

# Four-fermion deformations of the massless Schwinger model and confinement

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**ABSTRACT:** We consider the massless charge- $N$  Schwinger model and its deformation with two four-fermion operators. Without the deformations, this model exhibits chiral symmetry breaking without confinement. It is usually asserted that the massless Schwinger model is always deconfined and a string tension emerges only when a mass for the fermion field is turned on. We show that in the presence of these four-fermion operators, the massless theory can in fact confine. One of the four-fermion deformations is chirally neutral, and is a marginal deformation. The other operator can be relevant or irrelevant, and respects a  $\mathbb{Z}_2$  subgroup of chiral symmetry for even  $N$ , hence forbidding a mass term. When it is relevant, even the exactly massless theory exhibits both confinement and spontaneous chiral symmetry breaking. The construction is analogous to QCD(adj) in 2d. While the theory without four-fermion deformations is deconfined, the theory with these deformations is generically in a confining phase. We study the model on  $\mathbb{R}^2$  using bosonization, and also analyze the mechanism of confinement on  $\mathbb{R} \times S^1$ , where we find that confinement is driven by fractional instantons.

**KEYWORDS:** Anomalies in Field and String Theories, Confinement, Wilson, 't Hooft and Polyakov loops

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**1 Introduction**

The Schwinger model — U(1) gauge theory coupled to a Dirac fermion in two spacetime dimensions — is a famous playground for the exploration of ideas about quark confinement and chiral symmetry breaking [1–3]. These topics are notoriously difficult to study in non-abelian gauge theories in four spacetime dimensions, so it is very helpful to be able to explore them in the calculable setting of the Schwinger model.

Historically, the Schwinger model was introduced as U(1) quantum electrodynamics with a single unit-charge Dirac fermion.<sup>1</sup> Here we will study some generalizations of this theory with the aim of making a better toy model for 4d gauge theory. Our starting point

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<sup>1</sup>The seminal works from the 1970s, e.g. [2, 3], considered the response of the theory to the introduction of test charges with irrational charge, so technically these works took the group manifold to be a real line  $\mathbb{R}$ .

will be the charge- $N$  Schwinger model, which was studied recently in e.g. refs. [4–10], see also [11] for an early analysis. This model has the Euclidean action

$$S_{\text{standard}} = \int d^2x \left( \frac{1}{4e^2} f_{\mu\nu}^2 + \bar{\psi} [\gamma^\mu (\partial_\mu + iNa_\mu)] \psi \right) + m_\psi \bar{\psi}_L \psi_R + \text{h.c.} . \quad (1.1)$$

Here  $a_\mu$  is a U(1) gauge field,  $\psi$  is a Dirac fermion field with chiral components  $\psi_L, \psi_R$ , the integer  $N$  is the charge of the fermion,  $e$  has unit mass dimension, and Hermitian conjugation is defined by analytic continuation from Minkowski space. The statement that  $a_\mu$  is a U(1) gauge field means that the gauge transformation functions  $\alpha(x)$  take values in U(1), meaning that there is  $2\pi$  periodicity. Physically, one could interpret this periodicity as arising from gauge transformations of a very heavy unit-charge test fermion field  $\psi_t$ ,

$$\psi_t \rightarrow e^{i\alpha} \psi_t,$$

while the gauge transformations of the other fields are

$$\begin{aligned} a_\mu &\rightarrow a_\mu - \partial_\mu \alpha, \\ \psi &\rightarrow e^{iN\alpha} \psi. \end{aligned}$$

Gauge invariance also implies that  $\int_{M_2} f \in 2\pi\mathbb{Z}$  where  $M_2$  is any closed smooth 2-manifold and  $f = da = \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu$  is the field strength 2-form.

We also add an explicit topological  $\theta$  term

$$S_\theta = \frac{i\theta}{2\pi} \int_{M_2} da \quad (1.2)$$

to the action, and assume that  $m_\psi \geq 0$ . The coefficient  $\theta$  is  $2\pi$  periodic. However, we will mostly focus on the physics at  $\theta = 0$ , where the theory has a  $\mathbb{Z}_2$  parity symmetry.

As defined above, the Schwinger model has a  $\mathbb{Z}_N$  1-form symmetry [12] for any value of  $m$ . This symmetry is generated by a collection of  $N$  local topological operators  $U_n(x)$  and has the effect of multiplying Wilson loops by  $\mathbb{Z}_N$  phases:

$$\langle U_n(x) e^{iq \int_C a_\mu dx^\mu} \rangle = \exp\left(\frac{2\pi i n q}{N} \ell(C, x)\right) \langle e^{iq \int_C a_\mu dx^\mu} \rangle \quad (1.3)$$

where  $\ell(C, x)$  is the linking number of  $C$  and  $x$ . The internal global symmetry of the  $m \neq 0$  Schwinger model — which is just the  $\mathbb{Z}_N$  1-form symmetry — coincides with the global symmetry of pure 4d  $SU(N)$  Yang-Mills theory. The existence of the 1-form  $\mathbb{Z}_N$  symmetry means that charge confinement is a sharply-defined concept in the Schwinger model when  $N > 1$ . The same is true in 4d  $SU(N)$  pure YM theory.<sup>2</sup>

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<sup>2</sup>In discussions of 4d YM theory it is common to call its  $\mathbb{Z}_N$  1-form symmetry “center symmetry” [13, 14]. This has some historical justification because the center subgroup of  $SU(N)$  is  $\mathbb{Z}_N$ , which happens to be the same as the 1-form symmetry group so long as all matter fields are in representations of  $N$ -ality zero. We won’t use this language here because the addition of matter reduces the 1-form symmetry to a discrete subgroup, while the center subgroup of the U(1) gauge group is U(1).

When  $m_\psi = 0$ , at the classical level the Schwinger model has a  $U(1)$  axial symmetry. As usual, the ABJ anomaly means that the chiral symmetry in the quantum theory is reduced, and is generated by

$$\psi(x) \rightarrow e^{2\pi i \gamma_5 / (2N)} \psi(x). \tag{1.4}$$

The faithfully-acting symmetry is  $\mathbb{Z}_N$ , and acts as  $\bar{\psi}_L \psi_R \rightarrow e^{2\pi i / N} \bar{\psi}_L \psi_R$ . The existence of the discrete chiral symmetry means that it is meaningful to discuss spontaneous chiral symmetry breaking, just as in e.g 4d  $SU(N)$   $\mathcal{N} = 1$  super-YM theory. The  $\mathbb{Z}_N$  0-form chiral symmetry and the  $\mathbb{Z}_N$  1-form symmetry have a mixed 't Hooft anomaly [4]. Indeed, the internal global symmetries and anomalies of the massless charge- $N$  Schwinger model coincide with the internal bosonic global symmetries and anomalies of 4d  $SU(N)$   $\mathcal{N} = 1$  super-YM theory.

From the perspective of the first paragraph of this introduction, it would be nice if the dynamics of 4d  $SU(N)$  gauge theories and the charge- $N$  Schwinger model looked similar. Unfortunately, the behavior of the 4d and 2d theories is very different! When  $m_\psi = 0$ , the good news is that the  $\mathbb{Z}_N$  chiral symmetry is spontaneously broken, just as in 4d  $\mathcal{N} = 1$  SYM. The bad news for the comparison with 4d gauge theory is that the  $\mathbb{Z}_N$  1-form symmetry of the Schwinger model with  $m_\psi = 0$  is spontaneously broken, and the expectation values of large ‘fundamental’ Wilson loops have a perimeter-law behavior

$$\langle e^{i \int_C a} \rangle \sim e^{-\mu P(C)}, \tag{1.5}$$

where  $C$  is e.g. a circular contour with perimeter  $P(C)$ ,  $\mu$  is a UV scale (of order the mass of a heavy test particle), and  $a = a_\mu dx^\mu$ . Confinement appears (and the 1-form symmetry is restored) when  $m_\psi \neq 0$ , but the string tension scales as  $T \sim m_\psi e$  for  $m_\psi \ll e$ . In 4d  $\mathcal{N} = 1$  SYM, in contrast, the 1-form  $\mathbb{Z}_N$  symmetry is not spontaneously broken, and large Wilson loops obey an area law.

There are three basic ways to understand the behavior of Wilson loops in the Schwinger model:

- (a) Solve the charge- $N$  Schwinger model exactly on  $\mathbb{R}^2$  using bosonization and compute the relevant expectation values. This has the virtue of using direct and relatively elementary arguments.
- (b) Relate deconfinement to the existence of a mixed 't Hooft anomaly between the  $\mathbb{Z}_N$  1-form symmetry and the  $\mathbb{Z}_N$  0-form chiral symmetry. This approach has the advantage that it uses only basic symmetry principles, and so it generalizes to theories which are not exactly solvable.
- (c) Solve the model on  $\mathbb{R} \times S^1$  with small  $S^1$  and extrapolate the phase structure to  $\mathbb{R}^2$ . Due to the 't Hooft anomaly, the quantum-mechanical EFT has  $N$  degenerate ground states. One can take any linear combination of them to be a ground state. The two most interesting choices result in either the chiral condensate being non-zero while the Polyakov loop has zero expectation value, or vice versa. Only the former choice

extrapolates nicely to  $\mathbb{R}^2$ . The issue with the other choice is that in the large  $S^1$  limit the Polyakov loop disappears as an observable. We should emphasize that in 2d gauge theories, the vanishing of the Polyakov loop expectation value on a cylinder  $S^1$  is not sufficient to conclude that the 1-form symmetry is unbroken on  $\mathbb{R}^2$ .

All of these approaches have been discussed in the literature [4–10].

The fact that charge- $q$  Wilson loops are deconfined for all  $q \in \mathbb{Z}$  in the charge- $N$  Schwinger model sharply contrasts with the expected behavior in 4d  $SU(N)$  gauge theory with adjoint fermions. The latter theory is expected to confine fundamental test charges even when the mass of the dynamical fermion goes to zero, at least as it is outside of the conformal window. Here we discuss a modification of the charge- $N$  Schwinger model which brings its dynamics much closer to the dynamics of 4d gauge theories. We will mostly focus on the Schwinger model with even  $N$  for reasons that will become clear shortly.

We now explain the basic idea of this paper. First, we recall that at high energies the standard Schwinger model (1.1) approaches a free-field CFT fixed point, which can be described as a free massless Dirac fermion. We then note that this CFT contains a unique exactly marginal operator

$$\mathcal{O}_{jj} = j_\mu j^\mu = \bar{\psi} \gamma_\mu \psi \bar{\psi} \gamma^\mu \psi = -4 \bar{\psi}_R \psi_L \bar{\psi}_L \psi_R. \tag{1.6}$$

This ‘Thirring model’ operator is exactly marginal,  $\Delta_{jj} = 2$ , and neutral under all symmetries of the model. Therefore we are free to add it to the action with a dimensionless coefficient  $g \in \mathbb{R}$ .<sup>3</sup> This yields a generalization of the Schwinger model which we will call the Schwinger-Thirring (ST) model:

$$S_{\text{ST}} = S_{\text{standard}} + g \int d^2x \mathcal{O}_{jj} \tag{1.7}$$

Since (1.7) is an interacting theory, in general one might expect a dimensionless parameter like  $g$  to run with the RG scale. However, when  $m_\psi = 0$  it is known that in fact  $g$  remains an exactly marginal parameter even after we take into account the gauge interaction [15–18]. This is easiest to see using bosonization, as we review below, but it can also be deduced directly in the fermionic variables, see appendix B. There is a minimum value of  $g$ ,  $g_* = -\pi/2$ , below which some operator scaling dimensions become negative, and the theory ceases to be unitary. We will assume that  $g > g_*$ . Turning on the perturbation by  $\mathcal{O}_{jj}$  perturbation does not affect the symmetries and anomalies of the massless Schwinger model. As a result, the massless Schwinger-Thirring model remains in a deconfined phase with a finite mass gap and spontaneous chiral symmetry breaking for  $g \in (g_*, \infty)$ , just like the original charge  $N$  Schwinger model.

We should emphasize that the high energy behavior of  $S_{\text{ST}}$  is *not* the same as that of the original Schwinger model, although it is continuously connected to it. Rather than approaching a free-field CFT fixed point, it approaches an *interacting* CFT fixed point

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<sup>3</sup>Such an operator can be radiatively generated in various UV modifications of the Schwinger model. For example, it will be generated if the fermions are coupled to a massive vector boson. It is also possible that some discretizations of (1.1) might flow to (1.7) in the continuum limit.

at high energies. In this UV CFT fixed point the scaling dimensions of operators do not coincide with their ‘engineering’ dimensions.

The lowest-dimension four-fermion operator which is invariant under parity but not  $\mathbb{Z}_N$  chiral symmetry is

$$\mathcal{O}_\chi = \bar{\psi}_L \psi_R (D_\mu \bar{\psi}_L) (D^\mu \psi_R). \tag{1.8}$$

At the free-field fixed point (that is, at high energies and with  $g = 0$ ), the scaling dimension of  $\mathcal{O}_\chi$  coincides with its engineering dimension, which is 4. So when  $g = 0$ , this operator is RG irrelevant. But the scaling dimension  $\Delta_\chi$  of  $\mathcal{O}_\chi$  depends on  $g$ , and we will show that  $\Delta_\chi$  decreases monotonically as  $g$  is increased. Specifically, bosonization implies that

$$\Delta = \frac{4}{1 + 2g/\pi}. \tag{1.9}$$

The scaling dimension of  $\mathcal{O}_\chi$  diverges as  $g \rightarrow g_* = -\pi/2$  from above, but it becomes relevant when  $g > \pi/2$ .

As a result, there is a critical value of  $g$  ( $g = \pi/2$ ) at which  $\mathcal{O}_\chi$  becomes marginal at the UV fixed point, and as  $g$  is increased further, the  $\mathcal{O}_\chi$  operator becomes relevant at the UV fixed point. At the same time, we note that the operator  $\mathcal{O}_\chi$  is the lowest-dimension operator with charge 2 under  $\mathbb{Z}_N$  chiral symmetry, and it is invariant under all other symmetries. In the rest of this paper, we will discuss<sup>4</sup> what happens to the low-energy physics once we add the  $\mathcal{O}_\chi$  operator to the UV action of the ST model as a perturbation:<sup>5</sup>

$$S = S_{\text{ST}} + \Lambda^{2-\Delta_\chi} \int d^2x (\mathcal{O}_\chi + \mathcal{O}_\chi^\dagger). \tag{1.10}$$

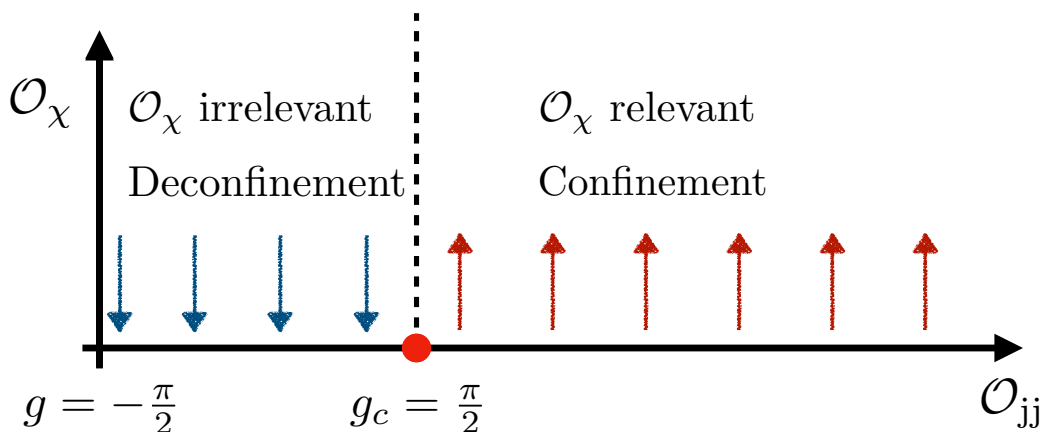
The parameter  $\Lambda$  is a new parameter with unit mass dimension. Its power is fixed from the scaling dimension of  $\mathcal{O}_\chi$  at the UV fixed point. Whether one should think of  $\Lambda$  as an IR or a UV energy scale depends on  $\Delta_\chi$ , and as we have already said  $\Delta_\chi$  depends on the marginal parameter  $g$ . If  $\Delta_\chi > 2$ , then  $\Lambda$  is a UV scale: the model defined by eq. (1.10) needs a UV completion at the scale  $\Lambda$ . In this case we will get a physically-interesting model if  $e \ll \Lambda$ . If  $\Delta_\chi < 2$ , then  $\Lambda$  is an IR mass scale in the same sense as the fermion mass parameter is an ‘IR scale’ of a free-fermion theory, and there is no a priori constraint on the ratio  $\Lambda/e$ . Nevertheless, we will see that the model is weakly coupled (in the bosonized duality frame) when  $\Lambda/e \ll 1$ , and so that is the regime we will focus on.

To get a feeling for the physics of the four-fermion deformed model (1.10) let us first suppose that  $\Delta_\chi > 2$ . This would be the case if e.g. we set  $g = 0$ . Then chiral symmetry is explicitly broken at the UV scale  $\Lambda$ . For even  $N$  it is broken to a  $\mathbb{Z}_2$  subgroup, while for odd  $N$  it is broken completely. One might therefore expect that the model becomes

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<sup>4</sup>An inspirational brief discussion of a very similar deformation was given in ref. [8]. However, the form of the deformation in the fermionic variables given in ref. [8] is the same as our  $\mathcal{O}_\chi$  without derivatives, which vanishes identically due to fermi statistics. The form of the deformation in the bosonic variables in ref. [8] coincides with our bosonized expressions, but the consequences of the technical irrelevance of this operator were not highlighted.

<sup>5</sup>Just as in Footnote 3, there are a variety of UV modification of (1.1) that can flow to (1.10), such as an appropriate coupling of  $\psi_L$  and  $\psi_R$  to a massive scalar field.



**Figure 1.** Behaviour of the massless Schwinger model in the presence of two four-fermion operators  $\mathcal{O}_{jj}$  and  $\mathcal{O}_\chi$  for  $N$ -even.  $\mathcal{O}_{jj}$  is a marginal deformation. When  $g > g_c$ ,  $\mathcal{O}_\chi$  becomes relevant and the massless theory becomes confining. The behaviour of the four-fermion-deformed Schwinger model is similar in this respect to four-fermion-deformed QCD(adj)<sub>2</sub>.

confining for odd  $N$ . However, this is not quite correct. At long distances  $\ell \gg 1/e \gg 1/\Lambda$ , the deformation by  $\mathcal{O}_\chi$  is irrelevant in the RG sense, and at distances large compared to  $1/\Lambda$  there can be an emergent  $\mathbb{Z}_N$  chiral symmetry.<sup>6</sup> Indeed, in this case the string tension induced by the  $\Lambda$  perturbation is proportional to e.g.  $e^4/\Lambda^2$  when  $g = 0$ , and so unit test charges separated by a distance  $L$  satisfying  $1/e \ll L \ll \Lambda^2/e^3$  will not feel a linear potential. In this sense the ST model with a deformation by  $\mathcal{O}_\chi$  with  $\Delta_\chi > 2$  is no more confining than U(1) QED in three spacetime dimensions, interpreted as a lattice gauge theory on a square Euclidean lattice with a Wilson action. In that model, at finite lattice spacing  $a$  there are finite-action monopole instantons which induce a finite string tension [13]. But the monopole-instanton action diverges in the continuum limit  $a \rightarrow 0$ , so the string tension also goes to zero in the continuum limit.<sup>7</sup>

We will show that when  $g \geq \pi/2$ , our deformed Schwinger model (1.10) confines fundamental (that is,  $q = \pm 1$ ) test charges when  $N > 2$ . When  $N$  is even the  $\mathbb{Z}_N$  1-form symmetry is spontaneously broken to  $\mathbb{Z}_{N/2}$ , so test charges with  $q = N/2 \bmod N$  are deconfined, while others are confined. Also, when  $N$  is even, the model has a  $\mathbb{Z}_2$  chiral symmetry, the fermion mass term is forbidden, and one can think of (1.10) as a variant of the massless charge- $N$  Schwinger model with *confinement* for  $N > 2$ . When  $N$  is odd and

<sup>6</sup>When the  $\mathcal{O}_\chi$  operator is irrelevant, we expect it to behave similarly to the Wilson term in the Wilson fermion action in 4d lattice gauge theory. In 4d, the Wilson term is an irrelevant dimension 5 operator, schematically  $\bar{\psi} D_\mu D^\mu \psi$ , which breaks chiral symmetry as well as the degeneracy between the 16 ‘doubler’ fermion modes. If the bare mass term is set to zero, then a lattice-scale mass term is induced by RG flow, and there is no emergent chiral symmetry in the infrared: the Wilson term is dangerously irrelevant. But one can get an emergent chiral symmetry in the infrared by tuning the bare quark mass term. We expect the same to be true for the  $\mathcal{O}_\chi$  operator when  $N$  is odd, but leave a detailed exploration of this feature of the model to future work.

<sup>7</sup>Recently it was understood that there are other lattice actions which flow to U(1) QED in three spacetime dimensions without any finite-action monopole-instantons even at finite lattice spacing [19], see also e.g. [20].

larger than 1, chiral symmetry is completely broken, and the  $\mathbb{Z}_N$  1-form symmetry is not spontaneously broken at all. The mass term can be generated by fluctuations. We will show how these features arise using bosonization on  $\mathbb{R}^2$ , as well as by an analysis on  $\mathbb{R} \times S^1$  when  $S^1$  is small. These analyses have complementary strengths, and combining them yields some interesting insights into the nature of the confinement mechanism in this model.

The behavior of the four-fermion-deformed charge- $N$  Schwinger model is much closer to the expected behavior of QCD-like 4d gauge theories. The four-fermion-deformed Schwinger model is also a nice toy model for the behavior of 2d  $SU(N)$  adjoint QCD. Adjoint QCD in 2d has only a  $\mathbb{Z}_2$  chiral symmetry when the quark mass is set to zero, and is also known to have two interesting four-fermion deformations [21] consistent with chiral symmetry. When these deformations are tuned to zero, 2d adjoint QCD deconfines on  $\mathbb{R}^2$  due to a mixed 't Hooft anomaly between its  $\mathbb{Z}_N$  1-form symmetry and an exotic non-invertible symmetry [8]. However, once the four-fermion deformations are turned on, at generic points in its parameter space 2d adjoint QCD confines [21].

## 2 Confinement from elementary considerations

### 2.1 Confinement in the standard charge $N$ Schwinger model

As discussed in the introduction, the massless charge- $N$  Schwinger model has a  $\mathbb{Z}_N^{(1)}$  1-form symmetry and  $\mathbb{Z}_N^{(0)}$  0-form chiral symmetry. It is often asserted that when the fermions are massless, the theory does not confine integer test charges, while with massive fermions, it does confine integer test charges. The common argument for this involves considering the topological  $\theta$  parameter of  $U(1)$  gauge theory, which enters the Euclidean action through

$$S_\theta = \frac{i\theta}{2\pi} \int_{M_2} da. \tag{2.1}$$

Coleman observed that changing  $\theta$  by  $2\pi$  corresponds to inserting a particle of charge  $\pm 1$  at  $x = \pm\infty$ . This means that the  $k$ -string tension can be written as

$$T_k(\theta) = \mathcal{E}(\theta + 2\pi k) - \mathcal{E}(\theta), \tag{2.2}$$

where  $\mathcal{E}(\theta)$  is the vacuum energy density as a function of  $\theta$ .

Of course, when  $m_\psi = 0$ , there is no  $\theta$  dependence in vacuum energy, because a chiral rotation can remove the  $\theta$  term from the action. This immediately implies that the massless theory does not confine integer test charges. Once a mass term for fermions is added, the  $\theta$  term can no longer be removed by chiral rotations: a transformation that would remove the topological term from the action reintroduces it in the mass term as  $m_\psi \bar{\psi}_L \psi_R \rightarrow m_\psi e^{i\theta/N} \bar{\psi}_L \psi_R$ . As a result, when  $m_\psi \neq 0$ , the degeneracy between the  $N$  chirally broken vacua is lifted. When  $m_\psi$  is small, the  $\theta$  dependence of the vacuum energy density emerges as  $\mathcal{E}_k(\theta) = -m_\psi \langle \bar{\psi}_L \psi_R \rangle + \text{c.c}$  where the chiral condensate is given by (see e.g. [22]):

$$\langle \bar{\psi}_L \psi_R \rangle = \frac{m_\psi e^\gamma}{4\pi} e^{i\frac{\theta+2\pi k}{N}}, \quad k = 1, \dots, N \tag{2.3}$$



and  $m_\gamma = Ne/\sqrt{\pi}$  is the mass gap in the theory.<sup>8</sup> The string tension for a charge- $k$  probe in the presence of the theta angle can be written as

$$T_k(\theta) = -m_\psi \mu \frac{Ne}{2\pi^{3/2}} \left[ \cos\left(\frac{\theta + 2\pi k}{N}\right) - \cos\left(\frac{\theta}{N}\right) \right] + O(m_\psi^2). \quad (2.4)$$

For  $\theta = 0$ , this expression can be simplified into

$$T_k(\theta = 0) = m_\psi \mu \frac{Ne}{\pi^{3/2}} \sin^2\left(\frac{\pi k}{N}\right) + O(m_\psi^2). \quad (2.5)$$

In these formulas  $\mu$  is a renormalization scale, which of course would cancel in appropriate ratios of dimensionful physical quantities. Clearly, there is a finite tension, and hence confinement, for charges  $k \neq 0 \pmod{N}$ , and charges that are multiples of  $N$  are screened:

$$\text{mass deformation : } \quad \langle W_k(C) \rangle = \begin{cases} e^{-T_k A(C)}, & k \neq 0 \pmod{N} \\ e^{-MP(C)}, & k = 0 \pmod{N} \end{cases}. \quad (2.6)$$

where  $A(C)$  is the area of the disk-like region enclosed by the curve  $C$ ,  $P(C)$  is the perimeter of  $C$ ,  $M$  is a non-universal mass scale, and we have assumed that  $A(C)$  is large compared to the microscopic scales of the theory, while at the same time it is small compared to the size of the spacetime manifold.

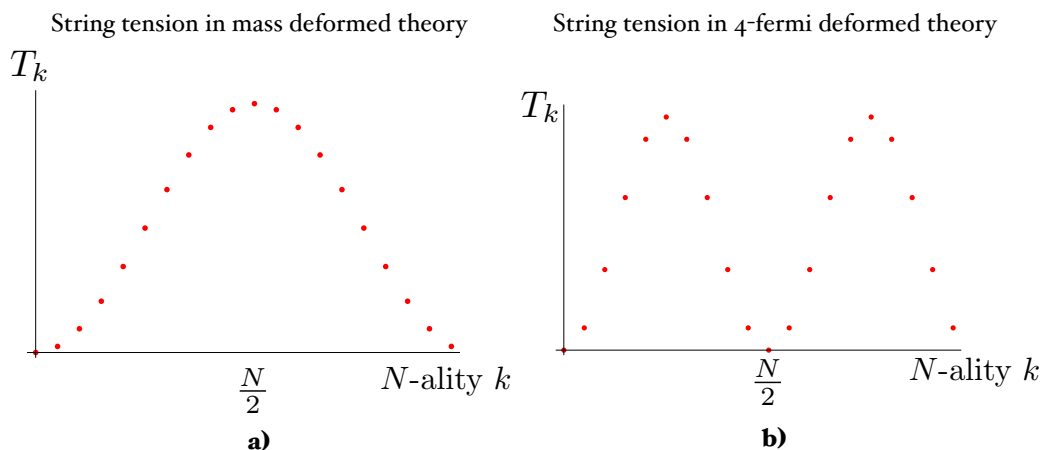
## 2.2 Confinement in the four-fermion deformed charge- $N$ Schwinger model

The discussion above may lead one to think that it is necessary to have massive fermions to achieve confinement in the Schwinger model. However, this is not true. All we need is for the vacuum energy density to have non-trivial  $\theta$ -dependence. In fact, even when the fermions are exactly massless and a chiral symmetry protects a mass term from being generated, the gauge interactions in the Schwinger model may lead to confinement of fundamental test charges. For this to be the case, what we need is a deformation which is chirally charged, so that with its inclusion, the  $\theta$  term cannot be removed, while at the same time the deformation preserves a non-trivial subgroup of the chiral symmetry so that the mass term is still forbidden. Finally, we want the deformation operator to be marginal or relevant, so that its effects survive at long distances. We will defer a discussion of this last point to the next section, and focus on the first point here.

Consider the chirally-charged four-fermion operator in eq. (1.8). The ABJ anomaly reduces  $U(1)_A$  down to  $\mathbb{Z}_{2N}$ , but the  $\mathbb{Z}_2$  part of this transformation is part of the gauge redundancy. Therefore the faithful symmetry is only  $\mathbb{Z}_N$ , as explained earlier, and the four-fermion deformation breaks the anomaly-free faithfully-acting  $\mathbb{Z}_N$  chiral symmetry down to  $\mathbb{Z}_2$  for  $N$  even and breaks it completely for  $N$  odd. Therefore, for  $N$  even, a mass term cannot be generated when the deformation is turned on. However, if  $N$  is odd, a mass term can be generated non-perturbatively. A heuristic argument for this goes as follows. Pure  $U(1)$  gauge theory on a torus has instantons with integer topological charge  $\frac{1}{2\pi} \int_{T^2} f \in \mathbb{Z}$ . If we add a massless charge  $N$  Dirac fermion, a charge 1 instanton has  $2N$

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<sup>8</sup>We use the same symbol  $e$  for the base of the natural logarithm and the gauge coupling, and hope that readers can distinguish them from context.



**Figure 2.** String tensions in the charge- $N$  Schwinger model for even  $N$  as a function of  $N$ -ality in a) the mass-deformed theory and b) the theory deformed by the two four-fermion operators. The first case exhibits a single hump structure, and the string tension is maximal at  $k = \frac{N}{2}$ , and there is generically a two-fold degeneracy of tensions. In the second case, the string tension vanishes at  $k = \frac{N}{2}$ , there is a double-hump structure, and the spectrum of string tensions is generically 4-fold degenerate.

zero modes. When  $N$  is odd we can soak up its fermion zero modes  $\frac{N-1}{2}$  times by using the four-fermion operators and generate a fermion mass term from a sum over a dilute gas of instantons. However, this picture is only heuristic because when  $T^2$  is large compared to the gauge coupling  $e$ , instantons are not localized, so a dilute instanton gas sum does not make sense. In section 5 we will discuss a regime where a semiclassical calculation involving finite-action field configuration does make sense, and make these remarks more precise.

If we do a chiral rotation to remove the topological term (1.2) in the action, we reintroduce  $\theta$  in the chirally-charged four-fermion operator as  $O_\chi \mapsto e^{2i\theta/N} O_\chi$ . Therefore, even in the absence of massless fermions, the vacuum energy density depends on  $\theta$ . Following the same steps as in the undeformed theory, we find that the string tension for a charge  $k$  probe is

$$T_k \sim -\Lambda^{2-\Delta_\chi} \left[ \cos\left(\frac{2(\theta + 2\pi k)}{N}\right) - \cos\left(\frac{2\theta}{N}\right) \right] + \mathcal{O}(\Lambda^{2(2-\Delta_\chi)}), \quad (2.7)$$

where  $\Delta_\chi$  is the scaling dimension of  $O_\chi$ . If  $\Delta_\chi > 2$ ,  $\Lambda$  is a UV scale, and  $T_k$  vanishes for all  $k$  as we take  $\Lambda/e \gg 1$ . If  $\Delta_\chi < 2$ , then  $\Lambda$  is an IR scale. The expression above assumes that  $\Lambda/e \ll 1$  when  $\Delta_\chi < 2$ . For  $\theta = 0$ , this expression can be simplified into

$$T_k \sim \Lambda^{2-\Delta_\chi} \sin^2\left(\frac{2\pi k}{N}\right) + \mathcal{O}(\Lambda^{2(2-\Delta_\chi)}). \quad (2.8)$$

Note that for  $N$  even, the string tension vanishes for charges  $k = 0, N/2 \pmod{N}$ , and is non-vanishing otherwise. For  $N$  odd, string tensions except for  $k = 0 \pmod{N}$  are non-zero. The basic fate of confinement is illustrated by the sketch in figure 1. An interesting feature in both cases is the double-hump structure of the tensions as a function of  $k$ . For

example, for odd  $N$ , the minimal tension is not  $T_1 = T_{N-1}$ , but  $T_{(N-1)/2} = T_{(N+1)/2}$ . This is illustrated in figure 2.

To summarize, in our version of the massless Schwinger model defined by (1.10) with  $g > \pi/2$ , large Wilson loops have the following behavior:

$$\langle W_k(C) \rangle = \begin{cases} e^{-T_k A(C)}, & k \neq 0, N/2 \pmod{N} \\ e^{-MP(C)}, & k = 0, N/2 \pmod{N} \end{cases}. \quad (2.9)$$

The probe charges  $k = 0, N/2 \pmod{N}$  are screened and the other probe charges are confined. The main distinction relative to the standard massive Schwinger model is the fact that  $k = N/2$  probe charge is confined in the massive model, and is screened in the four-fermion deformed model. In the semi-classical domain, we will see the microscopic difference between these two versions of confinement.

### 3 Bosonization

It is famously useful to treat the Schwinger model using bosonization, and in this section we describe the bosonized form of the model. This will allow us to understand the interplay of the two four-fermion deformations of the model, and will be very useful both for the analysis of the dynamics on  $\mathbb{R}^2$ , as well as to understand some subtleties that arise in our analysis of the physics on  $\mathbb{R} \times S^1$ .

Bosonization amounts to a ‘change of variables’ in the path integral from the fermion field  $\psi$  to a scalar field  $\varphi$ . The scalar is circle-valued,  $\varphi(x) \equiv \varphi(x) + 2\pi$ , so e.g.  $d\varphi$  and  $e^{ik\varphi}$  with integer  $k$  are good local operators, but  $\varphi$  itself is not. To write down the bosonized theory, consider a free massless fermion with gauged fermion parity. It has two  $U(1)$  global symmetries, the vector-like symmetry  $U(1)_V$  and the axial symmetry  $U(1)_A$ , with conserved current 1-forms  $j_V, j_A$  respectively. The bosonic action corresponding to the free-fermion theory is

$$S_{\varphi, \text{free}} = \int_{M_2} \frac{1}{8\pi} \|d\varphi\|^2 \quad (3.1)$$

where  $\|C\|^2 \equiv C \wedge \star C$  for any differential form  $C$ . The conserved currents of the bosonic and fermionic theories are related via

$$j_V \leftrightarrow -\frac{1}{2\pi} \star d\varphi, \quad j_A \leftrightarrow \frac{i}{4\pi} d\varphi. \quad (3.2)$$

Chiral symmetry acts on  $\varphi$  via  $e^{i\varphi} \rightarrow e^{2\pi i/N} e^{i\varphi}$ . The bosonic operator corresponding to the fermion bilinears is

$$\bar{\psi}_L(x)\psi_R(x) \leftrightarrow -\frac{\mu e^\gamma}{2\pi} e^{i\varphi(x)} \quad (3.3)$$

and  $\mu$  is the renormalization scale. This scale appears on the bosonic side of the mapping because the two-point function of  $\varphi$  calculated from the action (3.1) has logarithmic UV/IR

sensitivity,  $\langle \varphi(x)\varphi(0) \rangle - \langle \varphi(0)^2 \rangle \sim \log(x\mu)$ . When the renormalization scale is changed from  $\mu$  to  $\mu'$ , exponentials of  $\varphi$  transform as [23]

$$e^{ik\varphi} \rightarrow \left| \frac{\mu'}{\mu} \right|^{k^2} e^{ik\varphi} \tag{3.4}$$

where on the left the renormalization scale of  $e^{ik\varphi}$  is  $\mu$  while on the right it is  $\mu'$ .

We will also need relations between the four-fermion operators  $\mathcal{O}_{jj}$ ,  $\mathcal{O}_\chi$  and bosonized quantities. First, note that

$$|j_V|^2 = \bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma^\mu\psi \leftrightarrow \frac{1}{4\pi^2}|d\varphi|^2. \tag{3.5}$$

This relation follows from the bosonic substitution (3.2) for the current  $(j_V)^\mu = \bar{\psi}\gamma^\mu\psi$ . It implies that the bosonic dual of the Schwinger-Thirring model with action (1.7) is

$$S_{\text{bosonized ST}} = \int_{M_2} \left[ \frac{1}{2e^2} \|da\|^2 + \frac{1}{2}R^2 \|d\varphi\|^2 - m\mu \cos(\varphi) - \frac{iN}{2\pi}d\varphi \wedge a \right], \tag{3.6}$$

where  $m = e^\gamma m_\psi/2\pi$ , and

$$R^2 = \frac{1}{4\pi} \left( 1 + \frac{2g}{\pi} \right). \tag{3.7}$$

The parameter  $R$  gives the ‘radius’ of the canonically-normalized scalar  $\tilde{\varphi}$  associated to  $\varphi$ , so that the periodicity of  $\tilde{\varphi}$  is  $2\pi R$ . When  $g = 0$ ,  $R = 1/\sqrt{4\pi}$ , so the periodicity of  $\tilde{\varphi}$  is  $\pi^{1/2}$ .

Note when  $m = 0$ , the action (3.6) is quadratic in the fields, and so it is a free field theory in the bosonic duality frame for any  $R$  (that is, any  $g$ ). This makes it clear that  $g$  does not run in the ST model: it is an exactly marginal parameter. The energy is bounded from below so long as  $g > -\pi/2$ . When  $g \neq 0$ , the renormalization scale-change relation becomes

$$e^{ik\varphi} \rightarrow \left| \frac{\mu'}{\mu} \right|^{k^2/4\pi R^2} e^{ik\varphi} \tag{3.8}$$

Equation (3.8) implies that the scaling dimension of  $e^{ik\varphi}$  is

$$\Delta_k \equiv \Delta[e^{ik\varphi}] = \frac{k^2}{1 + 2g/\pi}, \tag{3.9}$$

so that the bosonization rule for e.g. the fermion bilinear becomes

$$\bar{\psi}_L(x)\psi_R(x) \leftrightarrow -\frac{\mu^{\Delta_1}e^\gamma}{2\pi}e^{i\varphi(x)}. \tag{3.10}$$

We can also state the bosonization rule for the operator

$$\mathcal{O}_\chi = \bar{\psi}_L\psi_R(D_\mu\bar{\psi}_L)(D^\mu\psi_R), \tag{3.11}$$

which is a scalar operator with chiral charge 2 and scaling dimension 4 at  $g = 0$ . The only such operator in the bosonic description is  $e^{2i\varphi}$ , so we conclude that

$$\mathcal{O}_\chi + \mathcal{O}_\chi^\dagger \longleftrightarrow c\mu^{\Delta_2} \cos(2\varphi) \tag{3.12}$$

where  $c$  is an  $\mathcal{O}(1)$  numerical constant.<sup>9</sup> This means that

$$\Lambda^{2-\Delta_2} \int d^2x (\mathcal{O}_\chi + \mathcal{O}_\chi^\dagger) \longleftrightarrow c\mu^{\Delta_2} \Lambda^{2-\Delta_2} \int d^2x \cos(2\varphi). \quad (3.13)$$

Since we will be working with the bosonized form of the theory from here onward, we will absorb  $c$  into the normalization of  $\Lambda$ , so it will not appear in our formulas. The bosonized action of the Schwinger model deformed by  $\mathcal{O}_{jj}$  and  $\mathcal{O}_\chi$  with  $m_\psi = 0$  is thus

$$S = \int_{M_2} \left[ \frac{1}{2} R^2 \|d\varphi\|^2 + \mu^{\Delta_2} \Lambda^{2-\Delta_2} \cos(2\varphi) + \frac{1}{2e^2} \|da\|^2 - \frac{i}{2\pi} N d\varphi \wedge a \right] \quad (3.14)$$

In what follows we will often integrate the axion-like gauge field coupling of  $\varphi$  by parts and drop the total derivative that appears in the process. Appendix A contains a discussion of some interesting global subtleties of axion interaction terms in 2d abelian gauge theories. Handling these subtleties is important in some of our calculations on  $\mathbb{R} \times S^1$ .

The action in (3.14) depends on the renormalization scale  $\mu$ . When  $\Delta_2 > 2$ , the  $\mathcal{O}_\chi$  deformation is irrelevant, and we should assume that  $\Lambda \gg m_\gamma \equiv \frac{eN}{2\pi R}$  to get a well-defined theory without needing to specify a detailed UV completion. Then the scale relevant to the low-energy physics will be  $\sim m_\gamma$ , see (4.3). When  $\Delta_2 < 2$ , the  $\mathcal{O}_\chi$  deformation is relevant, and  $\Lambda$  could be larger or smaller than  $m_\gamma$ . We will focus on the situation where  $\Lambda \ll m_\gamma$  when  $\Delta_2 < 2$ . Given this assumption, it will be useful to shift the renormalization scale to  $m_\gamma$  in (3.14) for any value of  $\Delta_2 > 0$ . Using (3.8), this gives the form of the action we will use from here onward:

$$S = \int_{M_2} \left[ \frac{1}{2} R^2 \|d\varphi\|^2 + m_\gamma^{\Delta_2} \Lambda^{2-\Delta_2} \cos(2\varphi) + \frac{1}{2e^2} \|da\|^2 - \frac{i}{2\pi} N d\varphi \wedge a \right]. \quad (3.15)$$

As we already mentioned in the introduction, when  $m = 0$ , and we do not turn on the  $\mathcal{O}_\chi$  deformation, the model has  $\mathbb{Z}_N$  1-form and 0-form symmetries. These two symmetries have a mixed 't Hooft anomaly. One simple way to see this is to analyze the theory on  $\mathbb{R} \times S^1$  [4]. Another way is to write explicit expressions for the topological operators that generate these symmetries [10]. To do this it is helpful to rewrite the action in first-order form as

$$S_{1^{\text{st order}}} = \int_{M_2} \left[ \frac{1}{2R^2} \|b^{(1)}\|^2 + ib^{(1)} \wedge d\varphi + m_\gamma^{\Delta_2} \Lambda^{2-\Delta_2} \cos(2\varphi) + \frac{e^2}{2} \|b^{(0)}\|^2 + ib^{(0)} \wedge da - \frac{i}{2\pi} N d\varphi \wedge a \right]. \quad (3.16)$$

The chiral symmetry is associated with the existence of topological line operators of the form

$$V_k(C) = \exp \left[ \frac{2\pi i k}{N} \int_C \left( b^{(1)} + \frac{N}{2\pi} a \right) \right]. \quad (3.17)$$

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<sup>9</sup>Our argument for eq. (3.12) is based on matching symmetries and scaling dimensions. We are not aware of a complete and explicit discussion of bosonization for this chirality-violating four-fermion operator in the literature for the operator  $\mathcal{O}_\chi$ , although see section 5.6 of ref. [24] for some interesting related discussion in a condensed-matter context.

The 1-form symmetry is generated by local topological operators  $U_n(x)$ . They take form

$$U_n(x) = \exp \left[ \frac{2\pi i n}{N} \left( b^{(0)} + \frac{N}{2\pi} \varphi \right) \right]. \quad (3.18)$$

The key thing to take from these expressions is that  $U_n(x)$  is charged under the  $\mathbb{Z}_N$  chiral symmetry, while  $V_k(C)$  is charged under the  $\mathbb{Z}_N$  1-form symmetry. This means that there is the 't Hooft anomaly between the two  $\mathbb{Z}_N$  global symmetries.

The expectation values of  $U_1(x)$  take the form

$$\langle U_1(x) \rangle = e^{2\pi i k/N}, \quad (3.19)$$

see e.g. [10] for an extensive discussion. The choice of  $k$  labels the  $N$  universes of the model. Domain walls between universes have infinite tension, and can be thought of as Wilson lines. The form of eq. (3.18) implies that chiral symmetry relates different values of  $k$ . The different universes have identical vacuum energy densities, which implies that chiral symmetry is spontaneously broken on  $\mathbb{R}^2$ . But the fact that domain walls between the chiral vacua can be thought of as Wilson lines implies that the 1-form symmetry is also spontaneously broken. To summarize, when  $m = 0$  and we do not turn on the  $\mathcal{O}_\chi$  perturbation, the theory does not confine test charges with  $q \in \mathbb{N}$ .

Another instructive perspective [8] on deconfinement in the massless non-deformed Schwinger model is offered by the fact that given a charge-1 Wilson loop  $W(C)$  on a contour  $C$ , we can always insert  $V_1(C')$ , where  $C'$  is a contour lying e.g. inside  $C$ , and has an opposite orientation to  $C$ . On the one hand, the operator  $V_1(C')$  is topological, so  $C'$  can be shrunk to arbitrarily small size, and  $V_1(C' \rightarrow 0) \rightarrow 1$ , so that

$$\langle W(C) \rangle = \langle V_1(C') W(C) \rangle. \quad (3.20)$$

But on the other hand, the operator  $V_k(C')$  looks like the world-line of a particle with charge  $k$ . If we take  $C' = \bar{C}$  to be the curve  $C$  traversed in the opposite direction, then

$$\langle V_1(C' = \bar{C}) W(C) \rangle = \left\langle \exp \left[ -\frac{2\pi i}{N} \int_C b^{(1)} \right] \right\rangle \quad (3.21)$$

and the expectation value on the right has a perimeter-law expectation value because  $b^{(1)}$  is not electrically charged. This argument leads to us to conclude that  $W(C)$  itself must have a perimeter-law expectation value.

If we turn on the  $\mathcal{O}_\chi$  deformation, so that  $\Lambda \neq 0$ , the exact chiral symmetry is reduced to  $\mathbb{Z}_2$ . (The approximate low-energy chiral symmetry can be larger.) There is now only one topological line operator,  $V_{N/2}(C)$ . Correspondingly, the 't Hooft anomaly is between the surviving  $\mathbb{Z}_2$  chiral symmetry and the  $\mathbb{Z}_2$  subgroup of center symmetry generated by

$$U_{N/2}(x) = \exp \left[ i\pi b^{(0)} + \frac{iN}{2} \varphi \right]. \quad (3.22)$$

This means that test charges with  $q = N/2$  should be deconfined. The fate of confinement for test charges in other representations depends on whether  $\mathcal{O}_\chi$  is relevant, and will be discussed below.

## 4 Dynamics on $\mathbb{R}^2$

In the following sections we examine the dynamics of the charge- $N$  Schwinger model with four-fermion deformations. We first discuss the physics on  $\mathbb{R}^2$ , where our analysis will be under analytic control so long as the coefficient of the  $\mathbb{Z}_2$ -invariant chiral symmetry-breaking four-fermion deformation is small enough.

First, suppose  $m = 0$ , and turn off the  $\mathcal{O}_\chi$  deformation. Let us view  $\mathbb{R}^2$  as the infinite-volume limit of some closed manifold  $M_2$ , such as a torus, and drop the boundary term in eq. (3.15). The gauge field  $a$  enters the action as  $f = da$ , so instead of integrating over  $a$  we can integrate over  $f$  as long as we ensure that the fluxes of  $f$  on  $M_2$  are properly quantized. We integrate out  $b^{(0)}, b^{(1)}$  in eq. (3.15) to get an action in terms of  $\varphi$  and  $a$ , and then note that the partition function can be written as

$$\begin{aligned} Z &= \int \mathcal{D}\varphi \mathcal{D}f \sum_{\nu \in \mathbb{Z}} \delta\left(\nu - \frac{1}{2\pi} \int_{M_2} f\right) e^{-S_{\text{Schwinger}}} \\ &= \int \mathcal{D}\varphi \mathcal{D}f \sum_{k \in \mathbb{Z}} \exp\left(ik \int_{M_2} f\right) e^{-S_{\text{Schwinger}}} \\ &= \sum_{k \in \mathbb{Z}} \mathcal{D}\varphi e^{-\int_{M_2} \mathcal{L}_k}, \end{aligned} \tag{4.1}$$

where

$$\mathcal{L}_k = \frac{1}{2} R^2 \|d\varphi\|^2 + \frac{1}{2} \left(\frac{eN}{2\pi}\right)^2 \left\| \varphi - \frac{2\pi k}{N} \right\|^2 = \frac{1}{2} \left[ \|d\tilde{\varphi}\|^2 + m_\gamma^2 \left\| \tilde{\varphi} - \frac{2\pi k R}{N} \right\|^2 \right]. \tag{4.2}$$

In the second equality above we switched to the canonically normalized field  $\tilde{\varphi}$ ,  $\varphi = \tilde{\varphi}/R$ , and defined

$$m_\gamma = \frac{eN}{2\pi R}, \tag{4.3}$$

which reduces to  $m_\gamma = eN/\sqrt{\pi}$  when  $g = 0$ . The sum over  $k$  in eq. (4.1) ensures that the path integral is invariant under  $\varphi \rightarrow \varphi + 2\pi$ . Finally, we take the limit  $M_2 \rightarrow \mathbb{R}^2$  in eq. (4.1).

The periodicity of  $\tilde{\varphi}$  is  $2\pi R$ . In the limit  $g \rightarrow \infty$  with  $eN$  fixed, the mass gap vanishes, and the long distance local physics is that of an  $\mathbb{R}$ -valued massless scalar. We can think of  $k$  as a universe label (mod  $N$ ), and the universes are all degenerate. So as long as the  $\mathcal{O}_\chi$  operator is turned off, the  $\mathbb{Z}_N$  1-form symmetry is spontaneously broken for any  $g$ , as is the  $\mathbb{Z}_N$  0-form chiral symmetry.

Now consider turning on the deformation by  $\mathcal{O}_\chi$ , while keeping  $m = 0$ . Consulting eq. (3.16) we get

$$\mathcal{L}_k = \frac{1}{2} R^2 \|d\varphi\|^2 + \frac{1}{2} \left(\frac{eN}{2\pi}\right)^2 \left\| \varphi - \frac{2\pi k}{N} \right\|^2 - m_\gamma^{\Delta_2} \Lambda^{2-\Delta_2} \cos(2\varphi) \tag{4.4}$$

When  $R^2 = 1/4\pi$  (the  $g = 0$  point), the dimensionless factor in the coefficient of  $\cos(2\varphi)$  is  $\left(\frac{eN}{\Lambda}\right)^2$ , so as we take  $\Lambda \gg eN$  the  $\cos(2\varphi)$  term becomes less and less important. This is consistent with the expectation that  $\cos(2\varphi)$  is irrelevant at  $g = 0$ . On the other hand,

if  $R^2 > 1/2\pi$  (corresponding to  $g > \pi/2$ ), the  $\cos(2\varphi)$  term becomes large when  $\Lambda \gg eN$ , which is consistent with the expectation that  $\cos(2\varphi)$  becomes relevant when  $g > \pi/2$ .

When  $\Delta_2 > 2$  the  $\mathcal{O}_\chi$  perturbation is not all that interesting. The scale  $\Lambda$  is a short-distance scale, past which the theory needs a UV completion. We get an emergent  $\mathbb{Z}_N$  chiral symmetry at long distances as we take the UV scale  $\Lambda$  to infinity. The considerations in section 2.2 imply that there is a non-vanishing string tension for  $m_\gamma/\Lambda \neq 0$ , but it goes to zero as  $\Lambda$  becomes large. In this sense the theory is no more (and no less) confining than compact U(1) QED in 2+1 dimensions on a Euclidean lattice with the Wilson gauge action.

For us the more interesting case is  $R^2 > 1/(2\pi)$ , where  $\mathcal{O}_\chi$  is relevant, and  $\Lambda$  should be interpreted as an infrared scale, which we assume is small compared to  $m_\gamma$ .<sup>10</sup> If  $N$  is even, then the chiral symmetry is  $\mathbb{Z}_2$  at long distances. We would like to determine the realization of this symmetry, as well as the realization of the  $\mathbb{Z}_N$  1-form symmetry. To do this it is important to understand when our Lagrangian is weakly coupled. The squared mass of the particles created by  $\varphi$  in the  $k$ th universe (that is, the coefficient of  $\frac{1}{2}\tilde{\varphi}^2$ ) is

$$m_{\text{eff}}^2 = m_\gamma^2 \left[ 1 + \left( \frac{\Lambda}{m_\gamma} \right)^{2-\Delta_2} \frac{4}{R^2} \cos\left(\frac{4\pi k}{n}\right) + \dots \right]. \quad (4.5)$$

while the coefficient of e.g. the quartic interaction  $\frac{\lambda}{4!}\tilde{\varphi}^4$  is

$$\lambda = m_\gamma^2 \left[ -\frac{16}{R^4} \left( \frac{\Lambda}{m_\gamma} \right)^{2-\Delta_2} \cos\left(\frac{4\pi k}{N}\right) + \dots \right]. \quad (4.6)$$

where the  $\dots$  represents terms that are higher-order in the small parameter  $\left(\frac{\Lambda}{m_\gamma}\right)^{2-\Delta_2}$ . The theory is weakly-coupled if e.g. the parameter  $|\lambda/m_{\text{eff}}^2| \ll 1$ , and

$$\left| \frac{\lambda}{m_{\text{eff}}^2} \right| = \frac{16 \left(\frac{\Lambda}{m_\gamma}\right)^{2-\Delta_2} \cos\left(\frac{4\pi k}{n}\right)}{R^4 \left[ 1 + \frac{4}{R^2} \left(\frac{\Lambda}{m_\gamma}\right)^{2-\Delta_2} \cos\left(\frac{4\pi k}{n}\right) \right]} \simeq \frac{16}{R^4} \left(\frac{\Lambda}{m_\gamma}\right)^{2-\Delta_2} \cos\left(\frac{4\pi k}{n}\right). \quad (4.7)$$

This illustrates the fact that the massless four-Fermi-deformed Schwinger model is weakly coupled when  $\left(\frac{\Lambda}{m_\gamma}\right)^{2-\Delta_2} \ll 1$ , just as the conventional massive Schwinger model is weakly coupled when  $m_\psi/e \ll 1$ .

When the model is weakly-coupled we can read off the vacuum structure and the confining string tensions by considering the vacuum energy densities of the universes in the Lagrangian. Indeed, this was already discussed in section 2, where we had already tacitly assumed that there is a duality frame where the theory is weakly coupled.

## 5 Dynamics on $\mathbb{R} \times S^1$

We now discuss the calculation of the string tension both in the mass-perturbed theory as well as the massless theory with four-fermion perturbations on  $\mathbb{R} \times S^1$  with small  $S^1$ , as

<sup>10</sup>If  $R^2 = 1/(2\pi)$ , then  $\mathcal{O}_\chi$  is marginal if the gauge interaction is turned off and  $m = 0$ . This means that  $\mathcal{O}_\chi$  enters the action with a dimensionless coefficient which we can call  $\lambda$ . It may be interesting to understand whether and how this parameter runs with the renormalization scale.



well as on  $T^2$ . The results below match the expectations established in our analysis of the physics on  $\mathbb{R}^2$ . The benefit of working out the physics on a small circle is that it allows us to establish a semiclassical picture of the confinement mechanism. In particular, we will show below that confinement is induced by the proliferation of fractional instantons with topological charge  $Q = \pm 1/N$  and action  $S_1 = 1/(4R^2 m_\gamma L)$ . These instantons all carry fermion zero modes when the theory has a  $\mathbb{Z}_N$  chiral symmetry. So when the complete chiral symmetry is present, the model does not confine  $q = \pm 1$  test charges. When chiral symmetry is explicitly broken, either by a mass term or by the  $\mathcal{O}_\chi$  deformation, some or all of the fermion zero modes get lifted, and then confinement sets in. For earlier discussions of instantons in various versions of the Schwinger model see e.g. refs. [7, 22, 25–31].

Our analysis below has several unusual features. The first one is already hinted at above: we will see that the massless Schwinger model on a cylinder has instanton solutions with topological charge  $|Q| = \ell/N$  and actions  $S = \ell^2 S_1$ , which carry  $2\ell$  fermion zero modes when the model has  $\mathbb{Z}_N$  symmetry. Remarkably, there are also exact solutions of the equations of motion with *zero* topological charge that carry *exact* fermion zero modes. These  $Q = 0$  solutions are obtained simply by adding together solutions with non-zero topological charge. The fact that one can obtain exact solutions this way comes from the fact that in the bosonized duality frame, the equations of motion satisfied by the instantons are *linear* in the fields. The fact that instantons with  $|Q| \geq 0$  carry fermion zero modes implies that they do not contribute at all to the partition function of the massless Schwinger model with  $\mathbb{Z}_N$  symmetry. Of course, as soon as we consider correlation functions of operators that involve the fermions, or add chiral-symmetry breaking deformations to the action, the story changes and the instantons do contribute.

### 5.1 Fractional instantons

We begin our analysis on  $\mathbb{R} \times S^1$  with the massless charge- $N$  Schwinger model with  $\mathbb{Z}_N$  chiral symmetry. We denote the coordinate of  $S^1$  by  $x$ , and denote the coordinate of  $\mathbb{R}$  by  $\tau$ . When the circumference of  $S^1$ ,  $L$ , is small enough compared to  $1/e$ , we can use an EFT on  $\mathbb{R}$  — that is, quantum mechanics — to describe the long-distance dynamics. To describe this EFT we need to understand the potential for the gauge field holonomy  $\Omega_x$  on  $S^1$ . If we choose periodic boundary conditions for the charge  $N$  fermion,<sup>11</sup> and take Coulomb gauge such that  $\Omega_x = e^{i \int dx a_x} \equiv e^{ih(\tau)}$ , where  $h \sim h + 2\pi$ , the 1-loop holonomy potential is given by

$$V(h) = \frac{2}{\pi L^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(Nnh) = \min_n \frac{N^2}{2\pi L^2} \left( h - \frac{2\pi}{N} \left( n + \frac{1}{2} \right) \right)^2 - \frac{\pi}{6L^2}. \quad (5.1)$$

The fundamental domain  $h \in [0, 2\pi)$  contains  $N$  harmonic minima located at

$$h_n = \frac{2\pi}{N} \left( n + \frac{1}{2} \right), \quad n = 0, 1, \dots, N - 1. \quad (5.2)$$

---

<sup>11</sup>Physically, the boundary condition on the fundamental fermion does not matter in the Schwinger model, because it is part of gauge redundancy. Any boundary condition  $\psi(x_1, x_2 + L_2) = e^{i\alpha} \psi(x_1, x_2)$  will yield the same physical result.

If we take eq. (5.1) as a quantum effective potential in the 1d EFT description of the system valid for small  $L$ , then naively this EFT will support  $N$  distinct instanton solutions interpolating between consecutive minima:

$$\mathcal{F}_{n+1} : \quad |n\rangle_B \rightarrow |n+1\rangle_B, \quad n = 0, 1, \dots, N-1, \quad N \equiv 0. \quad (5.3)$$

To understand these instantons, we first recall that the topological charge of U(1) gauge theory is defined as

$$Q = \frac{1}{2\pi} \int_{M_2} f = \frac{1}{4\pi} \int d^2x \epsilon^{\mu\nu} f_{\mu\nu}. \quad (5.4)$$

In pure U(1) gauge theory  $Q$  is an integer, and on e.g.  $T^2 = S^1_\beta \times S^1_L$  we can write instanton solutions as e.g.  $a_\tau = 0$ ,  $a_x = \frac{2\pi Q \tau}{L\beta}$ . The action of a charge  $Q$  instanton is

$$S = \frac{1}{4e^2} 2L\beta \left( \frac{2\pi^2 Q}{L\beta} \right)^2 = \frac{2\pi Q^2}{e^2 L\beta}. \quad (5.5)$$

These  $Q \in \mathbb{Z}$  instantons are of course also present in U(1) gauge theory with matter. However, note that in a  $Q \in \mathbb{Z}$  instanton,  $\int_{S^1_L} a$  evolves from 0 to  $2\pi Q$  as  $\tau$  goes from 0 to  $\beta$ . This means that  $Q \in \mathbb{Z}$  instantons cannot represent tunneling events between the nearest-neighbor vacua of (5.1).

What we need are instantons with fractional topological charge, with  $Q = \ell/N$ , with  $\ell \in \mathbb{Z}$ . When  $\ell = 1$ , these fractional instantons are precisely the tunneling solutions of eq. (5.2). In the absence of fermion zero modes, these fractional instantons can only directly contribute to the partition function if one gauges the  $\mathbb{Z}_N$  1-form symmetry. But when the  $\mathbb{Z}_N$  1-form symmetry is *global*, the topological charge of admissible field configurations that contribute to the partition function must be an integer, and the partition function only receives contributions from ‘composite’ instantons with  $Q \in \mathbb{Z}$  built from the field configurations with the fractional charges  $Q = \ell/N$ . For a general discussion of how fractional-charge instantons contribute to gauge-theory path integrals see ref. [32].

However, the effective potential in eq. (5.1) has cusps, so we have to be careful when using it to study tunneling solutions. Equation (5.1) should be understood as the 1-loop vacuum free energy density in a constant holonomy background, extracted by taking a large-volume limit. This is to be distinguished from the 1-loop effective potential in the small- $L$  quantum mechanical theory describing the *dynamics* of the holonomy. Indeed, the fact that the energy density has cusps indicates that the effective 1d EFT fails to capture important physics — in this case, the existence of fermion zero modes. This follows from the standard ABJ anomaly which is responsible for the chiral symmetry being  $\mathbb{Z}_N$  rather than U(1). This anomaly implies that field configurations with topological charge  $Q$  have  $2N$  fermion zero modes.

To better understand the tunneling solutions and their zero modes, we can analyze the small- $L$  limit using bosonization, following ref. [7]. We work on  $T^2 = S^1_\beta \times S^1_L$ , with  $L \ll 1/e \ll \beta$ , and take the coordinates on the torus to be  $(\tau, x) \sim (\tau + \beta, x + L)$ . Our goal is to understand the effective action for  $\Omega_x = e^{i \int dx a_x}$ , which is derived in detail in appendix A. Just as above, we take Coulomb gauge such that  $\Omega_x \equiv e^{ih(\tau)}$ , where  $h \sim h + 2\pi$ .

The interesting contribution to the physics comes from the sum over the scalar winding number  $\oint_{S^1} d\varphi \in 2\pi\mathbb{Z}$ . The small- $L$  path integral capturing the dynamics of the holonomy becomes

$$Z \sim \int \mathcal{D}h \exp \left[ - \int d\tau \frac{1}{2e^2 L} \left( \frac{dh}{d\tau} \right)^2 \right] \sum_{n \in \mathbb{Z}} \exp \left[ - \int d\tau \frac{m_\gamma^2}{2e^2 L} \left( h - \frac{2\pi n}{N} \right)^2 \right]. \quad (5.6)$$

There are two things to notice about the above expression. First, if we take  $h = \text{constant}$  and take the large-volume limit  $\beta \rightarrow \infty$ , the above effective potential reduces to the vacuum free energy density in eq. (5.1), up to an  $h$ -independent shift. Second, and more importantly, the effective potential for the holonomy consists of  $N$  distinct branches. Each of the  $N$  minima, located at  $h = 2\pi n/N$ , lie in distinct branches of the effective potential.

An immediate consequence of the discussion above is that tunneling configurations  $h(\tau)$  between minima of the effective potential simply do not exist. In the fermionic duality frame, this statement maps to the fact that instantons carry robust fermion zero modes, and so they cannot contribute to the partition function. To allow tunneling events between minima, we must insert an operator charged under chiral symmetry. This is the bosonic analog of ‘soaking up’ the fermion zero modes. The necessary operator is simply  $e^{i\ell\varphi}$ , which carries charge  $\ell$  under the  $\mathbb{Z}_N$  chiral symmetry. The fermionic-variable image of  $e^{i\ell\varphi}$  can be thought of as either a point-split version of  $(\bar{\psi}_L \psi_R)^\ell$ , or its local operator analogue with derivatives.

Repeating the derivation of the holonomy effective potential in the presence of the insertion  $e^{i\ell\varphi(\tau_0, x_0)}$ , one finds (see appendix A.2)

$$\tilde{Z} \sim \int \mathcal{D}h \exp \left[ - \int d\tau \frac{1}{2e^2 L} \left( \frac{dh}{d\tau} \right)^2 \right] \sum_{n \in \mathbb{Z}} \exp \left[ - \int d\tau \frac{m_\gamma^2}{2e^2 L} \left( h - \frac{2\pi(n + \ell\Theta(\tau - \tau_0))}{N} \right)^2 \right]. \quad (5.7)$$

The equation of motion for  $h$  in the presence of the insertion becomes

$$\frac{d^2 h}{d\tau^2} = m_\gamma^2 \left( h - \frac{2\pi}{N} (n + \ell\Theta(\tau - \tau_0)) \right), \quad (5.8)$$

in the  $n$ th branch. This equation has finite-action solutions interpolating between  $h = 2\pi n/N$  and  $2\pi(n + \ell)/N$  which are illustrated in figure 3 and take the form

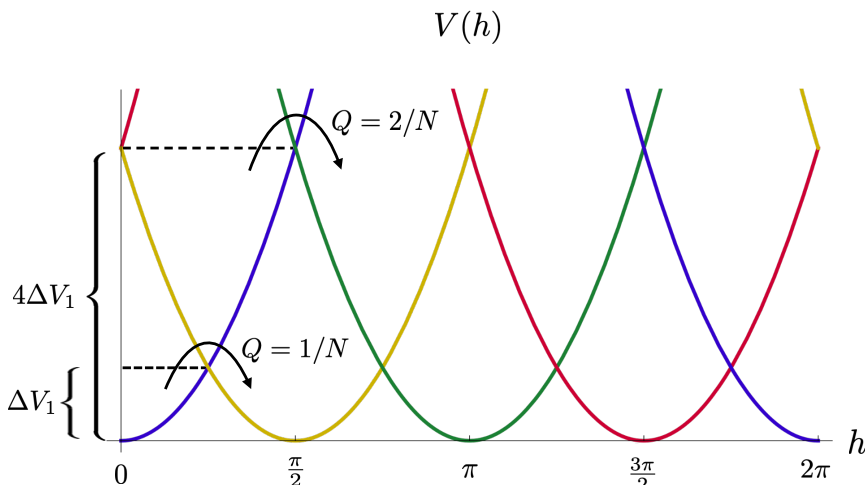
$$h_{n,\ell}(\tau; \tau_0) = \frac{2\pi n}{N} + \begin{cases} \frac{2\pi\ell}{N} \frac{\cosh(m_\gamma(\tau_0 - \beta))}{\sinh(m_\gamma\beta)} \sinh(m_\gamma\tau) & \tau \leq \tau_0 \\ \frac{2\pi\ell}{N} \left[ 1 + \frac{\cosh(m_\gamma\tau_0)}{\sinh(m_\gamma\beta)} \sinh(m_\gamma(\tau - \beta)) \right] & \tau > \tau_0 \end{cases}. \quad (5.9)$$

The topological charge and action of such an instanton in the  $\beta \rightarrow \infty$  limit are

$$Q = \frac{\ell}{N}, \quad S_\ell = \frac{\ell^2}{4R^2 m_\gamma L}. \quad (5.10)$$

The  $h$  equation of motion (5.8) with a finite number of insertions of  $e^{i\varphi}$  is linear, so sums of solutions are also solutions. For example,

$$h_{n;Q=0} = h_{n,\ell}(\tau; \tau_0) + h_{n+1,-\ell}(\tau + \tau'; \tau_0 + \tau') + \frac{2\pi\ell}{N} \quad (5.11)$$

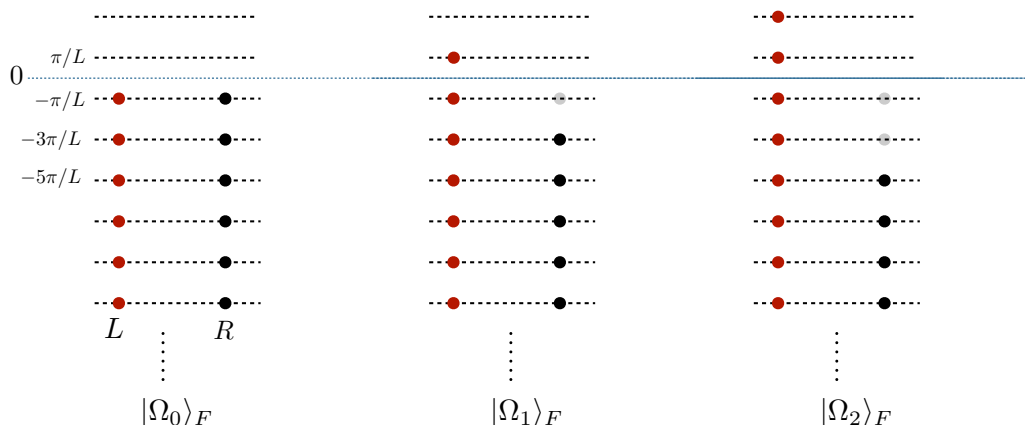


**Figure 3.** In the bosonized description, tunneling events connecting different branches of the holonomy effective potential are only possible in the presence of insertions carrying chiral charge. In the presence of the insertion  $e^{i\ell\varphi(\tau_0, x_0)}$ , a tunneling configuration in the  $n$ th branch starts at  $h = 2\pi n/N$  and travels up the potential until it meets the  $(n + \ell)$ th branch at time  $\tau_0$ . At this point, the insertion of the chirally-charged operator rearranges the branches and the tunneling configuration descends down to  $2\pi(n + \ell)/NL$ . In the above figure, we took  $N = 4$  and the cases  $\ell = 1, 2$  are indicated. The height that the instanton has to traverse scales as  $\ell^2$ , as does the instanton action (5.10).

is an exact solution of the  $h$  equation of motion with an insertion of  $e^{i\ell\varphi(\tau_0, x_0)}$  and  $e^{-i\ell\varphi(\tau_0 + \tau', x_0)}$  with  $Q = 0$  and an action  $S = \frac{\ell^2}{2R^2 m_\gamma L}$  when  $\beta$  is large. This solution has two unusual features. First, it is already quite peculiar to have an exact real solution of the equations of motion with  $Q = 0$ . Note that eq. (5.11) is *not* a saddle point at infinity (see e.g. [33]), since  $\tau'$  is finite. The other unusual feature is that this  $Q = 0$  solution has exact fermion zero modes, which are localized at  $\tau = \tau_0$  and  $\tau_0 + \tau'$ . This means that despite having vanishing topological charge it does not contribute to the partition function of the massless charge- $N$  Schwinger model with  $\mathbb{Z}_N$  chiral symmetry.

### 5.1.1 Fermion zero modes and confinement

Let us pass back to the fermionic description of the Schwinger model. The fermion zero modes of the fractional instantons discussed above play a crucial role in determining the dynamics of the charge- $N$  Schwinger model with a  $\mathbb{Z}_N$  chiral symmetry. The holonomy effective potential has  $N$  degenerate minima, so that the theory has an  $N$ -fold degenerate vacuum on a cylinder at the classical level. Normally one might expect this degeneracy to be lifted due to tunneling events connecting the classical vacua. However, our analysis above shows that tunneling events between these minima are forbidden, and the  $N$ -fold degeneracy of vacua persists when quantum effects are taken into account. This is due to the mixed 't Hooft anomaly between center-symmetry and chiral symmetry, and in practice, the absence of tunneling is enforced via the fermionic zero mode structure of the fractional instanton events.



**Figure 4.** The fermionic vacua  $|\Omega_n\rangle_F$  of the theory corresponding to harmonic bosonic ground state  $|n\rangle_B$  where  $h_n = \frac{2\pi}{N} (n + \frac{1}{2})$ , and  $\Delta h = \frac{2\pi}{N}$ . The change in the bosonic background sources is  $\Delta Q_5 = 2$ .

In the fermionic description, the harmonic perturbative vacua are not only described via the bosonic states  $|n\rangle_B$ , but also via degenerate fermionic states, which change between adjacent harmonic minima. The state  $|n\rangle$  associated with the  $n$ -th vacuum is a vector in the tensor product of states in bosonic and fermionic Hilbert spaces:

$$|n\rangle \equiv |n\rangle_B \otimes |\Omega_n\rangle_F \tag{5.12}$$

We can take  $|\Omega_0\rangle_F$  to be all L- and R-handed negative energy levels filled up as shown in figure 4.  $|\Omega_1\rangle_F$  differs from it by  $\Delta Q_5 = 2$  where all L states are shifted up by one unit and all R states are shifted down by one unit, i.e. a R-handed state is removed and a L-handed state is created. This is due to the ABJ anomaly and the consequent zero mode structure of fractional instantons. The usual instantons in the theory have  $2N$  fermion zero modes. However, the fractional instantons interpolating between consecutive vacua (5.3) have topological charge  $Q = 1/N$ , leading to  $\Delta Q_5 = 2$ .<sup>12</sup> As a result, the degenerate fermionic state  $|\Omega_n\rangle_F$  can be written as:

$$|\Omega_n\rangle_F = \prod_{k=-n}^{\infty} |k\rangle_L \otimes \prod_{k=-\infty}^{-n-1} |k\rangle_R \tag{5.13}$$

as shown in figure 4.

The amplitudes of the fractional instantons are of the form

$$\mathcal{F}_j \sim e^{-S_1} e^{i\theta/N} \bar{\psi}_L \psi_R, \quad j = 1, \dots, N. \tag{5.14}$$

The fermion fields in the prefactor of the instanton amplitude represent the fermionic zero modes. In the bosonized description, these zero modes are represented by the need to insert

<sup>12</sup>A Hilbert hotel analogy is useful. One can think of both  $L$  and  $R$  as Hilbert hotels. Both are infinite, and can always make a vacancy by moving all visitors to the next room. Or when a visitor is pushed out, the hotel still remains full.

the operator  $e^{i\varphi}$  at a spacetime point  $x$  to enable a  $Q = 1/N$  tunneling event centered at  $x$ . Thanks to the fermionic zero modes, the  $Q = 1/N$  instantons do not lift the degeneracy between the  $N$ -vacua, since the transition amplitude

$$\langle n + 1 | e^{-\beta H} | n \rangle = 0 \tag{5.15}$$

remains zero. Hence, as stated earlier, charge- $N$  QED with massless fermions and no four-fermion deformations has  $N$  exactly degenerate bosonic vacua on  $\mathbb{R}^1 \times S^1$ . This can be viewed as a consequence of the mixed 't Hooft anomaly between the  $\mathbb{Z}_N$  chiral symmetry and the  $\mathbb{Z}_N$  1-form symmetry, see e.g. appendix D of [34] for a simple example of this sort of phenomenon.

## 5.2 Mass perturbation

It has long been known that adding a mass term  $m_\psi \bar{\psi}\psi$  induces confinement. Of course, it also lifts the degeneracies between the  $N$  ground states of the  $m_\psi = 0$  theory. Indeed, when  $S^1$  is small, and  $0 < m \ll e$ , the nearest-neighbor matrix elements become non-zero, because the mass term can be used to “soak up” the fermion zero modes,

$$\langle n \pm 1 | e^{-\beta(H + \Delta H_{\text{mass}})} | n \rangle = \mu m L \beta K_1 e^{-S_1 \pm i \frac{\theta}{N}} \tag{5.16}$$

where  $K_1 = \frac{1}{2}$ , and we have set  $g = 0$  for simplicity. More generally the powers of the parameters in the prefactor in the tunneling amplitude depend on the scaling dimension of  $\bar{\psi}\psi$ , which depends on  $g$ . Indeed, in the bosonized description of the model, we can formally expand the

$$\exp \left[ \int d^2x \mu m \left( e^{i\varphi(x)} + e^{-i\varphi(x)} \right) \right] \tag{5.17}$$

term in the exponentiated Euclidean action in powers of  $m$ , and then note that this produces a sum of powers of  $e^{\pm i\varphi(x)}$  summed over the insertion points  $x$ . These insertions induce tunneling events from  $\varphi = 2\pi n/N$  to  $\varphi = 2\pi(n \pm 1)/N$ . Then the analysis in the preceding section implies that the  $\ell = \pm 1$  events contribute to the path integral with the weights given in (5.16).

To see the effect of the breaking of degeneracy of the vacua on confinement, let us consider the partition function  $Z(\beta)$  of the system in Born-Oppenheimer approximation. To calculate it, we need to sum over all periodic paths,  $a(\beta) = a(0)$ . These paths are described by maps  $S^1 \rightarrow S^1$ , and are classified by the winding number  $W \in \pi_1(S^1) = \mathbb{Z}$ , which is the integer valued topological charge. On the other hand, the physical system possess topological configurations with fractional topological charge  $Q = \pm 1/N$ . They are seeded by the  $e^{\pm i\varphi(x)}$  operators that come from the expansion of the mass term. If  $n$  ( $\bar{n}$ ) denotes the number of fractional instantons and anti-instantons, the configurations that contribute to  $Z(\beta)$  must satisfy

$$n - \bar{n} = WN, W \in \mathbb{Z}. \tag{5.18}$$

This condition is enforced by the constraint on the winding number. It is also enforced by the fact that when we expand the mass term in powers of  $m$  to produce a sum over insertions

of  $e^{\pm i\varphi}$ , the path integral measure is invariant under  $\mathbb{Z}_N$  chiral symmetry. Equation (5.18) allows configurations with integer topological charge but fractional action to contribute, e.g.,  $W = \frac{1}{N} - \frac{1}{N} = 0, S = 2S_1$  where  $S_1 \sim 1/N$ . As a result, the partition function can be expressed as:

$$\begin{aligned} Z(\beta, \theta) &= N \sum_{W \in \mathbb{Z}} \sum_{n=0}^{\infty} \sum_{\bar{n}=0}^{\infty} \frac{1}{n!} \frac{1}{\bar{n}!} \left( \beta L \mu m K_1 e^{-S_1 + i\theta/N} \right)^n \left( \beta L \mu m K_1 e^{-S_1 - i\theta/N} \right)^{\bar{n}} \delta_{n - \bar{n} - WN, 0} \\ &= \sum_{k=0}^{N-1} \sum_{n=0}^{\infty} \sum_{\bar{n}=0}^{\infty} \frac{1}{n!} \frac{1}{\bar{n}!} \left( \beta L \mu m K_1 e^{-S_1 + i\frac{\theta + 2\pi k}{N}} \right)^n \left( \beta L \mu m K_1 e^{-S_1 - i\frac{\theta + 2\pi k}{N}} \right)^{\bar{n}} \\ &= \sum_{k=0}^{N-1} e^{2\beta K_1 e^{-S_1} \cos \frac{\theta + 2\pi k}{N}} = \sum_{k=0}^{N-1} e^{-\beta E_k(\theta)}. \end{aligned} \tag{5.19}$$

Here we have assumed the dilute-instanton-gas limit, where  $S_1 \gg 1$ . To pass to the second line we converted the sum over winding number  $W \in \mathbb{Z}$  into a sum over lattice momenta  $k$  that form eigenstates of  $\mathbb{Z}_N$  translation symmetry by using the identity  $N \sum_{W \in \mathbb{Z}} \delta_{n - \bar{n} - WN} = \sum_{k=0}^{N-1} e^{i2\pi k(n - \bar{n})/N}$ . This can be used to do the sum over  $n$  and  $(\bar{n})$ , and yields the energy eigenvalues of the  $N$  branches of the vacuum energy in the small  $m_\psi$  limit:

$$E_k(\theta) = -\mu m L e^{-S_1} \cos \left( \frac{\theta + 2\pi k}{N} \right). \tag{5.20}$$

The corresponding eigenstates of the Hamiltonian are given by

$$|\theta, k\rangle = \sum_{n \in \mathbb{Z}} e^{i\frac{\theta + 2\pi k}{N} n} |n\rangle. \tag{5.21}$$

The ground state energy in a given range of theta is found by minimizing over the branches, and is given by  $E_{\text{gr}}(\theta) = \text{Min}_k E_k(\theta)$ . In the  $-\pi/N < \theta < \pi/N$  range, the ground state is the  $k = 0$  branch.

We can now compute the string tension. Computing the two-point function of charge- $k$  Polyakov loops at a large separation  $\tau$  in the vacuum for  $-\pi < \theta < \pi$  which correspond to  $|\theta, k' = 0\rangle$ , we obtain

$$\langle \theta, 0 | P^{-k}(\tau) P^k(0) | \theta, 0 \rangle \sim \exp[-\tau(E_k(\theta) - E_0(\theta))]. \tag{5.22}$$

In the semi-classical regime, we obtain

$$T_k(\theta) = -\mu m e^{-\pi/m_\gamma L} \left\{ \cos \left( \frac{\theta + 2\pi k}{N} \right) - \cos \left( \frac{\theta}{N} \right) \right\}. \tag{5.23}$$

Note that in the semi-classical domain, as on  $\mathbb{R}^2$ , the string tension vanishes for massless fermions. The benefit of working on  $\mathbb{R} \times S^1$  is that we can see the mechanism of confinement in the massive charge- $N$  Schwinger model: it is induced by fractional instantons with  $Q = 1/N$ , provided that their fermion zero modes are lifted.

### 5.3 $\mathcal{O}_\chi$ perturbation

We now discuss the other perturbation that can lift the vacuum degeneracies of the charge- $N$  Schwinger model: the perturbation by the operator  $\mathcal{O}_\chi$ . When  $N$  is even, this operator breaks the  $\mathbb{Z}_N$  chiral symmetry to  $\mathbb{Z}_2$ . In the fermionic description, this is a four-fermion operator. This operator can be relevant or irrelevant depending on the coefficient of the only chirally-invariant perturbation  $\mathcal{O}_{\text{jj}}$  of the model. Of course, the deformation by  $\mathcal{O}_\chi$  is most interesting when it is relevant.

When  $\mathcal{O}_\chi$  is added to the action and  $N$  is even, then

$$\langle n \pm 1 | e^{-\beta(H+\Delta H_\chi)} | n \rangle = 0, \tag{5.24}$$

$$\langle n \pm 2 | e^{-\beta(H+\Delta H_\chi)} | n \rangle = \beta L K_2 \Lambda^{2-\Delta_\chi} \mu^{\Delta_\chi} e^{-S_2 \pm 2i \frac{\theta}{N}}. \tag{5.25}$$

where  $K_2$  is a dimensionless constant. To see this we can formally expand the  $\exp \left[ \int d^2x \Lambda^{2-\Delta_\chi} \mathcal{O}_\chi \right]$  term in the exponentiated Euclidean action in powers of  $\Lambda$ , and then note that this produces a sum over insertions of  $e^{\pm i \ell \varphi(x)}$  with  $\ell \in 2\mathbb{Z}$ , which induce tunneling events localized at the points  $x$ . The absence of insertions with  $\ell = \pm 1 \pmod N$  implies (5.24), while the presence of insertions with  $\ell = \pm 2$  implies (5.25).

If  $N$  is odd, then after adding the  $\mathcal{O}_\chi$  deformation to the action, the matrix elements we discussed above become

$$\langle n \pm 1 | e^{-\beta(H+\Delta H_\chi)} | n \rangle = \left( \beta L K_2 \Lambda^{2-\Delta_\chi} \mu^{\Delta_\chi} e^{-S_2 \mp 2i \frac{\theta}{N}} \right)^{(N-1)/2}, \tag{5.26}$$

$$\langle n \pm 2 | e^{-\beta(H+\Delta H_\chi)} | n \rangle = \beta L K_2 \Lambda^{2-\Delta_\chi} \mu^{\Delta_\chi} e^{-S_2 \pm 2i \frac{\theta}{N}}. \tag{5.27}$$

The reason the matrix element (5.26) is non-zero is that if one starts in vacuum  $n$ , then tunneling to the next-to-nearest-neighbor on the ‘right’  $(N-1)/2$  times puts one in the vacuum with label  $n-1$ , due to the fact that the label  $n$  has a periodicity of  $N$ .

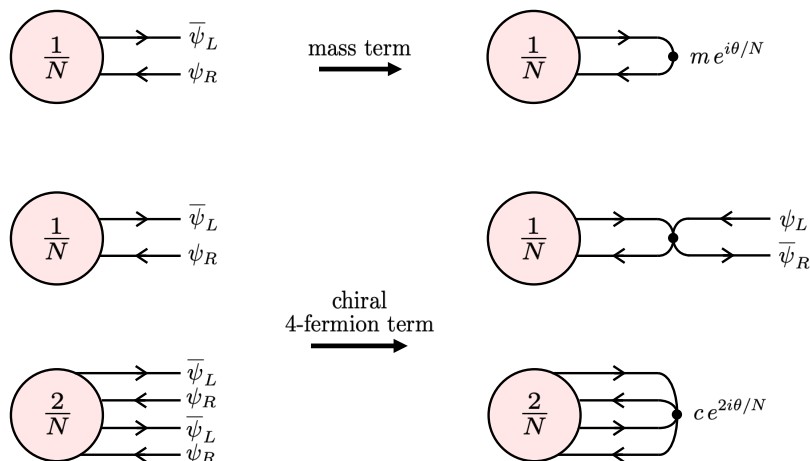
We can summarize this by observing that for even  $N$ , there is no tunneling to nearest neighbor vacua at all. The holonomy effective potential vacua split into two sets that never mix with each other under time evolution. This is a consequence of the mixed ’t Hooft anomaly between the  $\mathbb{Z}_2$  subgroup of the  $\mathbb{Z}_N$  1-form symmetry and the unbroken  $\mathbb{Z}_2$  subgroup of the 0-form chiral symmetry. However, for odd  $N$ , the amplitude to move to nearest neighbor minima is non-zero: it occurs at order  $\frac{N-1}{2}$  in semiclassics, and is suppressed by  $e^{-\frac{N-1}{2} S_2}$ . Of course, the disparity between even and odd  $N$  disappears at large  $N$ .

Next we consider the behavior of the string tension. In the Born-Oppenheimer approximation, the tunneling Hamiltonian can be written as

$$H_{\text{BO}} = - \sum_{n=1}^N K_2 \Lambda^{2-\Delta_\chi} \mu^{\Delta_\chi} e^{-S_2 + 2i \frac{\theta}{N}} |n+2\rangle \langle n| + \text{h.c.} \tag{5.28}$$

For even  $N$ , this Hamiltonian decompose to two decoupled Hamiltonians, one with a sum over  $n \in 0, 2, \dots, N-2$  and the other with a sum over  $n \in 1, 3, \dots, N-1$ . These sets of states remain unmixed at any non-perturbative order. For odd  $N$ , there is no such decomposition.





**Figure 5.** (Top) In the semi-classical domain of massive Schwinger model, confinement is generated by fractional instantons where zero modes are soaked up by mass term. (Middle) In massless Schwinger model with four-fermion deformations, it is not possible to lift the zero modes of a fractional instanton, but just flip its chirality. (Bottom) But it is possible to lift up the zero modes of a fractional instanton with topological charge  $Q = 2/N$ . These configurations causes confinement for external probe charges.

The energy spectrum can be obtained by diagonalizing  $H_{\text{BO}}$  and is given by

$$E_k(\theta) = -K_2 \Lambda^{2-\Delta_x} \mu^{\Delta_x} e^{-S_2} \cos\left(\frac{2(\theta + 2\pi k)}{N}\right). \tag{5.29}$$

Therefore, using the correlator (5.22), we can deduce that the string tensions are:

$$T_k = -K_2 \Lambda^{2-\Delta_x} \mu^{\Delta_x} e^{-S_2} \left[ \cos\left(\frac{2(\theta + 2\pi k)}{N}\right) - \cos\left(\frac{2\theta}{N}\right) \right] + O(c^2). \tag{5.30}$$

This result also extrapolates to the result we obtained on  $\mathbb{R}^2$  in the decompactification limit. As discussed in the previous section, for  $-\pi < \theta < \pi$ , it leads to confinement for all external charges for which  $k \neq 0, N/2 \pmod{N}$  and screening for  $k = 0, N/2 \pmod{N}$ .

## 6 Conclusions

At first glance the charge- $N$  Schwinger model is an attractive solvable toy model for questions about quark confinement and chiral symmetry breaking: it has a 1-form  $\mathbb{Z}_N$  chiral symmetry as well as a (discrete) chiral symmetry, just as 4d gauge theories with massless vector-like fermions. But it has long been known that the dynamics of this 2d model is radically different from 4d gauge theories: it does not confine fundamental-representation test charges in the chiral (vanishing charged fermion mass) limit. Indeed, by now it is common wisdom that 2d gauge theories with massless fermions do not confine, with both abelian examples like the charge- $N$  Schwinger model and non-abelian examples such as 2d  $SU(N)$  QCD with one Majorana fermion (see e.g. [8, 35–45]), among others [46].

In view of the very sharp difference of this behavior of 2d gauge theories with massless fermions from naive expectations based on more familiar 4d examples, it is important to

understand what drives these differences in a precise way. A natural but naive guess based on the paragraph above is that this behavior is driven by masslessness of the fermions, along with peculiarities of confinement in 2d. This is not quite correct. Instead, the crucial issue<sup>13</sup> is the presence (or absence) of appropriate mixed 't Hooft anomalies involving the 1-form symmetry. When appropriate 't Hooft anomalies are present the 2d gauge theories lie in deconfined phases on  $\mathbb{R}^2$ , while when appropriate 't Hooft anomalies are absent, they confine.

Our discussion in this paper gives a sharp illustration of this point in the context of the charge- $N$  Schwinger model. We have analyzed a modified version of the Schwinger model with dynamics that are closer to those of 4d gauge theories. The standard massless Schwinger model has a mixed 't Hooft anomaly between the  $\mathbb{Z}_N$  1-form symmetry and the  $\mathbb{Z}_N$  chiral symmetry, which leads to deconfinement. Our modification involves turning on deformations of the action by two four-fermion operators  $\mathcal{O}_{ij}$  and  $\mathcal{O}_\chi$ . The  $\mathcal{O}_{ij}$  operator is neutral under all of the symmetries of the model, while  $\mathcal{O}_\chi$  has charge 2 under chiral symmetry. The basic idea is that when it is relevant, the  $\mathcal{O}_\chi$  deformation reduces the chiral symmetry. However, since  $\mathcal{O}_\chi$  has chiral charge 2, it can preserve a  $\mathbb{Z}_2$  chiral symmetry, and the charged fermions do not get a mass term.<sup>14</sup> At the same time, the 't Hooft anomaly structure is altered, and so the theory should confine.

Let us review this in a little bit more detail. The mass operator has chiral charge 1. When  $N$  is even, and the story is simplest, the effect of these deformations is to break the chiral symmetry from  $\mathbb{Z}_N$  to  $\mathbb{Z}_2$  when  $N$  is even. Provided the coefficient of  $\mathcal{O}_{ij}$  is positive and large enough, the  $\mathcal{O}_\chi$  operator is relevant, and so the chiral symmetry remains  $\mathbb{Z}_2$  deep in the infrared. This symmetry is enough to forbid the fermion mass term, so the deformed theory must be viewed as a variant of the massless Schwinger model. We analyzed the behavior of this theory on  $\mathbb{R}^2$  and  $\mathbb{R} \times S^1$ , and showed that it confines fundamental test charges when  $N$  is even and is larger than 2. Test charges with charge  $q = N/2$  remain deconfined to a residual mixed 't Hooft anomaly between the  $\mathbb{Z}_2$  subgroup of the 1-form symmetry and the unbroken chiral symmetry. When  $N$  is odd, the  $\mathcal{O}_\chi$  perturbation breaks chiral symmetry completely. Depending on whether it is relevant or irrelevant, it may or may not be possible to get an emergent chiral symmetry in the infrared, see Footnote 6. When there is an emergent chiral symmetry in the infrared, the odd  $N$  model remains deconfined even with  $\mathcal{O}_\chi$ . Of course, when the  $\mathcal{O}_\chi$  operator is relevant and  $N$  is odd, test charges of all representations with non-zero  $N$ -ality are confined.

Our results on relevance versus irrelevance of the operator  $\mathcal{O}_\chi$  follow from standard properties of the abelian bosonization dictionary. We reached our results on confinement in three basic ways. First, they essentially follow from the totalitarian principle of QFT combined with symmetry arguments: in the absence of an 't Hooft anomaly forcing deconfinement, and without fine-tuning of relative vacuum energies of universes in this model (which could be done using deformations by local topological operators without breaking any additional symmetries), nothing forbids an area law for large Wilson loops, so one-form symmetries should not be spontaneously broken in 1 + 1d QFTs. Second, we took

<sup>13</sup>At least in the absence of deformations of the action by local topological operators — such deformations make the story more complicated, see ref. [10] for an extensive discussion.

<sup>14</sup>This assumes that we use a regulator that preserves  $\mathbb{Z}_2$  chiral symmetry.

advantage of bosonization to explicitly analyze the behavior of the theory on  $\mathbb{R}^2$ . For suitable values of deformation parameters bosonization allows one to systematically calculate observables like string tensions in abelian 2d gauge theories, which would normally require an intractable strong-coupling analysis in more complicated theories. We then examined the behavior of theory on  $\mathbb{R} \times S^1$  for small  $S^1$  (compared to e.g.  $1/(eN)$ ). This has the advantage that one can use either the bosonized or fermionic duality frames, and allowed us to explore the confinement mechanism in some detail. Confinement is driven by instantons with fractional topological charge, with a number of interesting parallels to confinement in 4d gauge theories on  $\mathbb{R}^3 \times S^1$ . The instanton physics in this model also has some unusual features: for example, there are finite-action configurations with zero topological charge with robust fermion zero modes.

Much of this has interesting parallels to  $SU(N)$  QCD with one adjoint Majorana fermion in 2d, see [21, 40]. It was pointed out in ref. [21] that 2d adjoint QCD also has two four-fermion deformations which preserve all of the standard symmetries of the model. In contrast to the 2d Schwinger model, the four-fermion deformations of 2d adjoint QCD are both classically marginal. Adjoint QCD in 2d can also be deformed by adding local topological operators to the action without breaking any standard symmetries [10]. At generic points in the parameter space of the model, there's no way to forbid an area law for large Wilson loops (the 't Hooft anomalies are not rich enough to do this in general), and so the model should confine at generic points in this parameter space. Note that this is a statement about the theory with the fermion mass term set to zero, where it is protected by symmetries from radiative corrections. Nevertheless, there is a corner in this parameter space where the model *does* deconfine, as explained in ref. [8]. In this corner of the parameter space, there is an extra unconventional symmetry which is generated by non-invertible topological line operators. These non-invertible lines are charged under the 1-form symmetry, which means that they participate in a mixed 't Hooft anomaly. This can be used to show that the area law term cannot arise in expectation values of large 't Hooft loops. However, just as in the charge- $N$  Schwinger model, the existence of chiral-symmetry-preserving deformations that drive the theory to confine means that keeping the charged fermion mass term set to zero is not *by itself* enough to drive deconfinement in 2d gauge theories.

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## A Bosonization and holonomy effective potential

In this appendix we use bosonization to compute the exact effective potential for the gauge field holonomy in the massless Schwinger model. This amounts to solving the theory on a torus. We emphasize certain aspects of the global structure of the bosonized theory that are particularly important in obtaining the correct result for the holonomy effective potential. A closely related discussion of axion-like couplings for abelian gauge fields appears in ref. [47].

We work on  $T^2 = S_L^1 \times S_\beta^1$ , with coordinates  $(\tau, x) \sim (\tau + \beta, x + L)$ . The gauge field satisfies the periodicity conditions

$$a(\tau + \beta, x) = a(\tau, x) + dh_\tau(\tau, x), \quad a(\tau, x + L) = a(\tau, x) + dh_x(\tau, x), \quad (\text{A.1})$$

where  $h_\tau, h_x$  are transition functions subject to the consistency condition

$$dh_\tau(\tau, x + L) + dh_x(\tau, x) = dh_\tau(\tau, x) + dh_x(\tau + \beta, x). \quad (\text{A.2})$$

In a U(1) gauge theory, we demand that the transition functions satisfy the cocycle condition

$$h_\tau(\tau, x + L) - h_\tau(\tau, x) - h_x(\tau + \beta, x) + h_x(\tau, x) = 2\pi Q, \quad (\text{A.3})$$

where  $Q \in \mathbb{Z}$ . The integer  $Q$  is in fact the topological charge on the torus,

$$\frac{1}{2\pi} \int_{T^2} da = 2\pi Q. \quad (\text{A.4})$$

In the path integral we sum over transition functions satisfying the cocycle condition for fixed  $Q$ , and sum over  $Q$ . Gauge transformations act as  $a \rightarrow a + d\lambda$  where  $\lambda(\tau, x)$  is an arbitrary real-valued function.<sup>15</sup> The transition functions themselves transform under gauge transformations (redundancies) as

$$h_\tau(\tau, x) \rightarrow h_\tau(\tau, x) + \lambda(\tau + \beta, x) - \lambda(\tau, x) + 2\pi m_\tau, \quad (\text{A.5})$$

$$h_x(\tau, x) \rightarrow h_x(\tau, x) + \lambda(\tau, x + L) - \lambda(\tau, x) + 2\pi m_x, \quad (\text{A.6})$$

which leaves the cocycle condition and the integer  $Q$  invariant.

The compact scalar  $\varphi$  can wind around the two cycles of the torus

$$\varphi(\tau + \beta, x) = \varphi(\tau, x) + 2\pi n_\tau, \quad \varphi(\tau, x + L) = \varphi(\tau, x) + 2\pi n_x. \quad (\text{A.7})$$

Naively, the bosonization rules suggest that the coupling between  $\varphi$  and  $a$  to be

$$\frac{iN}{2\pi} \int_{\tau_\star}^{\tau_\star + \beta} d\tau \int_{x_\star}^{x_\star + L} dx a_\mu(\tau, x) \epsilon^{\mu\nu} \partial_\nu \varphi(\tau, x). \quad (\text{A.8})$$

---

<sup>15</sup>We are used to requiring that gauge transformations are single-valued on the torus mod  $2\pi$ . In the current presentation, this is an unnecessary assumption. If we add charge-1 matter, for instance, it obeys  $\phi(\tau + \beta, x) = e^{ih_\tau(\tau, x)} \phi(\tau, x)$ ,  $\phi(\tau, x + L) = e^{ih_x(\tau, x)} \phi(\tau, x)$ . These boundary conditions are gauge-invariant with respect to (A.5), even for gauge functions which are not single-valued on the torus.

Here we have chosen an arbitrary basepoint  $(\tau_*, x_*)$  of the torus. The above integral suffers from two problems: it is not gauge invariant mod  $2\pi i$ , and it depends explicitly on the choice of reference point.<sup>16</sup> Following refs. [47, 49], we remedy these issues by including correction terms involving the transition functions and topological charge:

$$\begin{aligned} \left\langle \frac{i}{2\pi} \int a \wedge d\varphi \right\rangle &= \frac{i}{2\pi} \int_{\tau_*}^{\tau_*+\beta} d\tau \int_{x_*}^{x_*+L} dx a_\mu \epsilon^{\mu\nu} \partial_\nu \varphi \\ &\quad - \frac{i}{2\pi} \int_{x_*}^{x_*+L} dx h_\tau(\tau_*, x) \partial_x \varphi(\tau_*, x) + \frac{i}{2\pi} \int_{\tau_*}^{\tau_*+\beta} d\tau h_x(\tau, x_*) \partial_\tau \varphi(\tau, x_*) \\ &\quad + iQ\varphi(\tau_*, x_*). \end{aligned} \tag{A.9}$$

One can readily verify that the above result is manifestly gauge invariant, and (by repeated use of the cocycle condition) that it is independent of the choice of  $\tau_*, x_*$ . We may also integrate by parts to find a consistent expression for the typical ‘BF’ form of the topological coupling,

$$\begin{aligned} \left\langle \frac{i}{2\pi} \int \varphi \wedge da \right\rangle &= \frac{i}{2\pi} \int_{\tau_*}^{\tau_*+\beta} d\tau \int_{x_*}^{x_*+L} dx \varphi (\partial_\tau a_x - \partial_x a_\tau) \\ &\quad + in_\tau \left[ h_x(\tau_*, x_*) - \int_{x_*}^{x_*+\beta} dx a_x(\tau_*, x) \right] \\ &\quad - in_x \left[ h_\tau(\tau_*, x_*) - \int_{\tau_*}^{\tau_*+\beta} d\tau a_\tau(\tau, x_*) \right]. \end{aligned} \tag{A.10}$$

The quantities in brackets in the above correction terms are the (gauge-invariant) holonomies of the gauge field around the two non-contractible cycles.

### A.1 Dimensional reduction

Taking inspiration from ref. [7] we now consider the dimensional reduction of the theory on a circle, taking  $eL \ll 1$ . We start with the usual expression for the bosonized action,

$$\mathcal{L} \doteq \frac{1}{2e^2} |da|^2 + \frac{R^2}{2} |d\varphi|^2 + \frac{iN}{2\pi} a \wedge d\varphi, \tag{A.11}$$

and consider the correction terms at the end. We take Coulomb gauge  $\partial_x a_x = 0$ . This implies that the transition functions satisfy  $\partial_x^2 h_\tau = 0$  and  $\partial_x h_x = 0$ . We have further gauge freedom with gauge functions satisfying  $\partial_x^2 \lambda = 0$ , which we can use to set  $h_x = 0$ , while the cocycle condition fixes

$$h_\tau(\tau, x) = \frac{2\pi Q x}{L} + g(\tau) \tag{A.12}$$

for some arbitrary real-valued function  $g(\tau)$ . Hence, the boundary conditions on the gauge field are

$$a_\tau(\tau + \beta, x) = a_\tau(\tau, x) + \partial_\tau g(\tau), \quad a_x(\tau + \beta) = a_x(\tau) + \frac{2\pi Q}{L}, \tag{A.13}$$

---

<sup>16</sup>The fact that directly using the bosonization rules gives rise to gauge non-invariant terms in the Schwinger model on compact spacetimes was noticed in ref. [48].

and  $a_\tau, a_x$  are periodic in the  $x$  direction. Varying  $a_\tau$ , we find the Gauss law

$$\partial_x^2 a_\tau = \frac{iNe^2}{2\pi} \partial_x \varphi. \quad (\text{A.14})$$

If we integrate both sides over  $x$ , we find that  $\varphi$  cannot wind around  $S_L^1$ . This sets  $n_x = 0$  in eq. (A.7), so that without loss of generality we can make the decomposition

$$\begin{aligned} \varphi(\tau, x) &= \sum_{\ell \in \mathbb{Z}} \varphi_\ell(\tau) e^{2\pi i \ell (x - x_*) / L}, \\ a_\tau(\tau, x) &= \sum_{\ell \in \mathbb{Z}} a_{\tau, \ell}(\tau) e^{2\pi i \ell (x - x_*) / L}, \end{aligned} \quad (\text{A.15})$$

and the Gauss law gives the relation  $a_{\tau, \ell}(\tau) = \frac{Ne^2 L}{4\pi^2 \ell} \varphi_\ell(\tau)$  (for the nonzero modes). Plugging in the Kaluza-Klein decomposition and integrating over  $x$  gives the naive form of the effective action on  $S_\beta^1$ ,

$$\begin{aligned} S \stackrel{!}{=} L \int_{\tau_*}^{\tau_* + \beta} d\tau \left\{ \frac{1}{2e^2} (\partial_\tau a_x)^2 + \frac{R^2}{2} (\partial_\tau \varphi_0)^2 - \frac{iN}{2\pi} a_x \partial_\tau \varphi_0 \right. \\ \left. + R^2 \sum_{\ell > 0} \left[ \partial_\tau \varphi_\ell \partial_\tau \varphi_{-\ell} + \left( \left( \frac{2\pi \ell}{L} \right)^2 + \frac{N^2 e^2}{4\pi^2 R^2} \right) \varphi_\ell \varphi_{-\ell} \right] \right\} \end{aligned} \quad (\text{A.16})$$

The correction terms from eq. (A.9) give

$$\begin{aligned} iNQ\varphi(\tau_*, x_*) - \frac{iN}{2\pi} \int_{x_*}^{x_* + L} dx \left[ \frac{2\pi Q x}{L} + g(\tau_*) \right] \partial_x \varphi(\tau_*, x) \\ = iNQ\varphi(\tau_*, x_*) - \frac{iN}{2\pi} 2\pi Q [\varphi(\tau_*, x_*) - \varphi_0(\tau_*)] = iNQ\varphi_0(\tau_*). \end{aligned} \quad (\text{A.17})$$

Therefore, ignoring the non-zero modes of  $\varphi$ , which decouple exactly, we are left with

$$S = iNQ\varphi_0(\tau_*) + L \int_{\tau_*}^{\tau_* + \beta} d\tau \left[ \frac{1}{2e^2} (\partial_\tau a_x)^2 + \frac{R^2}{2} (\partial_\tau \varphi_0)^2 - \frac{iN}{2\pi} a_x \partial_\tau \varphi_0 \right]. \quad (\text{A.18})$$

The additional term involving the topological charge, which descended from the construction on  $T^2$ , matches the same term used in ref. [50] to properly define the Lagrangian describing the above quantum mechanical theory of  $N$  degenerate states.

## A.2 Holonomy potential

We are now in a position to derive the holonomy effective potential. Recall that in our chosen gauge, the holonomy around  $S_L^1$  is

$$e^{-ih_x(\tau, x_*)} e^{i \int_{x_*}^{x_* + L} dx a_x(\tau, x)} = e^{iLa_x(\tau)} \equiv e^{ia(\tau)}. \quad (\text{A.19})$$

For convenience, we drop the subscript on  $\varphi_0$ , choose  $\tau_* = 0$ , and replace  $n_\tau \rightarrow P$ . The boundary conditions obeyed by the fields are

$$a(\tau + \beta) = a(\tau) + 2\pi Q, \quad \varphi(\tau + \beta) = \varphi(\tau) + 2\pi P, \quad (\text{A.20})$$

with the values of  $Q, P$  summed over in the path integral. For completeness, we derive the holonomy effective potential in the presence of an insertion  $e^{i\ell\varphi(\tau_0)}$ . In order to get a non-vanishing result we have to modify the boundary conditions to

$$a(\tau + \beta) = a(\tau) + 2\pi \left( Q + \frac{\ell}{N} \right). \quad (\text{A.21})$$

The action in that case is

$$S = \int_0^\beta d\tau \left[ \frac{1}{2e^2 L} \dot{a}^2 + \frac{LR^2}{2} \dot{\varphi}^2 - \frac{iN}{2\pi} a \dot{\varphi} \right] + iN \left( Q + \frac{\ell}{N} \right) \varphi(0) - i\ell \varphi(\tau_0). \quad (\text{A.22})$$

Note that we had to modify the correction term to account for the new boundary conditions. We perform a mode decomposition consistent with the boundary conditions,

$$\varphi(\tau) = \frac{2\pi P}{\beta} \tau + \sum_{k \in \mathbb{Z}} \varphi_k e^{i\frac{2\pi k}{\beta} \tau} \quad a(\tau) = \frac{2\pi \left( Q + \frac{\ell}{N} \right)}{\beta} \tau + \sum_{k \in \mathbb{Z}} a_k e^{i\frac{2\pi k}{\beta} \tau}. \quad (\text{A.23})$$

The various terms in the action become, after completing the square,

$$\begin{aligned} S = & \frac{2\pi^2}{e^2 L \beta} \left( Q + \frac{\ell}{N} \right)^2 + \sum_{k \neq 0} \frac{2\pi^2 k^2}{e^2 L \beta} |a_k|^2 + \frac{2\pi^2 LR^2}{\beta} P^2 \\ & - iN\pi \left( Q + \frac{\ell}{N} \right) P - iNP \left( a_0 + \frac{2\pi\ell}{N} \frac{\tau_0}{\beta} \right) + iNQ\varphi_0 \\ & + \sum_{k \neq 0} \frac{\beta N^2}{8\pi^2 LR^2} \left| a_k + \frac{i\ell \omega^{-k}}{kN} \right|^2 - \frac{2\pi^2 LR^2}{\beta} \sum_{k \neq 0} k^2 \left| \varphi_k - \frac{\beta N}{4\pi^2 LR^2} \frac{(a_k + \frac{i\ell \omega^{-k}}{kN})}{k} \right|^2. \end{aligned} \quad (\text{A.24})$$

where we defined  $\omega = e^{2\pi i \tau_0 / \beta}$  and we have used the fact that  $a(t)$  is real so  $a_{-k} = a_k^*$ . Integrating over  $\varphi_0$  sets  $Q = 0$ , and integrating over  $\varphi_k$  for  $k \neq 0$  gives an overall holonomy-independent constant. Poisson resummation on  $P$  gives

$$\sum_{P \in \mathbb{Z}} e^{-\frac{2\pi^2 LR^2}{\beta} P^2 - iNP(a_0 + \frac{\pi\ell}{N} + \frac{2\pi\ell}{N} \frac{\tau_0}{\beta})} = \sqrt{\frac{\beta}{2\pi LR^2}} \sum_{n \in \mathbb{Z}} e^{-\frac{\beta N^2}{8\pi^2 LR^2} \left( a_0 - \frac{2\pi\ell}{N} \left( \frac{1}{2} - \frac{\tau_0}{\beta} \right) - \frac{2\pi n}{N} \right)^2}. \quad (\text{A.25})$$

Dropping the overall multiplicative factor, we have

$$\begin{aligned} S_{\text{eff},n} = & \beta \frac{2\pi^2}{e^2 L \beta^2} \left( \frac{\ell}{N} \right)^2 + \beta \sum_{k \neq 0} \frac{1}{2e^2 L} \left( \frac{2\pi k}{\beta} \right)^2 |a_k|^2 \\ & + \beta \frac{N^2}{8\pi^2 LR^2} \left( a_0 - \frac{2\pi\ell}{N} \left( \frac{1}{2} - \frac{\tau_0}{\beta} \right) - \frac{2\pi n}{N} \right)^2 + \beta \sum_{k \neq 0} \frac{N^2}{8\pi^2 LR^2} \left| a_k + \frac{i\ell \omega^{-k}}{kN} \right|^2. \end{aligned} \quad (\text{A.26})$$

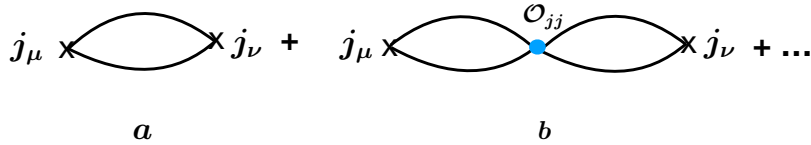
Noting that the Heaviside step function  $\Theta$  can be written as

$$\Theta(\tau - \tau_0) = \frac{1}{2} + \frac{\tau - \tau_0}{\beta} - \frac{i}{2\pi} \sum_{k \neq 0} \frac{1}{k} e^{2\pi i k (\tau - \tau_0) / \beta}, \quad (\text{A.27})$$

the above expression is equivalent to (using  $m_\gamma = \frac{eN}{2\pi R}$ )

$$S_{\text{eff},n} = \int_0^\beta d\tau \left[ \frac{1}{2e^2 L} \dot{a}^2 + \frac{m_\gamma^2}{2e^2 L} \left( a - \frac{2\pi n}{N} - \frac{2\pi\ell}{N} \Theta(\tau - \tau_0) \right)^2 \right], \quad (\text{A.28})$$

where the function  $a$  satisfies  $a(\beta) = a(0) + \frac{2\pi\ell}{N}$ . This gives eq. (5.7) in the main text.



**Figure 6.** Corrections to polarization due to  $\mathcal{O}_{jj}$ .

## B Scaling dimensions

In this appendix we calculate the scaling dimensions of operators as functions of the marginal parameter  $g$ , which is the coefficient of  $\mathcal{O}_{jj}$  operator, by summing some simple classes of diagrams in the fermionic presentation of our model. This reproduces the results from our analysis using bosonization in the main text. The fact that just summing these simple classes of diagrams already reproduces the exact results from bosonization means that all of the Feynman diagrams that we did not consider conspire to cancel, but showing this explicitly is a non-trivial project which did not manage to finish. Given the well-established nature of the bosonization approach, we do not pursue this more ambitious project here. Before coming to the calculations, we note that we will use Minkowski notation in this appendix, apart from using Euclidean rotations to evaluate the final momentum integrals.

Let us first consider the normalization in the relation (3.2) for bozonization of the current  $j_\mu = \bar{\psi}\gamma_\mu\psi$ . So called Bjorken limit is very instrumental in this respect. For two operators,  $A$  and  $B$ , the limit  $q_0 \rightarrow \infty$  in  $\int d^2x e^{iqx} \text{T}\{A(x)B(0)\}$  gives the equal-time commutator of  $A$  and  $B$ ,

$$\lim_{q_0 \rightarrow \infty} q_0 \int d^2x e^{iqx} \text{T}\{A(x)B(0)\} = i \int d^2x e^{iqx} [A(x), B(0)] \delta(x_0). \quad (\text{B.1})$$

When operators  $A$  and  $B$  are components of the vector current  $j^\mu = \bar{\psi}\gamma^\mu\psi$ , the leading contribution to this correlation function ( a polarization operator) is given by the diagram  $a$  in figure 6:

$$-i\Pi_{\mu\nu}(q) = \int d^2x e^{iqx} \langle \text{T}\{j_\mu(x)j_\nu(0)\} \rangle = \frac{i}{\pi q^2} (q_\mu q_\nu - g_{\mu\nu} q^2). \quad (\text{B.2})$$

In  $q_0 \rightarrow \infty$  limit, this defines the  $c$ -number part of  $[j_0(x), j_1(0)]\delta(x_0)$  commutator,

$$\int d^2x e^{iqx} [j_0(x), j_1(0)]\delta(x_0) = \frac{q_1}{\pi}, \quad (\text{B.3})$$

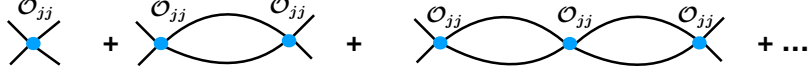
and fixes the coefficient in bozonization of the current  $j_\mu$ ,

$$j_\mu \implies -\frac{1}{2\pi} \epsilon_{\mu\nu} \partial^\nu \phi. \quad (\text{B.4})$$

Of course, the same result (B.2) for the polarization operator arises from the bosonic form of the current when the kinetic term of the  $\phi$  field is  $(1/8\pi)\partial_\mu\phi\partial^\mu\phi$ .

Switching on the  $\mathcal{O}_{jj}$  operator leads to appearance of the second and higher loops for the correlator (B.2). In the first order in  $g$  it is given by the two-loop diagram  $b$  in





**Figure 7.** Iterations of the operator  $\mathcal{O}_{jj}$ .

figure 6. This is a product,  $(-2g)\Pi_{\mu\gamma}\Pi_{\nu}^{\gamma} = (-2g/\pi)\Pi_{\mu\nu}$ , of two one-loop polarization operators resulting in the factor  $(-2g/\pi)$ . Accounting for higher loops gives a geometrical progression so we come to overall factor  $1/(1+2g/\pi)$ , consistent with bosonic considerations where the kinetic term becomes  $(R^2/2)\partial_{\mu}\phi\partial^{\mu}\phi$  with  $R^2 = (1+2g/\pi)/4\pi$ .

This phenomenon of finite renormalization of the current also shows up for the operator  $\mathcal{O}_{jj}$ . Iterations of this operator are illustrated by diagrams in figure 7. The same geometrical progression appears what could be interpreted as the effective substitution for the coupling  $g$ ,

$$g \Longrightarrow \frac{g}{1+2g/\pi}. \quad (\text{B.5})$$

A word of caution should be added here. The above consideration of loop corrections for the operator  $\mathcal{O}_{jj}$  refers to the pairing of bilinear operators  $j_{\mu} = \bar{\psi}\gamma_{\mu}\psi$  in the loop. However, the loop corrections for  $\mathcal{O}_{jj}$  also include other bilinears, namely,  $\bar{\psi}_L\psi_R, \psi_L\psi_L, \psi_R\psi_R$  paired with corresponding Hermitian conjugated ones. The individual loops are logarithmically divergent, so it could lead to breaking of the marginal nature of the operator  $\mathcal{O}_{jj}$ . Interestingly enough there is a cancellation between channels with fermion charge 0, like  $\bar{\psi}_L\psi_R$ , and double fermion charge, like  $\psi_L\psi_L$ . Altogether, the marginality of  $\mathcal{O}_{jj}$  is preserved.<sup>17</sup>

Let us consider now bosonization of the operator  $\bar{\psi}_L(x)\psi_R(x)$ . Its scaling dimension is clearly equal to 1 for free fermions. Bosonization relates this operator to  $e^{i\phi(x)}$ . To see that this bosonic operator has the same scaling dimension one can calculate the tadpole graphs. To this end let us start with one tadpole loop.

$$e^{i\phi} = \sum_{n=0}^{\infty} \frac{1}{n!} (i\phi)^n; \quad (i\phi)^n \Longrightarrow \frac{n!}{2(n-2)!} (i\phi)^{n-2} \langle (i\phi)(i\phi) \rangle; \\ e^{i\phi} \Longrightarrow e^{i\phi} \frac{1}{2} \langle (i\phi)(i\phi) \rangle. \quad (\text{B.6})$$

Here  $\langle (i\phi)(x)(i\phi)(x) \rangle$  is the propagator of the field  $i\phi$  from the point  $x$  to the same point. In Euclidean momentum space

$$\langle (i\phi)(x)(i\phi)(x) \rangle = - \int \frac{d^2k}{(2\pi)^2} \frac{4\pi}{k^2} = -2 \int_{\mu_1}^{\mu_2} \frac{dk}{k} = 2 \log \frac{\mu_1}{\mu_2}, \quad (\text{B.7})$$

where  $\mu_1$  and  $\mu_2$  are lower and upper cut-off in the momentum integration. Proceeding in the same fashion with next tadpoles we get exponentiation of the log,

$$e^{i\phi} \Longrightarrow e^{i\phi} e^{\langle (i\phi)(i\phi) \rangle / 2} = \frac{\mu_1}{\mu_2} e^{i\phi}. \quad (\text{B.8})$$

Now let us switch on the operator  $\mathcal{O}_{jj}$ . On bosonic side it is just a change of the kinetic term coefficient and the corresponding change of  $\langle (i\phi)(x)(i\phi)(x) \rangle$  by the factor

<sup>17</sup>We are thankful to A. Tselik for explaining this phenomenon to us.

$1/(1 + 2g/\pi)$ . This immediately shows that the scaling dimension instead of 1 becomes

$$\Delta_{e^{i\phi}} = \frac{1}{1 + 2g/\pi}. \quad (\text{B.9})$$

On the fermion side one have to consider the loop diagram generated by the initial operator  $\bar{\psi}_L(x)\psi_R(x)$  and the operator  $\mathcal{O}_{jj}$ . In the first order in  $g$

$$\begin{aligned} \bar{\psi}_L(0)\psi_R(0) i \int d^2x (-g) \mathcal{O}_{jj}(x) &= 4ig \bar{\psi}_L(0)\psi_R(0) \int d^2x \bar{\psi}_R(x)\psi_L(x)\bar{\psi}_L(x)\psi_R(x) \\ \implies -4ig \int d^2x \langle \psi_L(x)\bar{\psi}_L(0) \rangle \langle \psi_R(x)\bar{\psi}_R(0) \rangle \bar{\psi}_L(x)\psi_R(x). \end{aligned} \quad (\text{B.10})$$

Here  $\langle \psi_L(x)\bar{\psi}_L(0) \rangle$  and  $\langle \psi_R(x)\bar{\psi}_R(0) \rangle$  denote fermionic propagators. In momentum space we come to

$$4ig \int \frac{d^2p}{(2\pi)^2} \frac{1}{p_R} \frac{1}{p_L} \bar{\psi}_L\psi_R = -\frac{2g}{\pi} \log \frac{\mu_1}{\mu_2} \bar{\psi}_L\psi_R \quad (\text{B.11})$$

Accounting for higher loops leads to two effects. First, it leads to the exponentiation of the one loop result, and second, it leads to the substitution (B.5) for the coupling  $g$ . Altogether, we get for the scaling dimension,

$$\Delta_{\bar{\psi}_L\psi_R} = 1 - \frac{2g}{1 + 2g/\pi} = \frac{1}{1 + 2g/\pi}, \quad (\text{B.12})$$

where 1 comes from canonical dimension. This coincides with (B.9).

Derivation of (1.9) for the scaling dimension of the operator  $\mathcal{O}_\chi$  on both bosonic and fermionic sides is similar the one given above for the  $\bar{\psi}_L\psi_R$  operator.

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