly by M_P in Eq. (1) for $n = 3$.

It is convenient to represent each of the three terms in Eq. (20) by a separate subdiagram, as shown on the right of Fig. 1. This pictorial representation makes it easier to distinguish different terms in a Feynman diagram.

The reason to use Eq. (19) also for $n > 3$ is the following. It turns out that no matter how η_{ij} is chosen, there is no way to convert all c'_{ij} into c_{ij} when $n > 3$, thereby enabling M_P to be the off-shell Feynman amplitude. For that reason any choice of η_{ij} is equally good, so we might as well use Eq. (19), which not only reproduces

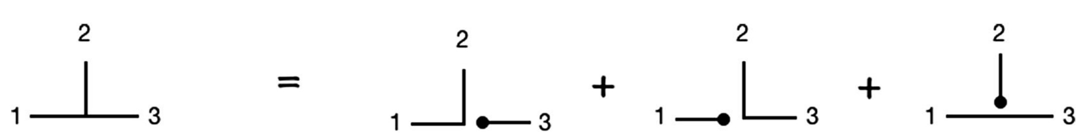

FIG. 1. Triple-gluon vertex and its three subdiagrams.

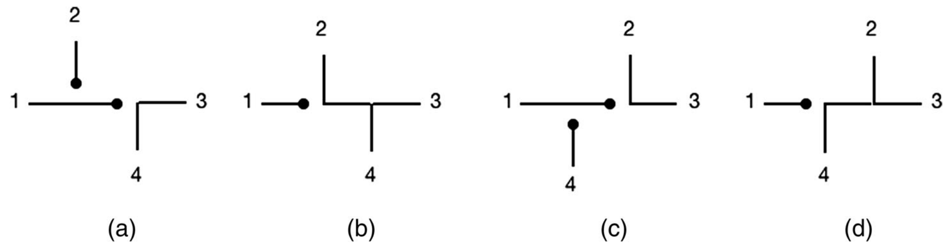

FIG. 2. Four $n = 4$ Feynman subdiagrams.

the triple-gluon vertex, but is also the simplest solution of Eq. (17).

To show that there is no way to convert all c'_{ij} into c_{ij}, consider $n = 4$. There are many Feynman subdiagrams but let us just look at the four shown in Fig. 2.

All four contain a factor involving some combination of c'_{1j}. That factor is $c'_{13} - c'_{14}$ in Fig. 2(a), $c'_{12} - (c'_{13} + c'_{14})$ in Fig. 2(b), $c'_{12} - c'_{13}$ in Fig. 2(c), and $(c'_{12} + c'_{13}) - c'_{14}$ in Fig. 2(d). To convert all these combinations of c' into the corresponding combinations of c, we must require

$$\begin{aligned} \eta_{13} - \eta_{14} &= 0, \\ \eta_{12} - (\eta_{13} + \eta_{14}) &= 0, \\ \eta_{12} - \eta_{13} &= 0, \\ (\eta_{12} + \eta_{13}) - \eta_{14} &= 0. \end{aligned} \tag{22}$$

Moreover, Eq. (17) also requires $\eta_{12} + \eta_{13} + \eta_{14} = d_1$. There are just too many equations for η_{1j} to have a solution. Thus it is not possible to convert all the c'_{ij} appearing in all the $n = 4$ Feynman diagrams into c_{ij}, no matter now η_{ij} are chosen. For a larger n, it is even worse because there will be more equations to satisfy.

M_P in Eq. (1) contains only a_{ij}, b_{ij}, c_{ij}, but no d_i; it clearly cannot be equal to the Feynman amplitude M_F for Yang-Mills theory which is a function of a'_{ij}, b_{ij}, c'_{ij}, unless all a' and c' can be converted into a and c without the appearance of k_i^2 and d_i. Since this is impossible for $n \geq 4$, an additional term $M_Q = M_F - M_P$ must be present.

V. METHOD TO COMPUTE M_Q ILLUSTRATED WITH $n = 4$

$M_Q = M_F(a', b, c') - M_P(a, b, c)$ can be obtained by using Feynman rules to compute M_F and Eq. (1) to compute M_P. Since there are many terms in M_F and many terms in M_P, this computation turns out to be quite tedious even for $n = 4$. It is much worse for larger n.

Fortunately, with the following observation there is a much simpler way to compute M_Q. For on-shell gluons with transverse polarization, where $a = a'$ and $c = c'$, we know that M_P gives the correct Yang-Mills amplitude:

$$M_F(a', b, c') = M_P(a', b, c'). \tag{23}$$

For off-shell kinematics, the Feynman rules remain the same, so M_F is not changed. If we use Eq. (15) to convert a' and c' in M_F into a and c, then Eq. (23) implies that those terms without the presence of any off-shell parameter k_i^2, d_i must add up to give $M_P(a, b, c)$. The remaining terms which contain at least one off-shell parameter must add up to give M_Q. Thus M_Q can be computed just by extracting those terms in M_F that contain off-shell parameters.

Let us illustrate how to do that for $n = 4$. The Feynman amplitude M_F has an s-channel diagram with nine terms, a t-channel diagram with nine terms, and a four-gluon diagram with three terms. The four-gluon terms consist of products $b_{ij}b_{kl}$, where $(ijkl)$ is a permutation of (1234). Since neither a' nor c' enters, it cannot contribute to M_Q, so we will ignore it from now on.

The 18 s-channel and t-channel subdiagrams are given in Fig. 3.

Using the recipe given above, M_Q turns out to be

$$\begin{aligned} M_Q = &\left(\sum_{i=1}^{4} k_i^2\right)\left(\frac{b_{12}b_{34}}{s} + \frac{b_{41}b_{23}}{t}\right) \\ &- \left[\frac{b_{12}}{s}(d_3c_{43} + d_4c_{34}) + \frac{b_{41}}{t}(d_2c_{32} + d_3c_{23}) \right. \\ &\left. + \frac{b_{23}}{t}(d_1c_{41} + d_4c_{14}) + \frac{b_{34}}{s}(d_1c_{21} + d_2c_{12})\right], \end{aligned} \tag{24}$$

where $s = s_{12} = (k_1 + k_2)^2 = s_{34} = (k_3 + k_4)^2$ and $t = s_{41} = (k_4 + k_1)^2 = s_{23} = (k_2 + k_3)^2$.

Note that there are ten terms in Eq. (24) but 18 diagrams in Fig. 3, so some of those diagrams must not contribute to M_Q. To identify the diagrams that do not contribute to M_Q, let us first recall the meaning of the graphical components in subdiagrams. A line ending with a heavy dot (which we shall refer to as a "hammer") represents $c'_{il} - c'_{ir}$, with i on the handle and l and r to the left and right, respectively, of the hammer head (the heavy dot). If k_l or k_r is an internal momentum, it must be converted into the appropriate sum of external momenta, and c'_{il}, c'_{ir} are then the corresponding sum of c' between i and these external momenta. With a similar notation, a heavy dot at both ends of a line (which we shall call a "dumbbell") represents the factor $a'_{l_1l_2} - a'_{l_1r_2} - a'_{r_1l_1} + a'_{r_1r_2}$, where l_i and r_i represent the

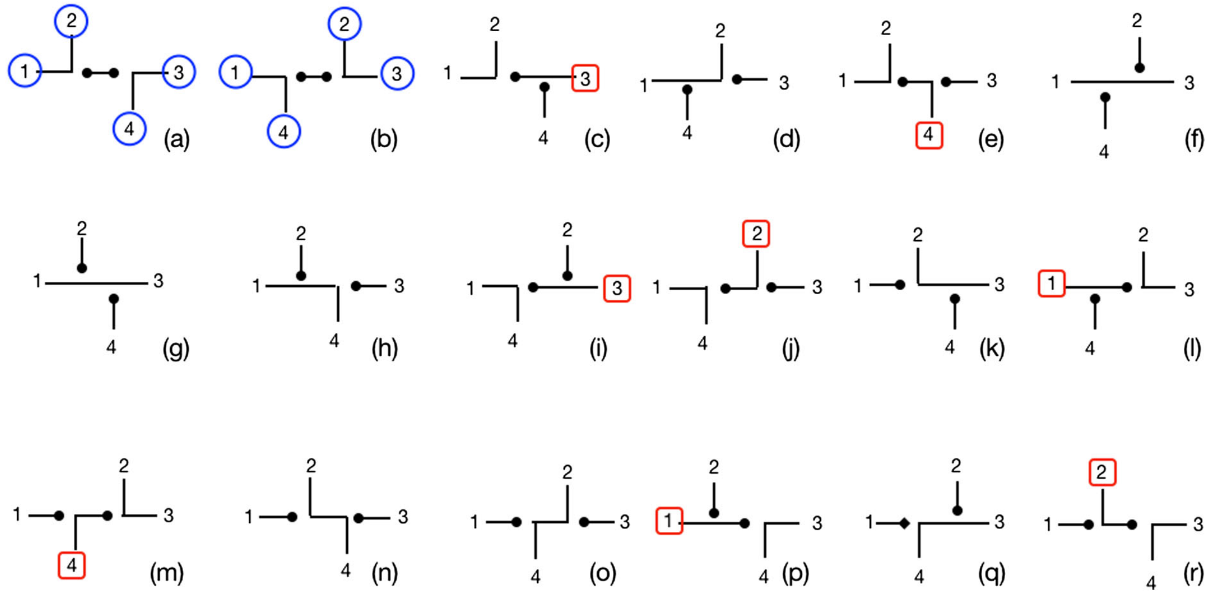

FIG. 3. The 18 s- and t-channel Feynman subdiagrams for $n = 4$. Line numbers enclosed by a box contribute to d_i, and line numbers enclosed by a circle contribute to k_i^2 in M_Q.

lines to the left and to the right, respectively, of the two dumbbells (heavy dots) $i = 1, 2$.

Graphically, the conversion equations (15), (18), and (19) say that $d_i/2$ appears at a hammer handle either when one and only one of its two neighboring lines appears in the hammer strike region, or, when both appear, they appear on the same side of the hammer head. For example, there are two hammers in Fig. 3(c), one at line 3 and one at line 4. The neighboring lines of 3 are 4 and 2; only one of them appears in the hammer strike region of 3, so d_3 appears. This is indicated in the diagram with a box around the number 3. The neighboring lines of 4 are 2 and 3; they appear in the hammer strike region of 4 on different sides, so d_4 does not enter, which is indicated in the diagram by the absence of a square box around the number 4. The emergence of k_i^2 in the dumbbell region, indicated by a circle around the line number, can be obtained similarly.

In this way we can see where d_i and k_i^2 appear in all the diagrams in Fig. 3. In particular, no d_i is present in Figs. 3(d), 3(f), 3(g), 3(h), 3(k), 3(n), 3(o), and 3(q), so these diagrams do not contribute to M_Q. The eight d_i terms in Eq. (24) come, respectively, from diagrams 3(c), 3(e), 3(j), 3(i), 3(l), 3(m), 3(p), and 3(r). Similar considerations applied to the dumbbell regions tell us where to put a circle to indicate the appearance of k_i^2.

Note that b_{13} comes from diagrams 3(f) and 3(g) and b_{24} comes from diagrams 3(n) and 3(o). The absence of these diagrams in M_Q is the reason why neither b_{13} nor b_{24} appears in Eq. (24).

Note also that M_Q is invariant under cyclic permutation. This should be the case because both M_F and M_P are invariant. When we permute Eq. (24) from (1234) to (2341), we get, for example,

$$\frac{b_{12}b_{34}}{s_{12}} \leftrightarrow \frac{b_{23}b_{41}}{s_{23}},$$
$$\frac{b_{12}}{s_{12}}(d_3c_{43}+d_4c_{34}) \to \frac{b_{23}}{s_{23}}(d_4c_{14}+d_1c_{41}),$$
$$\frac{b_{41}}{s_{41}}(d_2c_{32}+d_3c_{23}) \to \frac{b_{12}}{s_{12}}(d_3c_{43}+d_4c_{34}), \text{etc.},$$

showing explicitly that Eq. (24) is cyclic permutation invariant.

It is amusing to find out whether M_Q can be written in the form of Eq. (1). Namely, whether there exists a Möbius covariant function $Q = Q(A_{ij}, B_{ij}, C_{ij}, d_i, k_i^2)$ which transforms with a weight factor $(\lambda_1\lambda_2\lambda_3\lambda_4)^{-2}$, such that

$$M_Q = \left(-\frac{2g}{2\pi i}\right)^{n-3} \oint_\Gamma \frac{\sigma_{(pqr)}^2}{\sigma_{(12\ldots n)}} \left(\prod_{i=1, i\neq p,q,r}^{n} \frac{d\sigma_i}{f_i}\right) Q. \tag{25}$$

Since the dependence of M_Q on a, b, c is assumed to arise from the dependence of Q on A, B, C, it is clear from Eq. (24) that, if such a Q exists, it must be

$$\begin{aligned} Q = \Bigg[&\left(\sum_{i=1}^{4} k_i^2\right)\left(\frac{b_{12}b_{34}}{\sigma_{12}\sigma_{34}} + \frac{b_{14}b_{23}}{\sigma_{14}\sigma_{23}}\right) - \frac{b_{12}}{\sigma_{12}}\left(-d_3\frac{c_{43}}{\sigma_{43}} + d_4\frac{c_{34}}{\sigma_{34}}\right) \\ &- \frac{b_{14}}{\sigma_{14}}\left(-d_2\frac{c_{32}}{\sigma_{32}} + d_3\frac{c_{23}}{\sigma_{23}}\right) - \frac{b_{23}}{\sigma_{23}}\left(-d_4\frac{c_{14}}{\sigma_{14}} + d_1\frac{c_{41}}{\sigma_{41}}\right) \\ &- \frac{b_{34}}{\sigma_{34}}\left(-d_1\frac{c_{21}}{\sigma_{21}} + d_2\frac{c_{12}}{\sigma_{12}}\right)\Bigg] \frac{1}{\sigma_{31}\sigma_{24}}. \end{aligned} \tag{26}$$

The extra factor $1/\sigma_{31}\sigma_{24}$ outside of the square brackets is there to enable Q to transform with the correct covariant weight, and the signs of the various terms are needed to ensure M_Q to be reproduced after the σ integrations. With this Q, it turns out that M_Q computed using Eq. (25) is indeed the correct M_Q given by Eq. (24).

Although Q exists for $n = 4$, Möbius invariance cannot determine its form nor that of M_Q, so its existence is merely of academic interest. Unlike P, where Möbius invariance, permutation symmetry, and dimensional analysis largely determine what it should be, nothing similar is available for Q. For example, without the Feynman diagrams and the discussion earlier in this section, there is no way even to know that neither B_{13} nor B_{24} is present in Q. For that reason we shall no longer discuss Q from now on.

VI. LOCAL GAUGE INVARIANCE

A. Slavnov-Taylor identity

The emergence of M_Q can be traced back to local gauge invariance, the hallmark of Yang-Mills theory. An amplitude possessing local gauge invariance must satisfy the Slavnov-Taylor identity [32,33], which relates the divergence of an n-gluon Green's function to the Green's function with $(n-2)$ gluons and a ghost-antighost pair:

$$-\frac{\partial}{\partial x_i^{\mu_i}}\langle A_{\mu_1}^{a_1}(x_1)A_{\mu_2}^{a_2}(x_2)\ldots A_{\mu_n}^{a_n}(x_n)\rangle = \sum_{k\neq i}\langle\bar{\omega}^{a_i}(x_i)A_{\mu_2}^{a_2}(x_2)\ldots D_{\mu_k}\omega^{a_k}(x_k)\ldots A_{\mu_n}^{a_n}(x_n)\rangle. \quad (27)$$

A is the gluon field, ω and $\bar{\omega}$ are the ghost and antighost fields, respectively, and $(D_\mu\omega)^a = \partial_\mu\omega^a + gf_{abc}A_\mu^b\omega^c$ is the covariant derivative of the ghost field. The corresponding relation for color-stripped amplitudes is shown in Fig. 4, where solid lines are gluons and dotted lines are ghosts. A cross ($\times$) at line j represents the factor $d_j = \epsilon_j\cdot k_j$, and a box (■) at line j represents the factor k_j^2. The cross comes from the derivative of the ghost field, and the box is there to amputate the external leg in the $A\omega$ term of $D\omega$.

In tree order, this relation can be derived directly from the gluon tree amplitude by replacing ϵ_i in a gluon line by k_i [34]. Let us illustrate how that is done for $n = 3$ and $i = 2$.

Using the notation $\delta_i(\mathcal{O})$ to indicate replacing ϵ_i in $\mathcal{O}$ by k_i, we get from Eq. (20) that

$$\begin{aligned}\delta_2(V) &= \epsilon_1\cdot k_2\epsilon_3\cdot(k_1-k_2) + k_2\cdot\epsilon_3\epsilon_1\cdot(k_2-k_3) \\ &\quad + \epsilon_3\cdot\epsilon_1 k_2\cdot(k_3-k_1) \\ &= -\epsilon_1\cdot k_1\epsilon_3\cdot k_1 + \epsilon_1\cdot k_3\epsilon_3\cdot k_3 + k_1^2\epsilon_1\cdot\epsilon_3 \\ &\quad - k_3^2\epsilon_1\cdot\epsilon_3, \end{aligned} \quad (28)$$

where momentum conservation has been used to obtain the second line. These four terms are depicted by the four diagrams in Fig. 5, where Figs. 5(a) and 5(b) correspond to the first diagram on the right of Fig. 4, respectively, for $j = 1$ and $j = 3$, and Figs. 5(c) and 5(d) correspond to the second diagram. The $\epsilon_3\cdot k_1$ factor in the first term comes from the gluon-ghost vertex in 5(a). The minus signs came from color ordering before color is stripped.

What is important for our subsequent discussion is that $\delta_i(M)$ for a local gauge-invariant amplitude M consists of terms proportional to d_j and k_j^2 for all $j \neq i$, but it does not contain terms involving k_i^2 in leading order of the off-shell parameters. We shall refer to this absence of k_i^2 as *partial gauge invariance*. It turns out that neither M_P nor M_Q is

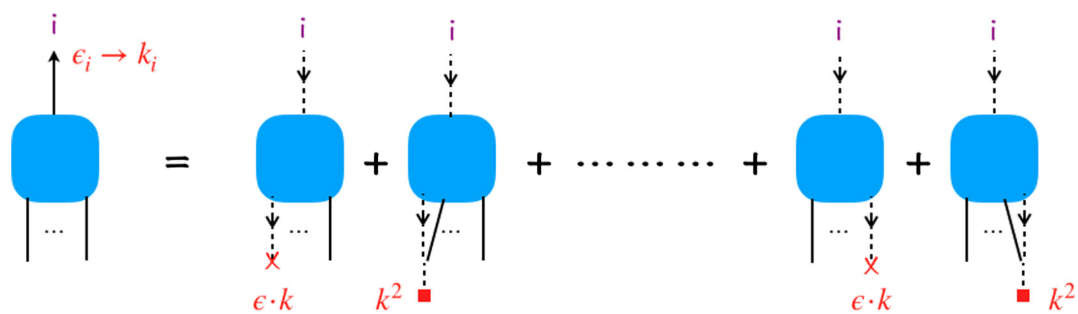

FIG. 4. The Slavnov-Taylor relation relating the divergence of a gluon amplitude to the covariant derivative on the ghost lines of gluon-ghost amplitudes.

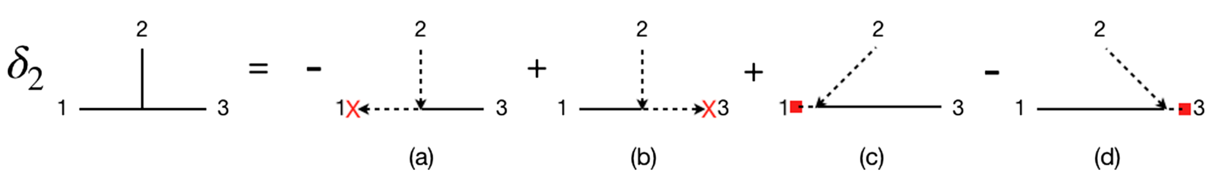

FIG. 5. The Slavnov-Taylor identity for $n = 3$.

locally gauge invariant, though their sum is, but both have partial gauge invariance. This property is useful in checking the calculations of M_P and M_Q and puts a constraint on the allowed forms of M_P and M_Q.

B. M_P does not have local gauge invariance but it is partially gauge invariant

Let us compute $\delta_2(\Psi_{13}^{13})$ to see whether $\delta_2(M_P)$ satisfies the Slavnov-Taylor identity. The change $\epsilon_2 \to k_2$ leads to $c'_{2j} \to a'_{2j}, b_{2j} = b_{j2} \to c'_{j2}$, which in turn leads to a change of Ψ_{13}^{13} in the (nth) row and column containing C_{2j} and B_{2j}. These changes are given by

$$\begin{aligned}
&\delta_2 d_2 = k_2^2,\\
&\delta_2 b_{2j} = \delta_2 b_{j2} = c'_{j2} = c_{j2} - \frac{1}{2} d_j \quad (j = 1, 3),\\
&\delta_2 b_{2j} = \delta_2 b_{j2} = c'_{j2} = c_{j2} \quad (j \neq 1, 2, 3),\\
&\delta_2 c_{2j} = \delta_2 c'_{2j} + \frac{1}{2} d_2 = a'_{2j} + \frac{1}{2} d_2 = a_{2j} - \frac{1}{2} k_j^2 \quad (j = 1, 3),\\
&\delta_2 c_{24} = \delta_2 c'_{24} = a'_{24} = a_{24} + \frac{1}{2} k_3^2,\\
&\delta_2 c_{2n} = c'_{2n} = a'_{2n} = a_{2n} + \frac{1}{2} k_1^2,\\
&\delta_2 c_{2j} = \delta_2 c'_{2j} = a'_{2j} = a_{2j} \quad (j \neq 1, 2, 3, 4, n).
\end{aligned} \tag{29}$$

All other elements of b_{ij}, c_{ij}, d_i, and all elements of a_{ij} remain the same.

We shall compute $\delta_2(M_P)$ using the property that subtracting the nth row (column) from the first row (column) of $\delta_2(\Psi_{13}^{13})$ does not change its Pfaffian. The first row of Ψ_{13}^{13} consists of

$$(0, A_{24}, A_{25}, \ldots, A_{2,n-1}, A_{2n}, -C_{12}, -C_{22}, -C_{32}, -C_{42}, \ldots, -C_{n2}),$$

none of which is affected by δ_2 except $-C_{22}$,

$$-\delta_2 C_{22} = \sum_{j\neq 2} \frac{\delta_2 c_{2j}}{\sigma_{2j}} = \left(\sum_{j\neq 2} A_{2j}\right) - \frac{1}{2} k_1^2 \left(\frac{1}{\sigma_{21}} - \frac{1}{\sigma_{2n}}\right) - \frac{1}{2} k_3^2 \left(\frac{1}{\sigma_{23}} - \frac{1}{\sigma_{24}}\right). \tag{30}$$

The nth row of Ψ_{13}^{13} consists of

$$(C_{22}, C_{24}, C_{25}, \ldots, C_{2,n-1}, C_{2n}, B_{21}, 0, B_{23}, B_{24}, \ldots, B_{2n}),$$

which under δ_2 is changed into

$$\begin{aligned}
&\left(\delta_2 C_{22}, \frac{\delta_2 c_{24}}{\sigma_{24}}, \frac{\delta_2 c_{25}}{\sigma_{25}}, \ldots, \frac{\delta_2 c_{2,n-1}}{\sigma_{2,n-1}}, \frac{\delta_2 c_{2n}}{\sigma_{2n}}, \frac{\delta_2 b_{21}}{\sigma_{21}}, 0, \frac{\delta_2 b_{23}}{\sigma_{23}}, \frac{\delta_2 b_{24}}{\sigma_{24}}, \ldots, \frac{\delta_2 b_{2n}}{\sigma_{2n}}\right)\\
&= (\delta_2 C_{22}, \hat{A}_{24}, A_{25}, \ldots, A_{2,n-1}, \hat{A}_{2n}, 0, -\hat{C}_{12}, 0, -\hat{C}_{32}, -C_{42}, \ldots, -C_{n2}),
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
\hat{A}_{24} &= A_{24} + \frac{1}{2} \frac{k_3^2}{\sigma_{24}},\\
\hat{A}_{2n} &= A_{2n} + \frac{1}{2} \frac{k_1^2}{\sigma_{2n}},\\
\hat{C}_{12} &= C_{12} - \frac{1}{2} \frac{d_1}{\sigma_{12}},\\
\hat{C}_{32} &= C_{32} - \frac{1}{2} \frac{d_3}{\sigma_{32}}.
\end{aligned} \tag{32}$$

Subtracting the nth row (column) from the first row (column) changes the first row into

$$-\frac{1}{2}\left(0, \frac{k_3^2}{\sigma_{24}}, 0, \ldots, 0, \frac{k_1^2}{\sigma_{2n}}, \frac{d_1}{\sigma_{21}}, 2\delta_2 C_{22}, \frac{d_3}{\sigma_{23}}, 0, \ldots, 0\right), \tag{33}$$

and the first column into the same thing with a minus sign, leaving the rest of $\delta_2(\Psi_{13}^{13})$ unchanged. The modified matrix contains only off-shell parameters d_j, k_j^2 in the first row (column), so every term in $\mathrm{Pf}(\delta_2(\Psi_{13}^{13}))$, and thus every term in $\delta_2(M_P)$, must be proportional to an off-shell parameter. Thus

(1) $\delta_2(M_P) = 0$ for on-shell gluons with transverse polarization, as we already know;
(2) k_2^2 and all d_j, k_j^2 for $j \geq 4$ are missing from $\delta_2(M_P)$, and hence M_P cannot satisfy the Slavnov-Taylor identity in which all k_j^2 and d_j for $j \neq 2$ must be present. This is why M_Q is needed to restore local gauge invariance of the amplitude;
(3) M_P is invariant under permutation of the particles, and thus if k_2^2 is absent from $\delta_2(M_P)$, k_i^2 must be absent from $\delta_i(M_P)$. By definition, M_P has partial gauge invariance;
(4) since both M_F and M_P have partial gauge invariance, M_Q must also have partial gauge invariance.

C. Partial gauge invariance of M_Q for $n = 4$

Partial gauge invariance is a useful tool for verifying calculations. Together with cyclic permutation invariance, it provides a nontrivial constraint on the allowed forms of M_Q. Let us illustrate these points with $n = 4$.

For convenience, Eq. (24) of M_Q for $n = 4$ is reproduced below:

$$M_Q = \left(\sum_{i=1}^{4} k_i^2\right)\left(\frac{b_{12}b_{34}}{s} + \frac{b_{41}b_{23}}{t}\right) - \left[\frac{b_{12}}{s}(d_3c_{43} + d_4c_{34}) + \frac{b_{41}}{t}(d_2c_{32} + d_3c_{23}) + \frac{b_{23}}{t}(d_1c_{41} + d_4c_{14}) + \frac{b_{34}}{s}(d_1c_{21} + d_2c_{12})\right].$$

Let us use it to verify partial gauge invariance. Since $\delta_2(d_2) = k_2^2$,

$$\delta_2(M_Q) = k_2^2\left[\left(\frac{c'_{12}b_{34}}{s} + \frac{c'_{32}b_{41}}{t}\right) - \left(\frac{b_{41}c_{32}}{t} + \frac{b_{34}c_{12}}{s}\right)\right] + \cdots = -\frac{1}{2}k_2^2\left[\frac{d_1b_{34}}{s} + \frac{d_3b_{41}}{t}\right] + \cdots,$$

where the ellipses represent terms without k_2^2. Thus the k_2^2 coefficient of $\delta_2(M_Q)$ vanishes in the zeroth order of the off-shell parameters. Similarly, the k_i^2 coefficients of the other $\delta_i(M_Q)$ also vanish in the zeroth order, thereby verifying that M_Q possesses partial gauge invariance.

Next, to illustrate the power of partial gauge invariance, we will use it to constrain the possible dependence of M_Q. For simplicity, let us assume the absence of b_{13} and b_{24}. On dimensional grounds, each term of M_Q must contain ϵ_1, ϵ_2, ϵ_3, ϵ_4 once and k twice in the numerator. The denominator could be either $s = s_{12} = s_{34}$ or $t = s_{41} = s_{23}$. The numerator must also contain at least one off-shell parameter; therefore, its allowed forms are confined to $b_{ij}b_{kl}k_m^2$ and $b_{ij}c_{kp}d_l$, with $(ijkl)$ being a permutation of (1234).

With b_{13} and b_{24} absent, (ij) in these terms must be either (12) or (34). First consider the term $b_{12}b_{34}k_m^2/s_{12}$. Since M_Q is cyclic permutation invariant, M_Q must consist of the combination

$$\alpha\left[\frac{b_{12}b_{34}}{s_{12}}k_m^2 + \frac{b_{23}b_{41}}{s_{23}}k_{m+1}^2 + \frac{b_{34}b_{12}}{s_{34}}k_{m+2}^2 + \frac{b_{41}b_{23}}{s_{41}}k_{m+3}^2\right] = \alpha\left[\frac{b_{12}b_{34}}{s}(k_m^2 + k_{m+2}^2) + \frac{b_{23}b_{41}}{t}(k_{m+1}^2 + k_{m+3}^2)\right]. \tag{34}$$

Under δ_i, to leading order b_{ij} turns into c_{ji}, so in order to have partial gauge invariance, the bcd terms in M_Q must be the following if $m = 1$ or 3:

$$-\frac{\alpha}{s}[b_{34}c_{21}d_1 + c_{43}b_{12}d_3] - \frac{\alpha}{t}[b_{41}c_{32}d_2 + b_{23}c_{14}d_4].$$

Applying a similar argument to the case when $m = 2$ or 4, and to the situations when the starting denominator is t rather than s, we conclude that M_Q must be equal to

$$\begin{aligned} M_Q = {} & \frac{\alpha_1}{s}[b_{12}b_{34}(k_1^2 + k_3^2) - b_{34}c_{21}d_1 - c_{43}b_{12}d_3] + \frac{\alpha_1}{t}[b_{23}b_{41}(k_2^2 + k_4^2) - b_{41}c_{32}d_2 - b_{23}c_{14}d_4] \\ & + \frac{\alpha_2}{s}[b_{12}b_{34}(k_2^2 + k_4^2) - b_{34}c_{21}d_1 - c_{43}b_{12}d_3] + \frac{\alpha_2}{t}[b_{23}b_{41}(k_1^2 + k_3^2) - b_{41}c_{32}d_2 - b_{23}c_{14}d_4] \\ & + \frac{\alpha_3}{t}[b_{12}b_{34}(k_1^2 + k_3^2) - b_{34}c_{21}d_1 - c_{43}b_{12}d_3] + \frac{\alpha_3}{s}[b_{23}b_{41}(k_2^2 + k_4^2) - b_{41}c_{32}d_2 - b_{23}c_{14}d_4] \\ & + \frac{\alpha_4}{t}[b_{12}b_{34}(k_2^2 + k_4^2) - b_{34}c_{21}d_1 - c_{43}b_{12}d_3] + \frac{\alpha_4}{s}[b_{23}b_{41}(k_1^2 + k_3^2) - b_{41}c_{32}d_2 - b_{23}c_{14}d_4]. \end{aligned}$$

The result agrees with Eq. (24) if we set $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = \alpha_4 = 0$.

VII. $n \geq 5$ AMPLITUDES

A. Organization of Feynman diagrams

Amplitudes of large n contain many Feynman diagrams, and each contains many terms. These terms can be organized in the following way.

A Feynman diagram without a four-gluon vertex contains n polarization vectors, $(n-2)$ triple-gluon vertices, and $(n-3)$ propagators, giving rise to a numerator of the form $b_{i_1i_2}b_{i_3i_4}\ldots b_{i_{2k-1},i_{2k}}c'_{i_{2k+1}j_{2k+1}}\ldots c'_{i_nj_n}a'_{j_1j_2}\ldots a'_{j_{2k-3}j_{2k-2}}$, where $I = (i_1i_2\ldots i_n)$ is a permutation of $(12\ldots n)$. Terms with different j_m's can mix through momentum conservation, but there is no way to combine terms with different k or different I; thus, it is useful to group together terms with the same k and I. A Feynman diagram contains terms with different k's and I's, but each of its subdiagrams contains a fixed k and a fixed I.

If four gluon vertices are present, each vertex simply eliminates a propagator and a pair of k's in the numerator.

For on-shell amplitudes, $M_F(a', b', c') = M_P(a', b', c')$. Instead of using Feynman rules and Feynman diagrams, the amplitude can also be computed using Pfaffian diagrams obtained from Eq. (1) [35,36]. Like the Feynman subdiagrams, each Pfaffian diagram has a fixed k and a unique I

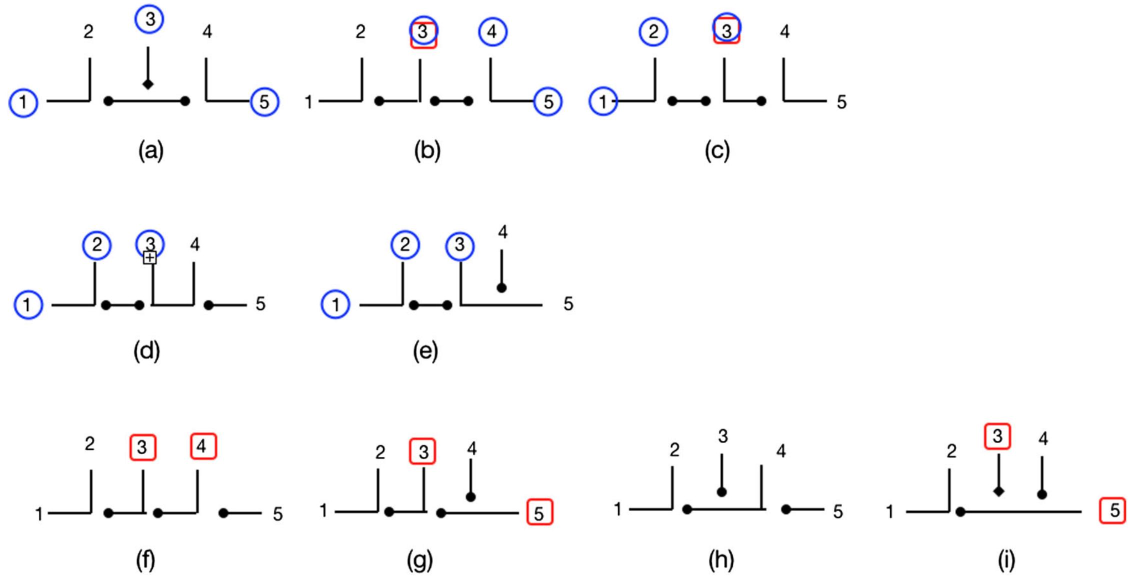

FIG. 6. Subdiagrams contributing to $b_{12}/s_{12}s_{45}$ terms of M_Q for $n = 5$.

structure, but unlike Feynman subdiagrams, Pfaffian diagrams do not contain internal momenta, so the necessity of expanding internal momenta into sums of external momenta is avoided, thereby resulting in fewer terms at the end [35,36].

For off-shell amplitudes, the decomposition $M_F = M_P + M_Q$ again results in fewer terms. M_P can be computed using Pfaffian diagrams as before, simply by replacing a' with a and c' with c. The computation of M_Q is relatively simple because many Feynman diagrams do not contribute to M_Q, and for those that do only some off-shell parameters appears. Furthermore, partial gauge invariance can be used to check the calculation. Thus both on shell and off shell, there is an advantage to use the CHY formalism to compute Yang-Mills amplitudes. It results in having fewer terms at the end.

We now illustrate the computation of part of M_Q for $n = 5$ and how partial gauge invariance can be used to check this calculation.

B. M_Q for $n = 5$

Figure 6 shows all the subdiagrams that contribute to terms proportional to $b_{12}/s_{12}s_{45}$. When d_i appears in a subdiagram, its i is surrounded by a square. When k_i^2 appears, its i is surrounded by a circle. For example, no line in subdiagram (h) has a square or a circle, so that diagram carries no off-shell parameter and does not contribute to M_Q. Lines 4 and 5 in Figs. 6(d) and 6(e) are not surrounded by a circle so k_4^2 and k_5^2 are not present in the M_Q of these diagrams.

The contributions to M_Q from diagrams 6(a)–6(c) are

$$
\begin{aligned}
-\frac{1}{2}b_{12}b_{45}[&(a_{13}-2a_{14}-a_{23}+2a_{25}+a_{34}-a_{35})d_3\\
&+(-6c_{34}-2c_{35}+d_3)k_1^2+(-2c_{34}+2c_{35}+d_3)k_2^2\\
&+(4c_{32}+2c_{34}+6c_{35})k_3^2+(4c_{32}+2c_{34}+2c_{35}-d_3)k_4^2\\
&+(4c_{32}-2c_{34}-2c_{35}-d_3)k_5^2], \qquad (35)
\end{aligned}
$$

and the contributions from diagrams 6(f), 6(g), and 6(i) are

$$
\begin{aligned}
\frac{1}{2}b_{12}[&4(c_{54}c_{43}-c_{45}c_{53})d_3+c_{54}(-2c_{31}+2c_{32}-d_3)d_4\\
&+c_{45}(2c_{31}+6c_{32}-d_3)d_5]. \qquad (36)
\end{aligned}
$$

Let us use these expressions to verify partial gauge invariance, which demands $\delta_i(M_Q)$ to contain no k_i^2 term in the zeroth order. This means that after we make the replacements $b_{ij} \to c_{ji}, c_{ij} \to a_{ji}, d_i \to k_i^2$, the coefficient of k_i^2 in M_Q without any off-shell parameters must be identically zero. This is true for all b_{ij} and all propagators, so those terms proportional to the same product of b with the same propagator in $\delta_i(Q)$ must be identically zero in the zeroth order as well.

The factor b_{12} in Fig. 6 will not be altered by $\delta_i(M_Q)$ only for $i = 3$, 4, 5, so without including more diagrams, we can only verify partial gauge invariance from Fig. 6 for $i = 3$, 4, 5. Diagrams 6(d) and 6(e) do not contain k_4^2 and k_5^2, so they can be ignored for the verification of $i = 4$ and $i = 5$. It is then easy to see from Eqs. (35) and (36) that partial gauge invariance is indeed valid for these two i's.

If we concentrate on terms of M_Q proportional to $b_{12}b_{45}/s_{12}s_{45}$, only diagrams 6(a)–6(c) contribute and only Eq. (35) is relevant. After applying δ_3 to it, the leading coefficient of $-\frac{1}{2}k_3^2 b_{12}b_{45}/s_{12}s_{45}$ is seen to be

$$
\begin{aligned}
&(a_{13}-2a_{14}-a_{23}+2a_{25}+a_{34}-a_{35})+(4a_{23}+2a_{43}+6a_{53})\\
&=2(a_{12}-a_{45})=s_{12}-s_{45},
\end{aligned}
$$

where $\sum_{j\neq i} a_{ij} = 0$ of Eq. (10), and the relations $2a_{12} = s_{12}, 2a_{45} = s_{45}$, have been used. Since the propagator for this term is $1/s_{12}s_{45}$, the resulting numerator above cancels one factor of the propagator, leaving the coefficient of the double pole to be zero, so the leading coefficient of $k_3^2 b_{12} b_{45}/s_{12}s_{45}$ is indeed zero, as demanded by partial gauge invariance.

VIII. CONCLUSION

It is difficult for an S-matrix theory to incorporate interaction because it knows nothing about the local space-time structure. An exception is the CHY theory, which with the guide of Möbius invariance is able to reproduce massless tree amplitudes for ϕ^3, Yang-Mills, gravity, and many other theories. In this article we investigated whether this invariance can also guide us to construct the correct off-shell amplitudes. For ϕ^3 interaction, we know that it is possible. For the Yang-Mills theory considered here, it turns out that the modified off-shell CHY amplitude M_P with Möbius invariance is not locally gauge invariant and therefore is not the correct Yang-Mills amplitude. A complementary amplitude M_Q must be added to restore local gauge invariance, but Möbius invariance is no longer a useful guide to its construction. Although neither M_P nor M_Q is locally gauge invariant, both are partially gauge invariant, a useful property that can be used to verify calculations and to simplify the Yang-Mills amplitude in the way discussed in the last section.

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