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Perturbative renormalization of the $T\bar{T}$-deformed free massive Dirac fermion

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Abstract: In this paper we explicitly carry out the perturbative renormalization of the $T\bar{T}$-deformed free massive Dirac fermion in two dimensions up to second order in the coupling constant. This is done by computing the two-to-two S-matrix using the LSZ reduction formula and canceling out the divergences by introducing counterterms. We demonstrate that the renormalized Lagrangian is unambiguously determined by demanding that it gives the correct S-matrix of a $T\bar{T}$-deformed integrable field theory. Remarkably, the renormalized Lagrangian is qualitatively very different from its classical counterpart.

Keywords: Field Theories in Lower Dimensions, Integrable Field Theories, Renormalization Regularization and Renormalons

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Contents

1 Introduction

Quantum field theory (QFT) is one of the most elegant and accurate frameworks which describes nature to a very high accuracy. QFTs admit deformations by operators which trigger a flow, known as the renormalization group (RG) flow, as we look into the theory at different scales [1, 2]. The fixed points of the RG-flow trajectory define conformal field theories (CFT) which are crucial in understanding aspects of condensed matter physics and string theory. However, it is equally significant to understand the RG flow away from the fixed points by deforming the QFT with a relevant or irrelevant operator. A relevant deformation stimulates the flow at lower energies (IR) while an irrelevant deformation drives the flow at higher energies (UV). Exploring the latter is more difficult as it requires infinitely many counterterms in computing physical quantities in such a theory. However, a special kind of irrelevant deformation, known as the $T\bar{T}$-deformation, has been introduced recently in two dimensions [3, 4] and has received wide attention in the past few years as the theory is solvable.

The $T\bar{T}$-deformation is generated by the irrelevant operator $\det(T_{\mu\nu})$, the determinant of the energy-momentum tensor $T_{\mu\nu}$ of the theory, and thus can be considered in any generic QFT. At the classical level, the Lagrangian of the deformed theory itself looks very interesting and was obtained in [5, 6]. For example, the Lagrangian of a $T\bar{T}$-deformed free massless boson is equivalent to the Nambu-Goto action for a string in three spacetime dimensions in the static gauge [5]. Although the deformation makes the theory non-local, often complicated and non-renormalizable, there are several remarkable features that make the deformed theory so compelling. One such feature is that the energy spectrum of the deformed theory can be derived non-perturbatively and in a compact form [3]. However, the spectrum appears to be sensitive to the sign of the $T\bar{T}$-coupling. For one of the signs, the highly excited states in the spectrum carry complex energies where the deformed theory becomes non-unitary. For this particular sign of the coupling, a holographic dual was proposed in [7] and was further investigated in [8]–[13]. For the other sign of the coupling, the spectrum is real and the deformed theory is unitary. Moreover, at high energies, the density of states exhibit Hagedorn behaviour like the two dimensional little string theory (LST). In the context of holography, it was argued that certain single-trace $T\bar{T}$-deformation of two dimensional CFTs corresponds to a two dimensional vacuum of LST [14], see [15]–[21] for related exciting works in this direction. Another interesting aspect of the two dimensional $T\bar{T}$-deformed theories is the partition function — one can compute the partition function of the deformed theory on a torus or a cylinder or a disk and each of them satisfy a linear diffusion-type differential equation [22]. In particular, $T\bar{T}$-deformation of a CFT was considered in [23] and the torus partition function of the deformed CFT was obtained to be modular invariant despite the fact that the deformed theory is not conformal. See [24, 25] for further exciting results concerning the partition functions in the $T\bar{T}$-deformed CFTs.

Although the $T\bar{T}$-deformation can be defined in any two dimensional QFT, a special interest has been taken in the study of integrable quantum systems [4, 26]–[29]. Integrable QFTs contain infinite number of conserved charges. The reason for this interest is that a $T\bar{T}$ deformation preserves the integrable structure of the theory [4]. In [4], it was also argued that the $T\bar{T}$-deformation modifies the S-matrix only by a CDD phase factor.

Being an irrelevant deformation, the Lagrangian of the $T\bar{T}$-deformed theory is apparently non-renormalizable as it involves an infinite number of counterterms. However, it is the integrable structure of the theory that enables one to obtain a renormalized Lagrangian in such a naively non-renormalizable theory. The $T\bar{T}$-deformed integrable theory provides an infinite number of constraints which uniquely fix the all counterterms that appear in the theory. In [30], the authors considered a $T\bar{T}$-deformed free scalar field theory and showed how to derive the renormalized Lagrangian perturbatively by demanding that it should produce the correct S-matrix. Obtaining a renormalized Lagrangian unambiguously allows one to compute all physical quantities of interest. This motivates us to consider the $T\bar{T}$-deformed free massive Dirac fermion in two dimensions and to compute the renormalized Lagrangian of such a theory perturbatively. As we will see, the renor-

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malized Lagrangian exists and is qualitatively very different from the classical Lagrangian. In particular, while the classical Lagrangian has only one scale, namely the bare coupling constant, the renormalized Lagrangian to second order reveals three different scales in the theory.

In this paper, the perturbative renormalization of the $T\bar{T}$-deformed free massive Dirac fermion in two dimensions will be explicitly performed. This will be done using the LSZ reduction formula to compute the two-to-two S-matrix and introducing counterterms to cancel divergences. The paper is organized as follows: in section 2 the basics of $T\bar{T}$-deformations as well as some related aspects of integrability will be reviewed. In section 3 the $T\bar{T}$-deformed free massive Dirac fermion in two dimensions will be considered. In section 4 the renormalized Lagrangian to second order in the $T\bar{T}$-coupling constant will be computed. Finally, in section 5 the main results and future directions will be discussed.

2 Integrability and $T\bar{T}$-deformations

In this section several aspects of integrability, two-dimensional quantum field theories and $T\bar{T}$-deformations are discussed.

In two dimensional integrable theories the S-matrix of any scattering process factorizes into two-to-two S-matrices and there is no particle production [31]–[37]. Consider a two-to-two scattering process of particles with identical mass m in two dimensions. The 2-momentum of the i^{th} particle can be parameterized by its rapidity, θ_i,[1] as,

$$p_i^\mu = \begin{pmatrix} m\cosh\theta_i \\ m\sinh\theta_i \end{pmatrix} . \tag{2.1}$$

Identify $p_i^0 = m\cosh\theta_i \equiv E_i(\theta_i)$ as the energy of the i^{th} particle and $p_i^1 = m\sinh\theta_i \equiv P_i(\theta_i)$ as the momentum of the i^{th} particle.

Two dimensional kinematics of the two-to-two scattering processes have the unique property that the incoming momenta of the particles equal the outgoing momenta of the particles. This fact and the Lorentz invariance implies that the S-matrix, denoted by $S(\theta)$, will only be a function of the difference of rapidities, $\theta = \theta_1 - \theta_2$.

In what follows the $T\bar{T}$-deformation of an integrable field theory will be considered. If a $T\bar{T}$-deformation is performed on an integrable field theory, the deformed theory is integrable as well [4]. As we mentioned earlier, there is no particle production in a scattering process in an integrable theory and hence, unitarity demands

$$|S(\theta)|^2 = 1 . \tag{2.2}$$

On the other hand, the crossing symmetry of the S-matrix[2] implies,

$$S(\theta) = S(i\pi - \theta) . \tag{2.3}$$

[1]Rapidity or the parameter of velocity is defined as, $\theta_i = \tanh^{-1}(v_i)$, where v_i is the velocity of the i-th particle and the speed of light is set to $c = 1$.

[2]It is the symmetry of the S-matrix under the interchange of the s and u-channels as will be shown later.

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The solution to (2.2) and (2.3) is simple and given by the CDD factor,

$$S_\alpha(\theta) = \frac{\sinh\theta - i\sin\alpha}{\sinh\theta + i\sin\alpha}\,, \tag{2.4}$$

where α is a real parameter related to the coupling constant. The product of $S_\alpha(\theta)$ over α, $\prod_\alpha S_\alpha(\theta)$, is also a solution to (2.2) and (2.3).

An alternative solution for the CDD factor was recently considered by Smirnov and Zamolodchikov [4], which admits the following representation

$$S_\alpha(\theta) = e^{i\alpha\sinh\theta}\,. \tag{2.5}$$

The authors considered $T\bar{T}$-deformed theories in two dimensions[3] where the deformation produces a one-parameter family of Lagrangians obeying the $T\bar{T}$ flow equation,

$$\frac{\partial\mathcal{L}(\lambda)}{\partial\lambda} = -4\Big(T^\lambda(z)\bar{T}^\lambda(z) - \theta^\lambda(z)\bar{\theta}^\lambda(z)\Big)\,, \tag{2.6}$$

λ being the $T\bar{T}$-coupling constant, $T^\lambda = T^\lambda_{zz}$, $\bar{T}^\lambda = T^\lambda_{\bar{z}\bar{z}}$ and $\theta^\lambda = \bar{\theta}^\lambda = T^\lambda_{z\bar{z}}$ are the components of the energy momentum tensor of the deformed theory and $\mathcal{L}(\lambda = 0)$ is the Lagrangian of the undeformed theory. It was argued that the deformed theory is integrable and the S-matrix of this theory can be expressed in a factorizable form, $\hat{S}'(\theta)S(\theta)$, where $S(\theta)$ is the CDD factor determined by unitarity and crossing-symmetry of the S-matrix,

$$S(\theta) = e^{i\lambda m^2\sinh\theta}\,. \tag{2.7}$$

The factor $\hat{S}'(\theta)$ signifies the presence of mass degeneracies in the spectrum and it satisfies the Yang-Baxter equation [37]. This typically fixes the "flavor" structure of the S-matrix. Simply, the $T\bar{T}$-deformation (2.6) corresponds to multiplying the S-matrix by the factor (2.7).

However, observe that (2.7) grows exponentially at large imaginary momenta. This behaviour is inconsistent with the analytic behaviour of S-matrices in a local QFT. Nevertheless, rather than just throwing out these theories, one can try to understand such theories as QFTs coupled to gravity [38]–[41]. In this paper, however, we will be restricted to the low-energy regime of the $T\bar{T}$-deformed theories. In this regime these are quantum field theories and can be studied perturbatively in the $T\bar{T}$-coupling λ around $\lambda = 0$ [8, 30, 42]–[46].[4]

[3]To be precise, the authors in [4] considered integrable quantum field theories (IQFT) and deformed them by generic scalars X_s such that the deformations preserve integrability. Being IQFT, these theories have an infinite number of conserved currents $(T_{s+1}(z), \Theta_{s-1}(z))$ and $(\bar{T}_{s+1}(z), \bar{\Theta}_{s-1}(z))$, where the index s represents the spin of the corresponding fields thus labeling the currents. The scalars X_s are defined in terms of these local currents. For example, X_1 is precisely the composite operator $T\bar{T}$ considered in our paper.

[4]For non-perturbative studies of the $T\bar{T}$-deformed theories, see [47, 48].

3 The $T\bar{T}$-deformed free massive Dirac fermion

Consider the Euclidean action for the free massive Dirac fermion with mass m,

$$I_0 = \int dx_1 dx_2 \, \mathcal{L}_0 = \int dx_1 dx_2 \, (i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi) \,. \tag{3.1}$$

Performing a Wick rotation to the Euclidean action $x_1 = it$ and $x_2 = x$, yields the Lorentzian action,

$$-I_0 = -\int dx_1 dx_2 \, \mathcal{L}_0 = i\int dt dx \, (-i\bar{\psi}\gamma^\mu\partial_\mu\psi + m\bar{\psi}\psi) \,, \tag{3.2}$$

where the γ^μs are the two dimensional gamma matrices satisfying the Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}\,\mathbb{I} \tag{3.3}$$

and $\eta_{\mu\nu}$ is the two dimensional Minkowski metric. We represent the gamma matrices in terms of the Pauli matrices: $\gamma_0 = \sigma_x$ and $\gamma_1 = -i\sigma_y$.

The equation of motion for the fermionic fields ψ and $\bar{\psi}$ are given by,

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0 \quad \text{and} \quad i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0. \tag{3.4}$$

To solve the Dirac equation (3.4), make the ansatz, $\psi(x) = u(k)e^{-ik.x}$. Plugging the ansatz into the Dirac equation and using the normalization $u^\dagger(k)u(k) = 1$, one finds the normalized positive energy plane wave solution,

$$u(k) = \begin{pmatrix} \sqrt{E_+ + k^1} \\ \frac{m}{\sqrt{E_+ + k^1}} \end{pmatrix} = \begin{pmatrix} \sqrt{k^0 + k^1} \\ \sqrt{k^0 - k^1} \end{pmatrix} \tag{3.5}$$

where, $E_+ = k^0 = \sqrt{(k^1)^2 + m^2}$.

Similarly, using $\psi(x) = v(k)e^{ik.x}$ one can obtain the negative energy plane wave solution,

$$v(k) = \begin{pmatrix} -\sqrt{E_- + k^1} \\ \frac{m}{\sqrt{E_- + k^1}} \end{pmatrix} = \begin{pmatrix} -\sqrt{k^0 + k^1} \\ \sqrt{k^0 - k^1} \end{pmatrix} \tag{3.6}$$

where, $E_- = k^0 = -\sqrt{(k^1)^2 + m^2}$.

Hence, the solution to the wave equation for the free massive Dirac fermion is,

$$\psi(x) = \int \frac{d\vec{k}}{2\pi}\frac{1}{\sqrt{2E}}\left(a(\vec{k})u(\vec{k})e^{-ik\cdot x} + b^\dagger(\vec{k})v(\vec{k})e^{ik\cdot x}\right) \,, \tag{3.7}$$

where $a^\dagger(\vec{k})$ and $a(\vec{k})$ are the fermion creation and annihilation operators respectively, while $b^\dagger(\vec{k})$ and $b(\vec{k})$ are the anti-fermion creation and annihilation operators respectively. The creation and annihilation operators satisfy the Clifford algebra,

$$\{a(\vec{k_1}), a^\dagger(\vec{k_2})\} = 2\pi\delta(\vec{k_1} - \vec{k_2}) \quad \text{and} \quad \{b(\vec{k_1}), b^\dagger(\vec{k_2})\} = 2\pi\delta(\vec{k_1} - \vec{k_2}) \,, \tag{3.8}$$

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while all other anti-commutators vanish. The Dirac adjoint of a spinor ψ is defined as, $\bar{\psi} = \psi^\dagger \gamma^0$.

Consider the $\bar{T}T$-deformation of the free massive Dirac fermion in two dimensional Euclidean spacetime,

$$I = \int d^2x\, \mathcal{L}(\lambda) = \int dx_1 dx_2\, (i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi) + \lambda \int dx_1 dx_2\, \mathcal{O}_{T\bar{T}}\,, \tag{3.9}$$

where $\mathcal{O}_{T\bar{T}}$ is the local $T\bar{T}$-operator given by the determinant of the energy-momentum tensor,

$$\mathcal{O}_{T\bar{T}} = \det(T^{(\lambda)}) = \frac{1}{2}\epsilon^{\mu\nu}\epsilon^{\rho\sigma} T^{(\lambda)}_{\mu\nu} T^{(\lambda)}_{\rho\sigma} = \frac{1}{2}\Big[(T_\mu{}^{\mu(\lambda)})^2 - T^{(\lambda)}_{\mu\nu} T^{\mu\nu(\lambda)} \Big]\,, \tag{3.10}$$

$\epsilon^{\mu\nu}$ the two-dimensional Levi-Civita tensor and $T^{(\lambda)}_{\mu\nu}$ the energy-momentum tensor of the finite-λ theory.

The canonical energy-momentum tensor of the undeformed theory is given by,

$$T^{(0)}_{\mu\nu(c)} = X_{\mu\nu} - \delta_{\mu\nu}(\mathrm{Tr}\, X - m\bar{\psi}\psi)\,, \tag{3.11}$$

where

$$X_{\mu\nu} = \frac{i}{2}(\bar{\psi}\gamma_\mu \partial_\nu \psi - \partial_\nu \bar{\psi}\gamma_\mu \psi)\,. \tag{3.12}$$

The above canonical energy-momentum tensor can be symmetrized using the Belinfante technique, yielding

$$T^{(0)}_{\mu\nu} = \tilde{X}_{\mu\nu} - \delta_{\mu\nu}(\mathrm{Tr}\, X - m\bar{\psi}\psi)\,, \tag{3.13}$$

where

$$\begin{aligned} \tilde{X}_{\mu\nu} &= \frac{i}{2}\Big(\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi - \partial_{(\mu}\bar{\psi}\gamma_{\nu)}\psi\Big) \\ &= \frac{i}{4}(\bar{\psi}\gamma_\mu \partial_\nu \psi + \bar{\psi}\gamma_\nu \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma_\nu \psi - \partial_\nu \bar{\psi}\gamma_\mu \psi)\,. \end{aligned} \tag{3.14}$$

The $T\bar{T}$-deformed Lagrangian of the free massive Dirac fermion can be obtained by solving the $T\bar{T}$ flow equation [6],

$$\frac{\partial \mathcal{L}(\lambda)}{\partial \lambda} = \mathcal{O}_{T\bar{T}}\,, \tag{3.15}$$

with the initial condition $\mathcal{L}(\lambda = 0) = \mathcal{L}_0$.

Solving (3.15) perturbatively in the $T\bar{T}$-coupling λ, one finds the $T\bar{T}$-deformed Lagrangian for the free massive Dirac fermion [6],

$$\begin{aligned} \mathcal{L}(\lambda) = {} & (i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi) - \frac{\lambda}{2}\Big(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2 + 2m\bar{\psi}\psi \tilde{X}_\mu{}^\mu - 2m^2(\bar{\psi}\psi)^2\Big) \\ & + \frac{\lambda^2}{2} m\bar{\psi}\psi\Big(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\Big)\,, \end{aligned} \tag{3.16}$$

where $\tilde{X}_{\mu\nu}$ is given by (3.14). It is noteworthy to mention that the $T\bar{T}$-deformed Lagrangian (3.16) is exact in λ, all the higher order terms in λ vanish identically due to the Grassmann nature of the fermion fields.[5]

[5] Although, terms proportional to $\tilde{X}^4$ can be present at third order in the $T\bar{T}$-coupling λ, the authors of [6] claimed that the $\mathcal{O}(\lambda^3)$ term vanishes by using Fierz identities. It is also possible to verify this claim directly by plugging the expression into MATHEMATICA.

Therefore, the $T\bar{T}$-deformed Lorentzian action of the free massive Dirac fermion is,

$$-I = -\int dx_1 dx_2\, \mathcal{L} = i \int dt dx \left[\left(-i\bar{\psi}\gamma^\mu \partial_\mu \psi + m\bar{\psi}\psi\right) + \frac{\lambda}{2}\left(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2 \right.\right.$$
$$\left.\left. + 2m\bar{\psi}\psi\tilde{X}_\mu{}^\mu - 2m^2(\bar{\psi}\psi)^2\right) - \frac{\lambda^2}{2} m\bar{\psi}\psi\left(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\right)\right]. \tag{3.17}$$

In what follows two-to-two scattering in the $T\bar{T}$-deformed free massive Dirac fermion theory will be considered. The S-matrix, $S(\theta)$, will be computed perturbatively to second order in the $T\bar{T}$-coupling λ. Finally the S-matrix will be compared with (2.7) and the renormalized Lagrangian will be constructed.

4 Renormalization of the $T\bar{T}$-deformed free massive Dirac fermion

In section 3, the classical Lagrangian of the $T\bar{T}$-deformed free massive Dirac fermion was stated. In this section the renormalized Lagrangian will be computed to second order.

In order to compute the renormalized Lagrangian similar methods to those found in [30] will be used. First, the S-matrix will be computed using the classical Lagrangian giving rise to UV divergences. Next, counterterms will be added to the Lagrangian in order to cancel the divergences and ensure that the final S-matrix is given by (2.7).

4.1 The S-matrix

Consider the two-to-two scattering of a fermion and anti-fermion in the $T\bar{T}$-deformed free massive Dirac theory,[6]

$$f_1 + \bar{f}_2 \to f_3 + \bar{f}_4$$

where f_1 represents the incoming fermion with momentum p_1, $\bar{f}_2$ the incoming anti-fermion with momentum p_2, f_3 the outgoing fermion with momentum p_3 and $\bar{f}_4$ the outgoing anti-fermion with momentum p_4.

Recall, due to the two dimensional kinematics of two-to-two scattering the momenta of the incoming particles equal the momenta of the outgoing particles. When considering fermion anti-fermion scattering the 0^{th}-order S-matrix is just the identity. Hence, the momentum of the incoming fermion must equal the momentum of the outgoing fermion and the momentum of the incoming anti-fermion must equal the momentum of the outgoing anti-fermion.[7] In particular,

$$p_1 = p_3 \qquad \text{and} \qquad p_2 = p_4. \tag{4.1}$$

[6] One can also consider the other possible two-to-two scattering, namely, the fermion-fermion scattering. However, the resulting renormalized Lagrangian would be the same as it does not depend on the particular scattering process. Because of this fact only the fermion anti-fermion scattering process is considered.

[7] If scattering in a scalar theory is considered [30], one can choose $p_1 = p_4$ and $p_2 = p_3$ also. However, it is easy to see that if one works with fermions, we must have $p_1 = p_3$ and $p_2 = p_4$ to have a non-zero scattering matrix.

By (2.1) and (4.1) the Mandelstam variables in the $(+,-)$ signature take the form,

$$\begin{aligned} s &= (p_1+p_2)^2 = 2m^2(1+\cosh\theta)\,, \\ t &= (p_1-p_3)^2 = 0\,, \\ u &= (p_1-p_4)^2 = 2m^2(1-\cosh\theta) = 4m^2 - s\,. \end{aligned} \tag{4.2}$$

Observe that the Mandelstam variables can be related to each other by the transformations,

$$u = s|_{\theta\to i\pi-\theta} \qquad \text{and} \qquad t = s|_{\theta\to i\pi}. \tag{4.3}$$

Finally due to (4.1), for the two-to-two scattering in an integrable theory the S-matrix, $S(\theta)$, is defined as,

$${}_{\text{out}}\langle p_3,p_4|p_1,p_2\rangle_{\text{in}} = (2\pi)^2\delta(p_1-p_3)\delta(p_2-p_4)2E(p_1)2E(p_2)S(\theta)\,, \tag{4.4}$$

where the zeroth order S-matrix in the $T\bar{T}$-coupling is $S^{(0)}(\theta)=1$. The S-matrix $S(\theta)$ is related to the scattering amplitude $\mathcal{A}$ by [30],

$$S(\theta) = \frac{\mathcal{A}}{4m^2\sinh\theta}\,. \tag{4.5}$$

However, before computing the S-matrix, express the deformed action (3.17) in a simpler form. By the field redefinitions,

$$\psi_a \to \psi_a + \lambda\frac{m}{2}\psi_a(\bar\psi\psi) \quad \text{and} \quad \bar\psi_a \to \bar\psi_a + \lambda\frac{m}{2}(\bar\psi\psi)\bar\psi_a\,, \tag{4.6}$$

where $a=\{1,2\}$ are the spinor indices, one can write the action (3.17) as,

$$-I = i\int dtdx\left[\left(-i\bar\psi\gamma^\mu\partial_\mu\psi + m\bar\psi\psi\right) + \frac{\lambda}{2}\left(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\right) + \frac{\lambda^2}{2}m\bar\psi\psi\left(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\right)\right]. \tag{4.7}$$

A Jacobian factor arises in the path integral due to the above field redefinition (4.6), however it does not produce any non-trivial contribution to the two-to-two S-matrix as will be shown later. For details of the field redefinition and the Jacobian see appendix A.

Observe that the new action (4.7) no longer contains linear terms of the form $\bar\psi\psi\tilde{X}_\mu{}^\mu$ and $(\bar\psi\psi)^2$, making it simpler to compute the one-loop bubble diagrams by decreasing the total number of terms. Also, the second order term in the redefined action (4.7) has a relative sign difference in comparison to the original action (3.17).

The two dimensional Dirac spinors $u(p_i)$, $\bar{u}(p_i)$, $v(p_i)$ and $\bar{v}(p_i)$ will arise in the computation of the S-matrix. Using (3.5), (3.6) and the fact that $p_3^\mu = p_1^\mu = (m\cosh\theta_1,\ m\sinh\theta_1)$ and $p_4^\mu = p_2^\mu = (m\cosh\theta_2,\ m\sinh\theta_2)$ the two dimensional Dirac spinors can be written as,

$$\begin{aligned} u(p_1) &= \sqrt{m}\begin{pmatrix}\sqrt{\cosh\theta_1+\sinh\theta_1}\\ \sqrt{\cosh\theta_1-\sinh\theta_1}\end{pmatrix} = u(p_3)\,, \quad u(p_2) = \sqrt{m}\begin{pmatrix}\sqrt{\cosh\theta_2+\sinh\theta_2}\\ \sqrt{\cosh\theta_2-\sinh\theta_2}\end{pmatrix} = u(p_4) \\ v(p_1) &= \sqrt{m}\begin{pmatrix}-\sqrt{\cosh\theta_1+\sinh\theta_1}\\ \sqrt{\cosh\theta_1-\sinh\theta_1}\end{pmatrix} = v(p_3)\,, \quad v(p_2) = \sqrt{m}\begin{pmatrix}-\sqrt{\cosh\theta_2+\sinh\theta_2}\\ \sqrt{\cosh\theta_2-\sinh\theta_2}\end{pmatrix} = v(p_4)\,. \end{aligned} \tag{4.8}$$

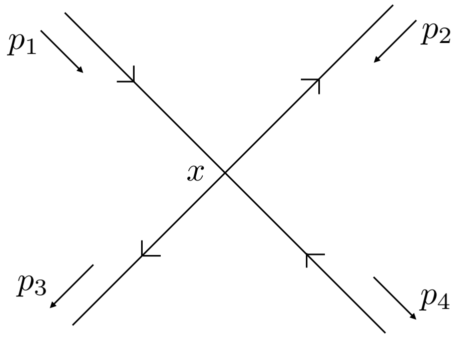

Figure 1. Tree-level diagram that contributes to the S-matrix at first order.

The Dirac adjoints $\bar{u}(p_i) = u(p_i)^\dagger\gamma^0$ and $\bar{v}(p_i) = v(p_i)^\dagger\gamma^0$ can be written as,

$$\begin{aligned}
\bar{u}(p_1) &= \sqrt{m}\left(\sqrt{\cosh\theta_1 - \sinh\theta_1}\sqrt{\cosh\theta_1 + \sinh\theta_1}\right) = \bar{u}(p_3)\,,\\
\bar{u}(p_2) &= \sqrt{m}\left(\sqrt{\cosh\theta_2 - \sinh\theta_2}\sqrt{\cosh\theta_2 + \sinh\theta_2}\right) = \bar{u}(p_4)\,,\\
\bar{v}(p_1) &= \sqrt{m}\left(\sqrt{\cosh\theta_1 - \sinh\theta_1} - \sqrt{\cosh\theta_1 + \sinh\theta_1}\right) = \bar{v}(p_3)\,,\\
\bar{v}(p_2) &= \sqrt{m}\left(\sqrt{\cosh\theta_2 - \sinh\theta_2} - \sqrt{\cosh\theta_2 + \sinh\theta_2}\right) = \bar{v}(p_4)\,.
\end{aligned} \tag{4.9}$$

It is finally time to compute the S-matrix perturbatively up to second order in the $T\bar{T}$-coupling λ using the redefined Lagrangian,

$$\mathcal{L}(\lambda) = \mathcal{L}_0 + \mathcal{L}_1(\lambda) + \mathcal{L}_2(\lambda)\,, \tag{4.10}$$

where,

$$\begin{aligned}
\mathcal{L}_0 &= -i\bar{\psi}\gamma^\mu\partial_\mu\psi + m\bar{\psi}\psi\,,\\
\mathcal{L}_1(\lambda) &= \frac{\lambda}{2}\Big(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\Big)\,,\\
\mathcal{L}_2(\lambda) &= \frac{\lambda^2}{2}m\bar{\psi}\psi\Big(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\Big).
\end{aligned} \tag{4.11}$$

4.1.1 First order S-matrix

At first order in the $T\bar{T}$-coupling λ, the S-matrix only gets a contribution from the tree-level diagrams of the form found in figure 1, where the vertex corresponds to the quartic couplings, $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$ or $(\tilde{X}_\mu{}^\mu)^2$. The total contribution to the tree-level amplitude of a fermion anti-fermion scatering process is,

$$\mathcal{A}^{(1)} = i\frac{\lambda}{2}\langle 0|b(p_4)a(p_3)\mathrm{T}\big[\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle. \tag{4.12}$$

In order to evaluate (4.12) compute the general contribution to the amplitude of four external fields with arbitrary indices for the fermion anti-fermion scattering process,

$$\begin{aligned}
\mathcal{A}^{(1)}_{\bar{\psi}_a\psi_b\bar{\psi}_c\psi_d} &= \langle 0|b(p_4)a(p_3)\mathrm{T}\big[\bar{\psi}_a\psi_b\bar{\psi}_c\psi_d\big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle\\
&= \bar{u}_a(p_3)v_b(p_4)\bar{v}_c(p_2)u_d(p_1) + \bar{v}_a(p_2)u_b(p_1)\bar{u}_c(p_3)v_d(p_4) - \bar{v}_a(p_2)v_b(p_4)\bar{u}_c(p_3)u_d(p_1)\\
&\quad - \bar{u}_a(p_3)u_b(p_1)\bar{v}_c(p_2)v_d(p_4)
\end{aligned} \tag{4.13}$$

where the four possible Wick contractions were performed, u_a, v_a, $\bar{u}_a$ and $\bar{v}_a$ are the Dirac spinors given by (4.8) and (4.9) and $a,b,c,d=\{1,2\}$ are the spinor indices.

The total first order amplitude can be computed by plugging each vertex into (4.13) and adding the results together. By (3.14), the quartic couplings $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$ and $(\tilde{X}_\mu{}^\mu)^2$ can be expressed as,

$$\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} = -\frac{1}{4}\Big(\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi - 2\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi + \partial_{(\mu}\bar{\psi}\gamma_{\nu)}\psi\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi\Big) \quad (4.14)$$

$$(\tilde{X}_\mu{}^\mu)^2 = -\frac{1}{4}\Big(\bar{\psi}\gamma^\mu\partial_\mu\psi\ \bar{\psi}\gamma^\nu\partial_\nu\psi - 2\bar{\psi}\gamma^\mu\partial_\mu\psi\ \partial_\nu\bar{\psi}\gamma^\nu\psi + \partial_\mu\bar{\psi}\gamma^\mu\psi\ \partial_\nu\bar{\psi}\gamma^\nu\psi\Big). \quad (4.15)$$

Using (4.13) the tree-level contribution from each of these terms in (4.14) can be computed and thus the contribution from $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$ to the tree-level amplitude can be determined. For example, the first term in (4.14), $\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi$, contributes as

$$\begin{aligned}
&\mathcal{A}^{(1)}_{\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi}\\
&= \gamma_{ab(\mu}\gamma^{(\mu}_{cd}\Big[\bar{u}_a(p_3)v_b(p_4)\bar{v}_c(p_2)u_d(p_1)ip_{4\nu)}(-i)p_1^{\nu)} + \bar{v}_a(p_2)u_b(p_1)\bar{u}_c(p_3)v_d(p_4)(-i)p_{1\nu)}ip_4^{\nu)}\\
&-\bar{v}_a(p_2)v_b(p_4)\bar{u}_c(p_3)u_d(p_1)ip_{4\nu)}(-i)p_1^{\nu)} - \bar{u}_a(p_3)u_b(p_1)\bar{v}_c(p_2)v_d(p_4)(-i)p_{1\nu)}ip_4^{\nu)}\Big]\\
&= \bar{u}(p_1)\cdot\gamma_{(\mu}p_{2\nu)}\cdot v(p_2)\ \bar{v}(p_2)\cdot\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1) + \bar{v}(p_2)\cdot\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)\ \bar{u}(p_1)\cdot\gamma^{(\mu}p_2^{\nu)}\cdot v(p_2)\\
&-\bar{v}(p_2)\cdot\gamma_{(\mu}p_{2\nu)}\cdot v(p_2)\ \bar{u}(p_1)\cdot\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1) - \bar{u}(p_1)\cdot\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)\ \bar{v}(p_2)\cdot\gamma^{(\mu}p_2^{\nu)}\cdot v(p_2)\\
&= -2m^4(1+3\cosh\theta+2\cosh 2\theta)
\end{aligned} \quad (4.16)$$

where on the first line the general amplitude, (4.13), was used, a factor of $\pm i$ appears whenever a derivative operator acts on ψ or $\bar{\psi}$.[8] (4.1) was used to get the second equality and on the final line the mathematical expressions for the Dirac spinors (4.8) and (4.9) were plugged in.

Similarly, the contributions from the other two terms in (4.14) can be computed. The computation yields,

$$\mathcal{A}^{(1)}_{\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi} = -2m^4(1-\cosh\theta-2\cosh 2\theta) \quad (4.17)$$

$$\mathcal{A}^{(1)}_{\partial_{(\mu}\bar{\psi}\gamma_{\nu)}\psi\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi} = -2m^4(1+3\cosh\theta+2\cosh 2\theta)\,. \quad (4.18)$$

Adding together (4.16), (4.17) and (4.18) according to (4.14) gives the contribution of the quartic coupling $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$,

$$\begin{aligned}
\mathcal{A}^{(1)}_{\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}} &= -\frac{1}{4}\Big(\mathcal{A}^{(1)}_{\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi} - 2\mathcal{A}^{(1)}_{\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi} + \mathcal{A}^{(1)}_{\partial_{(\mu}\bar{\psi}\gamma_{\nu)}\psi\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi}\Big)\\
&= 4m^4(\cosh\theta+\cosh 2\theta).
\end{aligned} \quad (4.19)$$

[8] $+i$ appears if the derivative operator acting on $\psi(x)$ gives an outgoing momentum p_3 or p_4, while $-i$ appears if the derivative operator produces an ingoing momentum p_1 or p_2 upon acting on $\psi(x)$.

One can compute the contributions from the three terms found in (4.15) to evaluate the contribution from $(\tilde{X}_\mu{}^\mu)^2$ to the tree level amplitude in a similar manner,

$$\begin{aligned}
\mathcal{A}^{(1)}_{\bar\psi\gamma^\mu\partial_\mu\psi\bar\psi\gamma^\nu\partial_\nu\psi} &= -4m^4(1+\cosh\theta) \\
\mathcal{A}^{(1)}_{\bar\psi\gamma^\mu\partial_\mu\psi\partial_\nu\bar\psi\gamma^\nu\psi} &= 4m^4(1+\cosh\theta) \\
\mathcal{A}^{(1)}_{\partial_\mu\bar\psi\gamma^\mu\psi\partial_\nu\bar\psi\gamma^\nu\psi} &= -4m^4(1+\cosh\theta).
\end{aligned} \tag{4.20}$$

Adding the terms in (4.20) according to (4.15) yields the contribution from the quartic coupling $(\tilde{X}_\mu{}^\mu)^2$,

$$\begin{aligned}
\mathcal{A}^{(1)}_{(\tilde{X}_\mu{}^\mu)^2} &= -\frac{1}{4}\Big(\mathcal{A}^{(1)}_{\bar\psi\gamma^\mu\partial_\mu\psi\bar\psi\gamma^\nu\partial_\nu\psi} - 2\mathcal{A}^{(1)}_{\bar\psi\gamma^\mu\partial_\mu\psi\partial_\nu\bar\psi\gamma^\nu\psi} + \mathcal{A}^{(1)}_{\partial_\mu\bar\psi\gamma^\mu\psi\partial_\nu\bar\psi\gamma^\nu\psi}\Big) \\
&= 4m^4(1+\cosh\theta).
\end{aligned} \tag{4.21}$$

Substituting (4.19) and (4.21) into (4.12) gives the total tree-level amplitude,

$$\mathcal{A}^{(1)} = i\frac{\lambda}{2}\Big[\mathcal{A}^{(1)}_{\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}} - \mathcal{A}^{(1)}_{(\tilde{X}_\mu{}^\mu)^2}\Big] = 4i\lambda m^4\sinh^2\theta. \tag{4.22}$$

Therefore, by (4.5) the first order S-matrix is,

$$S^{(1)}(\theta) = \frac{\mathcal{A}^{(1)}}{4m^2\sinh\theta} = i\lambda m^2\sinh\theta\,. \tag{4.23}$$

This result exactly matches with what one would expect from (2.7) at linear order in the $T\bar{T}$-coupling λ.

4.1.2 Second order S-matrix

At second order the S-matrix gets contribution from one-loop diagrams. At one-loop, two types of diagrams can contribute to the second order amplitude: the first is the tadpole diagram while the second is the bubble diagram. Both of these diagrams will be computed in this section.

Contribution from tadpole diagrams

The second order Lagrangian of the $T\bar{T}$-deformed free massive Dirac fermion can be written as,

$$i\mathcal{L}_2(\lambda) = i\left[\frac{\lambda^2}{2}m\bar\psi\psi\left(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\right)\right] \tag{4.24}$$

where,

$$\bar\psi\psi\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} = -\frac{1}{4}\bar\psi\psi\Big(\bar\psi\gamma_{(\mu}\partial_{\nu)}\psi\bar\psi\gamma^{(\mu}\partial^{\nu)}\psi - 2\bar\psi\gamma_{(\mu}\partial_{\nu)}\psi\partial^{(\mu}\bar\psi\gamma^{\nu)}\psi + \partial_{(\mu}\bar\psi\gamma_{\nu)}\psi\partial^{(\mu}\bar\psi\gamma^{\nu)}\psi\Big)\,, \tag{4.25}$$

$$\bar\psi\psi(\tilde{X}_\mu{}^\mu)^2 = -\frac{1}{4}\bar\psi\psi\Big(\bar\psi\gamma^\mu\partial_\mu\psi\ \bar\psi\gamma^\nu\partial_\nu\psi - 2\bar\psi\gamma^\mu\partial_\mu\psi\ \partial_\nu\bar\psi\gamma^\nu\psi + \partial_\mu\bar\psi\gamma^\mu\psi\ \partial_\nu\bar\psi\gamma^\nu\psi\Big)\,. \tag{4.26}$$

Since the interaction vertices contain six fields, two internal fields must be contracted resulting in loops. The sextic couplings give rise to the one-loop tadpole diagrams shown in figure 2 and contribute to the second order S-matrix. The corresponding amplitude is given by,

$$\mathcal{A}^{\rm tad} = i\frac{\lambda^2}{2}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[m\bar{\psi}\psi\Big(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\Big)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle\,. \tag{4.27}$$

Just as in the first order case one can evaluate (4.27) by first deriving the general amplitude of a general sextic coupling,

$$\begin{aligned}\mathcal{A}^{\rm tad}_{\bar\psi_a\psi_b\bar\psi_c\psi_d\bar\psi_e\psi_f} &= \langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\bar\psi_a\psi_b\bar\psi_c\psi_d\bar\psi_e\psi_f\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle \\ &= -\big\langle\psi_b(x)\bar\psi_a(x)\big\rangle\mathcal{A}^{(1)}_{\bar\psi_c\psi_d\bar\psi_e\psi_f} + \big\langle\psi_d(x)\bar\psi_a(x)\big\rangle\mathcal{A}^{(1)}_{\bar\psi_c\psi_b\bar\psi_e\psi_f} - \big\langle\psi_f(x)\bar\psi_a(x)\big\rangle\mathcal{A}^{(1)}_{\bar\psi_c\psi_b\bar\psi_e\psi_d} \\ &\quad + \big\langle\psi_b(x)\bar\psi_c(x)\big\rangle\mathcal{A}^{(1)}_{\bar\psi_a\psi_d\bar\psi_e\psi_f} - \big\langle\psi_b(x)\bar\psi_e(x)\big\rangle\mathcal{A}^{(1)}_{\bar\psi_a\psi_d\bar\psi_c\psi_f} - \big\langle\psi_d(x)\bar\psi_c(x)\big\rangle\mathcal{A}^{(1)}_{\bar\psi_a\psi_b\bar\psi_e\psi_f} \\ &\quad + \big\langle\psi_f(x)\bar\psi_c(x)\big\rangle\mathcal{A}^{(1)}_{\bar\psi_a\psi_b\bar\psi_e\psi_d} + \big\langle\psi_d(x)\bar\psi_e(x)\big\rangle\mathcal{A}^{(1)}_{\bar\psi_a\psi_b\bar\psi_c\psi_f} - \big\langle\psi_f(x)\bar\psi_e(x)\big\rangle\mathcal{A}^{(1)}_{\bar\psi_a\psi_b\bar\psi_c\psi_d}\end{aligned} \tag{4.28}$$

where all possible Wick contractions of fermionic fields were performed and $\mathcal{A}^{(1)}_{\bar\psi_a\psi_b\bar\psi_c\psi_d}$ represents the amplitude from a general quartic coupling $\bar\psi_a\psi_b\bar\psi_c\psi_d$ given by (4.13). The internal propagators give rise to loops whose values are derived in appendix B,

$$\langle\psi_a(x)\bar\psi_b(x)\rangle = N_0\delta_{ab}, \tag{4.29}$$

$$\langle\partial^\mu\psi_a(x)\bar\psi_b(x)\rangle = -\langle\psi_a(x)\partial^\mu\bar\psi_b(x)\rangle = N_1\gamma^\mu_{ab}, \tag{4.30}$$

$$\langle\partial^\mu\psi_a(x)\partial^\nu\bar\psi_b(x)\rangle = imN_1\eta^{\mu\nu}\delta_{ab} \tag{4.31}$$

where,

$$N_0 = -\frac{m}{4\pi}\log\left(\frac{m^2}{\Lambda^2}\right)\,, \tag{4.32}$$

$$N_1 = \frac{i\Lambda^2}{8\pi}\left[1 + \frac{m^2}{\Lambda^2}\log\left(\frac{m^2}{\Lambda^2}\right)\right]\,, \tag{4.33}$$

and Λ is the UV cut-off.

Using the general expression for the amplitude of a sextic coupling, (4.28), one can compute the contribution of $\bar\psi\psi\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$ by evaluating the contributions from each term of (4.25). The contribution of the first term in (4.25) is explicitly computed in appendix D.1 to be,

$$\mathcal{A}^{\rm tad}_{\bar\psi\psi\bar\psi\gamma_{(\mu}\partial_{\nu)}\psi\bar\psi\gamma^{(\mu}\partial^{\nu)}\psi} = 0\,, \tag{4.34}$$

where (4.29), (4.30) and (4.31) were used to obtain the above result.

Notice that the last term in (4.25) is the complex conjugate of the first term and will hence give a vanishing contribution to the amplitude as well,

$$\mathcal{A}^{\rm tad}_{\bar\psi\psi\partial_{(\mu}\bar\psi\gamma_{\nu)}\psi\partial^{(\mu}\bar\psi\gamma^{\nu)}\psi} = 0. \tag{4.35}$$

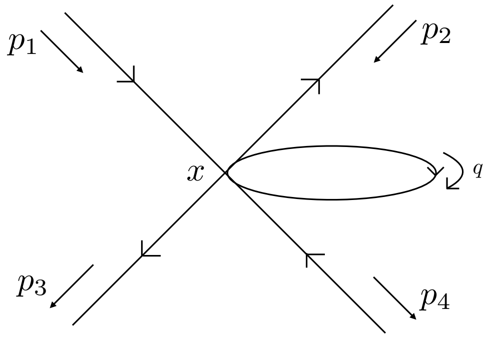

Figure 2. Tadpole diagram that contributes to the S-matrix at second order.

The second term in (4.25) gives a non-zero contribution to the amplitude and can be computed in a similar way,

$$\mathcal{A}^{\text{tad}}_{\bar{\psi}\psi\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi} = -\frac{m^3\Lambda^2}{\pi}\left[1+4\left(2-\cosh\theta\right)\frac{m^2}{\Lambda^2}\log\frac{m^2}{\Lambda^2}\right]\cosh^2\frac{\theta}{2}. \tag{4.36}$$

Combining (4.34), (4.35) and (4.36) according to (4.25), the total contribution from the sextic coupling $\bar{\psi}\psi\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$ is,

$$\begin{aligned}\mathcal{A}^{\text{tad}}_{\bar{\psi}\psi\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}} &= -\frac{1}{4}\left[\mathcal{A}^{\text{tad}}_{\bar{\psi}\psi\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi} - 2\mathcal{A}^{\text{tad}}_{\bar{\psi}\psi\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi} + \mathcal{A}^{\text{tad}}_{\bar{\psi}\psi\partial_{(\mu}\bar{\psi}\gamma_{\nu)}\psi\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi}\right]\\ &= -\frac{m^3\Lambda^2}{2\pi}\left[1+4\left(2-\cosh\theta\right)\frac{m^2}{\Lambda^2}\log\frac{m^2}{\Lambda^2}\right]\cosh^2\frac{\theta}{2}. \end{aligned} \tag{4.37}$$

Following the same steps as in (D.1) one can compute the contribution from the other sextic coupling $\bar{\psi}\psi(\tilde{X}_{\mu}{}^{\mu})^2$ by determining the contributions from each term in (4.26),

$$\begin{aligned}\mathcal{A}^{\text{tad}}_{\bar{\psi}\psi\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi\bar{\psi}\gamma^{\nu}\partial_{\nu}\psi} &= 0\\ \mathcal{A}^{\text{tad}}_{\bar{\psi}\psi\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi\partial_{\nu}\bar{\psi}\gamma^{\nu}\psi} &= \frac{m^3\Lambda^2}{\pi}\left(1+\cosh\theta\right)\\ \mathcal{A}^{\text{tad}}_{\bar{\psi}\psi\partial_{\mu}\bar{\psi}\gamma^{\mu}\psi\partial_{\nu}\bar{\psi}\gamma^{\nu}\psi} &= 0. \end{aligned} \tag{4.38}$$

Combining the individual amplitudes according to (4.26) the total contribution from the sextic coupling $\bar{\psi}\psi(\tilde{X}_{\mu}{}^{\mu})^2$ is,

$$\begin{aligned}\mathcal{A}^{\text{tad}}_{\bar{\psi}\psi(\tilde{X}_{\mu}{}^{\mu})^2} &= -\frac{1}{4}\left[\mathcal{A}^{\text{tad}}_{\bar{\psi}\psi\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi\bar{\psi}\gamma^{\nu}\partial_{\nu}\psi} - 2\mathcal{A}^{\text{tad}}_{\bar{\psi}\psi\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi\partial_{\nu}\bar{\psi}\gamma^{\nu}\psi} + \mathcal{A}^{\text{tad}}_{\bar{\psi}\psi\partial_{\mu}\bar{\psi}\gamma^{\mu}\psi\partial_{\nu}\bar{\psi}\gamma^{\nu}\psi}\right]\\ &= \frac{m^3\Lambda^2}{2\pi}\left(1+\cosh\theta\right). \end{aligned} \tag{4.39}$$

Finally, substituting (4.37) and (4.39) into (4.27) yields the contribution to the amplitude from the tadpole diagrams,

$$\mathcal{A}^{\text{tad}} = i\frac{\lambda^2}{2}m\left(\mathcal{A}^{\text{tad}}_{\bar{\psi}\psi\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}} - \mathcal{A}^{\text{tad}}_{\bar{\psi}\psi(\tilde{X}_{\mu}{}^{\mu})^2}\right) = -i\frac{\lambda^2m^6}{4\pi}\cosh^2\frac{\theta}{2}\left[3\frac{\Lambda^2}{m^2}+4\left(2-\cosh\theta\right)\log\frac{m^2}{\Lambda^2}\right]. \tag{4.40}$$

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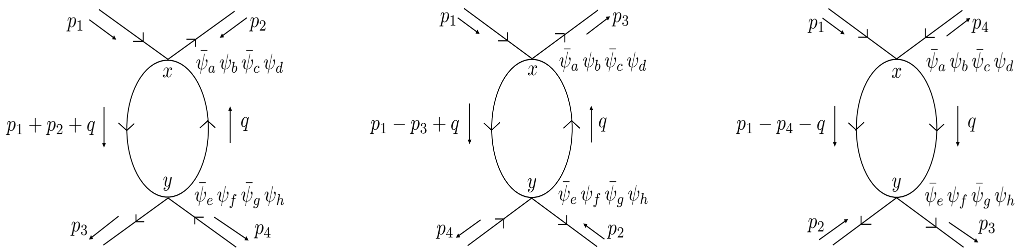

(a) s-channel. **(b)** t-channel. **(c)** u-channel.

Figure 3. The s, t and u-channels of a typical bubble diagram that contribute to the S-matrix at second order.

Converting from amplitude to S-matrix using (4.5) yields the tadpole contribution to the S-matrix,

$$S^{\text{tad}}(\theta) = \frac{\mathcal{A}^{\text{tad}}}{4m^2 \sinh\theta} = -i\frac{\lambda^2 m^4}{32\pi} \coth\frac{\theta}{2}\left[3\frac{\Lambda^2}{m^2} + 4\,(2-\cosh\theta)\log\frac{m^2}{\Lambda^2}\right]. \tag{4.41}$$

4.1.3 Contribution from bubble diagrams

The final and most complicated contribution to the S-matrix is the contribution that arises from the first order term in the Lagrangian squared,

$$\frac{1}{2!}\left(i\mathcal{L}_1(\lambda)\right)^2 = \frac{1}{2!}\left(\frac{i\lambda}{2}\right)^2\left(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}^{\mu}_{\mu})^2\right)^2. \tag{4.42}$$

These interaction vertices give rise to one-loop bubble diagrams of the form figure 3.

From (4.42) there are three different kinds of bubble diagrams:

(a) both vertices contain $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$ interactions,

(b) one vertex contains interaction $\tilde{X}_{\rho\lambda}\tilde{X}^{\rho\lambda}$ while the other contains $(\tilde{X}_{\mu}{}^{\mu})^2$ and

(c) both vertices contain $(\tilde{X}_{\mu}{}^{\mu})^2$ interactions.

For the bubble diagram case the amplitudes will be split up into s, t and u-channel contributions to the amplitude, as shown in figure 3. As done in the previous cases, begin by writing down the general expressions for the amplitude where both vertices x and y contain arbitrary non-derivative quartic couplings. These expressions will be the building blocks for the computation of the amplitude from the bubble diagrams.

In order to compute the general contributions of the s, t and u-channels to the amplitude consider the case when the vertex x involves a quartic coupling $\bar\psi_a(x)\psi_b(x)\bar\psi_c(x)\psi_d(x)$ and vertex y contains another quartic coupling $\bar\psi_e(y)\psi_f(y)\bar\psi_g(y)\psi_h(y)$.

The general contribution of the s-channel (figure 3a) to the amplitude is given by,

$$\begin{aligned}
\mathcal{B}^{(s)}_{abcdefgh} &= \langle 0|b(p_4)a(p_3)\mathrm{T}\left[\bar\psi_e(y)\psi_f(y)\bar\psi_g(y)\psi_h(y)\bar\psi_a(x)\psi_b(x)\bar\psi_c(x)\psi_d(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} \\
&= -\bar v_c(p_2)u_d(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar u_e(p_3)v_f(p_4)G_{ha}(\xi+q)G_{bg}(q) - \bar u_e(p_3)v_h(p_4)G_{fa}(\xi+q)G_{bg}(q) \\
&\quad -\bar u_g(p_3)v_f(p_4)G_{ha}(\xi+q)G_{be}(q) + \bar u_g(p_3)v_h(p_4)G_{fa}(\xi+q)G_{be}(q)\Big) \\
&\quad -\bar v_a(p_2)u_b(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar u_e(p_3)v_f(p_4)G_{hc}(\xi+q)G_{dg}(q) - \bar u_e(p_3)v_h(p_4)G_{fc}(\xi+q)G_{dg}(q)
\end{aligned}$$

$$-\bar{u}_g(p_3)v_f(p_4)G_{hc}(\xi+q)G_{de}(q)+\bar{u}_g(p_3)v_h(p_4)G_{fc}(\xi+q)G_{de}(q)\Big)$$
$$+\bar{v}_a(p_2)u_d(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{u}_e(p_3)v_f(p_4)G_{hc}(\xi+q)G_{bg}(q)-\bar{u}_e(p_3)v_h(p_4)G_{fc}(\xi+q)G_{bg}(q)$$
$$-\bar{u}_g(p_3)v_f(p_4)G_{hc}(\xi+q)G_{be}(q)+\bar{u}_g(p_3)v_h(p_4)G_{fc}(\xi+q)G_{be}(q)\Big)$$
$$+\bar{v}_c(p_2)u_b(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{u}_e(p_3)v_f(p_4)G_{ha}(\xi+q)G_{dg}(q)-\bar{u}_e(p_3)v_h(p_4)G_{fa}(\xi+q)G_{dg}(q)$$
$$-\bar{u}_g(p_3)v_f(p_4)G_{ha}(\xi+q)G_{de}(q)+\bar{u}_g(p_3)v_h(p_4)G_{fa}(\xi+q)G_{de}(q)\Big)\,, \tag{4.43}$$

where $\xi^2=(p_1+p_2)^2=s$ and $G_{ab}(q)=\frac{i(\gamma\cdot q+m)_{ab}}{q^2-m^2+i\epsilon}$ is the fermionic propagator in the free theory.

Next consider the contribution of the t-channel (figure 3b) to the amplitude. Since $p_1=p_3$, the general t-channel contribution to the amplitude is given by,

$$\begin{aligned}\mathcal{B}^{(t)}_{abcdefgh} &= \langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\bar{\psi}_e(y)\psi_f(y)\bar{\psi}_g(y)\psi_h(y)\bar{\psi}_a(x)\psi_b(x)\bar{\psi}_c(x)\psi_d(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)}\\
&= \bar{u}_c(p_3)u_d(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{v}_g(p_2)v_h(p_4)G_{fa}(q)G_{be}(q)-\bar{v}_g(p_2)v_f(p_4)G_{ha}(q)G_{be}(q)\\
&\quad -\bar{v}_e(p_2)v_h(p_4)G_{fa}(q)G_{bg}(q)+\bar{v}_e(p_2)v_f(p_4)G_{ha}(q)G_{bg}(q)\Big)\\
&\quad +\bar{u}_a(p_3)u_b(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{v}_g(p_2)v_h(p_4)G_{fc}(q)G_{de}(q)-\bar{v}_g(p_2)v_f(p_4)G_{hc}(q)G_{de}(q)\\
&\quad -\bar{v}_e(p_2)v_h(p_4)G_{fc}(q)G_{dg}(q)+\bar{v}_e(p_2)v_f(p_4)G_{hc}(q)G_{dg}(q)\Big)\\
&\quad -\bar{u}_a(p_3)u_d(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{v}_g(p_2)v_h(p_4)G_{fc}(q)G_{be}(q)-\bar{v}_g(p_2)v_f(p_4)G_{hc}(q)G_{be}(q)\\
&\quad -\bar{v}_e(p_2)v_h(p_4)G_{fc}(q)G_{bg}(q)+\bar{v}_e(p_2)v_f(p_4)G_{hc}(q)G_{bg}(q)\Big)\\
&\quad -\bar{u}_c(p_3)u_b(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{v}_g(p_2)v_h(p_4)G_{fa}(q)G_{de}(q)-\bar{v}_g(p_2)v_f(p_4)G_{ha}(q)G_{de}(q)\\
&\quad -\bar{v}_e(p_2)v_h(p_4)G_{fa}(q)G_{dg}(q)+\bar{v}_e(p_2)v_f(p_4)G_{ha}(q)G_{dg}(q)\Big)\,.\end{aligned} \tag{4.44}$$

Finally consider the contribution of the u-channel (figure 3c) to the amplitude. The general u-channel contribution to the amplitude is given by,

$$\begin{aligned}\mathcal{B}^{(u)}_{abcdefgh} &= \langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\bar{\psi}_e(y)\psi_f(y)\bar{\psi}_g(y)\psi_h(y)\bar{\psi}_a(x)\psi_b(x)\bar{\psi}_c(x)\psi_d(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}\\
&= \Big(u_d(p_1)v_b(p_4)\bar{u}_e(p_3)\bar{v}_g(p_2)-u_d(p_1)v_b(p_4)\bar{u}_g(p_3)\bar{v}_e(p_2)-u_b(p_1)v_d(p_4)\bar{u}_e(p_3)\bar{v}_g(p_2)\\
&\quad +u_b(p_1)v_d(p_4)\bar{u}_g(p_3)\bar{v}_e(p_2)\Big)\int\frac{d^2q}{(2\pi)^2}\Big(G_{hc}(\zeta-q)G_{fa}(q)-G_{ha}(\zeta-q)G_{fc}(q)\Big)\,,\end{aligned} \tag{4.45}$$

where $\zeta^2=(p_1-p_4)^2=(p_1-p_2)^2=u$ as $p_2=p_4$.

Notice that a general s-channel amplitude (4.43) involves evaluation of the one-loop integral,

$$\begin{aligned} I^{(s)}_{abcd}(\xi) &= \int \frac{d^2q}{(2\pi)^2} G_{ab}(\xi+q) G_{cd}(q) = \int \frac{d^2q}{(2\pi)^2} \frac{i(\gamma\cdot(\xi+q)+m)_{ab}}{(\xi+q)^2-m^2} \frac{i(\gamma\cdot q+m)_{cd}}{q^2-m^2} \\ &= -\int \frac{d^2q}{(2\pi)^2} \frac{\gamma^\mu_{ab}\gamma^\nu_{cd}(\xi+q)_\mu q_\nu + m\delta_{ab}\gamma^\mu_{cd}q_\mu + m\gamma^\mu_{ab}(\xi+q)_\mu\delta_{cd} + m^2\delta_{ab}\delta_{cd}}{[(\xi+q)^2-m^2](q^2-m^2)} \\ &= -L^{(s)}(\xi)\Big(m\not{\xi}_{ab}\delta_{cd} + m^2\delta_{ab}\delta_{cd}\Big) - \Big(\not{\xi}_{ab}\not{L}^{(s)}_{cd}(\xi) + m\delta_{ab}\not{L}^{(s)}_{cd}(\xi) + m\not{L}^{(s)}_{ab}(\xi)\delta_{cd}\Big) \\ &\quad -\gamma^\mu_{ab}\gamma^\nu_{cd}L^{(s)}_{\mu\nu}(\xi)\,, \end{aligned} \tag{4.46}$$

where $\xi^\mu = (p_1+p_2)^\mu$, $\not{L}^{(s)}_{ab} = \gamma^\mu_{ab}L^{(s)}_\mu$, $\not{\xi}_{ab} = \gamma^\mu_{ab}\xi_\mu$ and

$$\begin{aligned} L^{(s)} &= \int \frac{d^2q}{(2\pi)^2} \frac{1}{[(\xi+q)^2-m^2]\,(q^2-m^2)} = -\frac{\pi+i\theta}{4\pi m^2\sinh\theta}\,, \\ L^{(s)}_\mu &= \int \frac{d^2q}{(2\pi)^2} \frac{q_\mu}{[(\xi+q)^2-m^2]\,(q^2-m^2)} = \frac{\pi+i\theta}{8\pi m^2\sinh\theta}\,\xi_\mu\,, \\ L^{(s)}_{\mu\nu} &= \int \frac{d^2q}{(2\pi)^2} \frac{q_\mu q_\nu}{[(\xi+q)^2-m^2]\,(q^2-m^2)} \\ &= -\frac{i}{8\pi}\left[1+\log\frac{\Lambda^2}{m^2}+(i\pi-\theta)\tanh\frac{\theta}{2}\right]\eta_{\mu\nu} + \frac{i}{16\pi m^2\cosh^2\frac{\theta}{2}}\left[1+(i\pi-\theta)\coth\theta\right]\xi_\mu\xi_\nu\,. \end{aligned} \tag{4.47}$$

The above integrals are derived in detail in appendix C.

Similar yet simpler integrals are involved in the evaluation of the general contribution to the amplitude of t-channel (4.44). In the t-channel case the integrals are much simpler as both the internal propagators carry momentum q since $p_1 = p_3$ and take the form,

$$\begin{aligned} I^{(t)}_{abcd} &= \int \frac{d^2q}{(2\pi)^2} G_{ab}(q) G_{cd}(q) \\ &= -m^2\delta_{ab}\delta_{cd}L^{(t)} - \Big(m\delta_{ab}\not{L}^{(t)}_{cd} + m\not{L}^{(t)}_{ab}\delta_{cd}\Big) - \gamma^\mu_{ab}\gamma^\nu_{cd}L^{(t)}_{\mu\nu}\,, \end{aligned} \tag{4.48}$$

where,

$$L^{(t)} = \int \frac{d^2q}{(2\pi)^2} \frac{1}{\left(q^2-m^2\right)^2}\,, \qquad L^{(t)}_\mu = \int \frac{d^2q}{(2\pi)^2} \frac{q_\mu}{\left(q^2-m^2\right)^2}\,, \qquad L^{(t)}_{\mu\nu} = \int \frac{d^2q}{(2\pi)^2} \frac{q_\mu q_\nu}{\left(q^2-m^2\right)^2}. \tag{4.49}$$

However, there is no need to explicitly compute $L^{(t)}$, $L^{(t)}_\mu$ and $L^{(t)}_{\mu\nu}$ as their values can easily be obtained from their s-channel counterparts (4.47). In section 4.1, it was shown that the t-channel corresponds to $\theta \to i\pi$. Thus, one can simply substitute $\theta \to i\pi$ and $\xi \to 0$ into (4.47) to obtain the corresponding t-channel loop integrals $L^{(t)}$, $L^{(t)}_\mu$ and $L^{(t)}_{\mu\nu}$. Hence,

$$L^{(t)} = \frac{i}{4\pi m^2}\,, \qquad L^{(t)}_\mu = 0\,, \qquad L^{(t)}_{\mu\nu} = \frac{i}{8\pi}\left(1-\log\frac{\Lambda^2}{m^2}\right)\eta_{\mu\nu}. \tag{4.50}$$

In the u-channel case (4.45) one must compute the following loop-integral similar to the s-channel case,

$$\begin{aligned}
I^{(u)}_{abcd}(\zeta) &= \int \frac{d^2q}{(2\pi)^2} G_{ab}(\zeta-q)G_{cd}(q) \\
&= -L^{(u)}(\zeta)\Big(m\not{\zeta}_{ab}\delta_{cd}+m^2\delta_{ab}\delta_{cd}\Big)-\Big(\not{\zeta}_{ab}\not{L}^{(u)}_{cd}(\zeta)+m\delta_{ab}\not{L}^{(u)}_{cd}(\zeta)-m\not{L}^{(u)}_{ab}(\zeta)\delta_{cd}\Big) \\
&\quad +\gamma^{\mu}_{ab}\gamma^{\nu}_{cd}L^{(u)}_{\mu\nu}(\zeta)\,, \qquad (4.51)
\end{aligned}$$

where $\zeta^\mu = (p_1-p_4)^\mu = (p_1-p_2)^\mu$ and

$$\begin{aligned}
L^{(u)} &= \int \frac{d^2q}{(2\pi)^2}\frac{1}{\big[(\zeta-q)^2-m^2\big](q^2-m^2)}\,, \\
L^{(u)}_{\mu} &= \int \frac{d^2q}{(2\pi)^2}\frac{q_\mu}{\big[(\zeta-q)^2-m^2\big](q^2-m^2)}\,, \\
L^{(u)}_{\mu\nu} &= \int \frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu}{\big[(\zeta-q)^2-m^2\big](q^2-m^2)}\,. \qquad (4.52)
\end{aligned}$$

Similarly to the t-channel case, it was shown in section 4.1 that the u-channel corresponds to $\theta \to i\pi-\theta$. Hence, naively one can evaluate the u-channel loop-integrals $L^{(u)}$, $L^{(u)}_\mu$ and $L^{(u)}_{\mu\nu}$ directly from their s-channel counterparts (4.47) by replacing $\theta \to i\pi-\theta$.[9] However, notice that in the u-channel loop integrals (4.52) the functional dependence of the integrands on ζ is of the form $\zeta-q$, while in the s-channel loop integrals (4.47) the corresponding functional dependence is $\xi+q$. Therefore, since the functional dependence of the integrands on ζ is $\zeta-q$, an extra minus sign must be included whenever the integrand is odd in q,

$$\begin{aligned}
L^{(u)} &= i\frac{\theta}{4\pi m^2\sinh\theta}\,, \\
L^{(u)}_{\mu} &= i\frac{\theta}{8\pi m^2\sinh\theta}\zeta_\mu\,, \\
L^{(u)}_{\mu\nu} &= -\frac{i}{8\pi}\left[1+\log\frac{\Lambda^2}{m^2}-\theta\coth\frac{\theta}{2}\right]\eta_{\mu\nu}-\frac{i}{16\pi m^2\sinh^2\frac{\theta}{2}}\left(1-\theta\coth\theta\right)\zeta_\mu\zeta_\nu\,. \qquad (4.53)
\end{aligned}$$

The derivations of the above integrals are given explicitly in appendix C.

When the vertices have derivative interactions the amplitudes involve more complicated one-loop integrals,

$$(I_{\mu_1\cdots\mu_n})_{abcd}(\xi) = \int \frac{d^2q}{(2\pi)^2}\prod_{i=1}^{n} q_{\mu_i} G_{ab}(\xi+q)G_{cd}(q)\,, \qquad (4.54)$$

which requires the computation of integrals of the form,

$$L_{\mu_1\cdots\mu_{n+2}} = \int \frac{d^2q}{2\pi^2}\frac{\prod_{i=1}^{n+2} q_{\mu_i}}{\big[(\xi+q)^2-m^2\big](q^2-m^2)} \qquad (4.55)$$

[9]Replacing θ by $i\pi-\theta$ is equivalent to replacing ξ^2 by ζ^2.

where $n \in \{0, \ldots, 4\}$. A detailed discussion on the evaluation of these one-loop integrals can be found in appendix C. At this point one can plug the one-loop integral expressions and vertices into MATHEMATICA to obtain the contribution to the amplitudes from the s, t and u-channels.

It is important, however, to mention that in the S-matrix computation for a $T\bar{T}$-deformed scalar [30], one does not need to calculate amplitudes for all the channels explicitly to compute the total amplitude. Rather, one can use the following trick: evaluate the s-channel amplitude explicitly, then consider the appropriate limits to obtain the t and u-channel amplitudes. However, such a procedure can not be performed when there are non-trivial polarization vectors that depend on the momenta or rapidities of the external particles. To compute the total amplitude of the $T\bar{T}$-deformed free massive Dirac fermionic theory one must take into account the external polarization vectors described by the Dirac spinors u_a, v_a, $\bar{u}_a$ and $\bar{v}_a$ which depend on the rapidities θ_i of the particles. Thus the naive substitution $\theta \to i\pi$ or $\theta \to i\pi - \theta$ into the final s-channel amplitude would produce incorrect results for the amplitude from the other channels and hence they must be explicitly computed.

(a) Bubble diagrams with both vertices containing interaction $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$

First consider the bubble diagrams where both vertices of figure 3 contain quartic couplings $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$. The amplitude due to these kinds of bubble diagrams is given by,

$$
\begin{aligned}
\mathcal{A}_{\rm a} &= \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\tilde{X}_{\mu\nu}(y)\tilde{X}^{\mu\nu}(y)\tilde{X}_{\rho\lambda}(x)\tilde{X}^{\rho\lambda}(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle \\
&= \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\gamma^{(\mu}\partial^{\nu)}\psi(y)\;\; \bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x) \\
&\quad -2\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\gamma^{(\mu}\partial^{\nu)}\psi(y)\;\; \bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x) \\
&\quad +\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\gamma^{(\mu}\partial^{\nu)}\psi(y)\;\; \partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x) \\
&\quad -2\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\;\; \bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x) \\
&\quad +4\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\;\; \bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x) \\
&\quad -2\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\;\; \partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x) \\
&\quad +\partial_{(\mu}\bar{\psi}(y)\gamma_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\;\; \bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x) \\
&\quad -2\partial_{(\mu}\bar{\psi}(y)\gamma_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\;\; \bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x) \\
&\quad +\partial_{(\mu}\bar{\psi}(y)\gamma_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\;\; \partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\;\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle\,.
\end{aligned}
\tag{4.56}
$$

Denote the contribution to the amplitude from the $i^{\rm th}$ term in (4.56) by $\mathcal{A}_{\rm a}^{\{(s_i),(t_i),(u_i)\}}$. The total s-channel contribution to the amplitude in the case where both vertices contain quartic couplings $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$ can be computed by plugging the terms found in (4.56) into the general s-channel amplitude (4.43). This computation is done for all nine terms found in (4.56) in appendix D.2.

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Combining the s-channel contributions, (D.3)–(D.11), gives the total s-channel amplitude from the s-channel diagram where both vertices contain quartic couplings $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$,

$$\begin{aligned}\mathcal{A}_{\rm a}^{(s)} &= 2\left(\mathcal{A}_{\rm a}^{(s_1)}+\mathcal{A}_{\rm a}^{(s_2)}+\cdots+\mathcal{A}_{\rm a}^{(s_9)}\right)\\ &= -\frac{\lambda^2 m^6}{38400\pi}\Bigg[9600\pi\,\mathrm{csch}\,\theta\,(10+2\cosh\theta+\cosh 2\theta+2\cosh 3\theta+\cosh 4\theta)\\ &\quad+i\left(49852-2520\frac{\Lambda^2}{m^2}-38400\log\frac{m}{\Lambda}+\cosh\theta\left(-21889-11040\frac{\Lambda^2}{m^2}-1350\frac{\Lambda^4}{m^4}\right.\right.\\ &\quad\left.+96000\log\frac{m}{\Lambda}\right)+24\cosh 2\theta\left(-1587-655\frac{\Lambda^2}{m^2}+3200\log\frac{m}{\Lambda}\right)+\cosh 3\theta\left(-14647\right.\\ &\quad\left.\left.+38400\log\frac{m}{\Lambda}\right)+9600\theta\,\mathrm{csch}\,\theta\,(10+2\cosh\theta+\cosh 2\theta+2\cosh 3\theta+\cosh 4\theta)\right)\Bigg]. \end{aligned} \tag{4.57}$$

An extra multiplicative factor 2 arises from the identical contribution of the diagram with the two vertices x and y exchanged. The same factor will be included while computing the t and u-channel amplitudes.

The t-channel contribution to the amplitude when both vertices contain the quartic coupling $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$ can be computed in a similar method to the s-channel contribution. The total t-channel contribution to the amplitude can be computed by plugging the terms found in (4.56) into the general t-channel amplitude (4.44). This computation is done for all nine terms found in (4.56) in appendix D.2.

Combining the t-channel contributions, (D.13)–(D.21), gives the total t-channel amplitude from the t-channel diagram with both vertices containing quartic couplings $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$,

$$\begin{aligned}\mathcal{A}_{\rm a}^{(t)} &= 2\left(\mathcal{A}_{\rm a}^{(t_1)}+\mathcal{A}_{\rm a}^{(t_2)}+\cdots+\mathcal{A}_{\rm a}^{(t_9)}\right)\\ &= i\frac{\lambda^2 m^6}{256\pi}\left[18\frac{\Lambda^2}{m^2}+9\frac{\Lambda^4}{m^4}+2\cosh\theta\left(31+33\frac{\Lambda^2}{m^2}+192\log\frac{m}{\Lambda}\right)-128\cosh 2\theta\left(1+2\log\frac{m}{\Lambda}\right)\right]. \end{aligned} \tag{4.58}$$

The total u-channel contribution to the amplitude from the same quartic couplings as above can be computed in a similar method to the s and t-channel contributions. This is evaluated by plugging the terms found in (4.56) into the general u-channel amplitude (4.45). See appendix D.2 for the u-channel contributions of all the individual terms in (4.56).

Combining the u-channel contributions, (D.23)–(D.31), gives the total u-channel amplitude from the u-channel diagram where both vertices contain quartic couplings $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$,

$$\begin{aligned}\mathcal{A}_{\rm a}^{(u)} &= 2\left(\mathcal{A}_{\rm a}^{(u_1)}+\mathcal{A}_{\rm a}^{(u_2)}+\cdots+\mathcal{A}_{\rm a}^{(u_9)}\right)\\ &= -i\frac{\lambda^2 m^6}{9600\pi}\cosh^2\frac{\theta}{2}\left[18011+7470\frac{\Lambda^2}{m^2}+675\frac{\Lambda^4}{m^4}+6\cosh\theta\left(-3843-2620\frac{\Lambda^2}{m^2}+3200\log\frac{m}{\Lambda}\right)\right.\\ &\quad\left.+\cosh 2\theta\left(14647-38400\log\frac{m}{\Lambda}\right)-9600\left(\theta\coth\frac{\theta}{2}+4\log\frac{m}{\Lambda}+2\theta\sinh 2\theta\right)\right]. \end{aligned} \tag{4.59}$$

Finally, adding together the contributions from the s, t and u-channels to the amplitude, (4.57), (4.58) and (4.59), one finds the total amplitude from the bubble diagrams where both vertices contain the quartic interaction $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$,

$$\begin{aligned}\mathcal{A}_{\rm a} &= \mathcal{A}_{\rm a}^{(s)}+\mathcal{A}_{\rm a}^{(t)}+\mathcal{A}_{\rm a}^{(u)}\\ &= -\frac{\lambda^2 m^6}{19200\pi\sinh\theta}\Bigg[4800\pi\left(10+2\cosh\theta+\cosh 2\theta+2\cosh 3\theta+\cosh 4\theta\right)+i\Bigg(\bigg(34571\\ &\quad +4860\frac{\Lambda^2}{m^2}\bigg)\sinh\theta-\sinh 2\theta\left(6659+9360\frac{\Lambda^2}{m^2}+9600\log\frac{m}{\Lambda}\right)+4800\left(8\theta-13\sinh\theta\log\frac{m}{\Lambda}\right)\\ &\quad -\sinh 3\theta\left(3163+7860\frac{\Lambda^2}{m^2}-14400\log\frac{m}{\Lambda}\right)\Bigg)\Bigg]\,. \end{aligned} \tag{4.60}$$

(b) Bubble diagrams with one vertex containing interaction $\tilde{X}_{\rho\lambda}\tilde{X}^{\rho\lambda}$ while the other vertex contains $(\tilde{X}_\mu{}^\mu)^2$

Consider the bubble diagrams of figure 3 with one vertex containing the quartic coupling $\tilde{X}_{\rho\lambda}\tilde{X}^{\rho\lambda}$ and the second vertex containing the quartic coupling $(\tilde{X}_\mu{}^\mu)^2$. The amplitude due to these kinds of bubble diagrams is given by,

$$\begin{aligned}\mathcal{A}_{\rm b} =& -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\left(\tilde{X}_\mu{}^\mu(y)\right)^2\tilde{X}_{\rho\lambda}(x)\tilde{X}^{\rho\lambda}(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle\\ =& -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\bar\psi(y)\gamma^\nu\partial_\nu\psi(y)\;\;\bar\psi(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar\psi(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\\ & -2\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\bar\psi(y)\gamma^\nu\partial_\nu\psi(y)\;\;\bar\psi(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar\psi(x)\gamma^{\lambda)}\psi(x)\\ & +\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\bar\psi(y)\gamma^\nu\partial_\nu\psi(y)\;\;\partial_{(\rho}\bar\psi(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar\psi(x)\gamma^{\lambda)}\psi(x)\\ & -2\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\;\;\bar\psi(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar\psi(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\\ & +4\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\;\;\bar\psi(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar\psi(x)\gamma^{\lambda)}\psi(x)\\ & -2\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\;\;\partial_{(\rho}\bar\psi(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar\psi(x)\gamma^{\lambda)}\psi(x)\\ & +\partial_\mu\bar\psi(y)\gamma^\mu\psi(y)\;\partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\;\;\bar\psi(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar\psi(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\\ & -2\partial_\mu\bar\psi(y)\gamma^\mu\psi(y)\;\partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\;\;\bar\psi(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar\psi(x)\gamma^{\lambda)}\psi(x)\\ & +\partial_\mu\bar\psi(y)\gamma^\mu\psi(y)\;\partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\;\;\partial_{(\rho}\bar\psi(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar\psi(x)\gamma^{\lambda)}\psi(x)\;\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle\,. \end{aligned} \tag{4.61}$$

The multiplicative factor of 2 arises from the two identical types of cross-terms: $\left(\tilde{X}_\mu{}^\mu(y)\right)^2\cdot\tilde{X}_{\rho\lambda}(x)\tilde{X}^{\rho\lambda}(x)$ and $\tilde{X}_{\mu\nu}(y)\tilde{X}^{\mu\nu}(y)\left(\tilde{X}_\rho{}^\rho(x)\right)^2$.

Denote the contribution to the amplitude from the $i^{\rm th}$ term in (4.61) by $\mathcal{A}_{\rm b}^{\{(s_i),(t_i),(u_i)\}}$. The s-channel contribution to the amplitude can be computed by evaluating the contributions from each of the above terms in (4.61) and adding them together. The contribution to the s-channel amplitude from each term found in (4.61) is computed in appendix D.2 by plugging each term into the general s-channel amplitude (4.43).

Combining the s-channel contributions to the amplitude, (D.32)–(D.40), gives the total s-channel contribution to the amplitude from the diagrams where one vertex contains the quartic coupling $\tilde{X}_{\rho\lambda}\tilde{X}^{\rho\lambda}$ while the other vertex contains the quartic coupling $(\tilde{X}_\mu{}^\mu)^2$,

$$\begin{aligned}\mathcal{A}_\text{b}^{(s)} &= 2\left(\mathcal{A}_\text{b}^{(s_1)}+\mathcal{A}_\text{b}^{(s_2)}+\cdots+\mathcal{A}_\text{b}^{(s_9)}\right)\\ &= \frac{\lambda^2 m^6}{19200\pi\sinh\theta}\Bigg[9600\pi\,(9+3\cosh\theta+3\cosh 2\theta+\cosh 3\theta)+i\Bigg(9600\ \theta\,(9+3\cosh\theta\\ &\quad +3\cosh 2\theta+\cosh 3\theta)+12\sinh\theta\left(4147-470\frac{\Lambda^2}{m^2}-4800\log\frac{m}{\Lambda}\right)-2\sinh 2\theta\Bigg(3347\\ &\quad -60\frac{\Lambda^2}{m^2}+225\frac{\Lambda^4}{m^4}-14400\log\frac{m}{\Lambda}\Bigg)+12\sinh 3\theta\left(157-70\frac{\Lambda^2}{m^2}+1600\log\frac{m}{\Lambda}\right)+251\sinh 4\theta\Bigg)\Bigg]\,.\end{aligned} \tag{4.62}$$

Just as in the computation of the first type of bubble diagram, (4.57), a factor of 2 arises in the first line of the above expression, (4.62), because the same contribution would be found if the vertices x and y were exchanged. A similar factor would appear while evaluating the t and u-channel amplitudes as well.

Similarly, the t-channel contribution to the amplitude can be computed by evaluating the contributions from each of the terms in (4.61) and adding them together. The contribution to the t-channel amplitude from each term found in (4.61) is computed in appendix D.2 by plugging each term into the general t-channel amplitude (4.44).

Combining the t-channel contributions to the amplitude, (D.41)–(D.49), gives the total t-channel contribution to the amplitude from the diagram where one vertex contains the quartic coupling $\tilde{X}_{\rho\lambda}\tilde{X}^{\rho\lambda}$ while the other vertex contains the quartic coupling $(\tilde{X}_\mu{}^\mu)^2$,

$$\begin{aligned}\mathcal{A}_\text{b}^{(t)} &= 2\left(\mathcal{A}_\text{b}^{(t_1)}+\mathcal{A}_\text{b}^{(t_2)}+\cdots+\mathcal{A}_\text{b}^{(t_9)}\right)\\ &= -i\frac{\lambda^2 m^6}{64\pi}\left[3\frac{\Lambda^4}{m^4}+6\frac{\Lambda^2}{m^2}+2\cosh\theta\left(15+17\frac{\Lambda^2}{m^2}+96\log\frac{m}{\Lambda}\right)\right]\,.\end{aligned} \tag{4.63}$$

Finally, the u-channel contribution to the amplitude can be computed by evaluating the contributions from each of the terms in (4.61) and adding them together. The contribution to the u-channel amplitude from each term found in (4.61) is computed in appendix D.2 by plugging each term into the general u-channel amplitude (4.45).

Combining the u-channel contributions to the amplitude, (D.50)–(D.58), gives the total u-channel contribution to the amplitude from the diagram where one vertex contains the quartic coupling $\tilde{X}_{\rho\lambda}\tilde{X}^{\rho\lambda}$ while the other vertex contains the quartic coupling $(\tilde{X}_\mu{}^\mu)^2$,

$$\begin{aligned}\mathcal{A}_\text{b}^{(u)} &= 2\left(\mathcal{A}_\text{b}^{(u_1)}+\mathcal{A}_\text{b}^{(u_2)}+\cdots+\mathcal{A}_\text{b}^{(u_9)}\right)\\ &= -i\frac{\lambda^2 m^6}{2400\pi}\cosh\frac{\theta}{2}\left[4800\ \theta\cosh\frac{3\theta}{2}\coth\frac{\theta}{2}+\cosh\frac{\theta}{2}\left(2063+3510\frac{\Lambda^2}{m^2}-225\frac{\Lambda^4}{m^4}+251\cosh 2\theta\right)\right.\\ &\quad\left.+\cosh\frac{\theta}{2}\cosh\theta\left(-7114+840\frac{\Lambda^2}{m^2}+28800\log\frac{m}{\Lambda}\right)\right]\,.\end{aligned} \tag{4.64}$$

Finally, adding together the contributions from the s, t and u-channels to the amplitude, (4.62), (4.63) and (4.64), one finds the total amplitude from the bubble diagrams

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where one vertex contains the quartic coupling $\tilde{X}_{\rho\lambda}\tilde{X}^{\rho\lambda}$ while the other vertex contains quartic coupling $(\tilde{X}_\mu{}^\mu)^2$,

$$\begin{aligned}\mathcal{A}_\mathrm{b} &= \mathcal{A}_\mathrm{b}^{(s)} + \mathcal{A}_\mathrm{b}^{(t)} + \mathcal{A}_\mathrm{b}^{(u)} \\ &= \frac{\lambda^2 m^6}{800\pi\sinh\theta}\Bigg[400\pi\left(9+3\cosh\theta+3\cosh 2\theta+\cosh 3\theta\right) - i\left(\sinh\theta\left(-2047+930\frac{\Lambda^2}{m^2}\right.\right.\\ &\quad\left. + 3600\log\frac{m}{\Lambda}\right) + \sinh 2\theta\left(\frac{91}{2}+570\frac{\Lambda^2}{m^2}+2400\log\frac{m}{\Lambda}\right) + 2\sinh 3\theta\left(-177+35\frac{\Lambda^2}{m^2}\right.\\ &\quad\left.\left. + 200\log\frac{m}{\Lambda}\right) - 3200\,\theta\right)\Bigg]\,. \end{aligned} \tag{4.65}$$

(c) Bubble diagrams with both vertices containing interactions $(\tilde{X}_\mu{}^\mu)^2$

Finally consider the bubble diagrams where both vertices of figure 3 contain quartic couplings $(\tilde{X}_\mu{}^\mu)^2$. The amplitude due to these kinds of bubble diagrams is given by,

$$\begin{aligned}\mathcal{A}_\mathrm{c} &= \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\left(\tilde{X}_\mu{}^\mu(y)\right)^2\left(\tilde{X}_\rho{}^\rho(x)\right)^2\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle \\ &= \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\,\big[\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\bar\psi(y)\gamma^\nu\partial_\nu\psi(y)\;\;\bar\psi(x)\gamma^\rho\partial_\rho\psi(x)\;\bar\psi(x)\gamma^\lambda\partial_\lambda\psi(x) \\ &\quad -2\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\bar\psi(y)\gamma^\nu\partial_\nu\psi(y)\;\;\bar\psi(x)\gamma^\rho\partial_\rho\psi(x)\;\partial_\lambda\bar\psi(x)\gamma^\lambda\psi(x) \\ &\quad +\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\bar\psi(y)\gamma^\nu\partial_\nu\psi(y)\;\;\partial_\rho\bar\psi(x)\gamma^\rho\psi(x)\;\partial_\lambda\bar\psi(x)\gamma^\lambda\psi(x) \\ &\quad -2\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\;\;\bar\psi(x)\gamma^\rho\partial_\rho\psi(x)\;\bar\psi(x)\gamma^\lambda\partial_\lambda\psi(x) \\ &\quad +4\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\;\;\bar\psi(x)\gamma^\rho\partial_\rho\psi(x)\;\partial_\lambda\bar\psi(x)\gamma^\lambda\psi(x) \\ &\quad -2\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\;\;\partial_\rho\bar\psi(x)\gamma^\rho\psi(x)\;\partial_\lambda\bar\psi(x)\gamma^\lambda\psi(x) \\ &\quad +\partial_\mu\bar\psi(y)\gamma^\mu\psi(y)\;\partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\;\;\bar\psi(x)\gamma^\rho\partial_\rho\psi(x)\;\bar\psi(x)\gamma^\lambda\partial_\lambda\psi(x) \\ &\quad -2\partial_\mu\bar\psi(y)\gamma^\mu\psi(y)\;\partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\;\;\bar\psi(x)\gamma^\rho\partial_\rho\psi(x)\;\partial_\lambda\bar\psi(x)\gamma^\lambda\psi(x) \\ &\quad +\partial_\mu\bar\psi(y)\gamma^\mu\psi(y)\;\partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\;\;\partial_\rho\bar\psi(x)\gamma^\rho\psi(x)\;\partial_\lambda\bar\psi(x)\gamma^\lambda\psi(x)\;\big]\,a^\dagger(p_1)b^\dagger(p_2)|0\rangle\,. \end{aligned} \tag{4.66}$$

Denote the contribution to the amplitude from the i^th term in (4.66) by $\mathcal{A}_\mathrm{c}^{\{(s_i),(t_i),(u_i)\}}$. The s-channel contribution to the amplitude can be computed by evaluating the contributions from each of the above terms in (4.66) and adding the results together. The contribution to the s-channel amplitude from each term found in (4.66) is computed in appendix D.2 by plugging each term into the general s-channel amplitude (4.43).

Combining the s-channel contributions to the amplitude, (D.59)–(D.67), gives the total s-channel contribution to the amplitude from the diagrams where both vertices contain interactions $(\tilde{X}_\mu{}^\mu)^2$,

$$\begin{aligned} \mathcal{A}_{\rm c}^{(s)} &= 2\left(\mathcal{A}_{\rm c}^{(s_1)}+\mathcal{A}_{\rm c}^{(s_2)}+\cdots+\mathcal{A}_{\rm c}^{(s_9)}\right) \\ &= -\frac{\lambda^2 m^6}{19200\pi\sinh\theta}\Bigg[4800\pi\,(11+4\cosh\theta+\cosh 2\theta)+i\Bigg(4800\;\theta\,(11+4\cosh\theta+\cosh 2\theta) \\ &\quad -8\sinh\theta\left(-5111+1785\frac{\Lambda^2}{m^2}+9600\log\frac{m}{\Lambda}\right)+2\sinh 2\theta\left(4811+120\frac{\Lambda^2}{m^2}-75\frac{\Lambda^4}{m^4}\right. \\ &\quad \left.+2400\log\frac{m}{\Lambda}\right)+24\sinh 3\theta\left(42+5\frac{\Lambda^2}{m^2}\right)+17\sinh 4\theta\Bigg)\Bigg]. \end{aligned} \tag{4.67}$$

Just as in the computation of the first and second type of bubble diagrams, (4.57) and (4.62), a factor of 2 arises in the first line of the above expression, (4.67), because the same contribution would be found if the vertices x and y were exchanged. A similar factor would appear while evaluating the t and u-channel amplitudes as well.

Similarly the t-channel contribution to the amplitude can be computed by evaluating the contributions from each of the terms in (4.66) and adding the results together. The contribution to the t-channel amplitude from each term found in (4.66) is computed in appendix D.2 by plugging each term into the general t-channel amplitude (4.44).

Combining the t-channel contributions to the amplitude, (D.68)–(D.76), gives the total t-channel contribution to the amplitude from the diagram where both vertices contain interactions $(\tilde{X}_\mu{}^\mu)^2$,

$$\begin{aligned} \mathcal{A}_{\rm c}^{(t)} &= 2\left(\mathcal{A}_{\rm c}^{(t_1)}+\mathcal{A}_{\rm c}^{(t_2)}+\cdots+\mathcal{A}_{\rm c}^{(t_9)}\right) \\ &= -i\frac{\lambda^2 m^6}{64\pi}\left[32+30\frac{\Lambda^2}{m^2}-\frac{\Lambda^4}{m^4}+192\log\frac{m}{\Lambda}-2\cosh\theta\left(7+9\frac{\Lambda^2}{m^2}+48\log\frac{m}{\Lambda}\right)\right]. \end{aligned} \tag{4.68}$$

Finally the u-channel contribution to the amplitude can be computed by evaluating the contributions from each of the terms in (4.66) and adding the results together. The contribution to the u-channel amplitude from each term found in (4.66) is computed in appendix D.2 by plugging each term into the general u-channel amplitude (4.45).

Combining the u-channel contributions to the amplitude, (D.77)–(D.85), gives the total u-channel contribution to the amplitude from the diagram where both vertices contain interactions $(\tilde{X}_\mu{}^\mu)^2$,

$$\begin{aligned} \mathcal{A}_{\rm c}^{(u)} &= 2\left(\mathcal{A}_{\rm c}^{(u_1)}+\mathcal{A}_{\rm c}^{(u_2)}+\cdots+\mathcal{A}_{\rm c}^{(u_9)}\right) \\ &= -i\frac{\lambda^2 m^6}{2400\pi}\cosh^2\frac{\theta}{2}\left[2179-2370\frac{\Lambda^2}{m^2}+75\frac{\Lambda^4}{m^4}-14400\log\frac{m}{\Lambda}+2\cosh\theta\left(119+60\frac{\Lambda^2}{m^2}\right)\right. \\ &\quad \left.-17\cosh 2\theta-2400\;\theta\coth\frac{\theta}{2}\right]. \end{aligned} \tag{4.69}$$

Finally, adding together the contributions from the s, t and u-channels to the amplitude, (4.67), (4.68) and (4.69), one finds the total amplitude from the bubble diagrams where both vertices contain interactions $(\tilde{X}_\mu{}^\mu)^2$,

$$\begin{aligned}\mathcal{A}_{\mathrm{c}} &= \mathcal{A}_{\mathrm{c}}^{(s)} + \mathcal{A}_{\mathrm{c}}^{(t)} + \mathcal{A}_{\mathrm{c}}^{(u)} \\ &= -\frac{\lambda^2 m^6}{2400\pi \sinh\theta}\left[600\pi\left(11+4\cosh\theta+\cosh 2\theta\right) - i\left(\sinh\theta\left(-7586+1800\frac{\Lambda^2}{m^2}+9600\log\frac{m}{\Lambda}\right)\right.\right. \\ &\left.\left. -3\sinh\theta\cosh 2\theta\left(101+20\frac{\Lambda^2}{m^2}\right)+\sinh 2\theta\left(-\frac{3089}{2}+870\frac{\Lambda^2}{m^2}+4800\log\frac{m}{\Lambda}\right)-4800\,\theta\right)\right]. \end{aligned} \tag{4.70}$$

Total contribution from the bubble diagrams to the amplitude

The total contribution to the amplitude from the bubble diagrams is the sum of (4.60), (4.65) and (4.70),

$$\begin{aligned}\mathcal{A}^{\text{bubble}} &= \mathcal{A}_{\mathrm{a}} + \mathcal{A}_{\mathrm{b}} + \mathcal{A}_{\mathrm{c}} \\ &= -2\lambda^2 m^6 \sinh^3\theta - i\frac{\lambda^2 m^6}{4800\pi}\cosh^2\frac{\theta}{2}\left[27683+9240\frac{\Lambda^2}{m^2}-38400\log\frac{m}{\Lambda}\right. \\ &\left. -2\cosh\theta\left(10447+5940\frac{\Lambda^2}{m^2}-24000\log\frac{m}{\Lambda}\right)\right]. \end{aligned} \tag{4.71}$$

Total second order S-matrix

Adding the tadpole contribution (4.40) and the bubble contribution (4.71) to the amplitude gives the total second order amplitude,

$$\begin{aligned}\mathcal{A}^{(2)} &= \mathcal{A}^{\text{tad}} + \mathcal{A}^{\text{bubble}} \\ &= -2\lambda^2 m^6 \sinh^3\theta - i\frac{\lambda^2 m^6}{4800\pi}\cosh^2\frac{\theta}{2}\left[27683+12840\frac{\Lambda^2}{m^2}-19200\log\frac{m}{\Lambda}\right. \\ &\left. -2\cosh\theta\left(10447+5940\frac{\Lambda^2}{m^2}-19200\log\frac{m}{\Lambda}\right)\right]. \end{aligned} \tag{4.72}$$

Therefore, by (4.5) the second order S-matrix is,

$$\begin{aligned}S^{(2)}(\theta) &= \frac{\mathcal{A}^{(2)}}{4m^2\sinh\theta} \\ &= -\frac{1}{2}\lambda^2 m^4 \sinh^2\theta - i\frac{\lambda^2 m^4}{38400\pi}\coth\frac{\theta}{2}\left[27683+12840\frac{\Lambda^2}{m^2}-19200\log\frac{m}{\Lambda}\right. \\ &\left. -2\cosh\theta\left(10447+5940\frac{\Lambda^2}{m^2}-19200\log\frac{m}{\Lambda}\right)\right]. \end{aligned} \tag{4.73}$$

It is interesting to note that at second order, the real part of the amplitude comes only from the s-channel, while the t and u-channels give purely imaginary contributions to the amplitude.

4.2 Renormalized Lagrangian

Now that the S-matrix has been computed the $T\bar{T}$-deformed theory can be renormalized by demanding that the S-matrix has the form (2.7).

Discarding the imaginary finite pieces of second order S-matrix (4.73) yields,[10]

$$\begin{aligned} S^{(2)}(\theta) &= -\frac{1}{2}\lambda^2 m^4 \sinh^2\theta - i\frac{\lambda^2 m^4}{38400\pi}\coth\frac{\theta}{2}\left[12840\frac{\Lambda^2}{m^2} - 19200\log\frac{m}{\Lambda} - 2\cosh\theta\left(5940\frac{\Lambda^2}{m^2}\right.\right.\\ &\qquad \left.\left. - 19200\log\frac{m}{\Lambda}\right)\right] \\ &= -\frac{1}{2}\lambda^2 m^4 \sinh^2\theta - i\frac{\lambda^2 m^4}{\pi}\coth\frac{\theta}{2}\left[\frac{107-99\cosh\theta}{320}\frac{\Lambda^2}{m^2} - \frac{1-2\cosh\theta}{2}\log\frac{m}{\Lambda}\right]. \end{aligned} \tag{4.74}$$

Observe how the first term in (4.74) is exactly the expected second order S-matrix in an integrable field theory, but (4.74) also contains an extra divergent imaginary part. The S-matrix was computed using the classical (bare) Lagrangian (3.16), so counterterms must be added to the classical Lagrangian in order to get rid of the divergent imaginary part of $S^{(2)}(\theta)$. In this process the Lagrangian will be perturbatively renormalized up to second order in the $T\bar{T}$-coupling λ.

It is important to mention that the integrable structure of the $T\bar{T}$-deformed free massive Dirac fermion theory is what enables the theory to be renormalized perturbatively. In general, the scattering amplitude of a quantum field theory may contain logarithms of functions of Mandelstam variables, which involves θ.[11] However, local counterterms can never cancel a term involving θ and can only cancel terms involving powers of $\cosh^2\left(\frac{\theta}{2}\right)$.[12] Due to the integrable structure of the theory, our final amplitude does not contain terms involving θ although individual contributions to the amplitude have terms involving θ. This is expected because the S-matrix of an integrable theory can not have branch cuts and can only have poles.

Discarding the imaginary finite pieces of the second order amplitude (4.72) yields,

$$\begin{aligned} \mathcal{A}^{(2)} &= -2\lambda^2 m^6 \sinh^3\theta - i\frac{8\lambda^2 m^6}{\pi}\cosh^2\frac{\theta}{2}\left[\frac{107-99\cosh\theta}{320}\frac{\Lambda^2}{m^2} - \frac{1-2\cosh\theta}{2}\log\frac{m}{\Lambda}\right] \\ &= -2\lambda^2 m^6 \sinh^3\theta - i\frac{8\lambda^2 m^6}{\pi}\left[\left(\frac{103}{160}\frac{\Lambda^2}{m^2} - \frac{3}{2}\log\frac{m}{\Lambda}\right)\cosh^2\frac{\theta}{2} + \left(-\frac{99}{160}\frac{\Lambda^2}{m^2}\right.\right. \\ &\qquad \left.\left. + 2\log\frac{m}{\Lambda}\right)\cosh^4\frac{\theta}{2}\right]. \end{aligned} \tag{4.75}$$

[10] The imaginary finite pieces are not essential as one can change them by rescaling the cut-off, their choices are just equivalent to using different regularization schemes.

[11] (C.14) is an expression for the logarithm of a function of Mandelstam variables which appear while computing the one-loop momentum integrals in the scattering amplitude. When expressed in terms of rapidity, it yields terms involving θ.

[12] A polynomial in Mandelstam variable s corresponds to a polynomial in $\cosh^2\left(\frac{\theta}{2}\right)$, when expressed in terms of the rapidity.

Let the renormalized Lagrangian be,

$$\mathcal{L}_{\text{ren}}(\lambda) = -i\bar{\psi}\gamma^\mu\partial_\mu\psi + m\bar{\psi}\psi + \frac{\lambda}{2}\Big(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\Big) + \frac{\lambda^2}{2}m\bar{\psi}\psi\Big(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\Big)$$
$$+\alpha\lambda^2(\tilde{X}_\mu{}^\mu)^2 + \beta\lambda^2\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} + \mathcal{O}(\lambda^3)\,, \tag{4.76}$$

where α and β are divergent coefficients which must be tuned to exactly cancel the imaginary divergent contributions to the second order S-matrix (4.74).

The term $\alpha\lambda^2(\tilde{X}_\mu{}^\mu)^2$ contributes to the amplitude as,

$$\mathcal{A}_\alpha^{\text{count}} = (i\alpha\lambda^2)4m^4(1+\cosh\theta) = 8i\alpha\lambda^2 m^4\cosh^2\frac{\theta}{2}\,. \tag{4.77}$$

While the term $\beta\lambda^2\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$ contributes to the amplitude as,

$$\mathcal{A}_\beta^{\text{count}} = (i\beta\lambda^2)4m^4(\cosh\theta + \cosh 2\theta) = -8i\beta\lambda^2 m^4\left(3\cosh^2\frac{\theta}{2} - 4\cosh^4\frac{\theta}{2}\right)\,. \tag{4.78}$$

Adding (4.77) and (4.78), gives the total contribution from the counterterms to the amplitude,

$$\mathcal{A}^{\text{count}} = \mathcal{A}_\alpha^{\text{count}} + \mathcal{A}_\beta^{\text{count}} = 8i\lambda^2 m^4(\alpha - 3\beta)\cosh^2\frac{\theta}{2} + 32i\beta\lambda^2 m^4\cosh^4\frac{\theta}{2}\,. \tag{4.79}$$

The condition that the sum amplitude of the counterterms, (4.79), must exactly cancel the imaginary part of (4.75) gives the conditions,

$$m^4(\alpha - 3\beta) = \frac{m^6}{\pi}\left(\frac{103}{160}\frac{\Lambda^2}{m^2} - \frac{3}{2}\log\frac{m}{\Lambda}\right)\,, \tag{4.80}$$
$$32\beta m^4 = \frac{8m^6}{\pi}\left(-\frac{99}{160}\frac{\Lambda^2}{m^2} + 2\log\frac{m}{\Lambda}\right)\,. \tag{4.81}$$

Solving (4.80) and (4.81) for α and β gives,

$$\alpha = \frac{23}{128\pi}\Lambda^2\,, \tag{4.82}$$
$$\beta = \frac{m^2}{4\pi}\left(-\frac{99}{160}\frac{\Lambda^2}{m^2} + 2\log\frac{m}{\Lambda}\right)\,. \tag{4.83}$$

Substituting the above values of α and β into the renormalized Lagrangian (4.76) yields,

$$\mathcal{L}_{\text{ren}}(\lambda) = -i\bar{\psi}\gamma^\mu\partial_\mu\psi + m\bar{\psi}\psi + \frac{\lambda}{2}\left(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\right) + \frac{\lambda^2}{2}m\bar{\psi}\psi\left(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\right)$$
$$+\frac{23\Lambda^2}{128\pi}\lambda^2(\tilde{X}_\mu{}^\mu)^2 + \frac{m^2}{4\pi}\left(-\frac{99}{160}\frac{\Lambda^2}{m^2} + 2\log\frac{m}{\Lambda}\right)\lambda^2\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} + \mathcal{O}(\lambda^3)\,. \tag{4.84}$$

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However, the above renormalized Lagrangian is written in terms of the redefined fields. The renormalized Lagrangian can be written in terms of the original fields as,

$$\begin{aligned}\mathcal{L}_{\rm ren}(\lambda) =& -i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + m\bar{\psi}\psi + \frac{\lambda}{2}\left(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_{\mu}{}^{\mu})^2 + 2m\bar{\psi}\psi\tilde{X}_{\mu}{}^{\mu} - 2m^2(\bar{\psi}\psi)^2\right)\\ &-\frac{\lambda^2}{2}m\bar{\psi}\psi\left(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_{\mu}{}^{\mu})^2\right)\\ &+\frac{23\Lambda^2}{128\pi}\lambda^2(\tilde{X}_{\mu}{}^{\mu})^2 + \frac{m^2}{4\pi}\left(-\frac{99}{160}\frac{\Lambda^2}{m^2} + 2\log\frac{m}{\Lambda}\right)\lambda^2\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} + \mathcal{O}(\lambda^3)\,. \end{aligned} \tag{4.85}$$

Finally, the renormalized Lagrangian of the $T\bar{T}$-deformed free massive Dirac fermion in two dimensional Euclidean spacetime is given by,

$$\begin{aligned}\mathcal{L}_{\rm ren}(\lambda) =& \, i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi - \frac{\lambda}{2}\Big(2m\bar{\psi}\psi\;\tilde{X}_{\mu}{}^{\mu} - 2m^2(\bar{\psi}\psi)^2\Big) - \frac{g}{2}\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} + \frac{h}{2}(\tilde{X}_{\mu}{}^{\mu})^2\\ &+\frac{\lambda^2}{2}m\bar{\psi}\psi\Big(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_{\mu}{}^{\mu})^2\Big) + \mathcal{O}(\lambda^3)\,,\end{aligned} \tag{4.86}$$

where the renormalized couplings are given by,

$$\begin{aligned} g &= \lambda - \frac{\lambda^2 m^2}{2\pi}\left(\frac{99}{160}\frac{\Lambda^2}{m^2} - 2\log\frac{m}{\Lambda}\right)\,,\\ h &= \lambda - \frac{23\lambda^2}{64\pi}\Lambda^2\,. \end{aligned} \tag{4.87}$$

It is important to notice the major qualitative difference between the renormalized Lagrangian (4.86) and the classical Lagrangian (3.16). The classical Lagrangian (3.16) has only one scale, λ, which appears to be the coupling for all the quartic terms. However, the renormalized Lagrangian contains three different couplings, λ, g and h. In the renormalized Lagrangian, the two quartic terms $\bar{\psi}\psi\;\tilde{X}_{\mu}{}^{\mu}$ and $(\bar{\psi}\psi)^2$ share the old classical coupling λ, where as the other two quartic terms $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$ and $(\tilde{X}_{\mu}{}^{\mu})^2$ have very different couplings g and h, respectively, given by (4.87).

5 Discussion

In this paper the $T\bar{T}$-deformed free massive Dirac fermion in two dimensions was studied. First, the Lagrangian of the deformed theory was stated and massaged into an easier form for amplitude calculations using a field redefinition. The two-to-two S-matrix of the fermion anti-fermion scattering process was computed to second order in the $T\bar{T}$-coupling λ. At first order, the S-matrix exactly matches the expected result for an integrable field theory (2.7). However, at second order the S-matrix matches the expected result up to some divergent imaginary second order terms. Counterterms were added to the Lagrangian to cancel these divergent pieces and ensure that the final second order S-matrix agrees with the expected result (2.7), in the process the renormalized Lagrangian was obtained. Amazingly, integrability allows the naively non-renormalizable theory to be renormalized perturbatively. The renormalized Lagrangian was qualitatively very different from the classical Lagrangian as

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there are three different coupling constants (λ, g and h) in the renormalized Lagrangian, while in the classical case there is only one coupling constant (λ). Thus, the quantum integrability here leads to a more complicated renormalized Lagrangian than the classical one. This is not what always happens in an integrable theory, for example, the integrable sinh-Gordon model gives rise to a renormalized Lagrangian which has the same functional form as the classical one [30]. Further, the existence of the renormalized Lagrangian means that all quantities of the theory may now be computed using the standard QFT techniques. For example, one can compute the correlation functions of local fields perturbatively using the renormalized Lagrangian.

In this paper, renormalization was performed by computing the two-to-two S-matrix for the fermion anti-fermion scattering: $f_1 + \bar{f}_2 \to f_3 + \bar{f}_4$, and adding counterterms to cancel the divergences. However, one may consider the other possible two-to-two scattering process in this theory, namely, the fermion-fermion scattering: $f_1 + f_2 \to f_3 + f_4$. If this process had been chosen the same renormalized Lagrangian would be expected. The calculation of the S-matrix would involve an almost identical computation to what was done here except that only one plane wave solution to the Dirac equation, $u(k)$, would be present in the expressions.

The form of the renormalized Lagrangian is already exciting at second order in the $T\bar{T}$-coupling. It would be interesting to see how the renormalized Lagrangian looks at higher orders, because of the simple structure of the S-matrix the renormalized Lagrangian may have a simple and compact form. Similar to [30], the second order real contribution to the S-matrix came from the s-channel only. It would be useful to understand why this occurs and whether this is a general property of the $T\bar{T}$-deformed integrable theories.

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A Field redefinitions

The Lagrangian of the $T\bar{T}$-deformed free massive Dirac fermion in two dimensional Euclidean spacetime is given by (see (3.16)),

$$\begin{aligned}\mathcal{L}(\lambda) = {}& (i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi) - \frac{\lambda}{2}\Big(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2 + 2m\bar{\psi}\psi\tilde{X}_\mu{}^\mu - 2m^2(\bar{\psi}\psi)^2\Big)\\ &+\frac{\lambda^2}{2}m\bar{\psi}\psi\Big(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\Big).\end{aligned} \tag{A.1}$$

One can perform the following field redefinitions to the Lagrangian,

$$\begin{aligned}\psi_a &= \psi'_a + \alpha\lambda\psi'_a(\bar{\psi}'\psi') \\ \bar{\psi}_a &= \bar{\psi}'_a + \alpha\lambda(\bar{\psi}'\psi')\bar{\psi}'_a\,,\end{aligned} \tag{A.2}$$

where $a = \{1, 2\}$ is the spinor index and 'α' is assumed to be a real constant with dimension 1.

Under the field redefinitions the following terms transform as,

$$\begin{aligned}\bar{\psi}\psi &= B\bar{\psi}'\psi' \\ \bar{\psi}\gamma^\mu\partial_\mu\psi &= B\;\bar{\psi}'\gamma^\mu\partial_\mu\psi' + \alpha\lambda(1+\alpha\lambda\bar{\psi}'\psi')\;\bar{\psi}'\gamma^\mu\psi'\partial_\mu(\bar{\psi}'\psi') \\ \partial_\mu\bar{\psi}\gamma^\mu\psi &= B\;\partial_\mu\bar{\psi}'\gamma^\mu\psi' + \alpha\lambda(1+\alpha\lambda\bar{\psi}'\psi')\;\bar{\psi}'\gamma^\mu\psi'\partial_\mu(\bar{\psi}'\psi')\,,\end{aligned} \tag{A.3}$$

where

$$B = 1 + 2\alpha\lambda\bar{\psi}'\psi' + \alpha^2\lambda^2(\bar{\psi}'\psi')^2. \tag{A.4}$$

By (3.14) and (A.3), $\tilde{X}_{\mu\nu}$ and $\tilde{X}_\mu{}^\mu$ transform as,

$$\tilde{X}_{\mu\nu} = B\;\tilde{X}'_{\mu\nu} \quad \text{and} \quad \tilde{X}_\mu{}^\mu = B\;\tilde{X}'_\mu{}^\mu\,, \tag{A.5}$$

where

$$\tilde{X}'_{\mu\nu} = \frac{i}{2}\Big(\bar{\psi}'\gamma_{(\mu}\partial_{\nu)}\psi' - \partial_{(\mu}\bar{\psi}'\gamma_{\nu)}\psi'\Big)\,. \tag{A.6}$$

Writing the original Lagrangian (A.1) in terms of these redefined fields, one obtains the following redefined Lagrangian,

$$\mathcal{L}'(\lambda) = \mathcal{L}'_0(\lambda) + \mathcal{L}'_1(\lambda) + \mathcal{L}'_2(\lambda) \tag{A.7}$$

where,

$$\begin{aligned}\mathcal{L}'_0(\lambda) &= (i\bar{\psi}'\gamma^\mu\partial_\mu\psi' - m\bar{\psi}'\psi') + 2\alpha\lambda\bar{\psi}'\psi'[\tilde{X}'_\mu{}^\mu - m\bar{\psi}'\psi'] + \alpha^2\lambda^2(\bar{\psi}'\psi')^2[\tilde{X}'_\mu{}^\mu - m\bar{\psi}'\psi'] \\ \mathcal{L}'_1(\lambda) &= \frac{\lambda}{2}[(\tilde{X}'_\mu{}^\mu)^2 - \tilde{X}'_{\mu\nu}\tilde{X}'^{\mu\nu}] - \lambda m\bar{\psi}'\psi'[\tilde{X}'_\mu{}^\mu - m\bar{\psi}'\psi'] + 2\alpha\lambda^2\bar{\psi}'\psi'[(\tilde{X}'_\mu{}^\mu)^2 - \tilde{X}'_{\mu\nu}\tilde{X}'^{\mu\nu}] \\ \mathcal{L}'_2(\lambda) &= \frac{\lambda^2}{2}m\bar{\psi}'\psi'\Big(\tilde{X}'_{\mu\nu}\tilde{X}'^{\mu\nu} - (\tilde{X}'_\mu{}^\mu)^2\Big)\,.\end{aligned} \tag{A.8}$$

Notice that since ψ' and $\bar{\psi}'$ are Grassmann variables in two dimensions, $(\bar{\psi}'\psi')^3$ and $(\bar{\psi}'\psi')^2\tilde{X}'_\mu{}^\mu$ vanish and hence the term proportional to $(\bar{\psi}'\psi')^3$ and $(\bar{\psi}'\psi')^2\tilde{X}'_\mu{}^\mu$ drop out from (A.8). In order to simplify the computation of the S-matrix contributions from the bubble diagrams it is useful to get rid of the linear terms $\bar{\psi}'\psi'\tilde{X}'_\mu{}^\mu$ and $(\bar{\psi}'\psi')^2$ from the new redefined Lagrangian (A.7). This can be done by choosing $\alpha = \frac{m}{2}$. Substituting $\alpha = \frac{m}{2}$ into the new redefined Lagrangian (A.8) gives,

$$\mathcal{L}'(\lambda) = (i\bar{\psi}'\gamma^\mu\partial_\mu\psi' - m\bar{\psi}'\psi') - \frac{\lambda}{2}\Big(\tilde{X}'_{\mu\nu}\tilde{X}'^{\mu\nu} - (\tilde{X}'_\mu{}^\mu)^2\Big) - \frac{\lambda^2}{2}m\bar{\psi}'\psi'\Big(\tilde{X}'_{\mu\nu}\tilde{X}'^{\mu\nu} - (\tilde{X}'_\mu{}^\mu)^2\Big). \tag{A.9}$$

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A Jacobian factor arises in the path-integral measure due to the field redefinitions and may contribute to the S-matrix. The field redefinition (A.2), with $\alpha = \frac{m}{2}$, can be expressed in terms of the spinor components as,

$$\begin{aligned}
\psi_1 &= \psi_1' + \lambda \frac{m}{2} \psi_1' \bar{\psi}_2' \psi_2' \\
\psi_2 &= \psi_2' + \lambda \frac{m}{2} \psi_2' \bar{\psi}_1' \psi_1' \\
\bar{\psi}_1 &= \bar{\psi}_1' + \lambda \frac{m}{2} \bar{\psi}_2' \psi_2' \bar{\psi}_1' \\
\bar{\psi}_2 &= \bar{\psi}_2' + \lambda \frac{m}{2} \bar{\psi}_1' \psi_1' \bar{\psi}_2' .
\end{aligned} \tag{A.10}$$

The Jacobian matrix, M, can be constructed as,

$$M = \begin{pmatrix} \frac{\partial \psi_1}{\partial \psi_1'} & \frac{\partial \psi_1}{\partial \psi_2'} & \frac{\partial \psi_1}{\partial \bar{\psi}_1'} & \frac{\partial \psi_1}{\partial \bar{\psi}_2'} \\ \frac{\partial \psi_2}{\partial \psi_1'} & \frac{\partial \psi_2}{\partial \psi_2'} & \frac{\partial \psi_2}{\partial \bar{\psi}_1'} & \frac{\partial \psi_2}{\partial \bar{\psi}_2'} \\ \frac{\partial \bar{\psi}_1}{\partial \psi_1'} & \frac{\partial \bar{\psi}_1}{\partial \psi_2'} & \frac{\partial \bar{\psi}_1}{\partial \bar{\psi}_1'} & \frac{\partial \bar{\psi}_1}{\partial \bar{\psi}_2'} \\ \frac{\partial \bar{\psi}_2}{\partial \psi_1'} & \frac{\partial \bar{\psi}_2}{\partial \psi_2'} & \frac{\partial \bar{\psi}_2}{\partial \bar{\psi}_1'} & \frac{\partial \bar{\psi}_2}{\partial \bar{\psi}_2'} \end{pmatrix} \tag{A.11}$$

yielding the following Jacobian factor,

$$(\det M)^{-1} = 1 - \lambda m \bar{\psi}' \psi' = e^{-\lambda m \bar{\psi}' \psi' - \lambda^2 \frac{m^2}{2} (\bar{\psi}' \psi')^2} , \tag{A.12}$$

where the properties of Grassmann variables have been used.

Hence, the path integral measure transforms as

$$\mathcal{D}\bar{\psi}\ \mathcal{D}\psi = \mathcal{D}\bar{\psi}'\ \mathcal{D}\psi'\ e^{\int dx_1 dx_2 \left(-\lambda m \bar{\psi}' \psi' - \lambda^2 \frac{m^2}{2} (\bar{\psi}' \psi')^2\right) \Lambda^2} \tag{A.13}$$

with Λ being the cut-off. Therefore the path integral transforms as,

$$\begin{aligned}
\int \mathcal{D}\bar{\psi}\ \mathcal{D}\psi\ e^{-\int dx_1 dx_2 \mathcal{L}(\lambda)} &= \int \mathcal{D}\bar{\psi}'\ \mathcal{D}\psi'\ e^{-\int dx_1 dx_2 \left[\mathcal{L}'(\lambda) + \lambda m \bar{\psi}' \psi' \Lambda^2 + \lambda^2 \frac{m^2}{2} (\bar{\psi}' \psi')^2 \Lambda^2\right]} \\
&= \int \mathcal{D}\bar{\psi}\ \mathcal{D}\psi\ e^{i \int dt dx \left[-\mathcal{L}'(\lambda) - \lambda m \bar{\psi} \psi \Lambda^2 - \lambda^2 \frac{m^2}{2} (\bar{\psi} \psi)^2 \Lambda^2\right]} ,
\end{aligned} \tag{A.14}$$

where in the last line the variables ψ' and $\bar{\psi}'$ were replaced by the old variables ψ and $\bar{\psi}$, and a Wick-rotation was performed.

Observe that the second term in the exponential in (A.14) does not contribute to the two-to-two scattering amplitude while the third term yields a contribution proportional to Λ^2, at second order in λ. However, this polynomial divergence is of no physical consequence as the third term can be removed from the Lagrangian by adding a counter term once and for all and one can forget about it. Therefore, the Jacobian factor in the path integral measure does not yield any non-trivial contribution to the amplitude and can be ignored.

Hence, the Lorentzian action in terms of the redefined fields can be written as,

$$-I = -\int dx_1 dx_2\, \mathcal{L}'(\lambda) = i\int dt dx \quad \left[\left(-i\bar{\psi}\gamma^\mu\partial_\mu\psi + m\bar{\psi}\psi\right) + \frac{\lambda}{2}\left(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\right)\right.$$
$$\left. + \frac{\lambda^2}{2} m\bar{\psi}\psi\left(\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu} - (\tilde{X}_\mu{}^\mu)^2\right)\right]. \tag{A.15}$$

B Propagators and some useful formula

In this section the values of all the propagators that arise in the computation of the amplitude from the tadpole diagrams will be derived. In two dimensions, the propagator of the free massive Dirac fermion is given by,

$$\left\langle \psi_a(x)\bar{\psi}_b(y)\right\rangle = \int \frac{d^2q}{(2\pi)^2}\frac{i(\gamma\cdot q+m)_{ab}}{q^2-m^2+i\epsilon}e^{-iq(x-y)} = \int \frac{d^2q}{(2\pi)^2}G_{ab}(q)e^{-iq(x-y)}\,, \tag{B.1}$$

where,

$$G_{ab}(q) = \frac{i(\gamma\cdot q+m)_{ab}}{q^2-m^2+i\epsilon}. \tag{B.2}$$

The value of the propagator needed in the computation of the tadpole diagrams can be computed as,

$$\begin{aligned}\left\langle \psi_a(x)\bar{\psi}_b(x)\right\rangle &= \int \frac{d^2q}{(2\pi)^2}\frac{i(\gamma\cdot q+m)_{ab}}{q^2-m^2+i\epsilon} \\ &= i\int \frac{d^2q}{(2\pi)^2}\left[\frac{\gamma_{ab}\cdot q}{q^2-m^2+i\epsilon} + \frac{m\delta_{ab}}{q^2-m^2+i\epsilon}\right].\end{aligned} \tag{B.3}$$

By Lorentz symmetry the first term in (B.3) vanishes, thus

$$\begin{aligned}\left\langle \psi_a(x)\bar{\psi}_b(x)\right\rangle &= im\delta_{ab}\int \frac{d^2q}{(2\pi)^2}\frac{1}{q^2-m^2+i\epsilon} \\ &= im\delta_{ab}(-i)\int_0^\Lambda \frac{d^2q_E}{(2\pi)^2}\frac{1}{q_E^2-m^2+i\epsilon} \\ &= \frac{m}{4\pi}\log\left(\frac{\Lambda^2+m^2}{m^2}\right)\delta_{ab} \approx -\frac{m}{4\pi}\log\left(\frac{m^2}{\Lambda^2}\right)\delta_{ab} = N_0\ \delta_{ab}\,,\end{aligned} \tag{B.4}$$

where $N_0 = -\frac{m}{4\pi}\log\left(\frac{m^2}{\Lambda^2}\right)$, in the second equality a Wick rotation to Euclidean signature was performed[13] and the integral was regulated by a hard cutoff Λ ($\Lambda \gg m$).

From (B.1) the values of propagators with derivatives can be derived. When there is a derivative acting on ψ one finds,

$$\left\langle \partial^\mu\psi_a(x)\bar{\psi}_b(y)\right\rangle = \frac{\partial}{\partial x_\mu}\left\langle \psi_a(x)\bar{\psi}_b(y)\right\rangle = -i\int \frac{d^2q}{(2\pi)^2}\frac{i(\gamma\cdot q+m)_{ab}}{q^2-m^2+i\epsilon}\, q^\mu e^{-iq(x-y)}. \tag{B.5}$$

[13] $q^0 = iq_E^0$, $q^2 = -q_E^2$ and $d^2q \to id^2q_E$.

Hence,

$$\left\langle \partial^\mu \psi_a(x)\bar{\psi}_b(x)\right\rangle = \int \frac{d^2q}{(2\pi)^2}\left[\frac{q^\mu(\gamma_{ab}\cdot q)}{q^2-m^2+i\epsilon}+\frac{m\delta_{ab}q^\mu}{q^2-m^2+i\epsilon}\right]. \tag{B.6}$$

By Lorentz symmetry the second term vanishes yielding,

$$\left\langle \partial^\mu \psi_a(x)\bar{\psi}_b(x)\right\rangle = \int \frac{d^2q}{(2\pi)^2}\frac{q^\mu q^\nu(\gamma_{ab})_\nu}{q^2-m^2+i\epsilon} = \frac{1}{2}\eta^{\mu\nu}(\gamma_{ab})_\nu\int \frac{d^2q}{(2\pi)^2}\frac{q^2}{q^2-m^2+i\epsilon}. \tag{B.7}$$

As before, perform the integral (B.7) using a Wick rotation,

$$\left\langle \partial^\mu \psi_a(x)\bar{\psi}_b(x)\right\rangle = \frac{i\Lambda^2}{8\pi}\left[1+\frac{m^2}{\Lambda^2}\log\left(\frac{m^2}{\Lambda^2}\right)\right]\gamma^\nu_{ab} = N_1\gamma^\nu_{ab}\,, \tag{B.8}$$

where $N_1 = \frac{i\Lambda^2}{8\pi}\left[1+\frac{m^2}{\Lambda^2}\log\left(\frac{m^2}{\Lambda^2}\right)\right]$.

Similarly when a derivative acts of $\bar{\psi}$,

$$\left\langle \psi_a(x)\partial^\mu\bar{\psi}_b(x)\right\rangle = -\left\langle \partial^\mu \psi_a(x)\bar{\psi}_b(x)\right\rangle = -N_1\delta_{ab}. \tag{B.9}$$

If one derivative acts on ψ and a second on $\bar{\psi}$ one finds,

$$\left\langle \partial^\mu \psi_a(x)\partial^\nu\bar{\psi}_b(y)\right\rangle = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu}\left\langle \psi_a(x)\bar{\psi}_b(y)\right\rangle = i\int \frac{d^2q}{(2\pi)^2}\frac{(\gamma\cdot q+m)_{ab}}{q^2-m^2+i\epsilon}\;q^\mu q^\nu e^{-iq(x-y)}. \tag{B.10}$$

Implementing Lorentz symmetry and performing a Wick rotation yields,

$$\left\langle \partial^\mu \psi_a(x)\partial^\nu\bar{\psi}_b(x)\right\rangle = -\frac{m\Lambda^2}{8\pi}\left(1+\frac{m^2}{\Lambda^2}\log\left(\frac{m^2}{\Lambda^2}\right)\right)\eta^{\mu\nu}\delta_{ab} = imN_1\eta^{\mu\nu}\delta_{ab}. \tag{B.11}$$

C One-loop integrals

In this section the values of the one-loop integrals that arise in the computation of the amplitude from the bubble diagrams will be derived.

Begin with the simplest integral,

$$\int \frac{d^2q}{(2\pi)^2}\frac{1}{(q^2-\alpha^2)^2} = i\int \frac{d^2q_E}{(2\pi)^2}\frac{1}{(q_E^2+\alpha^2)^2} = \frac{i}{4\pi\alpha^2}\,, \tag{C.1}$$

where a Wick rotation was performed and the integral was converted to polar coordinates.

Next evaluate the following divergent integral which will be used in the computation of future integrals,

$$\int \frac{d^2q}{(2\pi)^2}\frac{q^2}{(q^2-\alpha^2)^2} = -\frac{i}{2\pi}\int_0^\Lambda dq_E\frac{q_E^3}{(q_E^2+\alpha^2)^2} = \frac{i}{4\pi}\left[\frac{\Lambda^2}{\Lambda^2+\alpha^2}-\log\left(\frac{\Lambda^2+\alpha^2}{\alpha^2}\right)\right] \tag{C.2}$$

where a Wick rotation was performed, the integral was converted to polar coordinates and the integral was regulated by a hard cut-off Λ ($\Lambda^2 \gg \alpha^2$).

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The following divergent integrals can be evaluated using the Lorentz symmetry and (C.2),

$$\int \frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu}{(q^2-\alpha^2)^2} = \frac{\eta_{\mu\nu}}{2}\int \frac{d^2q}{(2\pi)^2}\frac{q^2}{(q^2-\alpha^2)^2} = \frac{i}{8\pi}\eta_{\mu\nu}\left[\frac{\Lambda^2}{\Lambda^2+\alpha^2} - \log\left(\frac{\Lambda^2+\alpha^2}{\alpha^2}\right)\right] \tag{C.3}$$

$$\begin{aligned}\int \frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu q_\rho q_\lambda}{(q^2-\alpha^2)^2} &= \frac{1}{8}\left(\eta_{\mu\nu}\eta_{\rho\lambda}+\eta_{\mu\rho}\eta_{\nu\lambda}+\eta_{\mu\lambda}\eta_{\nu\rho}\right)\int \frac{d^2q}{(2\pi)^2}\frac{(q^2)^2}{(q^2-\alpha^2)^2} \\ &= \frac{i}{16\pi}C_{\mu\nu\rho\lambda}\left[\frac{\Lambda^2(\Lambda^2+2\alpha^2)}{2(\Lambda^2+\alpha^2)} - \alpha^2\log\left(\frac{\Lambda^2+\alpha^2}{\alpha^2}\right)\right],\end{aligned} \tag{C.4}$$

where,

$$C_{\mu\nu\rho\lambda} = \eta_{\mu\nu}\eta_{\rho\lambda}+\eta_{\mu\rho}\eta_{\nu\lambda}+\eta_{\mu\lambda}\eta_{\nu\rho}. \tag{C.5}$$

Lastly,

$$\begin{aligned}\int \frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu q_\rho q_\lambda q_\delta q_\omega}{(q^2-\alpha^2)^2} &= \frac{1}{48}\left[\eta_{\mu\nu}\left(\eta_{\rho\lambda}\eta_{\delta\omega}+\eta_{\rho\delta}\eta_{\lambda\omega}+\eta_{\rho\omega}\eta_{\lambda\delta}\right)+(\nu\leftrightarrow\rho)+(\nu\leftrightarrow\lambda)\right. \\ &\quad \left.+(\nu\leftrightarrow\delta)+(\nu\leftrightarrow\omega)\right]\int\frac{d^2q}{(2\pi)^2}\frac{(q^2)^3}{(q^2-\alpha^2)^2} \\ &= -\frac{i}{96\pi}F_{\mu\nu\rho\lambda\delta\omega}\left[\frac{\Lambda^4\left(\Lambda^2-3\alpha^2-\frac{6}{\Lambda^2}\alpha^4\right)}{4(\Lambda^2+\alpha^2)}+\frac{3}{2}\alpha^4\log\left(\frac{\Lambda^2+\alpha^2}{\alpha^2}\right)\right],\end{aligned} \tag{C.6}$$

where,

$$F_{\mu\nu\rho\lambda\delta\omega} = \eta_{\mu\nu}(\eta_{\rho\lambda}\eta_{\delta\omega}+\eta_{\rho\delta}\eta_{\lambda\omega}+\eta_{\rho\omega}\eta_{\lambda\delta})+(\nu\leftrightarrow\rho)+(\nu\leftrightarrow\lambda)+(\nu\leftrightarrow\delta)+(\nu\leftrightarrow\omega). \tag{C.7}$$

In the case where there is an odd number of momenta in the numerator, the integral vanishes as it is an odd function of the integration variable,

$$\int\frac{d^2q}{(2\pi)^2}\frac{\prod_{i=1}^{n+1}q_{\mu_i}}{(q^2-\alpha^2)^2} = 0, \tag{C.8}$$

where $n=\{0,2,4\}$.

The values of the above integrals, (C.1)–(C.8), will be used to derive useful results for evaluating the one-loop integrals that appear in section 4.1.3.

s-channel one-loop integrals

First, evaluate the one-loop integrals that appear while evaluating the s-channel amplitudes. Evaluate the following finite integral,

$$\textbf{(i)}\qquad L^{(s)}(\xi) = \int\frac{d^2q}{(2\pi)^2}\frac{1}{\left[(\xi+q)^2-m^2\right](q^2-m^2)} \tag{C.9}$$

where $\xi^\mu = (p_1 + p_2)^\mu$. By introducing Feynman parametrization (C.9) can be written as,

$$\begin{aligned} L^{(s)}(\xi) &= \int \frac{d^2q}{(2\pi)^2} \int_0^1 dx \, \frac{1}{\Big[(1-x)((\xi+q)^2 - m^2) + x(q^2 - m^2)\Big]^2} \\ &= \int_0^1 dx \int \frac{d^2q}{(2\pi)^2} \, \frac{1}{\Big[\big(q + \xi(1-x)\big)^2 + \xi^2 x(1-x) - m^2\Big]^2} \\ &= \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \, \frac{1}{\Big[k^2 + \xi^2 x(1-x) - m^2\Big]^2} \\ &= \frac{i}{4\pi} \int_0^1 dx \frac{1}{\Big(m^2 - \xi^2 x(1-x)\Big)} \, , \end{aligned} \tag{C.10}$$

where in the third line the momentum was shifted $q \to k = q + \xi(1-x)$ and in the fourth line (C.1) was used to evaluate the k integral. Evaluating the x-integral yields,

$$L^{(s)}(\xi) = \frac{i}{2\pi\xi^2\sqrt{1-\frac{4m^2}{\xi^2}}} \log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right). \tag{C.11}$$

It is useful to express the integral (C.11) in terms of the rapidity difference θ. Recall that the Mandelstam variables $s = (p_1+p_2)^2 = \xi^2$, $t = 0$ and $u = 4m^2 - s$ can be written in terms of θ as,

$$\begin{aligned} s &= 2m^2(1+\cosh\theta) = 4m^2\cosh^2\frac{\theta}{2} \\ u &= 4m^2 - s = -4m^2\sinh^2\frac{\theta}{2}. \end{aligned} \tag{C.12}$$

Hence,

$$\sqrt{1-\frac{4m^2}{\xi^2}} = \sqrt{1-\frac{4m^2}{s}} = \sqrt{-\frac{u}{s}} = \tanh\frac{\theta}{2} \tag{C.13}$$

$$\Rightarrow \log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right) = \log\left(\frac{\sqrt{-u/s}-1}{\sqrt{-u/s}+1}\right) = \log\left(\frac{\tanh\frac{\theta}{2}-1}{\tanh\frac{\theta}{2}+1}\right) = \log\left(-e^{-\theta}\right) = i\pi - \theta \tag{C.14}$$

where we used that s is really $s + i\epsilon$, in an attempt to pick the correct sheet.

Plugging everything into $L^{(s)}(\theta)$ gives,

$$L^{(s)}(\theta) = -\frac{\pi + i\theta}{4\pi m^2 \sinh\theta}. \tag{C.15}$$

$$\textbf{(ii)} \quad L^{(s)}_\mu(\xi) = \int \frac{d^2q}{(2\pi)^2} \frac{q_\mu}{\left[(\xi+q)^2 - m^2\right](q^2 - m^2)} \, . \tag{C.16}$$

Introduce the Feynman parameter x as before. Performing the q-integral and then the x-integral gives,

$$L_\mu^{(s)}(\xi) = -\frac{i\xi_\mu}{4\pi\xi^2\sqrt{1-\frac{4m^2}{\xi^2}}}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right). \tag{C.17}$$

Using (C.13) and (C.14) in (C.17) one obtains the final expression for $L_\mu^{(s)}(\theta)$,

$$L_\mu^{(s)}(\theta) = \xi_\mu \frac{\pi + i\theta}{8\pi m^2 \sinh\theta}. \tag{C.18}$$

In a similar manner, using the Feynman parametrization one can derive the following integrals,

$$\textbf{(iii)} \quad L_{\mu\nu}^{(s)}(\xi) = \int \frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu}{[(\xi+q)^2-m^2]\,(q^2-m^2)}$$
$$= -\frac{i}{8\pi}\left[1+\log\frac{\Lambda^2}{m^2}+\sqrt{1-\frac{4m^2}{\xi^2}}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)\right]\eta_{\mu\nu}$$
$$+\frac{i}{4\pi\xi^2}\left[1+\frac{1-2\frac{m^2}{\xi^2}}{\sqrt{1-\frac{4m^2}{\xi^2}}}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)\right]\xi_\mu\xi_\nu\,. \tag{C.19}$$

By (C.13), (C.14) and (C.19),

$$L_{\mu\nu}^{(s)}(\theta) = -\frac{i}{8\pi}\left[1+\log\frac{\Lambda^2}{m^2}+(i\pi-\theta)\tanh\frac{\theta}{2}\right]\eta_{\mu\nu}+\frac{i}{16\pi m^2\cosh^2\frac{\theta}{2}}\left[1+(i\pi-\theta)\coth\theta\right]\xi_\mu\xi_\nu \tag{C.20}$$

$$\textbf{(iv)} \quad L_{\mu\nu\rho}^{(s)}(\xi) = \int \frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu q_\rho}{[(\xi+q)^2-m^2]\,(q^2-m^2)}$$
$$= \frac{i}{16\pi}\left[1+\log\frac{\Lambda^2}{m^2}+\sqrt{1-\frac{4m^2}{\xi^2}}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)\right](\eta_{\mu\nu}\xi_\rho+\eta_{\nu\rho}\xi_\mu+\eta_{\rho\mu}\xi_\nu)$$
$$-\frac{i}{4\pi\xi^2}\left[\frac{3}{2}+\frac{1-3\frac{m^2}{\xi^2}}{\sqrt{1-\frac{4m^2}{\xi^2}}}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)\right]\xi_\mu\xi_\nu\xi_\rho\,. \tag{C.21}$$

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By (C.13), (C.14) and (C.21),

$$\begin{aligned} L^{(s)}_{\mu\nu\rho}(\theta) &= \frac{i}{16\pi}\left[1+\log\frac{\Lambda^2}{m^2}+(i\pi-\theta)\tanh\frac{\theta}{2}\right](\eta_{\mu\nu}\xi_\rho+\eta_{\nu\rho}\xi_\mu+\eta_{\rho\mu}\xi_\nu) \\ &\quad -\frac{i}{16\pi m^2\cosh^2\frac{\theta}{2}}\left[\frac{3}{2}+(i\pi-\theta)\left(\coth\theta-\frac{1}{2\sinh\theta}\right)\right]\xi_\mu\xi_\nu\xi_\rho \end{aligned} \tag{C.22}$$

(v)
$$\begin{aligned} L^{(s)}_{\mu\nu\rho\lambda}(\xi) &= \int\frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu q_\rho q_\lambda}{[(\xi+q)^2-m^2](q^2-m^2)} \\ &= \frac{i}{16\pi}\left[\frac{\Lambda^2}{2}+\frac{\xi^2}{6}\left(\frac{5}{3}-8\frac{m^2}{\xi^2}+\left(1-6\frac{m^2}{\xi^2}\right)\log\frac{\Lambda^2}{m^2}+\left(1-4\frac{m^2}{\xi^2}\right)^{3/2}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)\right)\right] \\ &\quad \times C_{\mu\nu\rho\lambda}+\frac{i}{8\pi}\left[-\frac{7}{18}+\frac{2}{3}\frac{m^2}{\xi^2}-\frac{1}{3}\left(1-\frac{m^2}{\xi^2}\right)\sqrt{1-\frac{4m^2}{\xi^2}}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)\right][\eta_{\mu\nu}\xi_\rho\xi_\lambda \\ &\quad +\eta_{\mu\rho}\xi_\nu\xi_\lambda+\eta_{\mu\lambda}\xi_\nu\xi_\rho+\eta_{\nu\rho}\xi_\mu\xi_\lambda+\eta_{\nu\lambda}\xi_\rho\xi_\mu+\eta_{\rho\lambda}\xi_\mu\xi_\nu]+\frac{i}{4\pi\xi^2}\left[\frac{11}{6}-\frac{m^2}{\xi^2}\right. \\ &\quad \left.+\frac{1-4\frac{m^2}{\xi^2}+2\frac{m^4}{\xi^4}}{\sqrt{1-\frac{4m^2}{\xi^2}}}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)\right]\xi_\mu\xi_\nu\xi_\rho\xi_\lambda, \end{aligned} \tag{C.23}$$

where $C_{\mu\nu\rho\lambda}$ is given in (C.5). By (C.13), (C.14) and (C.23),

$$\begin{aligned} L^{(s)}_{\mu\nu\rho\lambda}(\theta) &= \frac{i}{16\pi}\left[\frac{\Lambda^2}{2}+\frac{2}{3}m^2\cosh^2\frac{\theta}{2}\left(\frac{5}{3}-\frac{2}{\cosh^2\frac{\theta}{2}}+\left(1-\frac{3}{2\cosh^2\frac{\theta}{2}}\right)\log\frac{\Lambda^2}{m^2}+(i\pi-\theta)\tanh^3\frac{\theta}{2}\right)\right] \\ &\quad \times C_{\mu\nu\rho\lambda}+\frac{i}{8\pi}\left[-\frac{7}{18}+\frac{1}{6\cosh^2\frac{\theta}{2}}-\frac{1}{3}(i\pi-\theta)\tanh\frac{\theta}{2}\left(1-\frac{1}{4\cosh^2\frac{\theta}{2}}\right)-\frac{1}{3}\log\frac{\Lambda^2}{m^2}\right][\eta_{\mu\nu}\xi_\rho\xi_\lambda \\ &\quad +\eta_{\mu\rho}\xi_\nu\xi_\lambda+\eta_{\mu\lambda}\xi_\nu\xi_\rho+\eta_{\nu\rho}\xi_\mu\xi_\lambda+\eta_{\nu\lambda}\xi_\rho\xi_\mu+\eta_{\rho\lambda}\xi_\mu\xi_\nu]+\frac{i}{16\pi m^2\cosh^2\frac{\theta}{2}}\left[\frac{11}{6}-\frac{1}{4\cosh^2\frac{\theta}{2}}\right. \\ &\quad \left.+(i\pi-\theta)\left(\coth\frac{\theta}{2}-\frac{2}{\sinh\theta}+\frac{1}{4\sinh\theta\cosh^2\frac{\theta}{2}}\right)\right]\xi_\mu\xi_\nu\xi_\rho\xi_\lambda. \end{aligned} \tag{C.24}$$

(vi)
$$\begin{aligned} L^{(s)}_{\mu\nu\rho\lambda\delta}(\xi) &= \int\frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu q_\rho q_\lambda q_\delta}{[(\xi+q)^2-m^2](q^2-m^2)} \\ &= -\frac{i}{16\pi}\left[\frac{\Lambda^2}{4}+\xi^2\left(\frac{5}{36}-\frac{2}{3}\frac{m^2}{\xi^2}+\frac{1}{12}\left(1-6\frac{m^2}{\xi^2}\right)\log\frac{\Lambda^2}{m^2}\right.\right. \\ &\quad \left.\left.+\frac{1}{12}\left(1-4\frac{m^2}{\xi^2}\right)^{3/2}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)\right)\right]D_{\mu\nu\rho\lambda\delta}-\frac{i}{8\pi}\left[-\frac{1}{3}+\frac{m^2}{\xi^2}-\frac{1}{4}\log\frac{\Lambda^2}{m^2}\right. \\ &\quad \left.-\frac{1}{4}\left(1-2\frac{m^2}{\xi^2}\right)\sqrt{1-\frac{4m^2}{\xi^2}}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)\right]B_{\mu\nu\rho\lambda\delta} \\ &\quad -\frac{i}{4\pi\xi^2}\left[\frac{25}{12}-\frac{5}{2}\frac{m^2}{\xi^2}+\frac{1-5\frac{m^2}{\xi^2}+5\frac{m^4}{\xi^4}}{\sqrt{1-\frac{4m^2}{\xi^2}}}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)\right]\xi_\mu\xi_\nu\xi_\rho\xi_\lambda\xi_\delta, \end{aligned} \tag{C.25}$$

where,

$$
\begin{aligned}
D_{\mu\nu\rho\lambda\delta} &= C_{\mu\nu\rho\lambda}\xi_\delta + C_{\nu\rho\lambda\delta}\xi_\mu + C_{\rho\lambda\delta\mu}\xi_\nu + C_{\lambda\delta\mu\nu}\xi_\rho + C_{\delta\mu\nu\rho}\xi_\lambda\,, \qquad \text{(C.26)}\\
B_{\mu\nu\rho\lambda\delta} &= \eta_{\mu\nu}\xi_\rho\xi_\lambda\xi_\delta + \eta_{\mu\rho}\xi_\nu\xi_\lambda\xi_\delta + \eta_{\mu\lambda}\xi_\nu\xi_\rho\xi_\delta + \eta_{\mu\delta}\xi_\nu\xi_\rho\xi_\lambda + \eta_{\nu\rho}\xi_\mu\xi_\lambda\xi_\delta\\
&\quad +\eta_{\nu\lambda}\xi_\mu\xi_\rho\xi_\delta + \eta_{\nu\delta}\xi_\mu\xi_\rho\xi_\lambda + \eta_{\rho\lambda}\xi_\mu\xi_\nu\xi_\delta + \eta_{\rho\delta}\xi_\mu\xi_\nu\xi_\lambda + \eta_{\lambda\delta}\xi_\mu\xi_\nu\xi_\rho\,. \qquad \text{(C.27)}
\end{aligned}
$$

By (C.13), (C.14) and (C.25),

$$
\begin{aligned}
L^{(s)}_{\mu\nu\rho\lambda\delta}(\theta) &= -\frac{i}{16\pi}\left[\frac{\Lambda^2}{4}+4m^2\cosh^2\frac{\theta}{2}\left(\frac{5}{36}-\frac{1}{6\cosh^2\frac{\theta}{2}}+\frac{1}{12}\left(1-\frac{3}{2\cosh^2\frac{\theta}{2}}\right)\log\frac{\Lambda^2}{m^2}\right.\right.\\
&\quad\left.\left.+\frac{1}{12}(i\pi-\theta)\tanh^3\frac{\theta}{2}\right)\right]D_{\mu\nu\rho\lambda\delta}-\frac{i}{8\pi}\left[-\frac{1}{3}+\frac{1}{4\cosh^2\frac{\theta}{2}}-\frac{\cosh\theta}{8\cosh^2\frac{\theta}{2}}(i\pi-\theta)\tanh\frac{\theta}{2}\right.\\
&\quad\left.-\frac{1}{4}\log\frac{\Lambda^2}{m^2}\right]B_{\mu\nu\rho\lambda\delta}-\frac{i}{16\pi m^2\cosh^2\frac{\theta}{2}}\left[\frac{25}{12}-\frac{5}{8\cosh^2\frac{\theta}{2}}+(i\pi-\theta)\left(\coth\frac{\theta}{2}-\frac{5}{2\sinh\theta}\right.\right.\\
&\quad\left.\left.+\frac{5}{8\sinh\theta\cosh^2\frac{\theta}{2}}\right)\right]\xi_\mu\xi_\nu\xi_\rho\xi_\lambda\xi_\delta. \qquad \text{(C.28)}
\end{aligned}
$$

(vii) $L^{(s)}_{\mu\nu\rho\lambda\delta\omega}(\xi) = \int\frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu q_\rho q_\lambda q_\delta q_\omega}{[(\xi+q)^2-m^2](q^2-m^2)}$

$$
\begin{aligned}
&=-\frac{i}{96\pi}\left[\frac{\Lambda^4}{4}+\frac{1}{20}\xi^2\Lambda^2+\frac{1}{10}\xi^4\left(1-\frac{4m^2}{\xi^2}\right)^2\left(1+\frac{1}{2}\sqrt{1-\frac{4m^2}{\xi^2}}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)\right)\right.\\
&\quad\left.+\xi^4\left(\frac{1}{20}-\frac{1}{2}\frac{m^2}{\xi^2}+\frac{3}{2}\frac{m^4}{\xi^4}\right)\log\frac{\Lambda^2}{m^2}\right]F_{\mu\nu\rho\lambda\delta\omega}+\frac{i\xi^2}{96\pi}\left[\frac{\Lambda^2}{\xi^2}+\frac{13}{25}-\frac{44}{15}\frac{m^2}{\xi^2}+\frac{8}{5}\frac{m^4}{\xi^4}\right.\\
&\quad\left.+\frac{1}{10}\left(3-14\frac{m^2}{\xi^2}+8\frac{m^4}{\xi^4}\right)\sqrt{1-\frac{4m^2}{\xi^2}}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)+\frac{1}{10}\left(3-20\frac{m^2}{\xi^2}\right)\log\frac{\Lambda^2}{m^2}\right]H_{\mu\nu\rho\lambda\delta\omega}\\
&\quad+\frac{i}{8\pi}\left[-\frac{89}{300}+\frac{37}{30}\frac{m^2}{\xi^2}-\frac{2}{5}\frac{m^4}{\xi^4}-\frac{1}{5}\left(1-3\frac{m^2}{\xi^2}+\frac{m^4}{\xi^4}\right)\sqrt{1-\frac{4m^2}{\xi^2}}\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)\right.\\
&\quad\left.-\frac{1}{5}\log\frac{\Lambda^2}{m^2}\right]M_{\mu\nu\rho\lambda\delta\omega}+\frac{i}{4\pi\xi^2}\left[\frac{137}{60}-\frac{13}{3}\frac{m^2}{\xi^2}+\frac{m^4}{\xi^4}+\frac{1}{\sqrt{1-\frac{4m^2}{\xi^2}}}\left(1-6\frac{m^2}{\xi^2}+9\frac{m^4}{\xi^4}\right.\right.\\
&\quad\left.\left.-2\frac{m^6}{\xi^6}\right)\log\left(\frac{\sqrt{1-\frac{4m^2}{\xi^2}}-1}{\sqrt{1-\frac{4m^2}{\xi^2}}+1}\right)\right]\xi_\mu\xi_\nu\xi_\rho\xi_\lambda\xi_\delta\xi_\omega \qquad \text{(C.29)}
\end{aligned}
$$

where,

$$
\begin{aligned}
H_{\mu\nu\rho\lambda\delta\omega} &= C_{\mu\nu\rho\lambda}\xi_\delta\xi_\omega + C_{\mu\nu\rho\delta}\xi_\lambda\xi_\omega + C_{\mu\nu\rho\omega}\xi_\lambda\xi_\delta + C_{\mu\nu\lambda\delta}\xi_\rho\xi_\omega + C_{\mu\nu\lambda\omega}\xi_\rho\xi_\delta + C_{\mu\rho\lambda\delta}\xi_\nu\xi_\omega\\
&\quad + C_{\mu\rho\lambda\omega}\xi_\nu\xi_\delta + C_{\nu\rho\lambda\delta}\xi_\mu\xi_\omega + C_{\nu\rho\lambda\omega}\xi_\mu\xi_\delta + C_{\mu\rho\delta\omega}\xi_\nu\xi_\lambda + C_{\mu\lambda\delta\omega}\xi_\nu\xi_\rho + C_{\lambda\rho\delta\omega}\xi_\nu\xi_\mu\\
&\quad + C_{\nu\rho\delta\omega}\xi_\mu\xi_\lambda + C_{\mu\nu\delta\omega}\xi_\rho\xi_\lambda + C_{\nu\lambda\delta\omega}\xi_\mu\xi_\rho\,, \qquad \text{(C.30)}
\end{aligned}
$$

$$
\begin{aligned}
M_{\mu\nu\rho\lambda\delta\omega} = {} & \xi_\mu\xi_\nu\xi_\rho\xi_\lambda\eta_{\delta\omega} + \xi_\mu\xi_\nu\xi_\rho\xi_\delta\eta_{\lambda\omega} + \xi_\mu\xi_\nu\xi_\rho\xi_\omega\eta_{\lambda\delta} + \xi_\mu\xi_\nu\xi_\lambda\xi_\delta\eta_{\rho\omega} + \xi_\mu\xi_\nu\xi_\lambda\xi_\omega\eta_{\rho\delta} \\
& + \xi_\mu\xi_\rho\xi_\lambda\xi_\delta\eta_{\nu\omega} + \xi_\mu\xi_\rho\xi_\lambda\xi_\omega\eta_{\nu\delta} + \xi_\nu\xi_\rho\xi_\lambda\xi_\delta\eta_{\mu\omega} + \xi_\nu\xi_\rho\xi_\lambda\xi_\omega\eta_{\mu\delta} + \xi_\mu\xi_\rho\xi_\delta\xi_\omega\eta_{\nu\lambda} \\
& + \xi_\mu\xi_\lambda\xi_\delta\xi_\omega\eta_{\nu\rho} + \xi_\lambda\xi_\rho\xi_\delta\xi_\omega\eta_{\nu\mu} + \xi_\nu\xi_\rho\xi_\delta\xi_\omega\eta_{\mu\lambda} + \xi_\mu\xi_\nu\xi_\delta\xi_\omega\eta_{\rho\lambda} + \xi_\nu\xi_\lambda\xi_\delta\xi_\omega\eta_{\mu\rho}
\end{aligned} \tag{C.31}
$$

and $F_{\mu\nu\rho\lambda\delta\omega}$ is given in (C.7). By (C.13), (C.14) and (C.29),

$$
\begin{aligned}
L^{(s)}_{\mu\nu\rho\lambda\delta\omega}(\theta) = {} & -\frac{i}{96\pi}\left[\frac{\Lambda^4}{4} + \frac{\Lambda^2 m^2}{5}\cosh^2\frac{\theta}{2} + 4m^4\cosh^4\frac{\theta}{2}\left(\frac{1}{5} - \frac{1}{2\cosh^2\frac{\theta}{2}} + \frac{3}{8\cosh^4\frac{\theta}{2}}\right)\log\frac{\Lambda^2}{m^2}\right. \\
& \left. + \frac{8}{5}m^4\sinh^4\frac{\theta}{2}\left(1 + \frac{1}{2}(i\pi-\theta)\tanh\frac{\theta}{2}\right)\right] F_{\mu\nu\rho\lambda\delta\omega} + \frac{im^2\cosh^2\frac{\theta}{2}}{24\pi}\left[\frac{13}{25} - \frac{11}{15\cosh^2\frac{\theta}{2}}\right. \\
& + \frac{1}{10\cosh^4\frac{\theta}{2}} + \frac{\Lambda^2}{4m^2\cosh^2\frac{\theta}{2}} + \frac{1}{10}(i\pi-\theta)\left(3 - \frac{7}{2\cosh^2\frac{\theta}{2}} + \frac{1}{2\cosh^4\frac{\theta}{2}}\right)\tanh\frac{\theta}{2} \\
& \left. + \frac{1}{10}\left(3 - \frac{5}{\cosh^2\frac{\theta}{2}}\right)\log\frac{\Lambda^2}{m^2}\right] H_{\mu\nu\rho\lambda\delta\omega} + \frac{i}{8\pi}\left[-\frac{89}{300} + \frac{37}{120\cosh^2\frac{\theta}{2}} - \frac{1}{40\cosh^4\frac{\theta}{2}}\right. \\
& \left. - \frac{1}{5}\log\frac{\Lambda^2}{m^2} - \frac{1}{5}(i\pi-\theta)\tanh\frac{\theta}{2}\left(1 - \frac{3}{4\cosh^2\frac{\theta}{2}} + \frac{1}{16\cosh^4\frac{\theta}{2}}\right)\right] M_{\mu\nu\rho\lambda\delta\omega} \\
& + \frac{i}{16\pi m^2\cosh^2\frac{\theta}{2}}\left[\frac{137}{60} - \frac{13}{12\cosh^2\frac{\theta}{2}} + \frac{1}{16\cosh^4\frac{\theta}{2}} + (i\pi-\theta)\coth\frac{\theta}{2}\left(1 - \frac{3}{2\cosh^2\frac{\theta}{2}}\right.\right. \\
& \left.\left. + \frac{9}{16\cosh^4\frac{\theta}{2}} - \frac{1}{32\cosh^6\frac{\theta}{2}}\right)\right]\xi_\mu\xi_\nu\xi_\rho\xi_\lambda\xi_\delta\xi_\omega .
\end{aligned} \tag{C.32}
$$

The above integrals (C.9)–(C.29) can be used to evaluate the one-loop integrals needed in the computation of the s-channel amplitudes in section 4.1.3. When no derivatives are present the loop integral looks like,

$$
\begin{aligned}
I^{(s)}_{abcd}(\xi) &= \int \frac{d^2q}{(2\pi)^2} G_{ab}(\xi+q)G_{cd}(q) \\
&= \int \frac{d^2q}{(2\pi)^2} \frac{i(\gamma\cdot(\xi+q)+m)_{ab}}{(\xi+q)^2-m^2}\frac{i(\gamma\cdot q+m)_{cd}}{q^2-m^2} \\
&= -\int \frac{d^2q}{(2\pi)^2} \frac{\gamma^\mu_{ab}\gamma^\nu_{cd}(\xi+q)_\mu q_\nu + m\delta_{ab}\gamma^\mu_{cd}q_\mu + m\gamma^\mu_{ab}(\xi+q)_\mu\delta_{cd} + m^2\delta_{ab}\delta_{cd}}{[(\xi+q)^2-m^2](q^2-m^2)} \\
&= -\Big(\gamma^\mu_{ab}\gamma^\nu_{cd}\xi_\mu L^{(s)}_\nu(\xi) + \gamma^\mu_{ab}\gamma^\nu_{cd}L^{(s)}_{\mu\nu}(\xi) + m\delta_{ab}\gamma^\mu_{cd}L^{(s)}_\mu(\xi) + m\gamma^\mu_{ab}\xi_\mu\delta_{cd}L^{(s)}(\xi) \\
&\quad + m\gamma^\mu_{ab}L^{(s)}_\mu(\xi)\delta_{cd} + m^2\delta_{ab}\delta_{cd}L^{(s)}(\xi)\Big) \\
&= -L^{(s)}(\xi)\Big(m\not{\xi}_{ab}\delta_{cd} + m^2\delta_{ab}\delta_{cd}\Big) - \Big(\not{\xi}_{ab}\not{L}^{(s)}_{cd}(\xi) + m\delta_{ab}\not{L}^{(s)}_{cd}(\xi) + m\not{L}^{(s)}_{ab}(\xi)\delta_{cd}\Big) \\
&\quad - \gamma^\mu_{ab}\gamma^\nu_{cd}L^{(s)}_{\mu\nu}(\xi)\,,
\end{aligned} \tag{C.33}
$$

where $\not{\xi}_{ab} = \gamma^\mu_{ab}\xi_\mu$ and $\not{L}^{(s)}_{ab} = \gamma^\mu_{ab}L^{(s)}_\mu$ with $L^{(s)}$, $L^{(s)}_\mu$ and $L^{(s)}_{\mu\nu}$ are given by (C.15), (C.18) and (C.20), respectively.

When it involves derivatives, a similar procedure can be performed to evaluate the one-loop integrals in the s-channel case,

$$\begin{aligned}(I_\mu)^{(s)}_{abcd}(\xi) &= \int \frac{d^2q}{(2\pi)^2} q_\mu G_{ab}(\xi+q)G_{cd}(q)\\ &= \int \frac{d^2q}{(2\pi)^2} q_\mu \frac{i(\gamma\cdot(\xi+q)+m)_{ab}}{(\xi+q)^2-m^2}\frac{i(\gamma\cdot q+m)_{cd}}{q^2-m^2}\\ &= -\left(m\not{\xi}_{ab}\delta_{cd}+m^2\delta_{ab}\delta_{cd}\right)L^{(s)}_\mu(\xi)-\left(\not{\xi}_{ab}\gamma^\nu_{cd}+m\delta_{ab}\gamma^\nu_{cd}+m\gamma^\nu_{ab}\delta_{cd}\right)L^{(s)}_{\mu\nu}(\xi)\\ &\quad -\gamma^\nu_{ab}\gamma^\rho_{cd}L^{(s)}_{\mu\nu\rho}(\xi) \end{aligned} \tag{C.34}$$

$$\begin{aligned}(I_{\mu\nu})^{(s)}_{abcd}(\xi) &= \int \frac{d^2q}{(2\pi)^2} q_\mu q_\nu G_{ab}(\xi+q)G_{cd}(q)\\ &= \int \frac{d^2q}{(2\pi)^2} q_\mu q_\nu \frac{i(\gamma\cdot(\xi+q)+m)_{ab}}{(\xi+q)^2-m^2}\frac{i(\gamma\cdot q+m)_{cd}}{q^2-m^2}\\ &= -\left(m\not{\xi}_{ab}\delta_{cd}+m^2\delta_{ab}\delta_{cd}\right)L^{(s)}_{\mu\nu}(\xi)-\left(\not{\xi}_{ab}\gamma^\rho_{cd}+m\delta_{ab}\gamma^\rho_{cd}+m\gamma^\rho_{ab}\delta_{cd}\right)L^{(s)}_{\mu\nu\rho}(\xi)\\ &\quad -\gamma^\rho_{ab}\gamma^\lambda_{cd}L^{(s)}_{\mu\nu\rho\lambda}(\xi) \end{aligned} \tag{C.35}$$

$$\begin{aligned}(I_{\mu\nu\rho})^{(s)}_{abcd}(\xi) &= \int \frac{d^2q}{(2\pi)^2} q_\mu q_\nu q_\rho G_{ab}(\xi+q)G_{cd}(q)\\ &= \int \frac{d^2q}{(2\pi)^2} q_\mu q_\nu q_\rho \frac{i(\gamma\cdot(\xi+q)+m)_{ab}}{(\xi+q)^2-m^2}\frac{i(\gamma\cdot q+m)_{cd}}{q^2-m^2}\\ &= -\left(m\not{\xi}_{ab}\delta_{cd}+m^2\delta_{ab}\delta_{cd}\right)L^{(s)}_{\mu\nu\rho}(\xi)-\left(\not{\xi}_{ab}\gamma^\lambda_{cd}+m\delta_{ab}\gamma^\lambda_{cd}+m\gamma^\lambda_{ab}\delta_{cd}\right)L^{(s)}_{\mu\nu\rho\lambda}(\xi)\\ &\quad -\gamma^\lambda_{ab}\gamma^\delta_{cd}L^{(s)}_{\mu\nu\rho\lambda\delta}(\xi) \end{aligned} \tag{C.36}$$

$$\begin{aligned}(I_{\mu\nu\rho\lambda})^{(s)}_{abcd}(\xi) &= \int \frac{d^2q}{(2\pi)^2} q_\mu q_\nu q_\rho q_\lambda G_{ab}(\xi+q)G_{cd}(q)\\ &= \int \frac{d^2q}{(2\pi)^2} q_\mu q_\nu q_\rho q_\lambda \frac{i(\gamma\cdot(\xi+q)+m)_{ab}}{(\xi+q)^2-m^2}\frac{i(\gamma\cdot q+m)_{cd}}{q^2-m^2}\\ &= -\left(m\not{\xi}_{ab}\delta_{cd}+m^2\delta_{ab}\delta_{cd}\right)L^{(s)}_{\mu\nu\rho\lambda}(\xi)-\left(\not{\xi}_{ab}\gamma^\delta_{cd}+m\delta_{ab}\gamma^\delta_{cd}+m\gamma^\delta_{ab}\delta_{cd}\right)L^{(s)}_{\mu\nu\rho\lambda\delta}(\xi)\\ &\quad -\gamma^\delta_{ab}\gamma^\omega_{cd}L^{(s)}_{\mu\nu\rho\lambda\delta\omega}(\xi) \end{aligned} \tag{C.37}$$

where $L^{(s)}_{\mu\nu}(\xi)$, $L^{(s)}_{\mu\nu\rho}(\xi)$, $L^{(s)}_{\mu\nu\rho\lambda}(\xi)$, $L^{(s)}_{\mu\nu\rho\lambda\delta}(\xi)$ and $L^{(s)}_{\mu\nu\rho\lambda\delta\omega}(\xi)$ are respectively given by (C.20), (C.22), (C.24), (C.28) and (C.32).

t-channel one-loop integrals

Next, evaluate the one-loop integrals that appear while evaluating the t-channel amplitudes. These integrals are much simpler as $t=(p_1-p_3)^2=0$,

$$I^{(t)}_{abcd}=\int \frac{d^2q}{(2\pi)^2} G_{ab}(q)G_{cd}(q)=-m^2\delta_{ab}\delta_{cd}L^{(t)}-m\left(\delta_{ab}\not{L}^{(t)}_{cd}+\not{L}^{(t)}_{ab}\delta_{cd}\right)-\gamma^\mu_{ab}\gamma^\nu_{cd}L^{(t)}_{\mu\nu}$$

$$(I_\mu)^{(t)}_{abcd}=\int \frac{d^2q}{(2\pi)^2}\; q_\mu\; G_{ab}(q)G_{cd}(q)=-m^2\delta_{ab}\delta_{cd}L^{(t)}_\mu-m\left(\delta_{ab}\gamma^\nu_{cd}+\gamma^\nu_{ab}\delta_{cd}\right)L^{(t)}_{\mu\nu}-\gamma^\nu_{ab}\gamma^\rho_{cd}L^{(t)}_{\mu\nu\rho}$$

$$(I_{\mu\nu})^{(t)}_{abcd} = \int \frac{d^2q}{(2\pi)^2}\; q_\mu q_\nu\; G_{ab}(q)G_{cd}(q) = -m^2\delta_{ab}\delta_{cd}L^{(t)}_{\mu\nu} - m\big(\delta_{ab}\gamma^\rho_{cd} + \gamma^\rho_{ab}\delta_{cd}\big)L^{(t)}_{\mu\nu\rho} - \gamma^\rho_{ab}\gamma^\lambda_{cd}L^{(t)}_{\mu\nu\rho\lambda}$$

$$(I_{\mu\nu\rho})^{(t)}_{abcd} = \int \frac{d^2q}{(2\pi)^2}\; q_\mu q_\nu q_\rho\; G_{ab}(q)G_{cd}(q) = -m^2\delta_{ab}\delta_{cd}L^{(t)}_{\mu\nu\rho} - m\big(\delta_{ab}\gamma^\lambda_{cd} + \gamma^\lambda_{ab}\delta_{cd}\big)L^{(t)}_{\mu\nu\rho\lambda} - \gamma^\lambda_{ab}\gamma^\delta_{cd}L^{(t)}_{\mu\nu\rho\lambda\delta}$$

$$(I_{\mu\nu\rho\lambda})^{(t)}_{abcd} = \int \frac{d^2q}{(2\pi)^2} q_\mu q_\nu q_\rho q_\lambda G_{ab}(q)G_{cd}(q) = -m^2\delta_{ab}\delta_{cd}L^{(t)}_{\mu\nu\rho\lambda} - m\big(\delta_{ab}\gamma^\delta_{cd} + \gamma^\delta_{ab}\delta_{cd}\big)L^{(t)}_{\mu\nu\rho\lambda\delta} - \gamma^\delta_{ab}\gamma^\omega_{cd}L^{(t)}_{\mu\nu\rho\lambda\delta\omega}\,, \tag{C.38}$$

where

$$L^{(t)} = \int \frac{d^2q}{(2\pi)^2}\frac{1}{(q^2-m^2)^2} = \frac{i}{4\pi m^2}$$

$$L^{(t)}_\mu = \int \frac{d^2q}{(2\pi)^2}\frac{q_\mu}{(q^2-m^2)^2} = 0$$

$$L^{(t)}_{\mu\nu} = \int \frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu}{(q^2-m^2)^2} = \frac{i}{8\pi}\left(1 - \log\frac{\Lambda^2}{m^2}\right)\eta_{\mu\nu}$$

$$L^{(t)}_{\mu\nu\rho} = \int \frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu q_\rho}{(q^2-m^2)^2} = 0$$

$$L^{(t)}_{\mu\nu\rho\lambda} = \int \frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu q_\rho q_\lambda}{(q^2-m^2)^2} = \frac{im^2}{32\pi}\left(\frac{\Lambda^2}{m^2} - 2\log\frac{\Lambda^2}{m^2}\right)(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\rho}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho})$$

$$L^{(t)}_{\mu\nu\rho\lambda\delta} = \int \frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu q_\rho q_\lambda q_\delta}{(q^2-m^2)^2} = 0$$

$$L^{(t)}_{\mu\nu\rho\lambda\delta\omega} = \int \frac{d^2q}{(2\pi)^2}\frac{q_\mu q_\nu q_\rho q_\lambda q_\delta q_\omega}{(q^2-m^2)^2} = -\frac{im^4}{384\pi}\left(\frac{\Lambda^4}{m^4} + 6\log\frac{\Lambda^2}{m^2}\right)F_{\mu\nu\rho\lambda\delta\omega}, \tag{C.39}$$

and $F_{\mu\nu\rho\lambda\delta\omega}$ is given by (C.7). The above integrals have been derived from the corresponding s-channel integrals (C.15), (C.18), (C.20), (C.22), (C.24), (C.28 and (C.32)) by taking the limit $\theta \to i\pi$ and setting $\xi \to 0$.

u-channel one-loop integrals

Finally, evaluate the one-loop integrals that appear while evaluating the u-channel amplitudes. When there are no derivatives the one-loop integral can be written as,

$$\begin{aligned} I^{(u)}_{abcd}(\zeta) &= \int \frac{d^2q}{(2\pi)^2}G_{ab}(\zeta-q)G_{cd}(q) = \int \frac{d^2q}{(2\pi)^2}\frac{i(\gamma\cdot(\zeta-q)+m)_{ab}}{(\zeta-q)^2-m^2}\frac{i(\gamma\cdot q+m)_{cd}}{q^2-m^2} \\ &= -\int \frac{d^2q}{(2\pi)^2}\frac{\gamma^\mu_{ab}\gamma^\nu_{cd}(\zeta-q)_\mu q_\nu + m\delta_{ab}\gamma^\mu_{cd}q_\mu + m\gamma^\mu_{ab}(\zeta-q)_\mu\delta_{cd} + m^2\delta_{ab}\delta_{cd}}{[(\zeta-q)^2-m^2](q^2-m^2)} \\ &= -L^{(u)}(\zeta)\Big(m\not{\zeta}_{ab}\delta_{cd} + m^2\delta_{ab}\delta_{cd}\Big) - \Big(\not{\zeta}_{ab}\not{L}^{(u)}_{cd}(\zeta) + m\delta_{ab}\not{L}^{(u)}_{cd}(\zeta) - m\not{L}^{(u)}_{ab}(\zeta)\delta_{cd}\Big) \\ &\quad + \gamma^\mu_{ab}\gamma^\nu_{cd}L^{(u)}_{\mu\nu}(\zeta)\,, \end{aligned} \tag{C.40}$$

where $\zeta_\mu = (p_1 - p_4)_\mu = (p_1 - p_2)_\mu$, $\not\zeta_{ab} = \gamma^\mu_{ab}\zeta_\mu$ and

$$\begin{aligned}
L^{(u)}(\zeta) &= \int \frac{d^2q}{(2\pi)^2} \frac{1}{\left[(\zeta-q)^2 - m^2\right](q^2 - m^2)}\,, \\
L^{(u)}_\mu(\zeta) &= \int \frac{d^2q}{(2\pi)^2} \frac{q_\mu}{\left[(\zeta-q)^2 - m^2\right](q^2 - m^2)}\,, \\
L^{(u)}_{\mu\nu}(\zeta) &= \int \frac{d^2q}{(2\pi)^2} \frac{q_\mu q_\nu}{\left[(\zeta-q)^2 - m^2\right](q^2 - m^2)}\,.
\end{aligned} \tag{C.41}$$

Comparing the above integrals with their s-channel counterparts (C.9), (C.16 and (C.19)) one sees that the structure of the integrals are identical apart from the fact that the u-channel integrands depends on $\zeta - q$ while the s-channel integrands depend on $\xi + q$.

Since the u-channel corresponds to $\theta \to i\pi - \theta$, the above integrals can be derived directly from the results of their s-channel counterparts (C.15), (C.18 and (C.20)) by replacing $\theta \to i\pi - \theta$. Notice that replacing θ by $i\pi - \theta$ is equivalent to replacing ξ^2 by ζ^2. However, since the functional dependence of the integrands on ζ is $\zeta - q$, an extra minus sign must be included whenever the integrand is odd in q. Thus,

$$\begin{aligned}
L^{(u)}(\theta) &= i\frac{\theta}{4\pi m^2 \sinh\theta} \\
L^{(u)}_\mu(\theta) &= i\frac{\theta}{8\pi m^2 \sinh\theta}\zeta_\mu \\
L^{(u)}_{\mu\nu}(\theta) &= -\frac{i}{8\pi}\left[1 + \log\frac{\Lambda^2}{m^2} - \theta\coth\frac{\theta}{2}\right]\eta_{\mu\nu} - \frac{i}{16\pi m^2 \sinh^2\frac{\theta}{2}}\left(1 - \theta\coth\theta\right)\zeta_\mu\zeta_\nu\,.
\end{aligned} \tag{C.42}$$

In a similar manner, one can derive the one-loop integrals in the u-channel case when there are derivatives,

$$\begin{aligned}
(I_\mu)^{(u)}_{abcd}(\zeta) &= \int \frac{d^2q}{(2\pi)^2}\, q_\mu G_{ab}(\zeta-q)G_{cd}(q) \\
&= -\Big(m\not\zeta_{ab}\delta_{cd} + m^2\delta_{ab}\delta_{cd}\Big)L^{(u)}_\mu(\zeta) - \Big(\not\zeta_{ab}\gamma^\nu_{cd} + m\delta_{ab}\gamma^\nu_{cd} + m\gamma^\nu_{ab}\delta_{cd}\Big)L^{(u)}_{\mu\nu}(\zeta) \\
&\quad -\gamma^\nu_{ab}\gamma^\rho_{cd}L^{(u)}_{\mu\nu\rho}(\zeta)
\end{aligned} \tag{C.43}$$

$$\begin{aligned}
(I_{\mu\nu})^{(u)}_{abcd}(\zeta) &= \int \frac{d^2q}{(2\pi)^2} q_\mu q_\nu G_{ab}(\zeta-q)G_{cd}(q) \\
&= -\Big(m\not\zeta_{ab}\delta_{cd} + m^2\delta_{ab}\delta_{cd}\Big)L^{(u)}_{\mu\nu}(\zeta) - \Big(\not\zeta_{ab}\gamma^\rho_{cd} + m\delta_{ab}\gamma^\rho_{cd} + m\gamma^\rho_{ab}\delta_{cd}\Big)L^{(u)}_{\mu\nu\rho}(\zeta) \\
&\quad -\gamma^\rho_{ab}\gamma^\lambda_{cd}L^{(u)}_{\mu\nu\rho\lambda}(\zeta)
\end{aligned} \tag{C.44}$$

$$\begin{aligned}
(I_{\mu\nu\rho})^{(u)}_{abcd}(\zeta) &= \int \frac{d^2q}{(2\pi)^2} q_\mu q_\nu q_\rho G_{ab}(\zeta-q)G_{cd}(q) \\
&= -\Big(m\not\zeta_{ab}\delta_{cd} + m^2\delta_{ab}\delta_{cd}\Big)L^{(u)}_{\mu\nu\rho}(\zeta) - \Big(\not\zeta_{ab}\gamma^\lambda_{cd} + m\delta_{ab}\gamma^\lambda_{cd} + m\gamma^\lambda_{ab}\delta_{cd}\Big)L^{(u)}_{\mu\nu\rho\lambda}(\zeta) \\
&\quad -\gamma^\lambda_{ab}\gamma^\delta_{cd}L^{(u)}_{\mu\nu\rho\lambda\delta}(\zeta)
\end{aligned} \tag{C.45}$$

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$$
\begin{aligned}
(I_{\mu\nu\rho\lambda})^{(u)}_{abcd}(\zeta) &= \int \frac{d^2q}{(2\pi)^2} q_\mu q_\nu q_\rho q_\lambda G_{ab}(\zeta-q) G_{cd}(q) \\
&= -\Big(m\not\zeta_{ab}\delta_{cd} + m^2\delta_{ab}\delta_{cd}\Big) L^{(u)}_{\mu\nu\rho\lambda}(\zeta) - \Big(\not\zeta_{ab}\gamma^\delta_{cd} + m\delta_{ab}\gamma^\delta_{cd} + m\gamma^\delta_{ab}\delta_{cd}\Big) L^{(u)}_{\mu\nu\rho\lambda\delta}(\zeta) \\
&\quad -\gamma^\delta_{ab}\gamma^\omega_{cd} L^{(u)}_{\mu\nu\rho\lambda\delta\omega}(\zeta)\,,
\end{aligned} \tag{C.46}
$$

where,

$$
\begin{aligned}
L^{(u)}_{\mu\nu\rho}(\theta) &= \int \frac{d^2q}{(2\pi)^2} \frac{q_\mu q_\nu q_\rho}{[(\zeta-q)^2-m^2]\,(q^2-m^2)} \\
&= -\frac{i}{16\pi}\left(1+\log\frac{\Lambda^2}{m^2} - \theta\coth\frac{\theta}{2}\right)(\eta_{\mu\nu}\zeta_\rho + \eta_{\nu\rho}\zeta_\mu + \eta_{\rho\mu}\zeta_\nu) \\
&\quad -\frac{i}{32\pi m^2\sinh^2\frac{\theta}{2}}\,(3-2\theta\coth\theta - \theta\,\mathrm{csch}\,\theta)\,\zeta_\mu\zeta_\nu\zeta_\rho
\end{aligned} \tag{C.47}
$$

$$
\begin{aligned}
L^{(u)}_{\mu\nu\rho\lambda}(\theta) &= \int \frac{d^2q}{(2\pi)^2} \frac{q_\mu q_\nu q_\rho q_\lambda}{[(\zeta-q)^2-m^2]\,(q^2-m^2)} \\
&= -\frac{im^2}{288\pi}\left[14 - 9\frac{\Lambda^2}{m^2} + 12\log\frac{\Lambda^2}{m^2} + 2\cosh\theta\left(5+3\log\frac{\Lambda^2}{m^2}\right) - 6\theta\left(2\coth\frac{\theta}{2}+\sinh\theta\right)\right] C_{\mu\nu\rho\lambda} \\
&\quad +\frac{i}{288\pi\sinh^2\frac{\theta}{2}}\left[1+6\log\frac{\Lambda^2}{m^2} - \cosh\theta\left(7+6\log\frac{\Lambda^2}{m^2}\right) + 3\theta\cosh\frac{3\theta}{2}\,\mathrm{csch}\,\frac{\theta}{2}\right][\eta_{\mu\nu}\zeta_\rho\zeta_\lambda \\
&\quad +\eta_{\mu\rho}\zeta_\nu\zeta_\lambda + \eta_{\mu\lambda}\zeta_\nu\zeta_\rho + \eta_{\nu\rho}\zeta_\mu\zeta_\lambda + \eta_{\nu\lambda}\zeta_\rho\zeta_\mu + \eta_{\rho\lambda}\zeta_\mu\zeta_\nu] + \frac{i}{192\pi m^2\sinh^4\frac{\theta}{2}}\,[8-11\cosh\theta \\
&\quad +3\theta\cosh 2\theta\,\mathrm{csch}\,\theta]\,\zeta_\mu\zeta_\nu\zeta_\rho\zeta_\lambda
\end{aligned} \tag{C.48}
$$

$$
\begin{aligned}
L^{(u)}_{\mu\nu\rho\lambda\delta}(\theta) &= \int \frac{d^2q}{(2\pi)^2} \frac{q_\mu q_\nu q_\rho q_\lambda q_\delta}{[(\zeta-q)^2-m^2]\,(q^2-m^2)} \\
&= -\frac{im^2}{576\pi}\left[14 - 9\frac{\Lambda^2}{m^2} + 12\log\frac{\Lambda^2}{m^2} + 2\cosh\theta\left(5+3\log\frac{\Lambda^2}{m^2}\right) - 6\theta\left(2\coth\frac{\theta}{2}+\sinh\theta\right)\right] D'_{\mu\nu\rho\lambda\delta} \\
&\quad -\frac{i}{192\pi}\left[8+6\log\frac{\Lambda^2}{m^2} + 6\,\mathrm{csch}^2\frac{\theta}{2} - \frac{3}{4}\theta\sinh 2\theta\,\mathrm{csch}^4\frac{\theta}{2}\right] B'_{\mu\nu\rho\lambda\delta} \\
&\quad +\frac{i}{384\pi m^2\sinh^4\frac{\theta}{2}}\,[10-25\cosh\theta + 3\theta\,(\mathrm{csch}\,\theta + 2\coth\theta + 2\cosh 2\theta\,\mathrm{csch}\,\theta)]\,\zeta_\mu\zeta_\nu\zeta_\rho\zeta_\lambda\zeta_\omega \\
L^{(u)}_{\mu\nu\rho\lambda\delta\omega}(\theta) &= \int \frac{d^2q}{(2\pi)^2} \frac{q_\mu q_\nu q_\rho q_\lambda q_\delta q_\omega}{[(\zeta-q)^2-m^2]\,(q^2-m^2)} \\
&= -\frac{im^4}{3840\pi}\left[4\frac{\Lambda^2}{m^2} + 10\frac{\Lambda^4}{m^4} + 4(8+\cosh 2\theta)\log\frac{\Lambda^2}{m^2} + 64\cosh^4\frac{\theta}{2} - 4\cosh\theta\left(\frac{\Lambda^2}{m^2} - 6\log\frac{\Lambda^2}{m^2}\right)\right. \\
&\quad \left. -\theta\sinh^5\theta\,\mathrm{csch}^6\frac{\theta}{2}\right] F_{\mu\nu\rho\lambda\delta\omega} - \frac{im^2}{2400\pi}\left[\frac{142}{3} - 25\frac{\Lambda^2}{m^2} + 5\log\frac{\Lambda^2}{m^2}(7+3\cosh\theta) + 26\cosh\theta\right. \\
&\quad \left. + 10\,\mathrm{csch}^2\frac{\theta}{2} + 5\theta(2-3\cosh\theta)\coth^3\frac{\theta}{2}\right] H'_{\mu\nu\rho\lambda\delta\omega} + \frac{i}{19200\pi\sinh^4\frac{\theta}{2}}\left[43 - 180\frac{\Lambda^2}{m^2}\right. \\
&\quad \left. -\cosh 2\theta\left(89+60\log\frac{\Lambda^2}{m^2}\right) - 2\cosh\theta\left(7-120\log\frac{\Lambda^2}{m^2}\right) + 30\theta\cosh\frac{5\theta}{2}\,\mathrm{csch}\,\frac{\theta}{2}\right] M'_{\mu\nu\rho\lambda\delta\omega} \\
&\quad +\frac{i}{15360\pi m^2\sinh^6\frac{\theta}{2}\sinh\theta}\,[60\theta\cosh 3\theta - 225\sinh\theta + 288\sinh 2\theta - 137\sinh 3\theta]\,\zeta_\mu\zeta_\nu\zeta_\rho\zeta_\lambda\zeta_\delta\zeta_\omega,
\end{aligned} \tag{C.49}
$$

$C_{\mu\nu\rho\lambda}$ and $F_{\mu\nu\rho\lambda\delta\omega}$ are given by (C.5) and (C.7) respectively and $D'_{\mu\nu\rho\lambda\delta}$, $B'_{\mu\nu\rho\lambda\delta}$, $H'_{\mu\nu\rho\lambda\delta\omega}$ and $M'_{\mu\nu\rho\lambda\delta\omega}$ are given by (C.26), (C.27), (C.30) and (C.31) with ζ_μ instead of ξ_μ.

D Computational details of the second order S-matrix

In this section the details of the second order S-matrix calculation will be presented. As discussed in 4.1.2, the S-matrix at second order gets contribution from tadpole diagrams as well as bubble diagrams.

D.1 Details of the contributions from the tadpole diagrams

The expression for the amplitude from a general sextic coupling was given in (4.28). Using this one can compute the contribution to the tadpole amplitude from each term in the second order Lagrangian (4.24).

The contribution to the amplitude due to the coupling $\bar{\psi}\psi\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$ can be evaluated by computing the contributions from each term of (4.25). The first term in (4.25) contributes,

$$\begin{aligned}
&\mathcal{A}^{\text{tad}}_{\bar{\psi}\psi\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi} \\
&= \langle 0|b(p_4)a(p_3)\mathrm{T}\left[\bar{\psi}\psi\ \bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle \\
&= \delta_{ab}\gamma_{cd(\mu}\gamma^{(\mu}_{ef}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\bar{\psi}_a\psi_b\bar{\psi}_c\partial_{\nu)}\psi_d\bar{\psi}_e\partial^{\nu)}\psi_f\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle \\
&= -2N_0\Big[\bar{u}(p_3)\cdot i\gamma_{(\mu}p_{4\nu)}\cdot v(p_4)\ \bar{v}(p_2)\cdot(-i)\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1)+\bar{v}(p_2)\cdot(-i)\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)\ \bar{u}(p_3)\cdot i\gamma^{(\mu}p_4^{\nu)}\cdot v(p_4) \\
&\quad -\bar{v}(p_2)\cdot i\gamma_{(\mu}p_{4\nu)}\cdot v(p_4)\ \bar{u}(p_3)\cdot(-i)\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1)-\bar{u}(p_3)\cdot(-i)\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)\ \bar{v}(p_2)\cdot i\gamma^{(\mu}p_4^{\nu)}\cdot v(p_4)\Big] \\
&\quad +N_1\Big[\bar{u}(p_3)\cdot\gamma_{(\mu}\gamma_{\nu)}\cdot v(p_4)\ \bar{v}(p_2)\cdot(-i)\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1)+\bar{v}(p_2)\cdot\gamma_{(\mu}\gamma_{\nu)}\cdot u(p_1)\ \bar{u}(p_3)\cdot i\gamma^{(\mu}p_4^{\nu)}\cdot v(p_4) \\
&\quad -\bar{v}(p_2)\cdot\gamma_{(\mu}\gamma_{\nu)}\cdot v(p_4)\ \bar{u}(p_3)\cdot(-i)\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1)-\bar{u}(p_3)\cdot\gamma_{(\mu}\gamma_{\nu)}\cdot u(p_1)\ \bar{v}(p_2)\cdot i\gamma^{(\mu}p_4^{\nu)}\cdot v(p_4)\Big] \\
&\quad -N_1\left[\bar{v}(p_2)\cdot\gamma_{(\mu}\gamma_{\nu)}\cdot v(p_4)\ \bar{u}(p_3)\cdot(-i)\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)+\bar{u}(p_3)\cdot\gamma_{(\mu}\gamma_{\nu)}\cdot u(p_1)\ \bar{v}(p_2)\cdot i\gamma_{(\mu}p_{4\nu)}\cdot v(p_4)\right. \\
&\quad \left.-\bar{u}(p_3)\cdot\gamma_{(\mu}\gamma_{\nu)}\cdot v(p_4)\ \bar{v}(p_2)\cdot(-i)\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)-\bar{v}(p_2)\cdot\gamma_{(\mu}\gamma_{\nu)}\cdot u(p_1)\ \bar{u}(p_3)\cdot i\gamma_{(\mu}p_{4\nu)}\cdot v(p_4)\right] \\
&\quad +N_0\Big[\bar{u}(p_3)\cdot i\gamma_{(\mu}p_{4\nu)}\cdot v(p_4)\ \bar{v}(p_2)\cdot(-i)\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1)+\bar{v}(p_2)\cdot(-i)\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)\ \bar{u}(p_3)\cdot i\gamma^{(\mu}p_4^{\nu)}\cdot v(p_4) \\
&\quad -\bar{v}(p_2)\cdot i\gamma_{(\mu}p_{4\nu)}\cdot v(p_4)\ \bar{u}(p_3)\cdot(-i)\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1)-\bar{u}(p_3)\cdot(-i)\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)\ \bar{v}(p_2)\cdot i\gamma^{(\mu}p_4^{\nu)}\cdot v(p_4)\Big] \\
&\quad -N_0\Big[\bar{v}(p_2)\cdot i\gamma_{(\mu}p_{4\nu)}\cdot v(p_4)\ \bar{u}(p_3)\cdot(-i)\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1)+\bar{u}(p_3)\cdot(-i)\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)\ \bar{v}(p_2)\cdot i\gamma^{(\mu}p_4^{\nu)}\cdot v(p_4) \\
&\quad -\bar{u}(p_3)\cdot i\gamma_{(\mu}p_{4\nu)}\cdot v(p_4)\ \bar{v}(p_2)\cdot(-i)\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1)-\bar{v}(p_2)\cdot(-i)\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)\ \bar{u}(p_3)\cdot i\gamma^{(\mu}p_4^{\nu)}\cdot v(p_4)\Big] \\
&\quad -N_1\ \operatorname{Tr}\left(\gamma_{(\mu}\gamma_{\nu)}\right)\Big[\bar{u}(p_3)\cdot v(p_4)\ \bar{v}(p_2)\cdot(-i)\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1)+\bar{v}(p_2)\cdot u(p_1)\ \bar{u}(p_3)\cdot i\gamma^{(\mu}p_4^{\nu)}\cdot v(p_4) \\
&\quad -\bar{v}(p_2)\cdot v(p_4)\ \bar{u}(p_3)\cdot(-i)\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1)-\bar{u}(p_3)\cdot u(p_1)\ \bar{v}(p_2)\cdot i\gamma^{(\mu}p_4^{\nu)}\cdot v(p_4)\Big] \\
&\quad +N_1\left[\bar{u}(p_3)\cdot v(p_4)\ \bar{v}(p_2)\cdot(-i)\gamma^{(\mu}\gamma^{\nu)}\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)+\bar{v}(p_2)\cdot u(p_1)\ \bar{u}(p_3)\cdot i\gamma^{(\mu}\gamma^{\nu)}\gamma_{(\mu}p_{4\nu)}\cdot v(p_4)\right. \\
&\quad \left.-\bar{v}(p_2)\cdot v(p_4)\ \bar{u}(p_3)\cdot(-i)\gamma^{(\mu}\gamma^{\nu)}\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)-\bar{u}(p_3)\cdot u(p_1)\ \bar{v}(p_2)\cdot i\gamma^{(\mu}\gamma^{\nu)}\gamma_{(\mu}p_{4\nu)}\cdot v(p_4)\right] \\
&\quad +N_1\Big[\bar{u}(p_3)\cdot v(p_4)\ \bar{v}(p_2)\cdot(-i)\gamma_{(\mu}\gamma_{\nu)}\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1)+\bar{v}(p_2)\cdot u(p_1)\ \bar{u}(p_3)\cdot i\gamma_{(\mu}\gamma_{\nu)}\gamma^{(\mu}p_4^{\nu)}\cdot v(p_4) \\
&\quad -\bar{v}(p_2)\cdot v(p_4)\ \bar{u}(p_3)\cdot(-i)\gamma_{(\mu}\gamma_{\nu)}\gamma^{(\mu}p_1^{\nu)}\cdot u(p_1)-\bar{u}(p_3)\cdot u(p_1)\ \bar{v}(p_2)\cdot i\gamma_{(\mu}\gamma_{\nu)}\gamma^{(\mu}p_4^{\nu)}\cdot v(p_4)\Big] \\
&\quad -N_1\operatorname{Tr}\left(\gamma^{(\mu}\gamma^{\nu)}\right)\left[\bar{u}(p_3)\cdot v(p_4)\ \bar{v}(p_2)\cdot(-i)\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)+\bar{v}(p_2)\cdot u(p_1)\ \bar{u}(p_3)\cdot i\gamma_{(\mu}p_{4\nu)}\cdot v(p_4)\right. \\
&\quad \left.-\bar{v}(p_2)\cdot v(p_4)\ \bar{u}(p_3)\cdot(-i)\gamma_{(\mu}p_{1\nu)}\cdot u(p_1)-\bar{u}(p_3)\cdot u(p_1)\ \bar{v}(p_2)\cdot i\gamma_{(\mu}p_{4\nu)}\cdot v(p_4)\right] \\
&= 0
\end{aligned} \tag{D.1}$$

where N_0 and N_1 are given by (4.32) and (4.33), respectively. (4.8) and (4.9) were used to obtain the final result. The contribution to the tadpole amplitude from the rest of the terms can be computed in a similar way and the results were given in section 4.1.2.

D.2 Details of the contributions from the bubble diagrams

In this subsection the contribution to the amplitude from each term found in (4.42) will be evaluated in detail. As mentioned before, these interaction vertices give rise to three different kinds of one-loop bubble diagrams.

(a) Both vertices contain interaction $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$

Begin with the case where both vertices of a bubble diagram contain interactions of the form $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$. The total s-channel amplitude can be computed by plugging each term of (4.56) into the general s-channel amplitude (4.43) and adding the results together.

The contribution to the amplitude from the first term in (4.56), $\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\cdot\gamma^{(\mu}\partial^{\nu)}\psi(y)\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)$, is given by,

$$
\begin{aligned}
\mathcal{A}_{\text{a}}^{(s_1)} &= \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\gamma^{(\mu}\partial^{\nu)}\psi(y) \\
&\qquad\qquad\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} \\
&= -\frac{\lambda^2}{128}\gamma_{ef(\mu}\gamma_{gh}^{(\mu}\gamma_{ab(\rho}\gamma_{cd}^{(\rho}\times \\
&\quad\Bigg[-\bar{v}_c(p_2)u_d(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{u}_e(p_3)v_f(p_4)G_{ha}(\xi+q)G_{bg}(q)ip_{4\nu)}(-i)(\xi+q)^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)} \\
&\quad-\bar{u}_e(p_3)v_h(p_4)G_{fa}(\xi+q)G_{bg}(q)(-i)(\xi+q)_{\nu)}ip_4^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)} \\
&\quad-\bar{u}_g(p_3)v_f(p_4)G_{ha}(\xi+q)G_{be}(q)ip_{4\nu)}(-i)(\xi+q)^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)} \\
&\quad+\bar{u}_g(p_3)v_h(p_4)G_{fa}(\xi+q)G_{be}(q)(-i)(\xi+q)_{\nu)}ip_4^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)}\Big) \\
&\quad-\bar{v}_a(p_2)u_b(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{u}_e(p_3)v_f(p_4)G_{hc}(\xi+q)G_{dg}(q)ip_{4\nu)}(-i)(\xi+q)^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)} \\
&\quad-\bar{u}_e(p_3)v_h(p_4)G_{fc}(\xi+q)G_{dg}(q)(-i)(\xi+q)_{\nu)}ip_4^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)} \\
&\quad-\bar{u}_g(p_3)v_f(p_4)G_{hc}(\xi+q)G_{de}(q)ip_{4\nu)}(-i)(\xi+q)^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)} \\
&\quad+\bar{u}_g(p_3)v_h(p_4)G_{fc}(\xi+q)G_{de}(q)(-i)(\xi+q)_{\nu)}ip_4^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)}\Big) \\
&\quad+\bar{v}_a(p_2)u_d(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{u}_e(p_3)v_f(p_4)G_{hc}(\xi+q)G_{bg}(q)ip_{4\nu)}(-i)(\xi+q)^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)} \\
&\quad-\bar{u}_e(p_3)v_h(p_4)G_{fc}(\xi+q)G_{bg}(q)(-i)(\xi+q)_{\nu)}ip_4^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)} \\
&\quad-\bar{u}_g(p_3)v_f(p_4)G_{hc}(\xi+q)G_{be}(q)ip_{4\nu)}(-i)(\xi+q)^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)} \\
&\quad+\bar{u}_g(p_3)v_h(p_4)G_{fc}(\xi+q)G_{be}(q)(-i)(\xi+q)_{\nu)}ip_4^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)}\Big) \\
&\quad+\bar{v}_c(p_2)u_b(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{u}_e(p_3)v_f(p_4)G_{ha}(\xi+q)G_{dg}(q)ip_{4\nu)}(-i)(\xi+q)^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)}
\end{aligned}
$$

$$-\bar{u}_e(p_3)v_h(p_4)G_{fa}(\xi+q)G_{dg}(q)(-i)(\xi+q)_{\nu)}ip_4^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)}$$
$$-\bar{u}_g(p_3)v_f(p_4)G_{ha}(\xi+q)G_{de}(q)ip_{4\nu)}(-i)(\xi+q)^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)}$$
$$+\bar{u}_g(p_3)v_h(p_4)G_{fa}(\xi+q)G_{de}(q)(-i)(\xi+q)_{\nu)}ip_4^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)}\Big)\Bigg], \tag{D.2}$$

where $\xi^\mu = (p_1+p_2)^\mu$. The above expression, (D.2), involves the one-loop integrals $(I_\mu)^{(s)}_{abcd}(\xi)$ and $(I_{\mu\nu})^{(s)}_{abcd}(\xi)$ whose values are given by (C.34) and (C.35), respectively. Plugging the values of the integrals and Dirac spinors, (4.8) and (4.9), into (D.2) one finds the s-channel contribution from the first term in (4.56) to be,

$$\begin{aligned}\mathcal{A}_{\rm a}^{(s_1)} = & -\frac{\lambda^2 m^6}{768\pi}\Bigg[\frac{3}{2}\pi\,\mathrm{csch}\,\theta\,(87+24\cosh\theta+17\cosh 2\theta+12\cosh 3\theta+4\cosh 4\theta)+i\left(95-6\frac{\Lambda^2}{m^2}\right.\\ & -24\log\frac{m}{\Lambda}+6\cosh\theta\left(1+25\log\frac{m}{\Lambda}\right)-6\cosh 2\theta\left(3+2\frac{\Lambda^2}{m^2}-12\log\frac{m}{\Lambda}\right)-2\cosh 3\theta\,\Big(5\\ & \left.\left.-12\log\frac{m}{\Lambda}\right)+\frac{3}{2}\theta\,\mathrm{csch}\,\theta\,(87+24\cosh\theta+17\cosh 2\theta+12\cosh 3\theta+4\cosh 4\theta)\right)\Bigg].\end{aligned} \tag{D.3}$$

The contributions to the s-channel amplitude from the eight additional terms in (4.56) can be obtained in a similar manner. Their values are given by,

$$\begin{aligned}\mathcal{A}_{\rm a}^{(s_2)} = & \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[-2\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\gamma^{(\mu}\partial^{\nu)}\psi(y)\\ & \times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)}\\ = & -\frac{\lambda^2 m^6}{1536\pi}\Bigg[6\pi\,\mathrm{csch}\,\theta\,(29+4\cosh\theta+3\cosh 2\theta+8\cosh 3\theta+4\cosh 4\theta)\\ & +i\left(98-6\frac{\Lambda^2}{m^2}+\cosh\theta\left(-47-42\frac{\Lambda^2}{m^2}+228\log\frac{m}{\Lambda}\right)+2\cosh 2\theta\left(-46-21\frac{\Lambda^2}{m^2}\right.\right.\\ & \left.+96\log\frac{m}{\Lambda}\right)+\cosh 3\theta\left(-37+96\log\frac{m}{\Lambda}\right)+6\theta\,\mathrm{csch}\,\theta\,(29+4\cosh\theta+3\cosh 2\theta\\ & \left.+8\cosh 3\theta+4\cosh 4\theta)\right)\Bigg],\end{aligned} \tag{D.4}$$

$$\begin{aligned}\mathcal{A}_{\rm a}^{(s_3)} = & \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\gamma^{(\mu}\partial^{\nu)}\psi(y)\\ & \times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)}\\ = & -\frac{\lambda^2 m^6}{1536\pi}\Bigg[3\pi\,\mathrm{csch}\,\theta\,(87+24\cosh\theta+17\cosh 2\theta+12\cosh 3\theta+4\cosh 4\theta)+i\left(70-240\log\frac{m}{\Lambda}\right.\\ & +2\cosh\theta\left(-74+15\frac{\Lambda^2}{m^2}+48\log\frac{m}{\Lambda}\right)+\cosh 2\theta\left(-82+144\log\frac{m}{\Lambda}\right)+4\cosh 3\theta\,\Big(-5\\ & \left.\left.+12\log\frac{m}{\Lambda}\right)+3\theta\,\mathrm{csch}\,\theta\,(87+24\cosh\theta+17\cosh 2\theta+12\cosh 3\theta+4\cosh 4\theta)\right)\Bigg],\end{aligned} \tag{D.5}$$

$$\mathcal{A}_{\rm a}^{(s_4)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} \;=\; \mathcal{A}_{\rm a}^{(s_2)}, \qquad \text{(D.6)}$$

$$\mathcal{A}_{\rm a}^{(s_5)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[4\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)}$$
$$= -\frac{\lambda^2 m^6}{76800\pi}\Bigg[600\pi\,\mathrm{csch}\,\theta\,(15-7\cosh 2\theta+4\cosh 3\theta+4\cosh 4\theta)$$
$$+i\Bigg(4252-120\frac{\Lambda^2}{m^2}-9600\log\frac{m}{\Lambda}+\cosh\theta\left(1111-5640\frac{\Lambda^2}{m^2}-1350\frac{\Lambda^4}{m^4}+10800\log\frac{m}{\Lambda}\right)$$
$$-8\cosh 2\theta\left(986+615\frac{\Lambda^2}{m^2}-1200\log\frac{m}{\Lambda}\right)+\cosh 3\theta\left(-3247+9600\log\frac{m}{\Lambda}\right)$$
$$+600\theta\,\mathrm{csch}\,\theta\,(15-7\cosh 2\theta+4\cosh 3\theta+4\cosh 4\theta)\Bigg)\Bigg], \qquad \text{(D.7)}$$

$$\mathcal{A}_{\rm a}^{(s_6)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\right.$$
$$\left.\times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} \;=\; \mathcal{A}_{\rm a}^{(s_2)}, \qquad \text{(D.8)}$$

$$\mathcal{A}_{\rm a}^{(s_7)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\partial_{(\mu}\bar{\psi}(y)\gamma_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} \;=\; \mathcal{A}_{\rm a}^{(s_3)}, \qquad \text{(D.9)}$$

$$\mathcal{A}_{\rm a}^{(s_8)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\partial_{(\mu}\bar{\psi}(y)\gamma_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} \;=\; \mathcal{A}_{\rm a}^{(s_2)}, \qquad \text{(D.10)}$$

$$\mathcal{A}_{\rm a}^{(s_9)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\partial_{(\mu}\bar{\psi}(y)\gamma_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\right.$$
$$\left.\times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} \;=\; \mathcal{A}_{\rm a}^{(s_1)}. \qquad \text{(D.11)}$$

Next, the t-channel amplitude in the case where both vertices contain interactions of the form $\tilde{X}_{\mu\nu}\tilde{X}^{\mu\nu}$ will be computed. By the general t-channel amplitude (4.44) the contribution from the first term in (4.56), $\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\gamma^{(\mu}\partial^{\nu)}\psi(y)\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)$, to the amplitude is given by,

$$\mathcal{A}_{\rm a}^{(t_1)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\gamma^{(\mu}\partial^{\nu)}\psi(y)$$
$$\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)}$$
$$= -\frac{\lambda^2}{128}\gamma_{ef(\mu}\gamma_{gh}^{(\mu}\gamma_{ab(\rho}\gamma_{cd}^{(\rho}\;\times$$
$$\Bigg[\bar{u}_c(p_3)u_d(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{v}_g(p_2)v_h(p_4)G_{fa}(q)G_{be}(q)(-i)q_{\nu)}ip_4^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)}$$

$$-\bar{v}_g(p_2)v_f(p_4)G_{ha}(q)G_{be}(q)(-i)q^{\nu)}ip_{4\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)}$$
$$-\bar{v}_e(p_2)v_h(p_4)G_{fa}(q)G_{bg}(q)(-i)q_{\nu)}ip_4^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)}$$
$$+\bar{v}_e(p_2)v_f(p_4)G_{ha}(q)G_{bg}(q)(-i)q^{\nu)}ip_{4\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)}\Big)$$
$$+\bar{u}_a(p_3)u_b(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{v}_g(p_2)v_h(p_4)G_{fc}(q)G_{de}(q)(-i)q_{\nu)}ip_4^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)}$$
$$-\bar{v}_g(p_2)v_f(p_4)G_{hc}(q)G_{de}(q)ip_{4\nu)}(-i)q^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)}$$
$$-\bar{v}_e(p_2)v_h(p_4)G_{fc}(q)G_{dg}(q)(-i)q_{\nu)}ip_4^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)}$$
$$+\bar{v}_e(p_2)v_f(p_4)G_{hc}(q)G_{dg}(q)ip_{4\nu)}(-i)q^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)}\Big)$$
$$-\bar{u}_a(p_3)u_d(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{v}_g(p_2)v_h(p_4)G_{fc}(q)G_{be}(q)(-i)q_{\nu)}ip_4^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)}$$
$$-\bar{v}_g(p_2)v_f(p_4)G_{hc}(q)G_{be}(q)ip_{4\nu)}(-i)q^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)}$$
$$-\bar{v}_e(p_2)v_h(p_4)G_{fc}(q)G_{bg}(q)(-i)q_{\nu)}ip_4^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)}$$
$$+\bar{v}_e(p_2)v_f(p_4)G_{hc}(q)G_{bg}(q)ip_{4\nu)}(-i)q^{\nu)}(-i)q_{\lambda)}(-i)p_1^{\lambda)}\Big)$$
$$-\bar{u}_c(p_3)u_b(p_1)\int\frac{d^2q}{(2\pi)^2}\Big(\bar{v}_g(p_2)v_h(p_4)G_{fa}(q)G_{de}(q)(-i)q_{\nu)}ip_4^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)}$$
$$-\bar{v}_g(p_2)v_f(p_4)G_{ha}(q)G_{de}(q)ip_{4\nu)}(-i)q^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)}$$
$$-\bar{v}_e(p_2)v_h(p_4)G_{fa}(q)G_{dg}(q)(-i)q_{\nu)}ip_4^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)}$$
$$+\bar{v}_e(p_2)v_f(p_4)G_{ha}(q)G_{dg}(q)ip_{4\nu)}(-i)q^{\nu)}(-i)p_{1\lambda)}(-i)q^{\lambda)}\Big)\Bigg]. \tag{D.12}$$

The above expression, (D.12), involves the one-loop integral $(I_{\mu\nu})^{(t)}_{abcd}$ given by (C.38). Plugging the value of this integral and Dirac spinors into (D.12) one finds the t-channel contribution from the first term in (4.56) to be,

$$\mathcal{A}_{\rm a}^{(t_1)}=-i\frac{\lambda^2m^6}{256\pi}\left[1+\frac{\Lambda^2}{m^2}+6\log\frac{m}{\Lambda}-\cosh\theta\left(4+5\frac{\Lambda^2}{m^2}+28\log\frac{m}{\Lambda}\right)+4\cosh 2\theta\left(1+2\log\frac{m}{\Lambda}\right)\right]. \tag{D.13}$$

The contributions to the t-channel amplitude from the eight additional terms in (4.56) can be obtained in a similar manner. Their values are given by,

$$\begin{aligned}\mathcal{A}_{\rm a}^{(t_2)}&=\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[-2\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\gamma^{(\mu}\partial^{\nu)}\psi(y)\\&\qquad\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)}\\&=-i\frac{\lambda^2m^6}{256\pi}\left[-2-2\frac{\Lambda^2}{m^2}-12\log\frac{m}{\Lambda}-3\cosh\theta\left(1+\frac{\Lambda^2}{m^2}+6\log\frac{m}{\Lambda}\right)+8\cosh 2\theta\left(1+2\log\frac{m}{\Lambda}\right)\right],\end{aligned} \tag{D.14}$$

$$\mathcal{A}_{\mathrm{a}}^{(t_3)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\bar\psi(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar\psi(y)\gamma^{(\mu}\partial^{\nu)}\psi(y)$$

$$\times\partial_{(\rho}\bar\psi(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar\psi(x)\gamma^{\lambda)}\psi(x)\Big]\, a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)}$$

$$= -i\frac{\lambda^2 m^6}{256\pi}\left[1+\frac{\Lambda^2}{m^2}+6\log\frac{m}{\Lambda}-\cosh\theta\left(5+4\frac{\Lambda^2}{m^2}+26\log\frac{m}{\Lambda}\right)+4\cosh 2\theta\left(1+2\log\frac{m}{\Lambda}\right)\right], \tag{D.15}$$

$$\mathcal{A}_{\mathrm{a}}^{(t_4)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[4\bar\psi(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\partial^{(\mu}\bar\psi(y)\gamma^{\nu)}\psi(y)$$

$$\times\bar\psi(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar\psi(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\Big]\, a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)} = \mathcal{A}_{\mathrm{a}}^{(t_2)}, \tag{D.16}$$

$$\mathcal{A}_{\mathrm{a}}^{(t_5)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[-2\bar\psi(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\partial^{(\mu}\bar\psi(y)\gamma^{\nu)}\psi(y)$$

$$\times\bar\psi(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar\psi(x)\gamma^{\lambda)}\psi(x)\Big]\, a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)}$$

$$= i\frac{\lambda^2 m^6}{512\pi}\Bigg[-8+10\frac{\Lambda^2}{m^2}+9\frac{\Lambda^4}{m^4}-48\log\frac{m}{\Lambda}+2\cosh\theta\left(1+3\frac{\Lambda^2}{m^2}+12\log\frac{m}{\Lambda}\right)$$

$$-32\cosh 2\theta\left(1+2\log\frac{m}{\Lambda}\right)\Bigg], \tag{D.17}$$

$$\mathcal{A}_{\mathrm{a}}^{(t_6)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[-2\bar\psi(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\partial^{(\mu}\bar\psi(y)\gamma^{\nu)}\psi(y)$$

$$\times\partial_{(\rho}\bar\psi(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar\psi(x)\gamma^{\lambda)}\psi(x)\Big]\, a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)} = \mathcal{A}_{\mathrm{a}}^{(t_2)}, \tag{D.18}$$

$$\mathcal{A}_{\mathrm{a}}^{(t_7)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\partial_{(\mu}\bar\psi(y)\gamma_{\nu)}\psi(y)\partial^{(\mu}\bar\psi(y)\gamma^{\nu)}\psi(y)$$

$$\times\bar\psi(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar\psi(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\Big]\, a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_{\mathrm{a}}^{(t_3)}, \tag{D.19}$$

$$\mathcal{A}_{\mathrm{a}}^{(t_8)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[-2\partial_{(\mu}\bar\psi(y)\gamma_{\nu)}\psi(y)\partial^{(\mu}\bar\psi(y)\gamma^{\nu)}\psi(y)$$

$$\times\bar\psi(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar\psi(x)\gamma^{\lambda)}\psi(x)\Big]\, a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_{\mathrm{a}}^{(t_2)}, \tag{D.20}$$

$$\mathcal{A}_{\mathrm{a}}^{(t_9)} = \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\partial_{(\mu}\bar\psi(y)\gamma_{\nu)}\psi(y)\partial^{(\mu}\bar\psi(y)\gamma^{\nu)}\psi(y)$$

$$\times\partial_{(\rho}\bar\psi(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar\psi(x)\gamma^{\lambda)}\psi(x)\Big]\, a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)} = \mathcal{A}_{\mathrm{a}}^{(t_1)}. \tag{D.21}$$

Lastly, the amplitude due to the same quartic interaction will be computed in the u-

channel case. By the general u-channel amplitude (4.45), the contribution from the first term in (4.56), $\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\gamma^{(\mu}\partial^{\nu)}\psi(y)\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)$, to the amplitude is given by,

$$\begin{aligned}\mathcal{A}_{\rm a}^{(u_1)} &= \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\gamma^{(\mu}\partial^{\nu)}\psi(y)\\ &\qquad\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}\\ &= -\frac{\lambda^2}{128}\gamma_{ef(\mu}\gamma_{gh}^{(\mu}\gamma_{ab(\rho}\gamma_{cd}^{(\rho}\times\\ &\quad\Big[u_d(p_1)v_b(p_4)\bar{u}_e(p_3)\bar{v}_g(p_2)ip_{4\lambda)}(-i)p_1^{\lambda)}-u_d(p_1)v_b(p_4)\bar{u}_g(p_3)\bar{v}_e(p_2)ip_{4\lambda)}(-i)p_1^{\lambda)}\\ &\quad-u_b(p_1)v_d(p_4)\bar{u}_e(p_3)\bar{v}_g(p_2)(-i)p_{1\lambda)}ip_4^{\lambda)}+u_b(p_1)v_d(p_4)\bar{u}_g(p_3)\bar{v}_e(p_2)(-i)p_{1\lambda)}ip_4^{\lambda)}\Big]\\ &\quad\times\int\frac{d^2q}{(2\pi)^2}\Big[G_{hc}(\zeta-q)G_{fa}(q)(-i)q_{\nu)}(-i)(\zeta-q)^{\nu)}-G_{ha}(\zeta-q)G_{fc}(q)(-i)q_{\nu)}(-i)(\zeta-q)^{\nu)}\Big],\end{aligned}\tag{D.22}$$

where $\zeta^\mu=(p_1-p_4)^\mu=(p_1-p_2)^\mu$. The above expression, (D.22), involves the one-loop integrals $(I_\mu)_{abcd}^{(u)}(\zeta)$ and $(I_{\mu\nu})_{abcd}^{(u)}(\zeta)$ whose values are given by (C.43) and (C.44), respectively. Plugging the values of the integrals and Dirac spinors into (D.22) one finds the u-channel contribution from the first term in (4.56) to be,

$$\begin{aligned}\mathcal{A}_{\rm a}^{(u_1)} &= -i\frac{\lambda^2m^6}{128\pi}\cosh^2\frac{\theta}{2}\left(1-4\cosh\theta\right)\Bigg[3\frac{\Lambda^2}{m^2}+10\log\frac{m}{\Lambda}+3\theta\coth\frac{\theta}{2}+4\theta\sinh\theta\\ &\qquad-\cosh\theta\left(3-8\log\frac{m}{\Lambda}\right)\Bigg].\end{aligned}\tag{D.23}$$

The contributions to the u-channel amplitude from the eight additional terms in (4.56) can be obtained in a similar manner. Their values are given by,

$$\begin{aligned}\mathcal{A}_{\rm a}^{(u_2)} &= \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[-2\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\gamma^{(\mu}\partial^{\nu)}\psi(y)\\ &\qquad\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}\\ &= i\frac{\lambda^2m^6}{192\pi}\cosh^2\frac{\theta}{2}\Bigg[-18-24\frac{\Lambda^2}{m^2}+9\theta\coth\frac{\theta}{2}+24\theta\sinh 2\theta+\cosh\theta\left(25+42\frac{\Lambda^2}{m^2}+24\log\frac{m}{\Lambda}\right)\\ &\quad+\cosh 2\theta\left(-25+48\log\frac{m}{\Lambda}\right)\Bigg],\end{aligned}\tag{D.24}$$

$$\begin{aligned}\mathcal{A}_{\rm a}^{(u_3)} &= \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\bar{\psi}(y)\gamma^{(\mu}\partial^{\nu)}\psi(y)\\ &\qquad\times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}\\ &= i\frac{\lambda^2m^6}{19200\pi}\cosh^2\frac{\theta}{2}\Bigg[-1311+30\frac{\Lambda^2}{m^2}-675\frac{\Lambda^4}{m^4}+8700\log\frac{m}{\Lambda}+150\theta\coth\frac{\theta}{2}\left(1-4\cosh\theta\right)^2\\ &\quad+2\cosh\theta\left(29+660\frac{\Lambda^2}{m^2}+1200\log\frac{m}{\Lambda}\right)+\cosh 2\theta\left(-1447+2400\log\frac{m}{\Lambda}\right)\Bigg],\end{aligned}\tag{D.25}$$

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$$
\begin{aligned}
\mathcal{A}_{\rm a}^{(u_4)} &= \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[-2\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\\
&\qquad\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}\\
&= i\frac{\lambda^2m^6}{64\pi}\cosh\frac{\theta}{2}(-1+4\cosh\theta)\left[\cosh\frac{3\theta}{2}\left(-1+4\log\frac{m}{\Lambda}\right)+\cosh\frac{\theta}{2}\left(1+\theta\coth\frac{\theta}{2}+4\theta\sinh\theta\right)\right],
\end{aligned}
\tag{D.26}
$$

$$
\begin{aligned}
\mathcal{A}_{\rm a}^{(u_5)} &= \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[4\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\\
&\qquad\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}\\
&= i\frac{\lambda^2m^6}{768\pi}\Bigg[-43-12\frac{\Lambda^2}{m^2}+72\log\frac{m}{\Lambda}+24\theta\cosh^2\frac{\theta}{2}\coth\frac{\theta}{2}(3-4\cosh\theta)^2\\
&\quad+4\cosh\theta\left(7+3\frac{\Lambda^2}{m^2}-6\log\frac{m}{\Lambda}\right)+\cosh 2\theta\left(31+24\frac{\Lambda^2}{m^2}\right)-8\cosh 3\theta\left(5-12\log\frac{m}{\Lambda}\right)\Bigg],
\end{aligned}
\tag{D.27}
$$

$$
\begin{aligned}
\mathcal{A}_{\rm a}^{(u_6)} &= \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[-2\bar{\psi}(y)\gamma_{(\mu}\partial_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\\
&\quad\times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)} \;=\; \mathcal{A}_{\rm a}^{(u_2)},
\end{aligned}
\tag{D.28}
$$

$$
\begin{aligned}
\mathcal{A}_{\rm a}^{(u_7)} &= \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\partial_{(\mu}\bar{\psi}(y)\gamma_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\\
&\qquad\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}\\
&= i\frac{\lambda^2m^6}{128\pi}\left(\cosh\frac{\theta}{2}+2\cosh\frac{3\theta}{2}\right)^2\left(\theta\coth\frac{\theta}{2}+2\log\frac{m}{\Lambda}\right),
\end{aligned}
\tag{D.29}
$$

$$
\begin{aligned}
\mathcal{A}_{\rm a}^{(u_8)} &= \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[-2\partial_{(\mu}\bar{\psi}(y)\gamma_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\\
&\quad\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)} \;=\; \mathcal{A}_{\rm a}^{(u_4)},
\end{aligned}
\tag{D.30}
$$

$$
\begin{aligned}
\mathcal{A}_{\rm a}^{(u_9)} &= \frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\partial_{(\mu}\bar{\psi}(y)\gamma_{\nu)}\psi(y)\partial^{(\mu}\bar{\psi}(y)\gamma^{\nu)}\psi(y)\\
&\quad\times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\Big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)} \;=\; \mathcal{A}_{\rm a}^{(u_1)}.
\end{aligned}
\tag{D.31}
$$

(b) One vertex contains $\tilde{X}_{\rho\lambda}\tilde{X}^{\rho\lambda}$ while the other contains $(\tilde{X}_\mu{}^\mu)^2$

Next consider the bubble diagrams where one vertex contains $\tilde{X}_{\rho\lambda}\tilde{X}^{\rho\lambda}$ while the other vertex contains $(\tilde{X}_\mu{}^\mu)^2$. Begin by considering the s-channel contributions to the amplitude. By the general s-channel amplitude (4.43), the contributions from all nine terms in (4.61) are,

$$\begin{aligned}\mathcal{A}_{\rm b}^{(s_1)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\big[\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\bar\psi(y)\gamma^\nu\partial_\nu\psi(y)\\ &\qquad\times\bar\psi(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar\psi(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)}\\ &= \frac{\lambda^2m^6}{192\pi\sinh\theta}\left[\frac{3\pi}{2}(29+10\cosh\theta+7\cosh2\theta+2\cosh3\theta)+i\left(\sinh\theta\left(36-6\frac{\Lambda^2}{m^2}-24\log\frac{m}{\Lambda}\right)\right.\right.\\ &\quad+\sinh\theta\cosh2\theta\left(1+12\log\frac{m}{\Lambda}\right)+5\sinh2\theta\left(1+3\log\frac{m}{\Lambda}\right)+\frac{3}{2}\theta(29+10\cosh\theta+7\cosh2\theta\\ &\quad\left.\left.+2\cosh3\theta)\right)\right],\end{aligned}\tag{D.32}$$

$$\begin{aligned}\mathcal{A}_{\rm b}^{(s_2)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\big[-2\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\bar\psi(y)\gamma^\nu\partial_\nu\psi(y)\\ &\qquad\times\bar\psi(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar\psi(x)\gamma^{\lambda)}\psi(x)\big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)}\\ &= \frac{\lambda^2m^6}{192\pi\sinh\theta}\left[3\pi(7+2\cosh\theta+5\cosh2\theta+2\cosh3\theta)+i\left(13\sinh\theta-\sinh2\theta\left(7+\frac{3}{2}\frac{\Lambda^2}{m^2}\right.\right.\right.\\ &\quad\left.\left.\left.-15\log\frac{m}{\Lambda}\right)+24\sinh2\theta\cosh\theta\log\frac{m}{\Lambda}+3\theta(7+2\cosh\theta+5\cosh2\theta+2\cosh3\theta)\right)\right],\end{aligned}\tag{D.33}$$

$$\begin{aligned}\mathcal{A}_{\rm b}^{(s_3)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\big[\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\bar\psi(y)\gamma^\nu\partial_\nu\psi(y)\\ &\qquad\times\partial_{(\rho}\bar\psi(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar\psi(x)\gamma^{\lambda)}\psi(x)\big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)}\\ &= \frac{\lambda^2m^6}{128\pi\sinh\theta}\left[\pi(29+10\cosh\theta+7\cosh2\theta+2\cosh3\theta)+i\left(8\sinh\theta\left(1-4\log\frac{m}{\Lambda}\right.\right.\right.\\ &\quad\left.\left.\left.+\cosh2\theta\log\frac{m}{\Lambda}\right)+\sinh2\theta\left(-6+3\frac{\Lambda^2}{m^2}+4\log\frac{m}{\Lambda}\right)+\theta(29+10\cosh\theta+7\cosh2\theta+2\cosh3\theta)\right)\right],\end{aligned}\tag{D.34}$$

$$\begin{aligned}\mathcal{A}_{\rm b}^{(s_4)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\big[-2\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\;\partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\\ &\qquad\times\bar\psi(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar\psi(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)}\\ &= \frac{\lambda^2m^6}{384\pi\sinh\theta}\left[6\pi(29+10\cosh\theta+7\cosh2\theta+2\cosh3\theta)+i\left(\sinh\theta\left(115-24\frac{\Lambda^2}{m^2}\right.\right.\right.\\ &\quad\left.-144\log\frac{m}{\Lambda}+\cosh3\theta\right)+\sinh\theta\cosh2\theta\left(13-6\frac{\Lambda^2}{m^2}+48\log\frac{m}{\Lambda}\right)+\sinh2\theta\left(\frac{13}{2}+42\log\frac{m}{\Lambda}\right)\\ &\quad\left.\left.+6\theta(29+10\cosh\theta+7\cosh2\theta+2\cosh3\theta)\right)\right],\end{aligned}\tag{D.35}$$

$$\begin{aligned}\mathcal{A}_{\rm b}^{(s_5)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\big[4\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y) \\ &\qquad \times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} \\ &= \frac{\lambda^2 m^6}{19200\pi\sinh\theta}\bigg[600\pi(7+2\cosh\theta+5\cosh 2\theta+2\cosh 3\theta)+i\bigg(12\sinh\theta\bigg(177+30\frac{\Lambda^2}{m^2} \\ &\quad +400\log\frac{m}{\Lambda}+\frac{151}{12}\cosh 3\theta\bigg)+48\sinh\theta\cosh 2\theta\bigg(8-5\frac{\Lambda^2}{m^2}+100\log\frac{m}{\Lambda}\bigg)-\sinh 2\theta\bigg(\frac{3343}{2} \\ &\quad +540\frac{\Lambda^2}{m^2}+225\frac{\Lambda^4}{m^4}-3000\log\frac{m}{\Lambda}\bigg)+600\ \theta(7+2\cosh\theta+5\cosh 2\theta+2\cosh 3\theta)\bigg)\bigg],\end{aligned} \tag{D.36}$$

$$\begin{aligned}\mathcal{A}_{\rm b}^{(s_6)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\big[-2\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y) \\ &\quad \times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_{\rm b}^{(s_4)},\end{aligned} \tag{D.37}$$

$$\begin{aligned}\mathcal{A}_{\rm b}^{(s_7)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\big[\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y) \\ &\quad \times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_{\rm b}^{(s_3)},\end{aligned} \tag{D.38}$$

$$\begin{aligned}\mathcal{A}_{\rm b}^{(s_8)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\big[-2\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y) \\ &\quad \times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_{\rm b}^{(s_2)},\end{aligned} \tag{D.39}$$

$$\begin{aligned}\mathcal{A}_{\rm b}^{(s_9)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\big[\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y) \\ &\quad \times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\big]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_{\rm b}^{(s_1)},\end{aligned} \tag{D.40}$$

where the computational method that was employed was exactly the same as when $\mathcal{A}_{\rm a}^{(s_1)}$ was computed in (D.2).

The contribution to the amplitude from the t-channel can be computed using the general t-channel amplitude, (4.44). The nine terms in (4.61) contribute to the amplitude as,

$$\begin{aligned}\mathcal{A}_{\rm b}^{(t_1)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\bar{\psi}(y)\gamma^\nu\partial_\nu\psi(y)\right. \\ &\qquad \left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)} \\ &= i\frac{\lambda^2 m^6}{64\pi}\left[1+\frac{\Lambda^2}{m^2}+6\log\frac{m}{\Lambda}-\cosh\theta\left(2+\frac{\Lambda^2}{m^2}+8\log\frac{m}{\Lambda}\right)\right],\end{aligned} \tag{D.41}$$

$$\begin{aligned}\mathcal{A}_{\rm b}^{(t_2)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \bar{\psi}(y)\gamma^\nu\partial_\nu\psi(y)\right. \\ &\qquad \left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)} \\ &= -i\frac{\lambda^2 m^6}{64\pi}(2+\cosh\theta)\left[1+\frac{\Lambda^2}{m^2}+6\log\frac{m}{\Lambda}\right],\end{aligned} \tag{D.42}$$

$$\mathcal{A}_{\rm b}^{(t_3)} = -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \bar{\psi}(y)\gamma^\nu\partial_\nu\psi(y)\right.$$
$$\left.\times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)}$$
$$= i\frac{\lambda^2 m^6}{64\pi}\left[1+\frac{\Lambda^2}{m^2}+6\log\frac{m}{\Lambda}-\cosh\theta\left(1+2\frac{\Lambda^2}{m^2}+10\log\frac{m}{\Lambda}\right)\right], \tag{D.43}$$

$$\mathcal{A}_{\rm b}^{(t_4)} = -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)}$$
$$= -i\frac{\lambda^2 m^6}{64\pi}(-2+3\cosh\theta)\left[1+\frac{\Lambda^2}{m^2}+6\log\frac{m}{\Lambda}\right], \tag{D.44}$$

$$\mathcal{A}_{\rm b}^{(t_5)} = -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[4\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)}$$
$$= -i\frac{\lambda^2 m^6}{128\pi}\left[8+14\frac{\Lambda^2}{m^2}+3\frac{\Lambda^4}{m^4}+48\log\frac{m}{\Lambda}+2\cosh\theta\left(1+3\frac{\Lambda^2}{m^2}+12\log\frac{m}{\Lambda}\right)\right], \tag{D.45}$$

$$\mathcal{A}_{\rm b}^{(t_6)} = -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)} = \mathcal{A}_{\rm b}^{(t_4)}, \tag{D.46}$$

$$\mathcal{A}_{\rm b}^{(t_7)} = -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_{\rm b}^{(t_3)}, \tag{D.47}$$

$$\mathcal{A}_{\rm b}^{(t_8)} = -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_{\rm b}^{(t_2)}, \tag{D.48}$$

$$\mathcal{A}_{\rm b}^{(t_9)} = -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_{\rm b}^{(t_1)}, \tag{D.49}$$

where the computational method that was employed was exactly the same as was used to evaluate $\mathcal{A}_{\rm a}^{(t_1)}$ in (D.12).

The contribution to the amplitude from the u-channel can be computed using the general u-channel amplitude, (4.45). The contributions from the nine terms in (4.61), to the amplitude, are given by,

$$\begin{aligned}\mathcal{A}_{\rm b}^{(u_1)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\bar{\psi}(y)\gamma^\nu\partial_\nu\psi(y)\right.\\ &\qquad\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}\\ &= i\frac{\lambda^2m^6}{96\pi}\cosh^2\frac{\theta}{2}\;(1-4\cosh\theta)\left[-4+3\frac{\Lambda^2}{m^2}+18\log\frac{m}{\Lambda}+\cosh\theta+3\theta\coth\frac{\theta}{2}\right],\end{aligned} \tag{D.50}$$

$$\begin{aligned}\mathcal{A}_{\rm b}^{(u_2)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\;\bar{\psi}(y)\gamma^\nu\partial_\nu\psi(y)\right.\\ &\qquad\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}\\ &= -i\frac{\lambda^2m^6}{96\pi}\coth\frac{\theta}{2}\left[\frac{3}{2}\sinh\theta\left(1+2\frac{\Lambda^2}{m^2}\right)-2\cosh\frac{3\theta}{2}\sinh\frac{\theta}{2}\left(8-3\frac{\Lambda^2}{m^2}-24\log\frac{m}{\Lambda}\right)\right.\\ &\qquad\left.+\cosh\frac{5\theta}{2}\sinh\frac{\theta}{2}-3\theta\,(1-\cosh\theta-2\cosh 2\theta)\right],\end{aligned} \tag{D.51}$$

$$\begin{aligned}\mathcal{A}_{\rm b}^{(u_3)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\;\bar{\psi}(y)\gamma^\nu\partial_\nu\psi(y)\right.\\ &\qquad\left.\times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}\\ &= -i\frac{\lambda^2m^6}{1600\pi}\left[25\theta\coth\frac{\theta}{2}\,(1+3\cosh\theta+2\cosh 2\theta)+\cosh^2\frac{\theta}{2}\left(-79+270\frac{\Lambda^2}{m^2}-75\frac{\Lambda^4}{m^4}\right.\right.\\ &\qquad\left.\left.+1100\log\frac{m}{\Lambda}+2\cosh\theta\left(-119-60\frac{\Lambda^2}{m^2}+200\log\frac{m}{\Lambda}\right)+17\cosh 2\theta\right)\right],\end{aligned} \tag{D.52}$$

$$\begin{aligned}\mathcal{A}_{\rm b}^{(u_4)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\;\partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.\\ &\qquad\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}\\ &= i\frac{\lambda^2m^6}{16\pi}\cosh^2\frac{\theta}{2}\;(1-4\cosh\theta)\left[4\log\frac{m}{\Lambda}+\theta\coth\frac{\theta}{2}\right],\end{aligned} \tag{D.53}$$

$$\begin{aligned}\mathcal{A}_{\rm b}^{(u_5)} &= -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2\frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[4\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\;\partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.\\ &\qquad\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right]a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}\\ &= -i\frac{\lambda^2m^6}{96\pi}\coth\frac{\theta}{2}\left[\sinh\theta\;\left(\frac{1}{2}+6\frac{\Lambda^2}{m^2}\right)-6\theta\,(1-\cosh\theta-2\cosh 2\theta)\right.\\ &\qquad\left.+\sinh\frac{\theta}{2}\cosh\frac{3\theta}{2}\left(-13+72\log\frac{m}{\Lambda}\right)\right],\end{aligned} \tag{D.54}$$

$$\mathcal{A}_{\rm b}^{(u_6)} = -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}$$
$$= -i\frac{\lambda^2 m^6}{16\pi}\cosh^2\frac{\theta}{2}\ \left[-1+6\frac{\Lambda^2}{m^2}+16\log\frac{m}{\Lambda}+\theta\left(4\sinh\theta+3\coth\frac{\theta}{2}\right)\right.$$
$$\left.+\cosh\theta\left(-5+8\log\frac{m}{\Lambda}\right)\right], \tag{D.55}$$

$$\mathcal{A}_{\rm b}^{(u_7)} = -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\bar{\psi}(x)\gamma^{(\rho}\partial^{\lambda)}\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}$$
$$= i\frac{\lambda^2 m^6}{32\pi}\cosh^2\frac{\theta}{2}\ \left(1-4\cosh\theta\right)\left[2\log\frac{m}{\Lambda}+\theta\coth\frac{\theta}{2}\right], \tag{D.56}$$

$$\mathcal{A}_{\rm b}^{(u_8)} = -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma_{(\rho}\partial_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}$$
$$= -i\frac{\lambda^2 m^6}{32\pi}\left[1+4\cosh\theta\ \log\frac{m}{\Lambda}-\cosh 2\theta\left(1-4\log\frac{m}{\Lambda}\right)\right.$$
$$\left.+\theta\left(5\sinh\theta+2\sinh 2\theta+2\coth\frac{\theta}{2}\right)\right], \tag{D.57}$$

$$\mathcal{A}_{\rm b}^{(u_9)} = -2\frac{1}{2!}\left(i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\partial_{(\rho}\bar{\psi}(x)\gamma_{\lambda)}\psi(x)\partial^{(\rho}\bar{\psi}(x)\gamma^{\lambda)}\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}$$
$$= -i\frac{\lambda^2 m^6}{32\pi}\cosh^2\frac{\theta}{2}\ \left[3\frac{\Lambda^2}{m^2}+10\log\frac{m}{\Lambda}+\cosh\theta\left(-3+8\log\frac{m}{\Lambda}\right)+\theta\left(4\sinh\theta+3\coth\frac{\theta}{2}\right)\right], \tag{D.58}$$

where the computational method that was employed was exactly the same as when $\mathcal{A}_{\rm a}^{(u_1)}$ was computed in (D.22).

(c) Both vertices contain interactions $(\tilde{X}_\mu{}^\mu)^2$

Finally consider the bubble diagrams where both vertices contain quartic couplings $(\tilde{X}_\mu{}^\mu)^2$. Begin by considering the s-channel contribution to the amplitude. By the general s-channel amplitude (4.43), the contributions from all nine terms in (4.66) are,

$$\mathcal{A}_{\rm c}^{(s_1)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \bar{\psi}(y)\gamma^\nu\partial_\nu\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma^\rho\partial_\rho\psi(x)\ \bar{\psi}(x)\gamma^\lambda\partial_\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)}$$
$$= -\frac{\lambda^2 m^6}{384\pi\sinh\theta}\left[3\pi\left(11+4\cosh\theta+\cosh 2\theta\right)+i\left(\sinh\theta\left(34-12\frac{\Lambda^2}{m^2}-48\log\frac{m}{\Lambda}\right)\right.\right.$$
$$\left.\left.+2\sinh 2\theta\left(4+3\log\frac{m}{\Lambda}\right)+3\theta\left(11+4\cosh\theta+\cosh 2\theta\right)\right)\right], \tag{D.59}$$

$$\mathcal{A}_c^{(s_2)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\ \bar\psi(y)\gamma^\nu\partial_\nu\psi(y)\right.$$
$$\left.\times\bar\psi(x)\gamma^\rho\partial_\rho\psi(x)\ \partial_\lambda\bar\psi(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)}$$
$$= -\frac{\lambda^2 m^6}{384\pi\sinh\theta}\left[6\pi\left(11+4\cosh\theta+\cosh 2\theta\right)+i\left(\sinh\theta\left(53-18\frac{\Lambda^2}{m^2}-96\log\frac{m}{\Lambda}\right)\right.\right.$$
$$\left.\left.+\sinh 2\theta\left(13+6\log\frac{m}{\Lambda}\right)+3\cosh 2\theta\sinh\theta+6\theta\left(11+4\cosh\theta+\cosh 2\theta\right)\right)\right], \qquad \text{(D.60)}$$

$$\mathcal{A}_c^{(s_3)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\ \bar\psi(y)\gamma^\nu\partial_\nu\psi(y)\right.$$
$$\left.\times\partial_\rho\bar\psi(x)\gamma^\rho\psi(x)\ \partial_\lambda\bar\psi(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)}$$
$$= -\frac{\lambda^2 m^6}{384\pi\sinh\theta}\left[3\pi\left(11+4\cosh\theta+\cosh 2\theta\right)+i\left(\sinh\theta\left(9-48\log\frac{m}{\Lambda}\right)+3\frac{\Lambda^2}{m^2}\sinh 2\theta\right.\right.$$
$$\left.\left.+\sinh 3\theta+3\theta\left(11+4\cosh\theta+\cosh 2\theta\right)\right)\right], \qquad \text{(D.61)}$$

$$\mathcal{A}_c^{(s_4)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\bar\psi(x)\gamma^\rho\partial_\rho\psi(x)\ \bar\psi(x)\gamma^\lambda\partial_\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_c^{(s_2)}, \qquad \text{(D.62)}$$

$$\mathcal{A}_c^{(s_5)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[4\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\bar\psi(x)\gamma^\rho\partial_\rho\psi(x)\ \partial_\lambda\bar\psi(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)}$$
$$= -\frac{\lambda^2 m^6}{19200\pi\sinh\theta}\left[600\pi\left(11+4\cosh\theta+\cosh 2\theta\right)+i\left(\sinh\theta\left(5948-2280\frac{\Lambda^2}{m^2}-9600\log\frac{m}{\Lambda}\right)\right.\right.$$
$$+\sinh 2\theta\left(\frac{2839}{2}-180\frac{\Lambda^2}{m^2}-75\frac{\Lambda^4}{m^4}+600\log\frac{m}{\Lambda}\right)+8\sinh\theta\cosh 2\theta\left(26+15\frac{\Lambda^2}{m^2}\right)$$
$$\left.\left.+17\sinh\theta\cosh 3\theta+600\ \theta\left(11+4\cosh\theta+\cosh 2\theta\right)\right)\right], \qquad \text{(D.63)}$$

$$\mathcal{A}_c^{(s_6)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar\psi(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\partial_\rho\bar\psi(x)\gamma^\rho\psi(x)\ \partial_\lambda\bar\psi(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_c^{(s_2)}, \qquad \text{(D.64)}$$

$$\mathcal{A}_c^{(s_7)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\partial_\mu\bar\psi(y)\gamma^\mu\psi(y)\ \partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\bar\psi(x)\gamma^\rho\partial_\rho\psi(x)\ \bar\psi(x)\gamma^\lambda\partial_\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_c^{(s_3)}, \qquad \text{(D.65)}$$

$$\mathcal{A}_c^{(s_8)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\partial_\mu\bar\psi(y)\gamma^\mu\psi(y)\ \partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\bar\psi(x)\gamma^\rho\partial_\rho\psi(x)\ \partial_\lambda\bar\psi(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_c^{(s_2)}, \qquad \text{(D.66)}$$

$$\mathcal{A}_c^{(s_9)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\partial_\mu\bar\psi(y)\gamma^\mu\psi(y)\ \partial_\nu\bar\psi(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\partial_\rho\bar\psi(x)\gamma^\rho\psi(x)\ \partial_\lambda\bar\psi(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(s)} = \mathcal{A}_c^{(s_1)}, \qquad \text{(D.67)}$$

where the employed computational method was exactly the same as when $\mathcal{A}_{\rm a}^{(s_1)}$ was computed in (D.2).

The contribution to the amplitude from the t-channel can be computed using the general t-channel amplitude (4.44). The nine terms in (4.66) yield the following contributions to the amplitude,

$$\begin{aligned}
\mathcal{A}_{\rm c}^{(t_1)} &= \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \bar{\psi}(y)\gamma^\nu\partial_\nu\psi(y)\right.\\
&\qquad\left.\times\bar{\psi}(x)\gamma^\rho\partial_\rho\psi(x)\ \bar{\psi}(x)\gamma^\lambda\partial_\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)}\\
&= -i\frac{\lambda^2 m^6}{64\pi}\left[1+\frac{\Lambda^2}{m^2}+6\log\frac{m}{\Lambda}-\cosh\theta\left(\frac{\Lambda^2}{m^2}+4\log\frac{m}{\Lambda}\right)\right], \qquad \text{(D.68)}\\
\mathcal{A}_{\rm c}^{(t_2)} &= \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \bar{\psi}(y)\gamma^\nu\partial_\nu\psi(y)\right.\\
&\qquad\left.\times\bar{\psi}(x)\gamma^\rho\partial_\rho\psi(x)\ \partial_\lambda\bar{\psi}(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)}\\
&= -i\frac{\lambda^2 m^6}{64\pi}(2-\cosh\theta)\left[1+\frac{\Lambda^2}{m^2}+6\log\frac{m}{\Lambda}\right], \qquad \text{(D.69)}\\
\mathcal{A}_{\rm c}^{(t_3)} &= \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \bar{\psi}(y)\gamma^\nu\partial_\nu\psi(y)\right.\\
&\qquad\left.\times\partial_\rho\bar{\psi}(x)\gamma^\rho\psi(x)\ \partial_\lambda\bar{\psi}(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)}\\
&= -i\frac{\lambda^2 m^6}{64\pi}\left[1+\frac{\Lambda^2}{m^2}+6\log\frac{m}{\Lambda}-\cosh\theta\left(1+2\log\frac{m}{\Lambda}\right)\right], \qquad \text{(D.70)}\\
\mathcal{A}_{\rm c}^{(t_4)} &= \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.\\
&\qquad\left.\times\bar{\psi}(x)\gamma^\rho\partial_\rho\psi(x)\ \bar{\psi}(x)\gamma^\lambda\partial_\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)} = \mathcal{A}_{\rm c}^{(t_2)}, \qquad \text{(D.71)}\\
\mathcal{A}_{\rm c}^{(t_5)} &= \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[4\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.\\
&\qquad\left.\times\bar{\psi}(x)\gamma^\rho\partial_\rho\psi(x)\ \partial_\lambda\bar{\psi}(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)}\\
&= -i\frac{\lambda^2 m^6}{128\pi}\left[8+6\frac{\Lambda^2}{m^2}-\frac{\Lambda^4}{m^4}+48\log\frac{m}{\Lambda}-2\cosh\theta\left(1+3\frac{\Lambda^2}{m^2}+12\log\frac{m}{\Lambda}\right)\right], \qquad \text{(D.72)}\\
\mathcal{A}_{\rm c}^{(t_6)} &= \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.\\
&\qquad\left.\times\partial_\rho\bar{\psi}(x)\gamma^\rho\psi(x)\ \partial_\lambda\bar{\psi}(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)} = \mathcal{A}_{\rm c}^{(t_2)}, \qquad \text{(D.73)}\\
\mathcal{A}_{\rm c}^{(t_7)} &= \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.\\
&\qquad\left.\times\bar{\psi}(x)\gamma^\rho\partial_\rho\psi(x)\ \bar{\psi}(x)\gamma^\lambda\partial_\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)} = \mathcal{A}_{\rm c}^{(t_3)}, \qquad \text{(D.74)}
\end{aligned}$$

$$\mathcal{A}_{\rm c}^{(t_8)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma^\rho\partial_\rho\psi(x)\ \partial_\lambda\bar{\psi}(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)} \;=\; \mathcal{A}_{\rm c}^{(t_2)}\,, \tag{D.75}$$
$$\mathcal{A}_{\rm c}^{(t_9)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\partial_\rho\bar{\psi}(x)\gamma^\rho\psi(x)\ \partial_\lambda\bar{\psi}(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(t)} \;=\; \mathcal{A}_{\rm c}^{(t_1)}\,, \tag{D.76}$$

where the employed computational method was exactly the same as was used in evaluating $\mathcal{A}_{\rm a}^{(t_1)}$ in (D.12).

The contribution to the amplitude from the u-channel can be computed using the general u-channel amplitude (4.45). The contributions to the amplitude from the nine terms in (4.66) are given by,

$$\mathcal{A}_{\rm c}^{(u_1)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \bar{\psi}(y)\gamma^\nu\partial_\nu\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma^\rho\partial_\rho\psi(x)\ \bar{\psi}(x)\gamma^\lambda\partial_\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}$$
$$= -i\frac{\lambda^2 m^6}{96\pi}\cosh^2\frac{\theta}{2}\left[4 - 3\frac{\Lambda^2}{m^2} - 18\log\frac{m}{\Lambda} - \cosh\theta - 3\theta\coth\frac{\theta}{2}\right], \tag{D.77}$$
$$\mathcal{A}_{\rm c}^{(u_2)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \bar{\psi}(y)\gamma^\nu\partial_\nu\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma^\rho\partial_\rho\psi(x)\ \partial_\lambda\bar{\psi}(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}$$
$$= -i\frac{\lambda^2 m^6}{48\pi}\cosh^2\frac{\theta}{2}\left[5 - 6\frac{\Lambda^2}{m^2} - 24\log\frac{m}{\Lambda} + \cosh\theta - 3\theta\coth\frac{\theta}{2}\right], \tag{D.78}$$
$$\mathcal{A}_{\rm c}^{(u_3)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \bar{\psi}(y)\gamma^\nu\partial_\nu\psi(y)\right.$$
$$\left.\times\partial_\rho\bar{\psi}(x)\gamma^\rho\psi(x)\ \partial_\lambda\bar{\psi}(x)\gamma^\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}$$
$$= -i\frac{\lambda^2 m^6}{4800\pi}\cosh^2\frac{\theta}{2}\left[279 - 270\frac{\Lambda^2}{m^2} + 75\frac{\Lambda^4}{m^4} - 1500\log\frac{m}{\Lambda} + 2\cosh\theta\left(19 + 60\frac{\Lambda^2}{m^2}\right)\right.$$
$$\left. - 17\cosh 2\theta - 150\ \theta\coth\frac{\theta}{2}\right], \tag{D.79}$$
$$\mathcal{A}_{\rm c}^{(u_4)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\left[-2\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)\right.$$
$$\left.\times\bar{\psi}(x)\gamma^\rho\partial_\rho\psi(x)\ \bar{\psi}(x)\gamma^\lambda\partial_\lambda\psi(x)\right] a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}$$
$$= i\frac{\lambda^2 m^6}{16\pi}\cosh^2\frac{\theta}{2}\left[4\log\frac{m}{\Lambda} + \theta\coth\frac{\theta}{2}\right], \tag{D.80}$$

$$\mathcal{A}_{\rm c}^{(u_5)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[4\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)$$

$$\times\bar{\psi}(x)\gamma^\rho\partial_\rho\psi(x)\ \partial_\lambda\bar{\psi}(x)\gamma^\lambda\psi(x)\Big]\,a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}$$

$$= -i\frac{\lambda^2 m^6}{48\pi}\cosh^2\frac{\theta}{2}\left[5-6\frac{\Lambda^2}{m^2}-36\log\frac{m}{\Lambda}+\cosh\theta-6\theta\coth\frac{\theta}{2}\right], \qquad \text{(D.81)}$$

$$\mathcal{A}_{\rm c}^{(u_6)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[-2\bar{\psi}(y)\gamma^\mu\partial_\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)$$

$$\times\partial_\rho\bar{\psi}(x)\gamma^\rho\psi(x)\ \partial_\lambda\bar{\psi}(x)\gamma^\lambda\psi(x)\Big]\,a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)} \;=\; \mathcal{A}_{\rm c}^{(u_2)}, \qquad \text{(D.82)}$$

$$\mathcal{A}_{\rm c}^{(u_7)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)$$

$$\times\bar{\psi}(x)\gamma^\rho\partial_\rho\psi(x)\ \bar{\psi}(x)\gamma^\lambda\partial_\lambda\psi(x)\Big]\,a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)}$$

$$= i\frac{\lambda^2 m^6}{32\pi}\cosh^2\frac{\theta}{2}\left[2\log\frac{m}{\Lambda}+\theta\coth\frac{\theta}{2}\right], \qquad \text{(D.83)}$$

$$\mathcal{A}_{\rm c}^{(u_8)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[-2\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)$$

$$\times\bar{\psi}(x)\gamma^\rho\partial_\rho\psi(x)\ \partial_\lambda\bar{\psi}(x)\gamma^\lambda\psi(x)\Big]\,a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)} \;=\; \mathcal{A}_{\rm c}^{(u_4)}, \qquad \text{(D.84)}$$

$$\mathcal{A}_{\rm c}^{(u_9)} = \frac{1}{2!}\left(-i\frac{\lambda}{2}\right)^2 \frac{1}{16}\langle 0|b(p_4)a(p_3)\mathrm{T}\Big[\partial_\mu\bar{\psi}(y)\gamma^\mu\psi(y)\ \partial_\nu\bar{\psi}(y)\gamma^\nu\psi(y)$$

$$\times\partial_\rho\bar{\psi}(x)\gamma^\rho\psi(x)\ \partial_\lambda\bar{\psi}(x)\gamma^\lambda\psi(x)\Big]\,a^\dagger(p_1)b^\dagger(p_2)|0\rangle^{(u)} \;=\; \mathcal{A}_{\rm c}^{(u_1)}. \qquad \text{(D.85)}$$

where the employed computational method was exactly the same as was used in evaluating $\mathcal{A}_{\rm a}^{(u_1)}$ in (D.22).

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