



# Universality class of alternative phase space and Van der Waals criticality



Amin Dehyadegari<sup>a</sup>, Bibhas Ranjan Majhi<sup>c</sup>, Ahmad Sheykhi<sup>a,b,\*</sup>, Afshin Montakhab<sup>a</sup>

<sup>a</sup> Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran

<sup>b</sup> Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), P.O. Box 55134-441, Maragha, Iran

<sup>c</sup> Department of Physics, Indian Institute of Technology Guwahati, Guwahati 781039, Assam, India

## ARTICLE INFO

### Article history:

Received 14 October 2018

Received in revised form 12 February 2019

Accepted 18 February 2019

Available online 20 February 2019

Editor: N. Lambert

### Keywords:

Universality

Phase space

Van der Waals

Transition criticality

## ABSTRACT

A new perspective toward thermodynamic phase space of Reissner-Nordstrom (RN) black holes in an anti-de-Sitter (AdS) spaces was recently proposed [1], where the square of the electric charge ( $Q^2$ ) of black hole was regarded as a thermodynamic variable and the cosmological constant (pressure) as a fixed quantity. In this paper, we address the universality class and critical properties of any AdS black hole in this alternative phase space. We disclose the critical behavior of AdS black hole in the alternative phase space in which a continuous phase transition happens and in a very general framework, independent of the spacetime metric. Based on the expansion of the equation of state and Landau thermodynamic potential in the neighborhood of a critical point in the alternative phase space, we confirm that the set of values for critical exponents for generic black hole is analogous to the Van der Waals fluid system. Finally, we reveal that the scalar curvature in geometry thermodynamic diverges at the critical point of black hole. Our study shows that the approach here is powerful enough to investigate the critical behavior of any black holes and further supports the viability of the alternative viewpoint toward phase space of black holes suggested in [1].

© 2019 Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP<sup>3</sup>.

## 1. Introduction

Thermodynamic phase transitions are one of the most intriguing phenomena in black hole physics which can provide insight into underlying structure of spacetime geometry. In particular, phase transition of anti-de-Sitter (AdS) black holes has received much attention since the discovery of the correspondence between gravity in an AdS spacetime and the conformal field theory (CFT) living on its boundary. The pioneering work in this regard was performed by Hawking and Page [2], who demonstrated the existence of a first-order phase transition in the Schwarzschild AdS black hole. According to the AdS/CFT duality, this phase transition can be interpreted as a confinement-deconfinement transition in the quark gluon plasma [3]. Later, the analogy between the small-large black hole phase transition and the liquid-gas Van der Waals

phase transition was reported in [4], where RN black hole was considered in asymptotic AdS spacetime. Later on, it was revealed that this similarity happens in the extended phase space of the RN-AdS black hole in which the cosmological constant ( $\Lambda$ ) is regarded as a thermodynamic variable corresponding to the thermodynamic pressure with a black hole's volume as a conjugate quantity [5, 6]. In the extended phase space, the cosmological constant as a dynamical quantity can take on arbitrary values in the first law of black hole thermodynamics where the mass of AdS black hole is interpreted as the enthalpy [7]. In recent years, various studies on black holes phase transition in an extended phase space have been carried out such as zeroth-order phase transition [8], reentrant phase transition [9] as well as superfluid-like phase transition [10] as well as study of triple points [11]. For more details, we refer to [12] and references therein.

Criticality is an interesting topic in phase transition context because thermodynamic quantities of the system show non-analytic behavior as one approaches the critical point, where a continuous phase transition occurs. This non-analytic behavior is expressed in the terms of power law functions which are governed by the critical exponents. The set of critical exponents define the universality

\* Corresponding author.

E-mail addresses: [amindehyadegari@gmail.com](mailto:amindehyadegari@gmail.com) (A. Dehyadegari), [bibhas.majhi@iitg.ac.in](mailto:bibhas.majhi@iitg.ac.in) (B.R. Majhi), [asheykhi@shirazu.ac.ir](mailto:asheykhi@shirazu.ac.ir) (A. Sheykhi), [montakhab@shirazu.ac.ir](mailto:montakhab@shirazu.ac.ir) (A. Montakhab).

class of a system, which are thought to obey the same symmetry principles. In case of charged AdS black hole, a continuous (second order) phase transition occurs between small-large black hole in an extended phase space [6]. Furthermore, the critical exponents associated with this transition are the same as those in the Van der Waals liquid-gas transition, i.e., both systems belong to the same universality class [6]. Critical behavior and universality class of AdS black holes in the extended phase space have been investigated in various spacetimes [13,14]. Recently, authors of Ref. [15] have shown that if there is a critical point in an extended phase space of general AdS black hole, the small-large black hole transition is in the Van der Waals universality class. Another interesting approach towards critical phenomena of AdS black hole is through variation of the electric charge of black hole in a fixed AdS background geometry, i.e. keeping the cosmological constant (pressure) as a fixed parameter. In this perspective, the charge of black hole is treated as a natural thermodynamic variable that can take on arbitrary values in the thermodynamic process. In this view, the thermodynamic behavior of AdS black hole is analysed in a thermodynamics phase space. The critical point and associated critical exponents in a phase space of black hole have been studied in a general way [16]. It was found that the values of critical exponents differ from those of the Van der Waals phase transition [16]. All these phase transitions are also discussed from the thermogeometrical point of view [17] where the phase transition is identified as the divergence of the Ricci scalar of the thermodynamic geometry. In additional, an interesting phenomenon of black hole reentrant phase transition has also been investigated in four-dimensional Born-Infeld AdS black in which the charge of system can vary and the cosmological constant is fixed [18]. In the context of Gauss-Bonnet-AdS, Van der Waals-like phase transition has been studied with a spherical horizon in the framework of holography [19]. It was shown that the nonlocal quantum objects can be used to determine the phase structure of Gauss-Bonnet-AdS black holes [19].

On the other hand, an alternative view of phase space has been proposed more recently where various thermodynamic quantities find more natural and physical meaning [1]. In this view, the square of the electric charge of black hole is considered as a thermodynamic variable and  $\Psi = 1/2r_+$  is subsequently considered as its conjugate, where  $r_+$  is the horizon radius. It was indeed shown that in this alternative phase space, phase transition and critical behavior of RN-AdS black hole, in four dimensions, occur in the  $Q^2$ - $\Psi$  plane, where relevant response function clearly signifies stable and unstable region [1]. Remarkably, in this viewpoint, the small-large black hole phase transition is quite similar to the Van der Waals liquid-gas system and belongs to the same universality class in contrast with previous study of [16]. Additionally, it would be interesting to study the universality class and critical properties for any AdS black hole in an alternative phase space where the cosmological constant ( $\Lambda$ ) is taken to be constant. Hence, in this paper, we provide a general framework, i.e. a metric independent way, for investigating the critical behavior of AdS black hole in the above-mentioned alternative phase space approach in which a continuous phase transition happens. In this analysis, the existence of the phase transition with respect to the alternative phase space variables is assumed to be there. Based on the expansion of the equation of state and Landau thermodynamic potential in the neighborhood of a critical point in the alternative phase space, we find that a set value of critical exponents for generic black hole is analogous to usual Van der Waals system. Also, we show that the scalar curvature in thermogeometric picture diverges at the critical point of black hole in an alternative phase space.

The present Article is structured as follows: In the next section, we obtain the critical exponents by using the equation of state of

black hole in a general scheme. In section 3, by considering the Landau thermodynamic potential, we study universality properties of black hole at phase transition. In section 4, we investigate thermodynamic geometry of the system at critical point in alternative phase space. The last section is devoted to concluding remarks.

## 2. Equation of state: general approach

Here, we intend to prove that the critical exponents of a continuous (second-order) phase transition are independent of the metric function (black hole) in an alternative phase space where  $Q^2$  is treated as a thermodynamic quantity and its conjugate is  $\Psi$  as proposed in [1]. For this purpose, the first law of black hole thermodynamics, for constant pressure, is written as<sup>1</sup>

$$dM = TdS + \Psi dq, \quad (1)$$

where  $q = Q^2$ . Here  $M$ ,  $S$  and  $T$  are the total mass, entropy and Hawking temperature of black hole, respectively. In general, the entropy only depends on the event horizon,  $r_+$ , i.e.  $S = S(r_+)$ . The Gibbs free energy is a thermodynamic potential that can be calculated by Legendre transform of the above equation

$$dG = -SdT + \Psi dq, \quad (2)$$

where  $G = G(T, q)$ . Thermodynamics of black hole may be described by equation of state  $q = q(\Psi, T)$  where  $q$  depends on  $\Psi$  and  $T$ . As we know, the critical point in  $q - \Psi$  plane is characterized by [1]

$$\left. \frac{\partial q}{\partial \Psi} \right|_{T_c} = 0, \quad \left. \frac{\partial^2 q}{\partial \Psi^2} \right|_{T_c} = 0, \quad (3)$$

where the subscript  $c$  refers to the critical point and critical quantities are  $T_c$ ,  $\Psi_c$  and  $q_c$ . The behavior of thermodynamic functions near the critical point is identified by the critical exponents which are defined for a Van der Waals system as [20]

$$\begin{aligned} C &\sim |T - T_c|^{-\alpha}, \\ |v_l - v_g| &\sim |T - T_c|^\beta, \\ P - P_c &\sim |v_l - v_g|^\delta, \\ \chi_T &= -\frac{1}{v} \left. \frac{\partial v}{\partial P} \right|_T \sim |T - T_c|^{-\gamma}. \end{aligned}$$

The first equation defines the exponent for thermal response function  $C$  (heat capacity), the second equation characterizes the non-analytic behavior of order parameter at the critical point, the third defines the critical isotherm, and the last equation defines the singularity in (mechanical) response function,  $\chi_T$  (isothermal compressibility). Here,  $v$  and  $P$  are specific volume and pressure, respectively, which define the thermodynamic phase space. Now, in order to find the critical exponents for a black hole, we expand  $q$  around the critical point

$$\begin{aligned} q(\Psi, T) &= a_{00} + a_{01}(T - T_c) + a_{02}(T - T_c)^2 \\ &\quad + a_{11}(T - T_c)(\Psi - \Psi_c) + a_{12}(T - T_c)^2 \\ &\quad \times (\Psi - \Psi_c) + a_{30}(\Psi - \Psi_c)^3 + \dots, \end{aligned} \quad (4)$$

<sup>1</sup> In the case of Gauss-Bonnet gravity, the first law of black hole thermodynamics is extended by considering the variation of Gauss-Bonnet coefficient [21]. Since we study critical behavior through the variation of charge only, the variation of the Gauss-Bonnet term in the first law is neglected. Hence, the main result of this paper does not change in the present of Gauss-Bonnet term.

where we have used the convention

$$a_{ij} \equiv (i!j!)^{-1} \partial^{i+j} q / \partial \Psi^i \partial T^j \Big|_{\Psi_c, T_c},$$

and use has been made of Eq. (3). It is worth noting that for function  $f = f(x, y)$  where  $x, y$  are independent variables, the Taylor series expansion around a point  $x = x_0, y = y_0$  will have the following form:

$$f(x, y) = f(x_0, y_0) + a_{10}(x - x_0) + a_{01}(y - y_0) + a_{11}(x - x_0)(y - y_0) + \dots, \quad (5)$$

in which all powers of  $(x - x_0)$  and  $(y - y_0)$  must be positive integer. The only assumption is that the function  $f(x, y)$  is continuous and  $n$  times differentiable within the interval  $[(x_0, y_0), (x, y)]$ . Therefore, since  $q(\Psi, T)$  was expanded into their derivatives, integral powers were appeared in the right hand of the expression. This will also be followed in the later analysis. For simplicity, we rewrite thermodynamic variables in dimensionless form as

$$\begin{aligned} q &= q_c(1 + \varrho), \\ T &= T_c(1 + t), \\ \Psi &= \Psi_c(1 + \psi), \end{aligned} \quad (6)$$

where  $\varrho, t$  and  $\psi$  are deviation from critical point. Substituting above expressions into Eq. (4), we have

$$\varrho = a'_{01}t + a'_{02}t^2 + a'_{11}t\psi + a'_{12}t^2\psi + a'_{30}\psi^3, \quad (7)$$

in which the prime quantities are equal to the rescaled coefficients in Eq. (4). Since phase transition occurs between small and large black hole under constant charge, one writes

$$\begin{aligned} \varrho &= a'_{01}t + a'_{02}t^2 + a'_{11}t\psi_s + a'_{12}t^2\psi_s + a'_{30}\psi_s^3 \\ &= a'_{01}t + a'_{02}t^2 + a'_{11}t\psi_l + a'_{12}t^2\psi_l + a'_{30}\psi_l^3, \end{aligned} \quad (8)$$

here,  $\psi_s$  ( $\psi_l$ ) stands for  $\psi$  at small (large) horizon. Also, applying the Maxwell's equal area law,  $\oint \Psi dq = 0$ , which is obtained from Gibbs free energy [1,20], and using Eq. (7) one can arrive at

$$\int_{\psi_l}^{\psi_s} \psi \left( a'_{11}t + a'_{12}t^2 + 3a'_{30}\psi^2 \right) d\psi = 0. \quad (9)$$

It is a matter of calculations to show that Eqs. (8) and (9) have the following solution

$$\psi_l = -\psi_s = \sqrt{-\frac{a'_{11}t + a'_{12}t^2}{a'_{30}}}. \quad (10)$$

Therefore, the order parameter near the critical point behaves as

$$|\psi_s - \psi_l| = 2\psi_s \sim t^{1/2} \implies \beta = 1/2. \quad (11)$$

In the vicinity of critical point, the shape of the critical isotherm,  $t = 0$ , is obtained by

$$\varrho = a'_{30}\psi^3 \implies \delta = 3. \quad (12)$$

According to equation (7), the behavior of response function,  $\chi_T = \partial \Psi / \partial q \Big|_T$ , is given by

$$\chi_T \sim \frac{1}{a'_{11}t} \implies \gamma = 1. \quad (13)$$

To denote the specific heat at fixed  $\Psi$  close to the critical point, we perform the expansion of entropy around the critical point

$$\begin{aligned} S(\Psi, T) &= s_{00} + s_{01}(T - T_c) + s_{10}(\Psi - \Psi_c) \\ &\quad + s_{11}(T - T_c)(\Psi - \Psi_c) + \dots, \end{aligned} \quad (14)$$

where  $s_{ij} \equiv (i!j!)^{-1} \partial^{i+j} S / \partial \Psi^i \partial T^j \Big|_{\Psi_c, T_c}$ . With  $S$  at hand, we can extract the critical exponent  $\alpha$ , as follows

$$C_\Psi = T \frac{\partial S}{\partial T} \Big|_\Psi = T_c s_{01} \implies \alpha = 0.$$

It is remarkable to note that  $S$  is only a function of  $\Psi$  for Maxwell electrodynamics, i.e.  $s_{ij} = 0$  for  $j \neq 0$  [1]. In this way we calculate the critical exponents of a black hole in a general framework without specifying the form of the metric, which coincide with those obtained for Van der Waals fluid system.

### 3. Phenomenological aspect: Landau function

Let us define the thermodynamic potential in this case as

$$K = M - TS - \Psi q. \quad (15)$$

Then use of first law Eq. (1) yields

$$dK = -SdT - qd\Psi. \quad (16)$$

The above states that  $K$  is function of both  $T$  and  $\Psi$  and

$$S = -\left(\frac{\partial K}{\partial T}\right)_\Psi; \quad q = -\left(\frac{\partial K}{\partial \Psi}\right)_T. \quad (17)$$

Consequently, the conditions Eq. (3) at the critical point take the following forms:

$$\frac{\partial^2 K}{\partial \Psi^2} \Big|_{T_c} = 0, \quad \frac{\partial^3 K}{\partial \Psi^3} \Big|_{T_c} = 0. \quad (18)$$

Now since  $K = K(T, \Psi)$ , Taylor expansion of it near the critical point is given by

$$\begin{aligned} K(T, \Psi) &= b_{00} + b_{10}(T - T_c) + b_{11}(T - T_c)(\Psi - \Psi_c) \\ &\quad + b_{01}(\Psi - \Psi_c) + b_{20}(T - T_c)^2 + b_{21}(T - T_c)^2(\Psi - \Psi_c) \\ &\quad + b_{22}(T - T_c)^2(\Psi - \Psi_c)^2 + b_{12}(T - T_c)(\Psi - \Psi_c)^2 \\ &\quad + b_{13}(T - T_c)(\Psi - \Psi_c)^3 + b_{04}(\Psi - \Psi_c)^4 + \dots \end{aligned} \quad (19)$$

In the above we have used  $b_{ij} = (i!j!)^{-1} (\partial^{i+j} K) / (\partial^i T \partial^j \Psi) \Big|_{\Psi_c, T_c}$  and the condition Eq. (18). Using Eq. (6), we obtain the near critical point  $K$  as

$$\begin{aligned} K(t, \psi) &= b_{00} + b'_{10}t + b'_{11}t\psi + b'_{01}\psi + b'_{20}t^2 \\ &\quad + b'_{12}t\psi^2 + b'_{21}t^2\psi + b'_{22}t^2\psi^2 + b'_{13}t\psi^3 + b'_{04}\psi^4. \end{aligned} \quad (20)$$

Here the prime quantities are the rescaled coefficients which appear in Eq. (19). Since their explicit forms are not needed, we do not mention their values. The same logic will be followed again and again. Using the first and second relations of Eq. (17), we obtain

$$\begin{aligned} S &= b''_{10} + b''_{11}\psi + b''_{20}t + b''_{21}t\psi + b''_{12}\psi^2 \\ &\quad + b''_{22}t\psi^2 + b''_{13}\psi^3, \end{aligned} \quad (21)$$

$$\begin{aligned} q &= a''_{01} + a''_{11}t + a''_{12}t\psi + a''_{21}t^2 + a''_{22}t^2\psi \\ &\quad + a''_{13}t\psi^2 + a''_{04}\psi^3, \end{aligned} \quad (22)$$

respectively. In the case of Maxwell electrodynamics, the coefficients of  $t$  in Eq. (21) is zero i.e.  $b''_{2i} = 0$  where  $i = 0, 1, 3$ .

Using the Maxwell construction (the equal-area law), one obtains

$$\int_{\psi_s}^{\psi_l} \psi dq = \int_{\psi_s}^{\psi_l} \psi (a''_{12}t + a''_{22}t^2 + 2a''_{13}t\psi + 3a''_{04}\psi^2) d\psi = 0. \quad (23)$$

By this construction, the unstable part of the isotherm is replaced by an isocharge which indicates that  $q$  is the same for  $\psi_s$  and  $\psi_l$  [1]. Thus, near the critical point, one can write

$$\begin{aligned} q &= a''_{01} + a''_{11}t + a''_{12}t\psi_s + a''_{21}t^2 + a''_{22}t^2\psi_s \\ &\quad + a''_{13}t\psi_s^2 + a''_{04}\psi_s^3, \\ q &= a''_{01} + a''_{11}t + a''_{12}t\psi_l + a''_{21}t^2 + a''_{22}t^2\psi_l \\ &\quad + a''_{13}t\psi_l^2 + a''_{04}\psi_l^3. \end{aligned} \quad (24)$$

Equating the above two and using Eq. (23), one obtains

$$\psi_{l,s} = \frac{-a''_{13}t \pm \sqrt{3t [a''_{13}t - 3a''_{04} (a''_{12} + a''_{22}t)]}}{3a''_{04}}, \quad (25)$$

which leads to

$$|\psi_s - \psi_l| = \frac{2\sqrt{3t [a''_{13}t - 3a''_{04} (a''_{12} + a''_{22}t)]}}{3a''_{04}}, \quad (26)$$

and to the lowest order in  $t$

$$|\psi_s - \psi_l| \sim (T - T_c)^{1/2}, \quad (27)$$

which yields  $\beta = 1/2$ . Next, Eq. (22) at  $T = T_c$  reduces to

$$q \sim \psi^3 \sim (\Psi - \Psi_c)^3, \quad (28)$$

and so  $\delta = 3$ .

The isothermal compressibility is defined as  $\chi_T = (\partial\Psi/\partial q)_T$ . Therefore differentiating Eq. (22) we find (to the lowest order in  $t$ ),  $\chi_T \sim t^{-1}$  which yields  $\gamma = 1$ . Similarly, the specific heat is given by  $C_\Psi = T(\partial S/\partial T)_\Psi$  and can be calculated using Eq. (21)

$$C_\Psi = (1+t) \left( \frac{\partial S}{\partial t} \right)_\Psi = (1+t) (b''_{20} + b''_{21}\psi + b''_{22}\psi^2), \quad (29)$$

which to the lowest order is just a constant. Therefore we find the critical exponent  $\alpha = 0$ .

#### 4. Thermogeometric description

The approach toward phase transition by considering thermodynamic geometry (thermogeometric) was first introduced by Weinhold [22] and Ruppeiner [23]. In this case the Ricci scalar of the metric diverges at the critical point or in other words divergence of Ricci scalar is the signature of the phase transition [24,25]. Both of these approaches are conformally related to each other by the temperature  $T$ . For recent studies on the Ruppeiner thermodynamic geometry, see [26–30] and references therein. This approach has different properties due to the different thermodynamic potentials, i.e. is not invariant under Legendre transformation [31,32]. This distinct problem was later remedied by Quevedo [32], who presented a Legendre invariant set of metrics in the phase space. In particular, a Legendre invariant metric has been investigated for black holes in [33] (see also [34]). Below, we follow procedure in [33], to construct the Legendre invariant metric in an alternative phase space.

The idea is as follows: First construct a thermodynamic phase space  $\mathcal{T}$  on which the coordinates are  $\mathcal{Z}^A = (\mathcal{F}, \mathcal{X}^a, \mathcal{P}^a)$  where  $\mathcal{F}$  is the thermodynamic potential and  $\mathcal{P}^a$  are the conjugate variables of the thermodynamic variables  $\mathcal{X}^a$ . In this representation the fundamental one form is given by

$$\theta_{\mathcal{F}} = d\mathcal{F} - \sum_{a,b} \delta_{ab} \mathcal{P}^a d\mathcal{X}^b, \quad (30)$$

where  $\delta_{ab}$  is the Kronecker delta. With the present setup, one can choose the thermodynamic geometry on  $\mathcal{T}$  as

$$\begin{aligned} \mathcal{G} &= \left( d\mathcal{F} - \sum_{a,b} \delta_{ab} \mathcal{P}^a d\mathcal{X}^b \right)^2 \\ &\quad + \lambda \left( \sum_{a,b} \xi_{ab} \mathcal{P}^a \mathcal{X}^b \right) \left( \sum_{c,d} \eta_{cd} d\mathcal{P}^c d\mathcal{X}^d \right), \end{aligned} \quad (31)$$

which is invariant under the following set of Legendre transformations:

$$\begin{aligned} \mathcal{F}_{old} &= \mathcal{F}_{new} - \delta_{ab} \mathcal{X}_{new}^a \mathcal{P}_{new}^b, \\ \mathcal{X}_{old}^a &= -\mathcal{P}_{new}^a; \quad \mathcal{P}_{old}^a = \mathcal{X}_{new}^a. \end{aligned} \quad (32)$$

Here  $\eta_{ab} = \text{diag}(-1, 1, 1, \dots)$  and  $\lambda$  is an arbitrary Legendre invariant function of  $\mathcal{X}^a$  while  $\xi_{ab}$  is an arbitrary diagonal constant matrix. The simplest choice is chosen as  $\lambda = 1$  and  $\xi_{ab} = \text{diag}(1, 1, \dots)$ . Therefore the general form of the simplest Legendre invariant metric is

$$\mathcal{G} = \theta_{\mathcal{F}}^2 + \left( \sum_{a,b} \xi_{ab} \mathcal{P}^a \mathcal{X}^b \right) \left( \sum_{c,d} \eta_{cd} d\mathcal{P}^c d\mathcal{X}^d \right). \quad (33)$$

We shall work with this particular form.

For the present case, we start with  $K$ , Eq. (15), as a thermodynamic potential. So according to Eq. (16), the coordinates of the thermodynamic phase space are  $\mathcal{Z}^A = (K, \mathcal{X}^a, \mathcal{P}^a)$  with  $\mathcal{X}^a = (\Psi, T)$  and  $\mathcal{P}^a = (-q, -S)$ . So the metric Eq. (33) takes the form:

$$\mathcal{G}_1 = \theta_K^2 + (-ST - q\Psi)(-dSdT + dqd\Psi), \quad (34)$$

with  $\theta_K = dK + qd\Psi + SdT$ . Use of Eqs. (16) and (17) leads to the following form:

$$\mathcal{G}_1 = (TK_T + \Psi K_\Psi)(-K_{\Psi\Psi}d\Psi^2 + K_{TT}dT^2), \quad (35)$$

where we use the conventions  $X_Y = \partial X/\partial Y$  and  $X_{YY} = \partial^2 X/\partial Y^2$ . The above metric is two dimensional which has the general form:

$$ds^2 = -f(x, y)dx^2 + g(x, y)dy^2. \quad (36)$$

The Ricci scalar of this metric is given by

$$R = \frac{1}{2f^2g^2} \left[ f(f_{yy}g_y - g_x^2) + g\{f_y^2 - f_xg_x - 2f(f_{yy} - g_{xx})\} \right]. \quad (37)$$

Here  $f = (TK_T + \Psi K_\Psi)K_{\Psi\Psi}$  and  $g = (TK_T + \Psi K_\Psi)K_{TT}$ . Therefore the Ricci scalar  $R$  diverges when  $K_{\Psi\Psi} = 0$  provided the numerator is finite. If both vanish, then one needs to be careful and use L'Hospital's rule to arrive at the same conclusion. This shows that the first condition (see Eq. (18)) at the critical point implies the divergence of Ricci tensor of the metric Eq. (35) at the critical point, i.e.  $K_{\Psi\Psi} = 2b_{02} = 0$ .

For the other condition we consider the original situation where  $q$  is expressed as function of  $\Psi$  and  $T$ . In this case  $dq =$

$q_\Psi d\Psi + q_T dT$ . Therefore the Legendre invariant metric can be chosen as

$$\mathcal{G}_2 = \theta_q^2 + (q_\Psi \Psi + q_T dT)(-d\Psi dq_\Psi + dT dq_T), \quad (38)$$

where we have considered the thermodynamic phase space as  $\mathcal{Z}^A = (q, \mathcal{X}^a, \mathcal{P}^a)$  with  $\mathcal{X}^a = (\Psi, T)$  and  $\mathcal{P}^a = (q_\Psi, q_T)$ . Here the fundamental one form is  $\theta_q = dq - q_\Psi d\Psi - q_T dT$ . Now since  $q_\Psi$  and  $q_T$  are functions of both  $\Psi$  and  $T$ , proceeding as before, the above reduces to the following form:

$$\mathcal{G}_2 = (q_\Psi \Psi + q_T dT)(-q_{\Psi\Psi} d\Psi^2 + q_{TT} dT^2). \quad (39)$$

This is again of the form Eq. (36) whose Ricci scalar is given by Eq. (37). Here  $f = (q_\Psi \Psi + q_T dT)q_{\Psi\Psi}$  and  $(q_\Psi \Psi + q_T dT)q_{TT}$ . Therefore the Ricci scalar for the present metric diverges at the critical point as  $q_{\Psi\Psi}$  vanishes at this point, i.e.  $q_{\Psi\Psi} = 2a_{20} = 0$ .

## 5. Concluding remarks

Choosing the correct independent thermodynamic variables is a key starting point in any thermodynamic treatment. The convenient choice of independent thermodynamic variables can lead to easier solutions, as is clearly demonstrated by usefulness of Legendre transform in thermodynamics. However, the “wrong” set of independent variables could lead to nonphysical results. In Ref. [1], an alternative thermodynamic phase space was proposed for RN black holes in AdS space. In this alternative view, the square of black hole electric charge (instead of the usual charge) was considered to be the independent thermodynamic variable. It was shown that the thermodynamic behavior in such an alternative view makes more physical sense and that the critical behavior resembled strongly with Van der Waals fluid, belonging to the same universality class. In the present work, we have approached the same problem by (i) generalizing to any AdS black hole independent of spacetime metric, and (ii) solving the problem both from the equation of state as well as thermodynamic potential approach. Both approaches lead clearly to the set of four critical exponents which are the same as the Van der Waals fluid system. This provides further evidence for the generality of Van der Waals universality class for AdS black holes on one hand, as well as indicating the validity of the alternative phase space proposed in [1]. Furthermore, we have also provided a thermogeometric approach where Ricci scalar has been calculated and shown to diverge at the critical point within the general alternative phase space.

## Acknowledgements

We thank Shiraz University Research Council. The work of AS has been supported financially by Research Institute for Astronomy and Astrophysics of Maragha, Iran. The work of BRM is supported by a START-UP RESEARCH GRANT (No. SG/PHY/P/BRM/01) from Indian Institute of Technology Guwahati, India.

## References

- [1] A. Dehyadegari, A. Sheykhi, A. Montakhab, Critical behaviour and microscopic structure of charged AdS black holes via an alternative phase space, *Phys. Lett. B* 768 (2017) 235, arXiv:1607.05333.
- [2] S. Hawking, D.N. Page, Thermodynamics of black holes in anti-de sitter space, *Commun. Math. Phys.* 87 (1983) 577.
- [3] E. Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, *Adv. Theor. Math. Phys.* 2 (1998) 505, arXiv:hep-th/9803131.
- [4] A. Chamblin, R. Emparan, C.V. Johnson, R.C. Myers, Charged AdS black holes and catastrophic holography, *Phys. Rev. D* 60 (1999) 064018, arXiv:hep-th/9902170; A. Chamblin, R. Emparan, C.V. Johnson, R.C. Myers, Holography, thermodynamics and fluctuations of charged AdS black holes, *Phys. Rev. D* 60 (1999) 104026, arXiv:hep-th/9904197.
- [5] B.P. Dolan, The cosmological constant and the black hole equation of state, *Class. Quantum Gravity* 28 (2011) 125020, arXiv:1008.5023; B.P. Dolan, Pressure and volume in the first law of black hole thermodynamics, *Class. Quantum Gravity* 28 (2011) 235017, arXiv:1106.6260.
- [6] D. Kubiznak, R.B. Mann, P-V criticality of charged AdS black holes, *J. High Energy Phys.* 1207 (2012) 033, arXiv:1205.0559.
- [7] D. Kastor, S. Ray, J. Traschen, Entropy and the mechanics of AdS black holes, *Class. Quantum Gravity* 26 (2009) 195011, arXiv:0904.2765.
- [8] A. Dehyadegari, A. Sheykhi, A. Montakhab, Novel phase transition in charged dilaton black holes, *Phys. Rev. D* 96 (2017) 084012, arXiv:1707.05307.
- [9] S. Gunasekaran, R.B. Mann, D. Kubiznak, Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization, *J. High Energy Phys.* 1211 (2012) 110, arXiv:1208.6251.
- [10] R.A. Hennigar, R.B. Mann, E. Tjoa, Superfluid black holes, *Phys. Rev. Lett.* 118 (2017) 021301, arXiv:1609.02564.
- [11] A.M. Frassino, D. Kubiznak, R.B. Mann, F. Simovic, Multiple reentrant phase transitions and triple points in Lovelock thermodynamics, *J. High Energy Phys.* 1409 (2014) 080, arXiv:1406.7015.
- [12] D. Kubiznak, R.B. Mann, M. Teo, Black hole chemistry: thermodynamics with Lambda, *Class. Quantum Gravity* 34 (6) (2017) 063001, arXiv:1608.06147 [hep-th].
- [13] S.H. Hendi, M.H. Vahidinia, Extended phase space thermodynamics and P-V criticality of black holes with a nonlinear source, *Phys. Rev. D* 88 (2013) 084045, arXiv:1212.6128; M.H. Dehghani, S. Kamrani, A. Sheykhi, P-V criticality of charged dilatonic black holes, *Phys. Rev. D* 90 (2014) 104020, arXiv:1505.02386; De.Ch. Zou, Sh.-J. Zhang, B. Wang, Critical behavior of Born-Infeld AdS black holes in the extended phasespace thermodynamics, *Phys. Rev. D* 89 (2014) 044002, arXiv:1311.7299.
- [14] S.H. Hendi, S. Panahiyan, B. Eslam Panah, P-V criticality and geometrical thermodynamics of black holes with Born-Infeld type nonlinear electrodynamics, *Int. J. Mod. Phys. D* 25 (2016) 1650010, arXiv:1410.0352; Z. Sherkatghanad, B. Mirza, Z. Mirzaeyan, S.A.H. Mansoori, Critical behaviors and phase transitions of black holes in higher order gravities and extended phase spaces, *Int. J. Mod. Phys. D* 26 (2017) 1750017, arXiv:1412.5028.
- [15] B.R. Majhi, S. Samanta, P-V criticality of AdS black holes in a general framework, *Phys. Lett. B* 773 (2017) 203, arXiv:1609.06224.
- [16] K. Bhattacharya, B.R. Majhi, S. Samanta, Van der Waals criticality in AdS black holes: a phenomenological study, *Phys. Rev. D* 96 (2017) 084037, arXiv:1709.02650.
- [17] K. Bhattacharya, B.R. Majhi, Thermogeometric description of the van der Waals like phase transition in AdS black holes, *Phys. Rev. D* 95 (2017) 104024, arXiv:1702.07174 [gr-qc].
- [18] A. Dehyadegari, A. Sheykhi, Reentrant phase transition of Born-Infeld-AdS black holes, *Phys. Rev. D* 98 (2018) 024011, arXiv:1711.01151.
- [19] S. He, L.F. Li, X.X. Zeng, Holographic Van der Waals-like phase transition in the Gauss-Bonnet gravity, *Nucl. Phys. B* 915 (2017) 243, arXiv:1608.04208.
- [20] H.B. Callen, 2nd ed., John Wiley & Sons, New York, 1985.
- [21] R.G. Cai, L.M. Cao, L. Li, R.Q. Yang, P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space, *J. High Energy Phys.* 1309 (2013) 005, arXiv:1306.6233.
- [22] F. Weinhold, *J. Chem. Phys.* 63 (1975) 2479; F. Weinhold, *J. Chem. Phys.* 63 (1975) 2484; F. Weinhold, *J. Chem. Phys.* 63 (1975) 2488; F. Weinhold, *J. Chem. Phys.* 63 (1975) 2496; F. Weinhold, *J. Chem. Phys.* 65 (1976) 559.
- [23] G. Ruppeiner, Thermodynamics: a Riemannian geometric model, *Phys. Rev. A* 20 (1979) 1608.
- [24] G. Ruppeiner, Riemannian geometry in thermodynamic fluctuation theory, *Rev. Mod. Phys.* 67 (1995) 605; G. Ruppeiner, *Rev. Mod. Phys.* 68 (1996) 313, Erratum.
- [25] J.D. Nulton, P. Salomon, Geometry of the ideal gas, *Phys. Rev. A* 31 (1985) 2520; D.A. Johnston, W. Janke, R. Kenna, Information geometry, one, two, three (and four), *Acta Phys. Pol. B* 34 (2003) 4923.
- [26] B. Mirza, H. Mohammadzadeh, Nonperturbative thermodynamic geometry of anyon gas, *Phys. Rev. E* 80 (2009) 011132.
- [27] B. Mirza, H. Mohammadzadeh, Ruppeiner geometry of anyon gas, *Phys. Rev. E* 78 (2008) 021127.
- [28] S.W. Wei, Y.X. Liu, Insight into the microscopic structure of an AdS black hole from thermodynamical phase transition, *Phys. Rev. Lett.* 115 (2015) 111302, arXiv:1502.00386.
- [29] M. Kord Zangeneh, A. Dehyadegari, A. Sheykhi, Comment on “Insight into the microscopic structure of an AdS black hole from a thermodynamical phase transition”, arXiv:1602.03711.
- [30] M. Kord Zangeneh, A. Dehyadegari, M.R. Mehdizadeh, B. Wang, A. Sheykhi, Thermodynamics, phase transitions and Ruppeiner geometry for Einstein-dilaton Lifshitz black holes in the presence of Maxwell and Born-Infeld electrodynamics, *Eur. Phys. J. C* 77 (2017) 423, arXiv:1610.06352.

- [31] R. Mrugala, J.D. Nulton, J.C. Schon, P. Salamon, Statistical approach to the geometric structure of thermodynamics, *Phys. Rev. A* 41 (1990) 3156.
- [32] H. Quevedo, Geometrothermodynamics, *J. Math. Phys.* 48 (2007) 013506, arXiv: physics/0604164.
- [33] J.L. Alvarez, H. Quevedo, A. Sanchez, Unified geometric description of black hole thermodynamics, *Phys. Rev. D* 77 (2008) 084004, arXiv:0801.2279.
- [34] R. Banerjee, B.R. Majhi, S. Samanta, Thermogeometric phase transition in a unified framework, *Phys. Lett. B* 767 (2017) 25, arXiv:1611.06701 [gr-qc].