BRST, Ward identities, gauge dependence, and a functional renormalization group

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Basic properties of gauge theories in the framework of the Faddeev-Popov (FP) method, Batalin-Vilkovisky (BV) formalism, and functional renormalization group (FRG) approach are considered. The FP and BV quantizations are characterized by the Becchi-Rouet-Stora-Tyutin (BRST) symmetry, while the BRST symmetry is broken in the FRG approach. It is shown that the FP method, the BV formalism, and the FRG approach can be provided with the Slavnov-Taylor identity, the Ward identity, and the modified Slavnov-Taylor identity, respectively. It is proven that using the background field method the background gauge invariance of the effective action within the FP and FRG quantization procedures can be achieved in nonlinear gauges. The gauge-dependence problem within the FP, BV, and FRG quantizations is studied. Arguments allowing us to state the existence of principal problems of the FRG in the case of gauge theories are given.

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I. INTRODUCTION

Over the past three decades, there has been increased interest in the nonperturbative approach in quantum field theory known as the functional renormalization group (FRG), which has been proposed in papers [1,2] and can be considered as a version of Wilson renormalization group [3,4]. The FRG approach has gotten further developments [5–11] and numerous applications [12–23]. There are many reviews devoted to detailed discussions of different aspects of the FRG approach, and among them, one can find Refs. [24–31] with qualitative references.

As a quantization procedure, the FRG belongs to covariant quantization schemes. In the case of gauge theories, any covariant quantization faces two principal problems: the unitarity of S matrix first formulated by Feynman [32] and the gauge dependence of results obtained. The study of the unitarity problem requires consideration of canonical formulation of a given theory on the quantum level and use of the Kugo-Ojima method in construction and analysis of physical state space with the help of the nilpotent Becchi-Rouet-Stora-Tyutin (BRST) operator [33] to discovery the criteria providing the unitarity. In the present paper, we will not touch the unitary problem in all covariant quantization approaches to gauge theories, restricting ourselves the gauge-dependence problem.

The gauge dependence is a problem in the quantum description of gauge theories beginning with famous papers by Jackiw [34] and Nielsen [35]. Study of the gauge-dependence problem can be directly performed in covariant quantization schemes, namely, in the Faddeev-Popov (FP) method [36], the Batalin-Vilkovisky (BV) formalism [37,38], and the FRG approach [1,2]. Analysis of the gauge dependence problem for Yang-Mills theories in the framework of the FP-method and for general gauge theories within the BV-formalism has been given in papers [39–41], respectively. Aspects of gauge invariance and related topics were always under close attention in the FRG [8,11,12,16,17,19,42–47]. Nevertheless, it seems a useful and important task to consider the gauge-dependence problem within the FRG approach for different types of gauge theories from general points of view.

We are going to compare with each other basic properties providing the FP method, BV formalism, and the FRG approach and find new features concerning the gauge-dependence problem in the FRG. Among the basic properties, it needs first of all to mention the BRST symmetry [48,49], which is considered a fundamental principle of modern quantum field theory allowing a suitable quantum description of a given dynamical system [50,51]. For the first time, the BRST symmetry was discovered as a global supersymmetry of quantum action (the Faddeev-Popov action) appearing in the process of quantization of Yang-Mills theories. In its turn, the BRST symmetry in the BV formalism is not the global supersymmetry of some action, but it is encoded into the quantum master equation. The role

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of BRST symmetry in the FP method and in the BV formalism is extremely important because it guarantees the gauge independence of the S-matrix elements. The BRST symmetry is broken in the FRG approach, which leads to the ill-defined S matrix [52].

The Ward identities in quantum theory of gauge fields are the next basic property. Their existence is a direct consequence of gauge invariance of the initial classical action underlying a given system with gauge freedom. The BRST transformations help to present the Ward identities in a unique form that sometimes causes incorrect conclusions concerning relations between the BRST symmetry and the Ward identities; namely, the Ward identities by themselves do not mean the existence of the BRST symmetry for a given gauge system. It is exactly the case of the FRG approach when it cannot be provided by the BRST symmetry in the presence of the modified Slavnov-Taylor (mST) identities.

In our investigation, we pay special attention to the gauge-dependence problem within the FP method, the BV formalism, and the FRG approach with or without using the background field method (BFM) [53–55] because of its importance for the physical interpretation of used schemes of quantization. Our interest in the background field method is caused by an important property of gauge invariance of the background effective action under gauge transformations of background fields helping to simplify quantum calculations in the Yang-Mills and gravity theories within the FP method. Unfortunately, this method does not help to improve the situation with the gauge-dependence problem in the FRG because the effective average action being a gauge-invariant functional remains a gauge-dependent object.

The paper is organized as follows. In Sec. II, a brief description of theories invariant under the gauge transformations from the point of view of the structure of corresponding gauge algebras is given. In Sec. III, the BRST symmetry in the context of the FP method, BV formalism, and FRG approach is discussed. In Sec. IV, the Slavnov-Taylor (ST) identity in the FP method, the Ward identity in BV formalism, and the mST identity in the FRG approach are studied. In Sec. V, the gauge-dependence problem is studied within quantization schemes mentioned above. In Sec. VI, the all basic properties of FP method and FRG approach are investigated for the Yang-Mills type of gauge theories within the BFM. Finally, in Sec. VII, the results obtained in the paper are discussed.

We use the DeWitt's condensed notations [56]. We employ the notation $\varepsilon(A)$ for the Grassmann parity of any quantity A. The right and left functional derivatives with respect to fields and antifields are marked by special symbols $\leftarrow$ and $\rightarrow$, respectively. Arguments of any functional are enclosed in square brackets [], and arguments of any function are enclosed in parentheses (). The symbol $F_{,A}[\phi,\ldots]$ means the right derivative of $F[\phi,\ldots]$ with respect to field ϕ^A.

II. GAUGE THEORIES

Let us start from some initial classical action $S_0[A]$ of the fields A^i, with Grassmann parities $\varepsilon(A^i)\equiv\varepsilon_i$, being invariant under the gauge transformations ($X_{,i}\equiv\delta X/\delta A^i$)

$$\delta A^i = R^i_\alpha(A)\xi^\alpha, \qquad S_{0,i}[A]R^i_\alpha(A)=0, \tag{2.1}$$

where ξ^α are arbitrary functions with Grassmann parities $\varepsilon(\xi^\alpha)\equiv\varepsilon_\alpha$, $\alpha=1,2,\ldots,m$, and $R^i_\alpha(A)$, $\varepsilon(R^i_\alpha(A))=\varepsilon_i+\varepsilon_\alpha$ are generators of gauge transformations. It is assumed the set of fields A^i is linear independent (in particular, it is not the case of higher-spin fields [57]). The general form of algebra of generators $R^i_\alpha(A)$ reads

$$\begin{aligned} &R^i_{\alpha,j}(A)R^j_\beta(A)-(-1)^{\varepsilon_\alpha\varepsilon_\beta}R^i_{\beta,j}(A)R^j_\alpha(A)\\ &\quad=-R^i_\gamma(A)F^\gamma_{\alpha\beta}(A)-S_{0,j}[A]M^{ij}_{\alpha\beta}(A), \end{aligned}\tag{2.2}$$

where $F^\gamma_{\alpha\beta}(A)=-(-1)^{\varepsilon_\alpha\varepsilon_\beta}F^\gamma_{\beta\alpha}(A)$ are structure functions depending, in general, on the fields A^i and $M^{ij}_{\alpha\beta}(A)$ satisfies the conditions $M^{ij}_{\alpha\beta}(A)=-(-1)^{\varepsilon_i\varepsilon_j}M^{ji}_{\alpha\beta}(A)=-(-1)^{\varepsilon_\alpha\varepsilon_\beta}M^{ij}_{\beta\alpha}(A)$.

If the structure functions do not depend on fields A^i, $M^{ij}_{\alpha\beta}(A)=0$ and, in addition, the generators $R^i_\alpha(A)$ form a set of linear independent operators with respect to the index α, then we have the case of the *Yang-Mills type of gauge theories* being very important for practical applications because all modern models of fundamental forces are described in terms of such a kind of theories.

For an example, let us consider the case of the pure Yang-Mills theory, defined by the action

$$S_{\rm YM}[A]=-\frac{1}{4}F^a_{\mu\nu}(A)F^a_{\mu\nu}(A), \tag{2.3}$$

where $F^a_{\mu\nu}(A)=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+f^{abc}A^b_\mu A^c_\nu$ is the field strength for the non-Abelian vector field A_μ, taking values in the adjoint representation of a compact semisimple Lie group with structure coefficients f^{abc}. We have the following identifications with previous notations:

$$A^i\mapsto A^a_\mu,\;\; F^\alpha_{\beta\gamma}\mapsto f^{abc},\;\; R^i_\alpha(A)\mapsto D^{ab}_\mu(A)=\delta^{ab}\partial_\mu+f^{acb}A^c_\mu. \tag{2.4}$$

Here, $D^{ab}_\mu(A)$ is the covariant derivative.

For a second example, consider the case of quantum gravity theories, defined by an action $S_0(g)$ of a Riemann metric $g=\{g_{\mu\nu}(x)\}$ with $\varepsilon(g)=0$[1] and which is invariant under general coordinate transformations. The generator of such transformation is linear in $g_{\mu\nu}$ and reads

[1]The standard example is Einstein gravity with a cosmological constant term, $S_0[g]=-\frac{1}{\kappa^2}\int dx\sqrt{-\det g}(R(g)+2\Lambda)$.

$$R_{\mu\nu\sigma}(x,y;g) = -\delta(x-y)\partial_\sigma g_{\mu\nu}(x) - g_{\mu\sigma}(x)\partial_\nu\delta(x-y) - g_{\sigma\nu}(x)\partial_\mu\delta(x-y). \quad (2.5)$$

Therefore, for an arbitrary gauge function ξ^α with $\varepsilon(\xi^\alpha)=0$, one has $\delta g_{\mu\nu} = R_{\mu\nu\sigma}(g)\xi^\sigma$, or, writing all the arguments explicitly,

$$\delta g_{\mu\nu}(x) = \int dy R_{\mu\nu\sigma}(x,y;g)\xi^\sigma(y). \quad (2.6)$$

In this case, the structure functions are given by

$$F^\alpha_{\beta\gamma}(x,y,z) = \delta(x-y)\delta^\alpha_\gamma\partial^{(x)}_\beta\delta(x-z) - \delta(x-z)\delta^\alpha_\beta\partial^{(x)}_\gamma\delta(x-y), \quad (2.7)$$

which satisfies the antisymmetry properties, $F^\alpha_{\beta\gamma}(x,y,z) = -F^\alpha_{\gamma\beta}(x,z,y)$, as usual.

In terms of the notation used, one has the correspondence

$$A^i \mapsto g_{\mu\nu}(x), \quad R^i_\alpha(A) \mapsto R_{\mu\nu\sigma}(x,y;g), \quad F^\alpha_{\beta\gamma} \mapsto F^\alpha_{\beta\gamma}(x,y,z). \quad (2.8)$$

In general, the structure functions may depend on fields A^i, $M^{ij}_{\alpha\beta}(A)$ may not be equal to zero (*open algebras*), and $R^i_\alpha(A)$ may not be linear independent in the index α (*reducible algebras*). In all these cases, we meet the so-called *general gauge theories* [37,38]. For our goals, a detailed description of structure of gauge algebras is not essential, and we omit their further discussions.

All results obtained below within the FP method and the FRG are valid for any Yang-Mills type of gauge theories in any admissible gauge. The same remark is valid for general gauge theories in the BV formalism.

III. BRST SYMMETRY

At present, the BRST symmetry is considered as a fundamental principle in the construction of the consistent quantization procedure for field and string theories [50,51]. In the next three subsections, we are going to discuss a status of the BRST symmetry for the Yang-Mills type of gauge theories within the FP method and the FRG approach and for the general gauge theories within the BV formalism.

A. BRST in FP method

Let $S_0[A]$ be an action of fields A^i which include Yang-Mills fields and, in general, multiplets of spinor and scalar fields. Vacuum functional for the Yang-Mills type of gauge theories is constructed by the Faddeev-Popov rules [36] in the form of functional integral

$$Z = \int D\phi \exp\left\{\frac{i}{\hbar}S_{\rm FP}[\phi]\right\} \quad (3.1)$$

over fields ϕ. In (3.1), $S_{\rm FP}[\phi]$ is the Faddeev-Popov action,

$$S_{\rm FP}[\phi] = S_0[A] + \bar{C}^\alpha(\chi_\alpha(A,B)\overleftarrow{\partial}_{A^i})R^i_\beta(A)C^\beta + B^\alpha\chi_\alpha(A,B), \quad (3.2)$$

where $\chi_\alpha(A,B)$ are functions lifting the degeneracy of the Yang-Mills action, $\phi = \{\phi^A\}$ is the set of all fields

$$\phi^A = (A^i, B^\alpha, C^\alpha, \bar{C}^\alpha), \qquad \varepsilon(\phi^A) = \varepsilon_A, \quad (3.3)$$

with the Faddeev-Popov ghost and antighost fields C^α and $\bar{C}^\alpha$ $[\varepsilon(C^\alpha) = \varepsilon(\bar{C}^\alpha) = 1, {\rm gh}(C^\alpha) = -{\rm gh}(\bar{C}^\alpha) = 1]$, respectively, and the Nakanishi-Lautrup auxiliary fields $B^\alpha[\varepsilon(B^\alpha)=0, {\rm gh}(B^\alpha)=0]$. A standard choice of linear and nondegenerate gauges $\chi_\alpha(A,B)$ reads

$$\chi_\alpha(A,\mathcal{B}) = F_{\alpha i}A^i + \frac{\xi}{2}B_\alpha, \quad (3.4)$$

where $F_{\alpha i}$, being some differential operations, do not depend on fields A^i and ξ is a constant gauge parameter. In what follows, we do not restrict ourselves to the case (3.4) and consider the gauge-fixing functions in general settings.

The action (3.2) is invariant under global supersymmetry (BRST symmetry) [48,49][2]

$$\delta_B A^i = R^i_\alpha(A)C^\alpha\mu, \qquad \delta_B C^\alpha = -\frac{1}{2}(-1)^{\varepsilon_\beta}F^\alpha_{\beta\gamma}C^\gamma C^\beta\mu,$$
$$\delta_B\bar{C}^\alpha = B^\alpha(-1)^{\varepsilon_\alpha}\mu, \qquad \delta_B B^\alpha = 0, \quad (3.5)$$

where μ is a constant anticommuting parameter or, in short,

$$\delta_B\phi^A = R^A(\phi)\mu, \qquad \varepsilon(R^A(\phi)) = \varepsilon_A + 1, \quad (3.6)$$

where

$$R^A(\phi) = \left(R^i_\alpha(A)C^\alpha, \quad 0, \quad -\frac{1}{2}(-1)^{\varepsilon_\beta}F^\alpha_{\beta\gamma}C^\gamma C^\beta, B^\alpha(-1)^{\varepsilon_\alpha}\right). \quad (3.7)$$

Introducing the gauge-fixing functional $\Psi = \Psi[\phi]$,

$$\Psi = \bar{C}^\alpha\chi_\alpha(A,B), \quad (3.8)$$

the action (2.7) is rewritten in the form

[2]For more compact presentation, we use the notation δ_B for $\delta_{\rm BRST}$.

$$S_{\rm FP}[\phi] = S_0[A] + \Psi[\phi]\hat{R}(\phi) = S_0[A] + \Psi[\phi]_{,A}R^A(\phi),$$
$$S_0[A]\hat{R}(\phi) = 0, \qquad (3.9)$$

where

$$\hat{R}(\phi) = \overleftarrow{\partial}_{\phi^A} R^A(\phi) \qquad (3.10)$$

is the generator of BRST transformations. Because of the nilpotency property of $\hat{R}$, $\hat{R}^2 = 0$, the BRST symmetry of $S_{\rm FP}$ follows from the presentation (3.9) immediately,

$$S_{\rm FP}[\phi]\hat{R}(\phi) = 0. \qquad (3.11)$$

The BRST symmetry of $S_{\rm FP}$ leads to a very important property of the vacuum functional (3.1), namely, its gauge independence. Indeed, let Z_ψ be vacuum functional corresponding to choice of gauge-fixing functional Ψ. Consider the vacuum functional for another choice of gauge condition $\Psi + \delta\Psi$, $Z_{\psi+\delta\Psi}$. Then, we have

$$Z_{\Psi+\delta\Psi} = \int D\phi \exp\left\{\frac{i}{\hbar}(S_{\rm FP}[\phi] + \delta\Psi[\phi]\hat{R}(\phi))\right\}. \qquad (3.12)$$

Making use of change of integration variables in the functional integral (3.12) in the form of the BRST transformations (3.6) but with parameter μ being an functional $\mu = \mu[\phi]$ with

$$\mu[\phi] = \frac{i}{\hbar}\delta\Psi[\phi] \qquad (3.13)$$

and taking into account that the Jacobian of the transformations is equal to

$$J = \exp\{-\mu[\phi]\hat{R}(\phi)\}, \qquad (3.14)$$

we obtain

$$Z_{\Psi+\delta\Psi} = Z_\Psi. \qquad (3.15)$$

In deriving (3.14), the relations

$$(-1)^{\varepsilon_i}\vec{\partial}_{A^i}R^i_\alpha(A) + (-1)^{\varepsilon_\beta+1}F^\beta_{\beta\alpha} = 0 \qquad (3.16)$$

were used. In Yang-Mills theories, for instance, the relations (3.16) are satisfied due to antisymmetry properties of the structure constants. The BRST transformations (3.5) obey the property of nilpotency, $\delta_B^2\phi^A = 0$. In terms of $R^A(\phi)$, this property means equalities

$$R^A_{,B}(\phi)R^B(\phi) = 0. \qquad (3.17)$$

In turn, the relations (3.16) are equivalent to

$$R^A_{,A}(\phi) = 0. \qquad (3.18)$$

We assume the validity of (3.17) and (3.18) in the case of any Yang-Mills type of gauge theories.

From (3.15), we conclude the gauge independence of the vacuum functional. It was the reason for us to drop subscript Ψ in the vacuum functional (3.1). The gauge independence of Z is closely related with the BRST symmetry of $S_{\rm FP}[\phi]$ and leads to the gauge independence of S-matrix elements due to the equivalence theorem [58].

B. BRST in BV formalism

Let $S_0[A]$ be an initial classical action belonging to the set of general gauge theories described in Sec. II. Quantization of this gauge theory can be performed in the BV formalism [37,38]. The vacuum functional can be presented in the form of functional integral

$$Z = \int D\phi D\phi^* d\lambda \exp\left\{\frac{i}{\hbar}(S[\phi,\phi^*] + (\phi^*_A - \Psi[\phi]\overleftarrow{\partial}_{\phi^A})\lambda^A)\right\}, \qquad (3.19)$$

where $S = S[\phi,\phi^*]$ is an action satisfying the quantum master equation

$$\frac{1}{2}(S,S) = i\hbar\Delta S \qquad (3.20)$$

and the boundary condition

$$S|_{\phi^*=\hbar=0} = S_0[A]. \qquad (3.21)$$

The total configuration space $\phi = \{\phi^A\}, \varepsilon(\phi^A) = \varepsilon_A$ is introduced. For irreducible theories, the set of fields ϕ^A coincides with (3.3). For reducible theories, the set of fields ϕ^A has more complicated structure [38] and contains main chains of the ghost, antighost, and auxiliary Nakanishi-Lautrup fields as well as pyramids of the ghosts for ghosts and auxiliary fields. For our goals here, the explicit structure of ϕ^A is not important; only its existence sufficient. To each field ϕ^A of the total configuration space, one introduces the corresponding antifield ϕ^*_A. The statistics of ϕ^*_A is opposite to the statistics of the corresponding fields ϕ^A, $\varepsilon(\phi^*_A) = \varepsilon_A + 1$. In the left-hand side of (3.20) on the space of the fields ϕ^A and antifields ϕ^*_A, the notation of antibracket

$$(F,G) = F(\overleftarrow{\partial}_{\phi^A}\vec{\partial}_{\phi^*_A} - \overleftarrow{\partial}_{\phi^*_A}\vec{\partial}_{\phi^A})G \qquad (3.22)$$

is used. In the right-hand side of (3.20), Δ means the second-order functional differential operator

$$\Delta = (-1)^{\varepsilon_A}\vec{\partial}_{\phi^A}\vec{\partial}_{\phi^*_A}, \qquad \varepsilon(\Delta) = 1, \qquad (3.23)$$

which obeys the nilpotency property

$$\Delta^2 = 0. \tag{3.24}$$

Additionally, in (3.19), the auxiliary fields $\lambda^A, \varepsilon(\lambda^A) = \varepsilon_A + 1$ are introduced. Finally, in (3.19), $\Psi = \Psi[\phi]$ is suitable odd gauge-fixing functional.

Note, first of all, that the integrand in (3.19) is invariant under the following global supertransformations:

$$\delta_B\phi^A = \lambda^A\mu, \quad \delta_B\phi^*_A = \mu(S[\phi,\phi^*]\overleftarrow{\partial}_{\phi^A}), \quad \delta_B\lambda^A = 0. \tag{3.25}$$

These transformations represent the BRST transformations in the space of variables ϕ, ϕ^*, λ. In the case of general gauge theories, the BRST symmetry is not the symmetry of some action in contrast with the FP method, but as in the case of the Yang-Mills type of gauge theories, they do not depend on the choice of the gauge-fixing condition. It is very important to realize that the existence of this symmetry is the consequence of the fact that the bosonic functional S satisfies the quantum master equation (3.21).

The role of this symmetry is the same as in the case of the Yang-Mills type of gauge theories; namely, it is responsible for the gauge independence of vacuum functional (3.19). Indeed, suppose $Z_\Psi \equiv Z$. We shall change infinitesimally the gauge $\Psi \to \Psi + \delta\Psi$. In the functional integral for $Z_{\Psi+\delta\Psi}$,

$$Z_{\Psi+\delta\Psi} = \int D\phi D\phi^* d\lambda \exp\Big\{\frac{i}{\hbar}(S[\phi,\phi^*] + (\phi^*_A - \Psi[\phi]\overleftarrow{\partial}_{\phi^A})\lambda^A - \delta\Psi[\phi]\overleftarrow{\partial}_{\phi^A}\lambda^A)\Big\}, \tag{3.26}$$

we make the change of variables in the form of (3.25) but with $\mu = \mu[\phi]$ being a functional of ϕ. The Jacobian of the transformations in lower order of $\mu[\phi]$ reads

$$J = \exp\{-\mu[\phi]\overleftarrow{\partial}_{\phi^A}\lambda^A + \mu[\phi]\Delta S[\phi,\phi^*]\}. \tag{3.27}$$

Then, we have

$$Z_{\Psi+\delta\Psi} = \int D\phi D\phi^* d\lambda J \exp\Big\{\frac{i}{\hbar}\Big(S[\phi,\phi^*] + (\phi^*_A - \Psi[\phi]\overleftarrow{\partial}_{\phi^A})\lambda^A - \delta\Psi[\phi]\overleftarrow{\partial}_{\phi^A}\lambda^A + \mu[\phi]\frac{1}{2}(S,S)\Big)\Big\}. \tag{3.28}$$

Choosing the functional $\mu[\phi]$ in the form

$$\mu[\phi] = -\frac{i}{\hbar}\delta\Psi[\phi] \tag{3.29}$$

and taking into account that $S[\phi,\phi^*]$ satisfies the quantum master equation (3.20), we obtain

$$Z_{\Psi+\delta\Psi} = Z_\Psi. \tag{3.30}$$

In turn, the gauge independence of vacuum functional (3.30) leads to the statement about the gauge independence of the S matrix due to the equivalence theorem [58]. Let us stress once more that the gauge independence of the vacuum functional (and S matrix) is a direct consequence of the BRST symmetry.

C. BRST in FRG

The recent development of quantum field theory is greatly related with attempts to study nonperturbative aspects of gauge theories. The request for such a nonperturbative treatment is related to nonperturbative nature of low-energy QCD and also an expectation to achieve a consistent theory of quantum gravity. One of the most promising approaches is related to different versions of the Wilson renormalization group approach [3,4]. The qualitative idea of this work can be formulated as follows: regardless, we do not know how to sum up the perturbative series; in some sense, there is a good qualitative understanding of the final output of such a summation for the propagator of the quantum field. A regularized propagator is supposed to have a singe pole and also provide some smooth behavior in the infrared (ir) region. It is possible to write a cutoff-dependent propagator which satisfies these requirements. Then, the cutoff dependence of the vertices can be established from the general scale dependence of the theory, which can be established by means of the functional methods. A compact and elegant formulation of the nonperturbative renormalization group has been proposed in Refs. [1,2] in terms of effective average action. The method was called the FRG approach for the effective average action; it is nowadays one of the most popular and developed methods, which can be seen from the review papers on the FRG approach [24–30].

The starting point of the FRG is the action

$$S_{Wk}[\phi] = S_{\rm FP}[\phi] + S_k[\phi], \tag{3.31}$$

where regulator action $S_k[\phi]$ is constructed by the rule

$$S_k[\phi] = \frac{1}{2}A^iR^{(1)}_{k|ij}A^j + \bar{C}^\alpha R^{(2)}_{k|\alpha\beta}C^\beta, \qquad R^{(1)}_{k|ij} = R^{(1)}_{k|ji}(-1)^{\varepsilon_i\varepsilon_j}. \tag{3.32}$$

In turn, regulator functions $R^{(1)}_{k|ij}$ and $R^{(2)}_{k|\alpha\beta}$ obey the properties

$$\lim_{k\to 0} R^{(1)}_{k|ij} = 0, \qquad \lim_{k\to 0} R^{(2)}_{k|\alpha\beta} = 0 \qquad \varepsilon(R^{(1)}_{k|ij}) = \varepsilon_i + \varepsilon_j,$$
$$\varepsilon(R^{(2)}_{k|\alpha\beta}) = \varepsilon_\alpha + \varepsilon_\beta. \tag{3.33}$$

It means that at vanishing regulators the action S_{Wk} coincides with the FP action,

$$\lim_{k\to 0} S_{Wk}[\phi] = S_{\rm FP}[\phi]. \tag{3.34}$$

The vacuum functional in the FRG approach is defined with the help of action $S_{Wk}[\phi]$ in the form of functional integral

$$Z_k = \int D\phi \exp\left\{\frac{i}{\hbar} S_{Wk}[\phi]\right\}. \tag{3.35}$$

By construction, the following relation exists,

$$\lim_{k\to 0} Z_k[\phi] = Z, \tag{3.36}$$

where Z is the well-defined vacuum functional in the FP method for any Yang-Mills type of gauge theories. The action $S_{Wk}[\phi]$ is not invariant under the BRST transformations,

$$\delta_B S_{Wk}[\phi] = \delta_B S_k[\phi] \neq 0, \tag{3.37}$$

where

$$\delta_B S_k[\phi] = (A^i R^{(1)}_{k|ij} R^j_\alpha(A) C^\alpha - B^\alpha R^{(2)}_{k|\alpha\beta} C^\beta$$
$$- \frac{1}{2} \bar{C}^\alpha R^{(2)}_{k|\alpha\beta} F^\beta_{\gamma\sigma} C^\sigma C^\gamma (-1)^{\varepsilon_\gamma})\mu. \tag{3.38}$$

Violation of the BRST symmetry leads to the gauge-dependence problem at least when $k \neq 0$. Indeed, let $Z_k = Z_{k|\Psi}$ be vacuum functional (3.35) corresponding to a choice of gauge fixing $\Psi = \Psi[\phi]$. Consider the vacuum functional when the gauge condition is described by functional $\Psi + \delta\Psi$,

$$Z_{k|\Psi+\delta\Psi} = \int D\phi \exp\left\{\frac{i}{\hbar}(S_{Wk}[\phi] + \delta\Psi[\phi]\hat{R}(\phi))\right\}. \tag{3.39}$$

Making use of the change of integration variables in the form of BRST transformation with $\mu[\phi]$ being as in (3.13), we obtain

$$Z_{k|\Psi+\delta\Psi} = \int D\phi \exp\left\{\frac{i}{\hbar}(S_{Wk}[\phi] + \delta_B S_k[\phi])\right\}. \tag{3.40}$$

We cannot propose a change of integration variables in (3.40) to reduce it to $Z_{k|\Psi}$ (see, for example, recent efforts to find a solution of the problem in gravity theories [59]). So,

$$Z_{k|\Psi+\delta\Psi} \neq Z_{k|\Psi}. \tag{3.41}$$

Therefore, in any case, the gauge-dependence problem exists within the FRG at the level when $k \neq 0$, and the corresponding S matrix does depend on gauges. Violation of the BRST symmetry entails an additional problem associated with unitarity since the usual solution assumes the existence of a nilpotent BRST charge [33]. Later on, we will return to discussion of this problem when studying the gauge dependence of the effective average action.

IV. WARD IDENTITIES

Quantization of gauge theories leads to very important understanding concerning the existence of relations between some Green's functions. These relations in the case of Yang-Mills theories are known as the Slavnov-Taylor identities [60,61]; for general gauge theories, they are named as the Ward identities in honor of John Ward who first discovered an identity in quantum electrodynamics providing the gradient invariance of the S-matrix elements [62]. In the FRG approach, the relations are refereed as the modified Slavnov-Taylor identities [8]. Notice that the ST identities are direct consequences of the gauge invariance of the Yang-Mills action, and they were introduced before the discovery of the BRST symmetry. In turn, the BRST symmetry helps to present the ST identities in a unique and compact form (see, for example, Ref. [52], in which this issue is presented and discussed in details). The latter circumstance is often the cause of misconception regarding the role of BRST symmetry in the existence of ST identities. Our interest in this issue is caused by the widespread opinion among the FRG community that these identities solve the problem of gauge dependence. Our point of view is completely different from this opinion. These identities are direct consequences of the gauge invariance of the initial classical action on the quantum level providing a correct solution to the renormalization procedure. Possible misunderstandings are caused by the fact that these identities can be represented in a universal form using the BRST transformations. But one must keep in mind that only in the case when the BRST transformations are transformations of global supersymmetry of a given gauge system the gauge independence of the S matrix can be confirmed. In particular, in the case of FRG approach, the mST identities do not guarantee the BRST symmetry.

A. ST identities in FP method

We begin our discussion of the ST identities appearing as a direct consequence of gauge invariance of initial classical action $S_0[A]$. For all practical goals of quantum calculations in the case of Yang-Mills type of gauge theories, it is sufficient to introduce the generating functional of Green's functions

$$Z[j] = \int D\phi \exp\left\{\frac{i}{\hbar}(S_{\rm FP}[\phi] + jA)\right\}, \tag{4.1}$$

where j_i, $\varepsilon(j_i) = \varepsilon_i$ are external sources to fields A^i. Thanks to the gauge invariance of the action $S_0[A]$ (2.1), the Green's functions of the theory obey the relations known as the ST identities [60,61]. These identities can be derived from (4.1) by means of the change of integration variables A^i, in the form of infinitesimal gauge transformations (2.1). The Jacobian of these transformations is equal to unity. Then, the basic ST identities for Yang-Mills fields can be written in the form

$$\begin{aligned} j_i\langle R^i_\alpha(A)\rangle_j &+ \langle B^\beta(\chi_\alpha(A,B)\overleftarrow{\partial}_{A^i})R^i_\alpha(A)\rangle_j \\ &+ \langle \bar{C}^\beta(\chi_\beta(A,B)\overleftarrow{\partial}_{A^i})R^i_{\gamma,k}(A)R^k_\alpha(A)C^\gamma\rangle_j(-1)^{\varepsilon_\alpha(\varepsilon_\gamma+1)} \\ &- \langle \bar{C}^\beta(\chi_\beta(A,B)\overleftarrow{\partial}_{A^i}\overleftarrow{\partial}_{A^k})R^k_\gamma(A)C^\gamma R^i_\gamma(A)\rangle_j(-1)^{\varepsilon_i+\varepsilon_j} \equiv 0, \end{aligned} \tag{4.2}$$

where the symbol $\langle G(\phi)\rangle_j$ means the vacuum expectation value of the quantity $G(\phi)$ in the presence of external sources j^a_μ,

$$\langle G(\phi)\rangle_j = \int D\phi G(\phi)\exp\left\{\frac{i}{\hbar}[S_{\mathrm{FP}}[\phi] + jA]\right\}. \tag{4.3}$$

The generating functional $Z[j]$ contains information about all Green's functions of the theory, which can be obtained by taking variational derivatives with respect to the sources. Similarly, the ST identities represent an infinite set of relations obtained from (4.2) by taking derivatives with respect to external sources j^a_μ. In the case of the linear gauge condition, the last summand in (4.2) disappears.

The form of the ST identities can be greatly simplified by introducing extra sources to the ghost, antighost, and auxiliary fields. In this case, one has to deal with the extended generating functional of the theory

$$Z[J] = \int D\phi\exp\left\{\frac{i}{\hbar}[S_{\mathrm{FP}}[\phi] + J\phi]\right\}. \tag{4.4}$$

The generating functional of connected Green's functions, $W[J]$, is defined by the relation

$$Z[J] = \exp\left\{\frac{i}{\hbar}W[J]\right\}. \tag{4.5}$$

Finally, the generating functional of the vertex Green's functions (effective action) is defined through the Legendre transformation of $W[J]$,

$$\Gamma[\Phi] = W[J] - J\Phi, \tag{4.6}$$

where the source fields J_A are solutions of the equations

$$\Phi^A = \overrightarrow{\partial}_{J_A}W[J]. \tag{4.7}$$

By means of (4.6) and (4.7), one can easily arrive at the relations

$$\Gamma[\Phi]\overleftarrow{\partial}_{\Phi^A} = -J_A. \tag{4.8}$$

The ST identities which are consequences of gauge symmetry of initial action can be rewritten with the help of the BRST symmetry of the Faddeev-Popov action. For this end, we make use of the change of variables in the functional integral (4.4) of the form (3.6). Because of the property (3.16) and nilpotency of μ, the Jacobian of this transformation is equal to 1. Using the invariance of the functional integral under change of integration variables, the following identity holds:

$$\int D\phi J\delta_B\phi\exp\left\{\frac{i}{\hbar}(S_{\mathrm{FP}}[\phi] + J\phi)\right\} \equiv 0. \tag{4.9}$$

Here, the nilpotency of BRST transformation and the consequent exact relation

$$\exp\left\{\frac{i}{\hbar}J\delta_B\phi\right\} = 1 + \frac{i}{\hbar}J\delta_B\phi \tag{4.10}$$

have been used.

From (4.5) and (4.8), it follows that

$$J_A R^A(-i\hbar\overrightarrow{\partial}_J)Z[J] \equiv 0, \quad J_A R^A(-i\hbar\overrightarrow{\partial}_J)W[J] \equiv 0, \tag{4.11}$$

which are the ST identities in a closed form for the functionals $Z[J]$ and $W[J]$. These identities, like those in (4.2), contain explicit information about gauge theory through generators of the BRST transformations. There exists a possibility to present the ST identities in a unique form with the introduction of a set of external sources (known as antifields in the BV formalism) $\Phi^*_A, \varepsilon(\Phi^*_A) = \varepsilon_A + 1$ to the BRST transformations and the extended generating functional of Green's functions

$$\begin{aligned} Z[J,\Phi^*] &= \int D\phi\exp\left\{\frac{i}{\hbar}[S_{\mathrm{FP}}[\phi] + J\phi + \Phi^*_A R^A(\phi)]\right\} \\ &= \exp\left\{\frac{i}{\hbar}W[J,\Phi^*]\right\}, \end{aligned} \tag{4.12}$$

where we used the notation for BRST transformations, $R^A(\phi)$, which was previously introduced in (3.6). It is clear that

$$Z[J,\Phi^*]|_{\Phi^*=0} = Z[J]. \tag{4.13}$$

Now, we can present the ST identities (4.11) in the following form:

$$J_A\overrightarrow{\partial}_{\Phi^*_A}Z[J,\Phi^*] \equiv 0, \qquad J_A\overrightarrow{\partial}_{\Phi^*_A}W[J,\Phi^*] \equiv 0. \tag{4.14}$$

In terms of the extended effective action, $\Gamma = \Gamma[\Phi, \Phi^*]$,

$$\Gamma[\Phi,\Phi^*] = W[J,\Phi^*] - J\Phi, \quad \Phi^A = \vec{\partial}_{J_A} W[J,\Phi^*],$$
$$\Gamma[\Phi,\Phi^*]\overleftarrow{\partial}_{\Phi^A} = -J_A, \tag{4.15}$$

the identities (4.14) is rewritten as

$$\Gamma\overleftarrow{\partial}_{\Phi^A}\vec{\partial}_{\Phi^*_A}\Gamma \equiv 0 \tag{4.16}$$

in the form of a nonlinear equation with respect to Γ (in the form of the Zinn-Justin equation [63]).

B. Ward identities in BV formalism

Now, we shall proceed with the derivation of the Ward identity for general gauge theories within the BV formalism. It is very useful from the beginning to work with the extended generating functional of Green's functions

$$Z[J,\phi^*] = \int D\phi \exp\left\{\frac{i}{\hbar}(S_{\rm ext}[\phi,\phi^*] + J_A\phi^A)\right\}$$
$$= \exp\left\{\frac{i}{\hbar}W[J,\phi^*]\right\}, \tag{4.17}$$

where $W[J, \phi^*]$ is the generating functional for connected Green's functions,

$$S_{\rm ext}[\phi,\phi^*] = S[\phi,\phi^* + \Psi[\phi]\overleftarrow{\partial}_\phi], \tag{4.18}$$

and functional $S[\phi, \phi^*]$ satisfies the quantum master equation (3.19) and the boundary condition (3.20). The gauge-fixing procedure (4.17) used in the BV formalism [37,38] can be described in terms of anticanonical transformation,

$$\phi'^A = \vec{\partial}_{\phi^{*\prime}_A} F[\phi,\phi^{*\prime}], \qquad \phi^*_A = F[\phi,\phi^{*\prime}]\overleftarrow{\partial}_{\phi^A}, \tag{4.19}$$

of a special form corresponding to the choice of generating functional $F[\phi, \phi^{*\prime}]$ in the form

$$F[\phi,\phi^{*\prime}] = \phi^{*\prime}_A\phi^A + \Psi[\phi], \qquad \varepsilon(\Psi) = 1, \tag{4.20}$$

as it was proposed for the first time in Ref. [41].

Notice that the action $S_{\rm ext}[\phi, \phi^*]$ satisfies the quantum master equation (3.19) as well. Indeed, the equality holds,[3]

$$\exp\left\{\frac{i}{\hbar}S_{\rm ext}[\phi,\phi^*]\right\} = \exp\{[\Psi,\Delta]\}\exp\left\{\frac{i}{\hbar}S[\phi,\phi^*]\right\}, \tag{4.21}$$

because

$$[\Psi,\Delta] = \Psi\overleftarrow{\partial}_{\phi^A}\vec{\partial}_{\phi^*_A}, \tag{4.22}$$

[3]For any two quantities F and H, the supercommutator is defined as $[F, H] = FH - HF(-1)^{\varepsilon(F)\varepsilon(H)}$.

and the operator $\exp\{[\Psi, \Delta]\}$ acts as the translation operator with respect to ϕ^*_A. Note that

$$[\Delta,[\Psi,\Delta]] = 0, \tag{4.23}$$

and therefore

$$\Delta\exp\left\{\frac{i}{\hbar}S_{\rm ext}\right\} = 0 \to \frac{1}{2}(S_{\rm ext},S_{\rm ext}) = i\hbar\Delta S_{\rm ext}. \tag{4.24}$$

Taking into account the equation (4.24), the explicit form of the operator Δ (3.23) and independence of operator $\vec{\partial}_{\phi^*_A}$ on the integration variables in functional integral we have the evident relation

$$0 = \int D\phi\exp\left\{\frac{i}{\hbar}J_A\phi^A\right\}\Delta\exp\left\{\frac{i}{\hbar}S_{\rm ext}[\phi,\phi^*]\right\}$$
$$= (-1)^{\varepsilon_A}\vec{\partial}_{\phi^*_A}\int D\phi\exp\left\{\frac{i}{\hbar}J_A\phi^A\right\}\vec{\partial}_{\phi^A}\exp\left\{\frac{i}{\hbar}S_{\rm ext}[\phi,\phi^*]\right\}. \tag{4.25}$$

Integrating by parts in the last integral, one finds that the theory in question satisfies the equality

$$J_A\vec{\partial}_{\phi^*_A}Z[J,\phi^*] = 0. \tag{4.26}$$

This is the Ward identity written for the extended generating functional of Green's functions. For the generating functional of connected Green's functions $W[J, \phi^*]$, the identity (4.26) is rewritten in the form

$$J_A\vec{\partial}_{\phi^*_A}W[J,\phi^*] = 0. \tag{4.27}$$

Introducing the generating functional of the vertex functions $\Gamma = \Gamma[\Phi, \Phi^*]$ (for uniformity of notations, we use $\phi^*_A = \Phi^*_A$) in a standard manner, through the Legendre transformation of $W[J, \Phi^*]$,

$$\Gamma[\Phi,\Phi^*] = W[J,\Phi^*] - J_A\Phi^A, \qquad \Phi^A = \vec{\partial}_{J_A}W[J,\Phi^*],$$
$$\Gamma[\Phi,\Phi^*]\overleftarrow{\partial}_{\phi^A} = -J_A. \tag{4.28}$$

the Ward identity (4.27) for $\Gamma = \Gamma[\Phi, \Phi^*]$ takes the form of the classical master equation,

$$(\Gamma,\Gamma) = 0. \tag{4.29}$$

The form (4.29) coincides with (4.16). The Ward identity (4.29) plays a crucial role in proving the gauge-invariant renormalizability of general gauge theories [41].

C. Modified Slavnov-Taylor identities in FRG

Although the BRST symmetry is broken in the FRG approach, nevertheless, certain relations between the

Green's functions known as the mST identities exist. It confirms that the existence of these relations is not related with the BRST symmetry but the main reason is gauge invariance of an initial classical action.

To discuss the mST identities, it is useful as in previous cases to introduce the average generating functional of Green's functions $Z_k = Z_k[J,\Phi^*]$ and the average generating functional of connected Green's functions $W_k = W_k[J,\Phi^*]$ in the FRG approach,

$$Z_k[J,\Phi^*] = \int D\phi \exp\left\{\frac{i}{\hbar}(S_0[A] + S_k[\phi] + \Psi[\phi]\hat{R}(\phi) + J_A\phi^A + \Phi^*_A R^A(\phi))\right\} = \exp\left\{\frac{i}{\hbar}W_k[J,\Phi^*]\right\}. \tag{4.30}$$

Making use of the change of integration variables in the sector of fields A^i in the form of gauge transformations

$$\delta A^i = R^i_\alpha(A)C^\alpha\mu = R^i(\phi)\mu, \tag{4.31}$$

taking into account the invariance of $S_0[A]$ under transformations (4.31) and the Jacobian of these transformations

$$J = 1 + (-1)^{\varepsilon_i}\vec{\partial}_{A^i}R^i_\alpha(A)C^\alpha\mu, \tag{4.32}$$

we arrive at the identity

$$\begin{aligned}(J_j\vec{\partial}_{\Phi^*_j} + S_{k,j}[-i\hbar\vec{\partial}_J]\vec{\partial}_{\Phi^*_j} + (-1)^{\varepsilon_j(\varepsilon_\alpha+1)}R^j_{\alpha,j}(-i\hbar\vec{\partial}_J)\vec{\partial}_{\bar{J}_\alpha} + \Phi^*_A R^A_{,j}(-i\hbar\vec{\partial}_J)\vec{\partial}_{\Phi^*_j}\\ + \Psi_{,A}[-i\hbar\vec{\partial}_J]R^A_{,i}(-i\hbar\vec{\partial}_J)\vec{\partial}_{\Phi^*_j} + (-1)^{\varepsilon_j}\Psi_{,jA}[-i\hbar\vec{\partial}_J]\vec{\partial}_{\Phi^*_A}\vec{\partial}_{\Phi^*_j})Z_k[J,\Phi^*] \equiv 0,\end{aligned} \tag{4.33}$$

which is nothing but the mST identity in the FRG approach and a direct consequence of gauge invariance of initial classical action $S_0[A]$ at the quantum level. Note that the mST identity in the case of pure Yang-Mills theory formulated in linear nonsingular Lorenz invariant gauges for the FRG approach was derived in Ref. [8].

One can present the mST identity (4.33) in a more compact form using additional information about invariance properties of quantities entering the exponent of the integrand (4.30). Consider the change of variables C^α, $\bar{C}^\alpha$,

$$\delta C^\alpha = -\frac{1}{2}(-1)^{\varepsilon_\beta}F^\alpha_{\beta\gamma}C^\gamma C^\beta\mu, \qquad \delta\bar{C}^\alpha = \mu B^\alpha \tag{4.34}$$

in the functional integral entering the identity (4.33). Then, the result

$$(J_A\vec{\partial}_{\Phi^*_A} + S_{k,A}[-i\hbar\vec{\partial}_J]\vec{\partial}_{\Phi^*_A})Z_k[J,\Phi^*] \equiv 0 \tag{4.35}$$

coincides with that obtained by making use the change of variables ϕ^A in the form of the BRST transformations, $\delta\phi^A = R^A(\phi)\mu$ in the functional (4.30). In terms of the average generating functional of connected Green's functions, $W_k = W_k[J,\Phi^*]$, the mST identity (4.35) is rewritten as

$$(J_A\vec{\partial}_{\Phi^*_A} + S_{k,A}[(\vec{\partial}_J W_k) - i\hbar\vec{\partial}_J]\vec{\partial}_{\Phi^*_A})W_k[J,\Phi^*] \equiv 0. \tag{4.36}$$

The effective average action, $\Gamma_k = \Gamma_k[\Phi,\Phi^*]$, is defined through the Legendre transformation of W_k,

$$\Gamma_k[\Phi,\Phi^*] = W_k[J,\Phi^*] - J\Phi, \quad \Phi^A = \vec{\partial}_{J_A}W_k[J,\Phi^*], \quad \Gamma_k[\Phi,\Phi^*]\overleftarrow{\partial}_{\Phi^A} = -J_A. \tag{4.37}$$

Then, the mST identity (4.36) can be presented in terms of Γ_k as

$$\Gamma_k\overleftarrow{\partial}_{\Phi^A}\vec{\partial}_{\Phi^*_A}\Gamma_k - S_{k,A}[\hat{\Phi}]\vec{\partial}_{\Phi^*_A}\Gamma_k \equiv 0, \tag{4.38}$$

or, using the antibracket,

$$\frac{1}{2}(\Gamma_k,\Gamma_k) - S_{k,A}[\hat{\Phi}]\vec{\partial}_{\Phi^*_A}\Gamma_k \equiv 0, \tag{4.39}$$

where the notations

$$\hat{\Phi}^A = \Phi^A + i\hbar(\Gamma''^{-1}_k)^{AB}\vec{\partial}_{\Phi^B}, \quad (\Gamma''_k)_{AB} = \vec{\partial}_{\Phi^A}\Gamma_k\overleftarrow{\partial}_{\Phi^B}, \quad (\Gamma''^{-1}_k)^{AC}\cdot(\Gamma''_k)_{CB} = \delta^A_B \tag{4.40}$$

are used. In the limit $k \to 0$, the mST identity (4.39) reduces to (4.29).

V. GAUGE DEPENDENCE

The gauge dependence is a problem in quantum description of gauge theories. Any covariant quantization scheme (FP method [36], BV formalism [37,38], FRG approach [1,2], and Gribov-Zwanziger theory [64–66]) for gauge theories meets with the gauge-dependence problem. Here, we remember the main aspects and solutions of the gauge-dependence problem in the FP method and the BV formalism. We obtain new results concerning the gauge-dependence problem of the effective average action precisely on the level of the flow equation.

A. Gauge dependence in FP method

It is well known that the Green's functions in gauge theories depend on the choice of gauge [17,34,35,39,40,53,67–71]. From the gauge independence of the S matrix [see Eq. (3.15)], it follows that the gauge dependence of the Green's functions in gauge theories must be of a special character. To study the character of this dependence, let us consider an infinitesimal variation of gauge-fixing functional $\Psi[\phi] \to \Psi[\phi] + \delta\Psi[\phi]$ in the functional integral (3.12). Then, we obtain

$$\delta Z[J,\Phi^*] = \frac{i}{\hbar}\int D\phi\,\delta\Psi_{,A}[\phi]R^A(\phi) \times \exp\left\{\frac{i}{\hbar}(S_{\rm PF}[\phi] + J_A\phi^A + \Phi^*_A R^A(\phi))\right\}. \quad (5.1)$$

Making use of the change of integration variables in the functional integral (5.1) in the form of the BRST transformations,

$$\delta\phi^A = R^A(\phi)\mu[\phi], \quad (5.2)$$

taking into account that due to (3.17) the corresponding Jacobian, J, is equal to

$$J = \exp\{-\mu[\phi]_{,A}R^A(\phi)\}, \quad (5.3)$$

choosing the functional $\mu[\phi]$ in the form $\mu[\phi] = (i/\hbar)\delta\Psi[\phi]$, the relation (5.1) is rewritten as

$$\delta Z[J,\Phi^*] = \frac{i}{\hbar}\int D\phi J_A R^A(\phi)\delta\Psi[\phi]\exp\left\{\frac{i}{\hbar}(S_{\rm PF}[\phi] + J_A\phi^A + \Phi^*_A R^A(\phi))\right\} = \frac{i}{\hbar}J_A R^A(-i\hbar\vec{\partial}_J)\delta\Psi[-i\hbar\vec{\partial}_J]Z[J,\Phi^*]. \quad (5.4)$$

The Eq. (5.1) can be equivalently presented in the form

$$\delta Z[J,\Phi^*] = \frac{i}{\hbar}\delta\Psi_{,A}[-i\hbar\vec{\partial}_J]R^A(-i\hbar\vec{\partial}_J)Z[J,\Phi^*]. \quad (5.5)$$

The relations (5.4) and (5.5) are equivalent due to the evident equality

$$\int D\phi\vec{\partial}_{\phi^B}\left(\Psi[\phi]R^B(\phi)\exp\left\{\frac{i}{\hbar}(S_{\rm PF}[\phi] + J_A\phi^A + \Phi^*_A R^A(\phi))\right\}\right) = 0, \quad (5.6)$$

where the equations

$$S_{{\rm PF},A}[\phi]R^A(\phi) = 0, \qquad R^A_{,A}(\phi) = 0, \qquad R^A_{,B}(\phi)R^B(\phi) = 0 \quad (5.7)$$

should be used. In terms of the functional $W[J,\Phi^*]$, the relations (5.4) and (5.5) are rewritten as

$$\delta W[J,\Phi^*] = J_A R^A(\vec{\partial}_J W - i\hbar\vec{\partial}_J)\delta\Psi[\vec{\partial}_J W - i\hbar\vec{\partial}_J]\cdot 1 \quad (5.8)$$

and

$$\delta W[J,\Phi^*] = \delta\Psi_{,A}[\vec{\partial}_J W - i\hbar\vec{\partial}_J]R^A(\vec{\partial}_J W - i\hbar\vec{\partial}_J)\cdot 1. \quad (5.9)$$

Finally, the gauge dependence of the effective action, $\Gamma = \Gamma[\Phi,\Phi^*]$, is described by the relation

$$\delta\Gamma[\Phi,\Phi^*] = -(\Gamma\overleftarrow{\partial}_{\Phi^A})R^A(\hat{\Phi})\delta\Psi[\hat{\Phi}]\cdot 1, \quad (5.10)$$

or

$$\delta\Gamma[\Phi,\Phi^*] = \delta\Psi_{,A}[\hat{\Phi}]R^A(\hat{\Phi})\cdot 1. \quad (5.11)$$

Calculating the effective action $\Gamma[\Phi,\Phi^*]$ on its extremals $\partial_{\Phi^A}\Gamma = 0$, from the equation (5.10) it follows that this action does not depend on the gauges,

$$\delta\Gamma|_{\partial_\Phi\Gamma=0} = 0, \quad (5.12)$$

making possible the physical interpretation of results obtained in the FP method.

B. Gauge dependence in BV formalism

Let us consider the gauge-dependence problem in the BV formalism. To do this, we make an infinitesimal variation of the gauge-fixing functional $\Psi[\phi] \to \Psi[\phi] + \delta\Psi[\phi]$. Then, due to (4.21), the variation of $\exp\{(i/\hbar)S_{\rm ext}\}$ reads

$$\delta\left(\exp\left\{\frac{i}{\hbar}S_{\rm ext}\right\}\right) = [\delta\Psi,\Delta]\exp\left\{\frac{i}{\hbar}S_{\rm ext}\right\} = \Delta\delta\Psi\exp\left\{\frac{i}{\hbar}S_{\rm ext}\right\} \quad (5.13)$$

because in the case, when Ψ and $\delta\Psi$ depend on the variables ϕ only, the operator $[\delta\Psi,\Delta]$ commutes with $[\Psi,\Delta]$.

Next, the corresponding variation of the functional $Z[J,\Phi^*]$ has the form

$$\begin{aligned}\delta Z[J,\phi^*] &= \int d\phi \exp\left\{\frac{i}{\hbar}J_A\phi^A\right\}\Delta\delta\Psi\exp\left\{\frac{i}{\hbar}S_{\rm ext}(\phi,\phi^*)\right\}\\ &= (-1)^{\varepsilon_A}\vec{\partial}_{\phi^*_A}\int d\phi\exp\left\{\frac{i}{\hbar}J_A\phi^A\right\}\vec{\partial}_{\phi^A}\delta\Psi\exp\left\{\frac{i}{\hbar}S_{\rm ext}(\phi,\phi^*)\right\}\\ &= -\vec{\partial}_{\phi^*_A}J_A\int d\phi\delta\Psi\exp\left\{\frac{i}{\hbar}[S_{\rm ext}(\phi,\phi^*)+J_A\phi^A]\right\}.\end{aligned} \tag{5.14}$$

Therefore,

$$\delta Z[J,\phi^*] = -\frac{i}{\hbar}J_A\vec{\partial}_{\phi^*_A}\delta\Psi[-i\hbar\vec{\partial}_J]Z[J,\phi^*]. \tag{5.15}$$

In terms of the generating functional $W = W[J,\phi^*]$ of connected Green's functions, we have

$$\delta W[J,\phi^*] = -J_A\vec{\partial}_{\phi^*_A}\Psi[(\vec{\partial}_J W) - i\hbar\vec{\partial}_J]\cdot 1. \tag{5.16}$$

In deriving the relation (5.16) describing the gauge dependence of functional W, the Ward identity (4.14) has been substantially used. This once again emphasizes that the gauge-dependence problem cannot be reduced to fulfilling Ward's identities. The variation of the generating functional of vertex functions $\Gamma = \Gamma[\Phi,\Phi^*]$, where $\Phi^*_A = \phi^*_A$, $\Phi^A = \vec{\partial}_{J_A}W[J,\Phi^*]$, can be written as

$$\delta\Gamma = \Gamma\overleftarrow{\partial}_{\Phi^A}(\vec{\partial}_{\Phi^*_A}\langle\delta\Psi\rangle + (\vec{\partial}_{\Phi^*_A}\Phi^B)\vec{\partial}_{\Phi^B}\langle\delta\Psi\rangle), \tag{5.17}$$

where we have used the equality

$$\vec{\partial}_{\Phi^*_A}|_J = \vec{\partial}_{\Phi^*_A}|_\Phi + (\vec{\partial}_{\Phi^*_A}\Phi^B)|_J\vec{\partial}_{\Phi^B}|_{\Phi^*} \tag{5.18}$$

and also introduced the notation $\langle\delta\Psi\rangle = \langle\delta\Psi\rangle[\Phi,\Phi^*]$ for the functional

$$\langle\delta\Psi\rangle = \delta\Psi[\hat{\Phi}]\cdot 1,\quad \hat{\Phi}^A = \Phi^A + i\hbar(\Gamma''^{-1})^{AB}\vec{\partial}_{\Phi^B}, \tag{5.19}$$

where

$$\Gamma''_{AB} = \vec{\partial}_{\Phi^A}\Gamma\overleftarrow{\partial}_{\Phi^B},\qquad (\Gamma''^{-1})^{AC}\cdot\Gamma''_{CB} = \delta^A_B. \tag{5.20}$$

Calculating the effective action $\Gamma[\Phi,\Phi^*]$ on its extremals $\partial_{\Phi^A}\Gamma = 0$, from the Eq. (5.17) it follows that this action does not depend on the gauges,

$$\delta\Gamma[\Phi,\Phi^*]|_{\partial_\Phi\Gamma=0} = 0. \tag{5.21}$$

There is another point of view related with this fact. Indeed, taking into account the Ward identity for the functional $W = W[J,\Phi^*]$ (4.14), we derive the relations

$$\begin{aligned}0 &= \vec{\partial}_{J_B}(J_A\vec{\partial}_{\Phi^*_A}W) = \vec{\partial}_{\Phi^*_A}W + (-1)^{\varepsilon_B}J_A\vec{\partial}_{\Phi^*_A}\vec{\partial}_{J_B}W,\\ J_A\vec{\partial}_{\Phi^*_A}\Phi^B &= J_A\vec{\partial}_{\Phi^*_A}\vec{\partial}_{J_B}W.\end{aligned} \tag{5.22}$$

Therefore, we can rewrite the equation (5.17) in the form

$$\delta\Gamma = \Gamma(\overleftarrow{\partial}_{\Phi^A}\vec{\partial}_{\Phi^*_A} - \overleftarrow{\partial}_{\Phi^*_A}\vec{\partial}_{\Phi^A})\langle\delta\Psi\rangle = (\Gamma,\langle\delta\Psi\rangle). \tag{5.23}$$

We see that the variation of the functional Γ under an infinitesimal change of gauge fixing may be expressed in the form of anticanonical transformation (4.19) of the fields and antifields with the generating function $F = F(\Phi,\Phi^*) = \Phi^*_A\Phi^A + \langle\delta\hat{\Psi}\rangle$,

$$\Phi'^A = \Phi^A + \vec{\partial}_{\Phi^*_A}\langle\delta\Psi\rangle,\qquad \Phi^{*'}_A = \Phi^*_A - \langle\delta\Psi\rangle\overleftarrow{\partial}_{\Phi^A}. \tag{5.24}$$

For the first time, such character of the gauge dependence of the effective action in the BV formalism has been described in Ref. [41], allowing one to prove gauge-invariant renormalizability of general gauge theories.

C. Gauge dependence in FRG

We consider the gauge-dependence problem within the FRG approach not restricting ourselves to special types of the initial classical action, $S_0[A]$, or gauge-fixing condition, $\Psi[\phi]$. We demonstrate that derivation of the flow equation and analysis of gauge dependence have the same level of accuracy.

The generating functional of the Green's functions has the form

$$\begin{aligned}Z_k[J,\Phi^*] &= \int D\phi\exp\left\{\frac{i}{\hbar}[S_{Wk}[\phi] + \Phi^*_A R^A(\phi) + J_A\phi^A]\right\}\\ &= \exp\left\{\frac{i}{\hbar}W_k[J,\Phi^*]\right\},\end{aligned} \tag{5.25}$$

where

$$S_{Wk}[\phi] = S_0[A] + S_k[\phi] + \Psi_{,A}[\phi]R^A(\phi). \tag{5.26}$$

Let us find the partial derivative of $Z_k[J,\Phi^*]$ with respect to ir cutoff parameter k. The result reads

$$\partial_k Z_k[J,\Phi^*] = \frac{i}{\hbar}\int D\phi\partial_k S_k[\phi]\exp\left\{\frac{i}{\hbar}[S_{Wk}[\phi] + \Phi^*_A R^A(\phi) + J_A\phi^A]\right\} = \frac{i}{\hbar}\partial_k S_k[-i\hbar\vec{\partial}_J]Z_k[J,\Phi^*]. \quad (5.27)$$

In deriving this result, the existence of the functional integral (5.25) is only used. In terms of generating functional of the connected Green's functions, we have

$$\partial_k W_k[J,\Phi^*] = \partial_k S_k[\vec{\partial}_J W_k - i\hbar\vec{\partial}_J]\cdot 1. \quad (5.28)$$

The basic equation (flow equation) of the FRG approach follows from (5.28)

$$\partial_k\Gamma_k[\Phi,\Phi^*] = \partial_k S_k[\hat{\Phi}]\cdot 1, \quad (5.29)$$

where $\hat{\Phi} = \{\hat{\Phi}^A\}$ is defined in (4.40). It follows from (4.40) that $\partial_k\hat{\Phi}^A \neq 0$. It is assumed that solutions to the flow equations (5.29) present the effective average action $\Gamma_k[\Phi,\Phi^*]$ beyond the usual perturbation calculations. In perturbation theory, the functional $\Gamma_k = \Gamma_k[\Phi,\Phi^*]$ is considered as a solution to the functional integrodifferential equation

$$\exp\left\{\frac{i}{\hbar}\Gamma_k[\Phi,\Phi^*]\right\} = \int D\phi\exp\left\{\frac{i}{\hbar}[S_{Wk}[\Phi+\phi] + \Phi^*_A R^A(\Phi+\phi) - \Gamma_k[\Phi,\Phi^*]\overleftarrow{\partial}_{\Phi^A}\phi^A]\right\}, \quad (5.30)$$

using in the functional integral the Taylor expansion for the exponent with respect to fields ϕ and then integrating over ϕ. Such a procedure is mathematically correct because the functional integral is well defined in the perturbation theory [72]. It is a known fact [52] that the effective average action found as a solution to the equation (5.30) depends on gauges even on shell.

Now, we analyze the gauge-dependence problem of the flow equation (5.29). Note that up to now this problem has never been discussed in the literature. To do this, we consider the variation of $\partial_k Z_k[J,\Phi^*]$ (5.27) under an infinitesimal change of the gauge-fixing functional, $\Psi[\phi]\to\Psi[\phi]+\delta\Psi[\phi]$. Taking into account that $\partial_k S_k$ does not depend on the gauge-fixing procedure, we obtain

$$\delta\partial_k Z_k[J,\Phi^*] = \left(\frac{i}{\hbar}\right)^2\partial_k S_k[-i\hbar\vec{\partial}_J]\delta\Psi_{,A}[-i\hbar\vec{\partial}_J] \times R^A(-i\hbar\vec{\partial}_J)Z_k[J,\Phi^*]. \quad (5.31)$$

In terms of the functional $W_k[J,\Phi^*]$, we have

$$\delta\partial_k W_k[J,\Phi^*] = \partial_k S_k[\vec{\partial}_J W_k - i\hbar\vec{\partial}_J]\delta\Psi_{,A}[\vec{\partial}_J W_k - i\hbar\vec{\partial}_J]R^A(\vec{\partial}_J W_k - i\hbar\vec{\partial}_J)\cdot 1. \quad (5.32)$$

Finally, the gauge dependence of the flow equation is described by the equation

$$\delta\partial_k\Gamma_k[\Phi,\Phi^*] = \partial_k S_k[\hat{\Phi}]\delta\Psi_{,A}[\hat{\Phi}]R^A(\hat{\Phi})\cdot 1. \quad (5.33)$$

Therefore, at any finite value of k, the effective average action depends on gauges. But what about the case when $k\to 0$? One can think that due to the property

$$\lim_{k\to 0}\Gamma_k = \Gamma, \quad (5.34)$$

where Γ is the standard effective action constructed by the Faddeev-Popov rules, the gauge dependence of the effective average action disappears at the fixed points (see, for example, Ref. [44]). It is not true because by itself the effective action Γ depends on gauges. Moreover, there exists an additional reason to doubt the gauge independence of the effective average action at the fixed points. Indeed, in the FRG, the effective average action Γ_k should be found as a solution to the flow equation (5.29), which includes the differential operation with respect to the ir parameter k. Let us present the effective average action in the form

$$\Gamma_k = \Gamma + kH_k, \quad (5.35)$$

where functional H_k obeys the property

$$\lim_{k\to 0}H_k = H_0 \neq 0. \quad (5.36)$$

Then, we have the relations

$$\partial_k\lim_{k\to 0}\Gamma_k = 0, \qquad \lim_{k\to 0}\partial_k\Gamma_k = H_0. \quad (5.37)$$

These two operation do not commute, and the gauge independence at the fixed points requires some additional study. Taking into account the commutativity of gauge variation and of the limit $k\to 0$ from Eqs. (5.33) and (5.37), it follows that

$$\lim_{k\to 0}\delta\partial_k\Gamma_k = \delta H_0. \quad (5.38)$$

If H_0 depends on gauges, then $\delta H_0 \neq 0$, and one meets the gauge-dependence problem at the fixed points. We are going to support the existence of this problem by explicit calculations of the effective average action for a toy gauge model based on electromagnetic field in the flat space-time.

The classical action of the model is

$$S_0(A) = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{5.39}$$

We choose the gauge-fixing function in the form corresponding to nonsingular gauges

$$\chi(A,B) = \frac{1}{\sqrt{1+\xi}}\partial^\alpha A_\alpha + B, \tag{5.40}$$

where B is an auxiliary field introducing the gauge and ξ is a gauge parameter. Integrating over field B in the functional integral yields the gauge-fixing action

$$S_{\mathrm{gf}}(A) = -\frac{1}{2(1+\xi)}\int d^4x(\partial^\alpha A_\alpha)^2. \tag{5.41}$$

The action for ghosts reads

$$S_{\mathrm{gh}}(\bar{C},C) = \frac{1}{\sqrt{1+\xi}}\int d^4x\bar{C}(\partial^\alpha\partial_\alpha)C. \tag{5.42}$$

Calculation of the effective average action of the model within the standard FRG method gives

$$\Gamma_k(\Phi) = S_0(A) + S_{\mathrm{gf}}(A) + S_{\mathrm{gh}}(\bar{C},C) + S_k(\Phi) + i\hbar\Gamma_k^{(1)}(\xi), \tag{5.43}$$

where the regulator action, $S_k(A,\bar{C},C)$, is

$$S_k(\Phi) = \frac{1}{2}\int d^4x A^\alpha(R_{k,A})_{\alpha\beta}A^\beta + \int d^4x\bar{C}R_{k,\mathrm{gh}}C, \tag{5.44}$$

and the function $\Gamma_k^{(1)}(\xi)$ has the form

$$\Gamma_k^{(1)}(\xi) = \frac{1}{2}\mathrm{Tr}\ \ln\left(\Box\delta^\alpha_\beta - \frac{\xi}{1+\xi}\partial^\alpha\partial_\beta + (R_{k,A})^\alpha_\beta\right) - \mathrm{Tr}\ \ln\left(\frac{1}{\sqrt{1+\xi}}\Box + R_{k,\mathrm{gh}}\right). \tag{5.45}$$

It is important to note that the action (5.43) is the exact solution to the flow equation without using any truncation schemes.

From (5.43)–(5.45), it follows that

$$\lim_{k\to 0}\Gamma_k(\Phi) = S_0(A) + S_{\mathrm{gf}}(A) + S_{\mathrm{gh}}(\bar{C},C) + i\hbar\frac{1}{2}\mathrm{Tr}\ \ln\left(\Box\delta^\alpha_\beta - \frac{\xi}{1+\xi}\partial^\alpha\partial_\beta\right) - i\hbar\mathrm{Tr}\ \ln\left(\frac{1}{\sqrt{1+\xi}}\Box\right), \tag{5.46}$$

and

$$\partial_k\Gamma_k(\Phi) = \partial_k S_k(\Phi) + i\hbar\frac{1}{2}\mathrm{Tr}[G^\alpha_\beta(\xi)\partial_k(R_{k,A})^\beta_\gamma] + i\hbar\mathrm{Tr}\left[\left(\frac{1}{\sqrt{1+\xi}}\Box + R_{k,\mathrm{gh}}\right)^{-1}\partial_k R_{k,\mathrm{gh}}\right], \tag{5.47}$$

where $G^\alpha_\beta(\xi)$ is an operator inverse to

$$M^\alpha_\beta(\xi) = \Box\delta^\alpha_\beta - \frac{\xi}{1+\xi}\partial^\alpha\partial_\beta + (R_{k,A})^\alpha_\beta, \quad M^\alpha_\beta(\xi)G^\beta_\gamma(\xi) = \delta^\alpha_\gamma. \tag{5.48}$$

Therefore, the relations (5.46) and (5.47) confirm main statements about gauge dependence in the FRG: the effective average action depends on gauges in the limit $k\to 0$, and the flow equation depends on gauges at any value of ir parameter k. Moreover, if the partial derivatives of regulator functions with respect to parameter k do not disappear in the limit $k\to 0$,

$$\lim_{k\to 0}\partial_k R_k \neq 0, \tag{5.49}$$

then in this case the second limit in Eq. (5.37) depends on gauges explicitly. Let us emphasize again that the toy model is useful in studying basic properties of effective average action in the FRG due to the its explicit form of this action. It allows to analyze the gauge dependence not only the effective average action but the flow equation at any value of ir parameter. In particular, this study indicates the existence of a real problem with gauge dependence even at the fixed points.

Quite recently by explicit calculations in the FRG approach, the gauge dependence of some mass parameters in gravity theories at the fixed points has been found [73]. It means that all general conclusions made in this subsection about gauge dependence in the FRG are true.

VI. BACKGROUND FIELD METHOD

The background field method (BFM) [53–55] presents a reformulation of quantization procedure for Yang-Mills theories allowing to work with the effective action invariant under the gauge transformations of background fields and to reproduce all usual physical results by choosing a special background field condition [50,55]. Application of the BFM simplifies essentially calculations of Feynman diagrams in gauge theories [74–78] (among recent applications of this approach see, for example, [79–83]). The gauge-dependence problem in this method remains very important matter although it does not discuss because standard considerations are restricted by the background field gauge condition only.

We study the gauge dependence of generating functionals of the Green's functions in the BFM for Yang-Mills theories in class of gauges depending on gauge and background vector fields. The background field gauge condition belongs them as a special choice. We prove that the gauge invariance can be achieved if the gauge-fixing functions satisfy a tensor transformation law. We consider the gauge dependence and gauge invariance problems within the background field formalism as two independent ones. To support this point of view we analyze the FRG approach [1,2] in the BFM. We find restrictions on tensor structure of the regulator functions which allow to construct a gauge invariant average effective action. Nevertheless, being gauge invariant this action remains a gauge-dependent quantity on-shell making impossible a physical interpretation of results obtained for gauge theories.

A. BFM in FP method

We consider any Yang-Mills type of gauge theory of fields A^i, with Grassmann parity $\varepsilon_i = \varepsilon(A^i)$. Application of the BFM requires specifying gauge fields of initial action $S_0[A]$ being invariant under gauge transformations, $\delta_\xi A^i = R^i_\alpha(A)\xi^\alpha$, $\varepsilon(R^i_\alpha) = \varepsilon_i + \varepsilon_\alpha, \varepsilon(\xi^\alpha) = \varepsilon_\alpha$. A complete set of fields $A^i = (A^{\alpha k}, A^m)$ includes fields $A^{\alpha k}$ of the gauge sector and also fields A^m of the matter sector of a given theory. We do not assume linearity in the fields of the gauge generators $R^i_\alpha(A)$ because quite recently generalization of the BFM for nonlinear gauge-fixing conditions and nonlinear realizations of the gauge generators has been found [84].

The BFM story begins with splitting the original fields A^i into two types of fields, through the substitution $A^i \longmapsto A^i + \mathcal{B}^i$ in the initial action $S_0[A]$. It is assumed that the fields $\mathcal{B}^i$ are not equal to zero *only* in the gauge sector. These fields form a classical background, while A^i are quantum fields, which means being subject of quantization; e.g., these fields are integration variables in functional integrals. It is clear that the total action satisfies

$$\delta_\omega S_0[A+\mathcal{B}] = 0 \tag{6.1}$$

under the transformation $A^i \longmapsto A'^i = A^i + R^i_\alpha(A+\mathcal{B})\omega^\alpha$. On the other hand, the new field $\mathcal{B}^i$ introduces extra new degrees of freedom and, thence, there is an ambiguity in the transformation rule for each of the fields A^i and $\mathcal{B}^i$. This ambiguity can be fixed in different ways, and in the BFM, it is done by choosing the transformation laws

$$\delta^{(q)}_\omega A^i = [R^i_\alpha(A+\mathcal{B}) - R^i_\alpha(\mathcal{B})]\omega^\alpha, \qquad \delta^{(c)}_\omega \mathcal{B}^i = R^i_\alpha(\mathcal{B})\omega^\alpha, \tag{6.2}$$

defining the *background field transformations* for the fields A^i and $\mathcal{B}^i$, respectively. In linear realization of gauge generators, the transformations (6.2) in the sector of fields A^i are just in the form $\delta^{(q)}_\omega A^i = R^i_\alpha(A)\omega^\alpha$. The superscript (q) indicates the transformation of the quantum fields, while that of the classical fields is labeled by (c). Thus, in Eq. (6.1), one has $\delta_\omega = \delta^{(q)}_\omega + \delta^{(c)}_\omega$. Indeed, the background field transformation rule for the field A^i was chosen so that

$$\delta^{(c)}_\omega \mathcal{B}^i + \delta^{(q)}_\omega A^i = R^i_\alpha(A+\mathcal{B})\omega^\alpha. \tag{6.3}$$

Quantization of gauge theory with action $S_0[A+\mathcal{B}]$ and gauge generators $R^i_\alpha(A+\mathcal{B})$ is performed in the FP method [36]. It means that one has to introduce a gauge-fixing condition for the quantum fields A^i and the set of all quantum fields $\phi = \{\phi^A\}$ as described in Sec. III. The corresponding Faddeev-Popov action in the BFM reads

$$S_{\rm FP}[\phi,\mathcal{B}] = S_0[A+\mathcal{B}] + \Psi[\phi,\mathcal{B}]\hat{R}(\phi,\mathcal{B}), \tag{6.4}$$

where the notations

$$\hat{R}(\phi,\mathcal{B}) = \overleftarrow{\partial}_{\phi^A} R^A(\phi,\mathcal{B}), \quad \Psi[\phi,\mathcal{B}] = \bar{C}^\alpha \chi_\alpha(A,B,\mathcal{B}), \tag{6.5}$$

$$R^A(\phi,\mathcal{B}) = (R^i_\alpha(A+\mathcal{B})C^\alpha, \qquad 0, \\ -(1/2)F^\alpha_{\beta\gamma}C^\gamma C^\beta(-1)^{\varepsilon_\beta}, \qquad (-1)^{\varepsilon_\alpha}B^\alpha) \tag{6.6}$$

are used. In (6.5), $\chi_\alpha(A,B,\mathcal{B})$ are gauge-fixing functions which may depend on fields B^α allowing us to introduce nonsingular gauges,

$$\chi_\alpha(A,B,\mathcal{B}) = \chi_\alpha(A,\mathcal{B}) + (\xi/2)g_{\alpha\beta}B^\alpha. \tag{6.7}$$

In this expression, ξ is a gauge parameter that has to be introduced in the case of a nonsingular gauge condition, and $g_{\alpha\beta}$ is an arbitrary invertible constant matrix such that $g_{\beta\alpha} = g_{\alpha\beta}(-1)^{\varepsilon_\alpha\varepsilon_\beta}$. The standard choice of $\chi_\alpha(A,\mathcal{B})$ in the BFM is of the type $\chi_\alpha(A,\mathcal{B}) = F_{\alpha i}(\mathcal{B})A^i$, which is a gauge-fixing condition linear in the quantum fields A^i. In what follows, consequent results do not require any kind of *a priori* specific dependence of the gauge-fixing functions $\chi_\alpha(A,B,\mathcal{B})$ on A^i, B^α, and $\mathcal{B}^i$.

The action (6.4) is invariant under the BRST transformations

$$\delta_B\phi^A = R^A(\phi,\mathcal{B})\mu, \qquad S_{\rm FP}[\phi,\mathcal{B}]\hat{R}(\phi,\mathcal{B}) = 0, \tag{6.8}$$

which do not depend on choice of the gauge-fixing condition. In (6.8), μ is a constant anticommuting parameter. The BRST transformations are applied only on quantum fields; thus, $\delta_B\mathcal{B}^i = 0$. Notice that the BRST operator is nilpotent,

$$\hat{R}^2(\phi,\mathcal{B}) = 0. \tag{6.9}$$

Apart from the global supersymmetry (BRST symmetry), a consistent formulation of the BFM requires that the Faddeev-Popov action be invariant under background field transformations. The former symmetry is ensured in the representation (6.4) of the Faddeev-Popov action, for any choice of gauge-fixing functional Ψ. Therefore, it is possible to extend considerations to a more general case in which $\Psi(\phi,\mathcal{B}) = \bar{C}^\alpha \chi_\alpha(\phi,\mathcal{B})$, where the gauge-fixing functions $\chi_\alpha(\phi,\mathcal{B})$ depend on *all* the fields under consideration and satisfy the condition $\varepsilon(\chi_\alpha) = \varepsilon_\alpha$. On the other hand, the presence of the background field symmetry is not immediate—especially in the case of nonlinear gauges—as the gauge-fixing functionals depend on the background fields. Below, we derive necessary conditions that the fermion gauge-fixing functional should satisfy to achieve the consistent application of the BFM.

Let us extend the transformation rule (6.2) to the whole set of quantum fields, as

$$\delta^{(q)}_\omega B^\alpha = -F^\alpha_{\gamma\beta} B^\beta \omega^\gamma, \qquad \delta^{(q)}_\omega C^\alpha = -F^\alpha_{\gamma\beta} C^\beta \omega^\gamma (-1)^{\varepsilon_\gamma},$$
$$\delta^{(q)}_\omega \bar{C}^\alpha = -F^\alpha_{\gamma\beta} \bar{C}^\beta \omega^\gamma (-1)^{\varepsilon_\gamma}. \tag{6.10}$$

Following the procedure used for the BRST symmetry, one can define the operator of background field transformations,

$$\hat{R}_\omega(\phi,\mathcal{B}) = \overleftarrow{\partial}_{\mathcal{B}^i} \delta^{(c)}_\omega \mathcal{B}^i + \overleftarrow{\partial}_{\phi^A} \delta^{(q)}_\omega \phi^A, \quad \varepsilon(\hat{R}_\omega) = 0. \tag{6.11}$$

The gauge invariance of the initial classical action implies that $S_0(A+\mathcal{B})\hat{R}_\omega(\phi,\mathcal{B}) = 0$. Furthermore, it is not difficult to verify that the background gauge operator, $\hat{R}_\omega = \hat{R}_\omega(\phi,\mathcal{B})$, commutes with the generator of BRST transformations, $\hat{R} = \hat{R}(\phi,\mathcal{B})$, i.e.,

$$[\hat{R}, \hat{R}_\omega] = 0. \tag{6.12}$$

Combining this result with the representation (6.4) of the Faddeev-Popov action, we get

$$\delta_\omega S_{\rm FP}(\phi,\mathcal{B}) = S_{\rm FP}(\phi,\mathcal{B}) \hat{R}_\omega(\phi,\mathcal{B}) = 0$$
$$\Leftrightarrow \Psi(\phi,\mathcal{B})\hat{R}_\omega(\phi,\mathcal{B}) = 0. \tag{6.13}$$

In other words, the Faddeev-Popov action is invariant under background field transformations if and only if the fermion gauge-fixing functional is a scalar with respect to this transformation. The condition (6.13) constrains the possible forms of the (extended) gauge-fixing function $\chi_\alpha(\phi,\mathcal{B})$, as the relation

$$\Psi(\phi,\mathcal{B})\hat{R}_\omega(\phi,\mathcal{B}) = \bar{C}^\alpha \delta_\omega \chi_\alpha(\phi,\mathcal{B}) - F^\alpha_{\gamma\beta} \bar{C}^\beta \omega^\gamma (-1)^{\varepsilon_\gamma} \chi_\alpha(\phi,\mathcal{B}) = 0 \tag{6.14}$$

fixes the transformation law for $\chi_\alpha(\phi,\mathcal{B})$,

$$\delta_\omega \chi_\alpha(\phi,\mathcal{B}) = -\chi_\beta(\phi,\mathcal{B}) F^\beta_{\alpha\gamma} \omega^\gamma. \tag{6.15}$$

Therefore, to have the invariance of the Faddeev-Popov action under background field transformations, it is necessary that the gauge function χ_α transforms as a tensor with respect to the gauge group. This requirement can be fulfilled, provided that $\chi_\alpha(\phi,\mathcal{B})$ is constructed only by using tensor quantities. Thus, Eq. (6.15) may impose a restriction on the form of gauge-fixing functions which are nonlinear on the fields A^i. In particular, if the gauge-fixing function $\chi_\alpha(\phi,\mathcal{B})$ is chosen in a form leading to invariance of the gauge-fixing action under the background gauge transformations, then the ghost action by itself will be invariant under these transformations as well.

At this point, we can conclude that (6.8) and (6.13) represent necessary conditions for the consistent application of the BFM. The first relation is associated to the gauge independence of the vacuum functional, which is needed for the gauge-independent S matrix and hence is a very important element for the consistent quantum formulation of a gauge theory [38,58], while the second relation is called to provide the invariance of the effective action in the BFM with respect to deformed (in the general case) background field transformations. In what follows, we shall consider these statements explicitly. To this end, it is convenient to introduce the extended action

$$S_{\rm ext}[\phi,\mathcal{B},\Phi^*] = S_{\rm FP}[\phi,\mathcal{B}] + \Phi^*_A R^A(\phi,\mathcal{B}), \tag{6.16}$$

where $\Phi^* = \{\Phi^*_A\}$ denote as usual the set of sources (antifields) to the BRST transformations, with the parities $\varepsilon(\Phi^*_A) = \varepsilon_A + 1$. The corresponding (extended) generating functional of the Green's functions reads

$$Z[J,\mathcal{B},\Phi^*] = \int \mathcal{D}\phi \exp\left\{\frac{i}{\hbar}(S_{\rm FP}[\phi,\mathcal{B}] + J_A\phi^A + \Phi^*_A R^A(\phi,\mathcal{B}))\right\} = \exp\left\{\frac{i}{\hbar} W[J,\mathcal{B},\Phi^*]\right\}, \tag{6.17}$$

where $J_A = (J_i, J^{(B)}_\alpha, \bar{J}_\alpha, J_\alpha)$ [with the parities $\varepsilon(J_A) = \varepsilon_A$] are the external sources for the fields ϕ^A. The BRST symmetry, together with the requirement that the generators R^i_α of gauge transformation satisfy

$$(-1)^{\varepsilon_i}\vec{\partial}_{A^i} R^i_\alpha(A+\mathcal{B}) + (-1)^{\varepsilon_\beta+1} F^\beta_{\beta\alpha} = 0 \Leftrightarrow R^A_{,A}(\phi,\mathcal{B}) = 0, \tag{6.18}$$

implies in the ST identity

$$J_A \vec{\partial}_{\Phi^*_A} Z[J,\mathcal{B},\Phi^*] = 0. \tag{6.19}$$

The relation (6.18) plays an important role in the derivation of the Ward identity insomuch as it ensures the triviality of the Berezenian related to the change of integration variables in the form of BRST transformations.

In terms of the generating functional $W[J,\mathcal{B},\Phi^*]$ of the connected Green's functions, the ST identity reads

$$J_A\vec{\partial}_{\Phi^*_A}W[J,\mathcal{B},\Phi^*) = 0. \tag{6.20}$$

The (extended) effective action is defined as

$$\Gamma = \Gamma[\Phi,\mathcal{B},\Phi^*] = W[J,\phi^*,\mathcal{B}] - J_A\Phi^A,$$
$$\Phi^A = \vec{\partial}_{J_A}W[J,\mathcal{B},\Phi^*], \tag{6.21}$$

and it satisfies the ST identity

$$\Gamma\overleftarrow{\partial}_{\Phi^A}\vec{\partial}_{\Phi^*_A}\Gamma = 0, \tag{6.22}$$

written in the form of the Zinn-Justin equation [63].

Let $Z_\Psi[\mathcal{B}] = Z[0,\mathcal{B},0]$ be the vacuum functional which corresponds to the choice of gauge-fixing functional $\Psi[\phi,\mathcal{B}]$ in the presence of external fields $\mathcal{B}$,

$$Z_\Psi[\mathcal{B}] = \int D\phi\exp\left\{\frac{i}{\hbar}S_{\rm FP}[\phi,\mathcal{B}]\right\}. \tag{6.23}$$

In turn, let $Z_{\Psi+\delta\Psi}$ be the vacuum functional corresponding to a gauge-fixing functional $\Psi[\phi,\mathcal{B}] + \delta\Psi[\phi,\mathcal{B}]$,

$$Z_{\Psi+\delta\Psi}[\mathcal{B}] = \int d\phi\exp\left\{\frac{i}{\hbar}(S_{\rm FP}[\phi,\mathcal{B}] + \delta\Psi[\phi,\mathcal{B}]\hat{R}(\phi,\mathcal{B}))\right\}. \tag{6.24}$$

Here, $\delta\Psi[\phi,\mathcal{B}]$ is an arbitrary infinitesimal odd functional which may in general have a form differing from (6.5). Making use of the change of variables ϕ^i in the form of BRST transformations but with replacement of the constant parameter μ by the functional

$$\mu = \mu[\phi,\mathcal{B}] = \frac{i}{\hbar}\delta\Psi[\phi,\mathcal{B}] \tag{6.25}$$

and taking into account that the Jacobian of transformations is equal to

$$J = \exp\{-\mu[\phi,\mathcal{B}]\hat{R}(\phi,\mathcal{B})\}, \tag{6.26}$$

we find the gauge independence of the vacuum functional

$$Z_\Psi[\mathcal{B}] = Z_{\Psi+\delta\Psi}[\mathcal{B}]. \tag{6.27}$$

The property (6.27) was a reason to omit the label Ψ in the definition of generating functionals (6.17), and it means that, due to the equivalence theorem [58], the physical S matrix does not depend on the gauge fixing.

The vacuum functional $Z[\mathcal{B}] = Z_\Psi[\mathcal{B}]$ obeys the very important property of gauge invariance with respect to gauge transformations of external fields,

$$\delta^{(c)}_\omega\mathcal{B}^i = R^i_\alpha(\mathcal{B})\omega^\alpha, \qquad \delta^{(c)}_\omega Z[\mathcal{B}] = 0. \tag{6.28}$$

It means the gauge invariance of functional $W[\mathcal{B}] = W[0,\mathcal{B},0]$, $\delta^{(c)}_\omega W[\mathcal{B}] = 0$, as well. The proof is based on using the change of variables $\phi^A \to \phi^A + \delta^{(q)}_\omega\phi^A$ in the functional integral (6.23) where $\delta^{(q)}_\omega\phi^A$ are defined in Eqs. (6.2) and (6.10) and taking into account that the Jacobian of these transformations is equal to a unit, and assuming the transformation law of gauge-fixing functions χ_α according to $\delta_\omega\chi_\alpha(\phi,\mathcal{B}) = -\chi_\beta(\phi,\mathcal{B})F^\beta_{\alpha\gamma}\omega^\gamma$. In particular, we can argue the invariance of $S_{\rm FP}[\phi,\mathcal{B}]$ under combined gauge transformations of external and quantum fields

$$\delta_\omega S_{\rm FP}[\phi,\mathcal{B}] = 0. \tag{6.29}$$

In its turn from the second in (6.28) and the relation $W[\mathcal{B}] = -i\hbar\ln Z[\mathcal{B}]$, it follows the invariance of functional $W[\mathcal{B}]$,

$$\delta^{(c)}_\omega W[\mathcal{B}] = 0 \tag{6.30}$$

under the background gauge transformations. Finally, the main object of the BFM, namely, the effective action of background fields, $\Gamma[\mathcal{B}]$, is invariant,

$$\delta^{(c)}_\omega\Gamma[\mathcal{B}] = 0, \tag{6.31}$$

under the background gauge transformations as well.

The relations between the standard generating functionals and the analogous quantities in the background field formalism are established with modification of gauge functions [55]. Here, for the sake of completeness, we compare the generating functionals in the BFM and in the traditional one—and, ultimately, their relations with $\Gamma[\mathcal{B}]$. To do this, we consider the generating functional of the Green's functions, which corresponds to the standard quantum field theory approach, but in a very special gauge fixing,

$$Z_2[J] = \int D\phi\exp\left\{\frac{i}{\hbar}(S_0[A] + \Psi[\phi-\mathcal{B},\mathcal{B}]\hat{R}(\phi) + J_A\phi^A)\right\}, \tag{6.32}$$

where $\hat{R}(\phi)$ is the generator of standard BRST transformations (3.10). In the last expression, all the dependence of the quantity $Z_2[J]$ on the external field is only through the gauge-fixing functional. Thus, this functional depends the external field $\mathcal{B}^i$, but since this dependence is not of the BFM type, $Z_2[J]$ is nothing else but the conventional generating functional of the Green's functions of the theory, defined by S_0 in a specific $\mathcal{B}^i$-dependent gauge. One of the consequences is that any kind of physical results does not depend on $\mathcal{B}^i$. The arguments of Ψ are written explicitly, showing that we assume that A^i only occurs in a specific combination with $\mathcal{B}^i$. We stress that, being formulated in the traditional way (i.e., not in the BFM), $Z_2[J]$ does not impose any constraint on the linearity of the gauge-fixing fermion Ψ with respect to the quantum field A^i.

Making some change of variables in the functional integral, it is easy to verify that there exists the relation

$$Z[J,\mathcal{B}] = Z_2[J]\exp\left\{-\frac{i}{\hbar}J_i\mathcal{B}^i\right\}, \tag{6.33}$$

where $Z[J,\mathcal{B}]$ is the functional $Z[J,\mathcal{B},\Phi^*]$ (6.17) restricted on hypersurface $\Phi^*_A = 0$. Accordingly, for the generating functional of the connected Green's functions, one has

$$W[J,\mathcal{B}] = W_2[J] - J_i\mathcal{B}^i, \tag{6.34}$$

where $W_2[J] = -i\hbar\ln Z_2[J]$. Recall that

$$\mathcal{A}^i = \vec{\partial}_{J_i}W[J,\mathcal{B}]. \tag{6.35}$$

Similarly,

$$\mathcal{A}^i_2 = \vec{\partial}_{J_i}W_2[J] = \mathcal{A}^i + \mathcal{B}^i. \tag{6.36}$$

Following the same line, let us define the effective action associated to $Z_2[J]$ as

$$\Gamma_2[\Phi_2] = W_2[J] - J_A\Phi^A_2. \tag{6.37}$$

A moment's reflection shows that

$$\Gamma[\Phi,\mathcal{B}] = \Gamma_2[\Phi_2]. \tag{6.38}$$

In other words, the effective action $\Gamma[\Phi,\mathcal{B}]$ in the background field formalism is equal to the initial effective action in a particular gauge with mean field $\mathcal{A}^i_2 = \mathcal{A}^i + \mathcal{B}^i$—or, switching off the mean fields,

$$\Gamma[\mathcal{B}] = \Gamma_2[\mathcal{A}_2]|_{\mathcal{A}_2=\mathcal{B}}. \tag{6.39}$$

We point out that the gauge is not associated to its linearity with respect to the quantum fields but to its dependence on the background field [see Eq. (6.32)].

Quantization of the Yang-Mills type of gauge theories in the BFM within the FP method provides very attractive features, namely, the BRST symmetry of the FP action, the background gauge invariance of effective action, and gauge independence of S-matrix elements.

B. BFM in FRG

Here, we discuss the background gauge invariance and gauge dependence of average effective action as well as violation of the BRST symmetry in the FRG [1,2] using the BFM. Of course, as to the background field symmetry, this issue is not new, see, for example, Refs. [7,16,43]), but we are going to remind the reader of the main results related to specific features of the FRG approach in the BFM. We pay special attention to the problem of gauge dependence of the flow equation as a new issue in our studies of the FRG.

Inclusion of the FRG in the BFM may be achieved in two ways with the help of special dependence of regulator functions on background fields [7,85] when the regulator action $S_k[\phi,\mathcal{B}]$ depends on background fields $\mathcal{B}$ or due to special tensor structure of regulator functions [86] when the regulator action $S_k[\phi]$ does not depend on $\mathcal{B}$. In both realizations, the regulator action S_k is invariant under background gauge transformations $\delta^{(q)}_\omega\phi^A = R^A_\omega(\phi,\mathcal{B})$, $\delta^{(c)}_\omega\mathcal{B}^i = R^i_\omega(\mathcal{B})$ [see the relations (6.2) and (6.10)],

$$\delta_\omega S_k = 0. \tag{6.40}$$

In what follows, we use the notation $S_k[\phi,\mathcal{B}]$ for definiteness. The full action of the FRG approach in the BFM has the form

$$S_{Wk}[\phi,\mathcal{B}] = S_{\rm FP}[\phi,\mathcal{B}] + S_k[\phi,\mathcal{B}] \tag{6.41}$$

and is invariant under background gauge transformations,

$$\delta_\omega S_{Wk}[\phi,\mathcal{B}] = 0. \tag{6.42}$$

Consider the generating functional of the Green's functions,

$$Z_k[J,\mathcal{B}] = \int D\phi\exp\left\{\frac{i}{\hbar}[S_{Wk}[\phi,\mathcal{B}] + J_A\phi^A]\right\} = \exp\left\{\frac{i}{\hbar}W_k[J,\mathcal{B}]\right\}, \tag{6.43}$$

and variation of this functional with respect to background gauge transformations of external fields $\mathcal{B}^i$. We have

$$\delta^{(c)}_\omega Z_k[J,\mathcal{B}] = \frac{i}{\hbar}\int D\phi\,\delta^{(c)}_\omega S_{Wk}[\phi,\mathcal{B}] \times\exp\left\{\frac{i}{\hbar}[S_{Wk}[\phi,\mathcal{B}] + J_A\phi^A]\right\}. \tag{6.44}$$

Making use the change of integration variables ϕ^A in the form of background gauge transformation in the functional integral (6.44) and taking into account the invariance of $S_{Wk}[\phi,\mathcal{B}]$ (6.42), we obtain

$$\delta^{(c)}_\omega Z_k[J,\mathcal{B}] = \frac{i}{\hbar}J_A R^A_\omega(-i\hbar\vec{\partial}_J,\mathcal{B})Z_k[J,\mathcal{B}]. \tag{6.45}$$

In terms of generating functional of the connected Green's functions $W_k[J,\mathcal{B}]$, the relation (6.45) is rewritten as

$$\delta^{(c)}_{\omega} W_k[J,\mathcal{B}] = J_A R^A_{\omega}(\vec{\partial}_J W_k - i\hbar\vec{\partial}_J, \mathcal{B})\cdot 1. \tag{6.46}$$

Because of the linearity of generators $R^A_{\omega}(\phi,\mathcal{B})$ with respect to ϕ, we have

$$R^A_{\omega}(\vec{\partial}_J W_k - i\hbar\vec{\partial}_J, \mathcal{B})\cdot 1 = R^A_{\omega}(\vec{\partial}_J W_k, \mathcal{B}), \tag{6.47}$$

and, therefore,

$$\delta^{(c)}_{\omega} W_k[J,\mathcal{B}] = J_A R^A_{\omega}(\vec{\partial}_J W_k, \mathcal{B}). \tag{6.48}$$

Introducing the effective average action $\Gamma_k[\Phi,\mathcal{B}]$ through the Legendre transformation of $W_k[J,\mathcal{B}]$,

$$\Gamma_k[\Phi,\mathcal{B}] = W_k[J,\mathcal{B}] - J_A\Phi^A, \qquad \Phi^A = \vec{\partial}_{J_A} W_k[J,\mathcal{B}],$$
$$\Gamma[\Phi,\mathcal{B}]\overleftarrow{\partial}_{\Phi^A} = -J_A, \tag{6.49}$$

from (6.48), it follows that

$$\delta^{(c)}_{\omega}\Gamma_k[\Phi,\mathcal{B}] = -\Gamma[\Phi,\mathcal{B}]\overleftarrow{\partial}_{\Phi^A} R^A_{\omega}(\Phi,\mathcal{B}), \tag{6.50}$$

or

$$\delta_{\omega}\Gamma_k[\Phi,\mathcal{B}] = 0. \tag{6.51}$$

The effective average action $\Gamma[\Phi,\mathcal{B}]$ is gauge invariant under the background gauge transformations of all fields Φ^A, $\mathcal{B}^i$. In particular, the functional $\Gamma_k[\mathcal{B}] = \Gamma_k[\Phi,\mathcal{B}]|_{\Phi=0}$,

$$\delta^{(c)}_{\omega}\Gamma_k[\mathcal{B}] = 0, \tag{6.52}$$

is invariant under the gauge transformations of external fields $\mathcal{B}^i$.

The BRST symmetry is broken on the level of action $\delta_B S_{Wk}[\phi,\mathcal{B}]$ (6.41),

$$\delta_B S_{Wk}[\phi,\mathcal{B}] = \delta_B S_k[\phi,\mathcal{B}] \neq 0, \tag{6.53}$$

On the quantum level, violation of the BRST symmetry leads to gauge dependence of the vacuum functional

$$Z_{k|\Psi}[\mathcal{B}] = \int D\phi \exp\left\{\frac{i}{\hbar} S_{Wk}[\phi,\mathcal{B}]\right\}. \tag{6.54}$$

Indeed, consider the vacuum functional corresponding to another choice of gauge-fixing functional, $\Psi[\phi] + \delta\Psi[\phi]$,

$$Z_{k|\Psi+\delta\Psi}[\mathcal{B}] = \int D\phi \exp\left\{\frac{i}{\hbar}(S_{Wk}[\phi,\mathcal{B}] + \delta\Psi_{,A}[\phi,\mathcal{B}]R^A(\phi,\mathcal{B}))\right\}. \tag{6.55}$$

Making use the change of integration variables ϕ^A in the form of BRST transformations with replacement constant parameter μ by functional $\mu[\phi,\mathcal{B}]$ and choosing this functional in the form

$$\mu[\phi,\mathcal{B}] = (i/\hbar)\delta\Psi[\phi,\mathcal{B}], \tag{6.56}$$

we obtain

$$Z_{k|\Psi+\delta\Psi}[\mathcal{B}] = \int D\phi \exp\left\{\frac{i}{\hbar}(S_{Wk}[\phi,\mathcal{B}] + \delta_B S_k[\phi,\mathcal{B}])\right\}. \tag{6.57}$$

We cannot propose any change of variables in the functional integral (6.57) to reduce it to $Z_{k|\Psi}[\mathcal{B}]$. Therefore,

$$Z_{k|\Psi+\delta\Psi}[\mathcal{B}] \neq Z_{k|\Psi}[\mathcal{B}], \tag{6.58}$$

and the vacuum functional of the FRG approach and the S matrix remain gauge dependent within the BFM as well.

To discuss the mST identity, it is useful, as we know from previous investigations, to introduce the extended generating functionals of the Green's functions $Z_k[J,\mathcal{B},\Phi^*]$ and connected Green's functions $W_k[J,\mathcal{B},\Phi^*]$,

$$Z_k[J,\mathcal{B},\Phi^*] = \int D\phi \exp\left\{\frac{i}{\hbar}[S_{Wk}[\phi,\mathcal{B}] + \Phi^*_A R^A(\phi,\mathcal{B}) + J_A\phi^A]\right\} = \exp\left\{\frac{i}{\hbar}W_k[J,\mathcal{B},\Phi^*]\right\}. \tag{6.59}$$

Using the change of variables ϕ^A in the form of BRST transformations (6.8) and taking into account the BRST invariance of $S_{\rm FP}[\phi,\mathcal{B}]$, we obtain

$$(J_A\vec{\partial}_{\Phi^*_A} + S_{k,A}[-i\hbar\vec{\partial}_J,\mathcal{B}]\vec{\partial}_{\Phi^*_A})Z_k[J,\mathcal{B},\Phi^*] \equiv 0, \tag{6.60}$$

which is the mST identity in the FRG within the BFM written for functional $Z_k[J,\mathcal{B},\Phi^*]$. It is clear that this identity coincides with the ST identity (6.19) in the limit $k\to 0$. In terms of the extended generating functional of the connected Green's functions, $W_k = W_k[J,\mathcal{B},\Phi^*]$, the identity (6.60) is rewritten as

$$(J_A\vec{\partial}_{\Phi^*_A} + S_{k,A}[(\vec{\partial}_J W_k) - i\hbar\vec{\partial}_J,\mathcal{B}]\vec{\partial}_{\Phi^*_A})W_k[J,\mathcal{B},\Phi^*] \equiv 0. \tag{6.61}$$

The extended effective average action, $\Gamma_k = \Gamma_k[\Phi,\mathcal{B},\Phi^*]$, is defined through the Legendre transformation of $W_k = W_k[J,\mathcal{B},\Phi^*]$,

$$\Gamma_k[\Phi,\mathcal{B},\Phi^*] = W_k[J,\mathcal{B},\Phi^*] - J\Phi, \quad \Phi^A = \vec{\partial}_{J_A} W_k[J,\Phi^*],$$
$$\Gamma_k[\Phi,\mathcal{B},\Phi^*]\overleftarrow{\partial}_{\Phi^A} = -J_A. \tag{6.62}$$

Then, the identity (6.61) can be presented in terms of Γ_k as

$$\Gamma_k \overleftarrow{\partial}_{\Phi^A} \overrightarrow{\partial}_{\Phi^*_A} \Gamma_k - S_{k,A}[\hat{\Phi}, \mathcal{B}] \overrightarrow{\partial}_{\Phi^*_A} \Gamma_k \equiv 0, \tag{6.63}$$

or, using the antibracket,

$$\frac{1}{2}(\Gamma_k, \Gamma_k) - S_{k,A}[\hat{\Phi}, \mathcal{B}] \overrightarrow{\partial}_{\Phi^*_A} \Gamma_k \equiv 0, \tag{6.64}$$

where the notations

$$\hat{\Phi}^A = \Phi^A + i\hbar (\Gamma''^{-1}_k)^{AB} \overrightarrow{\partial}_{\Phi^B}, \qquad (\Gamma''_k)_{AB} = \overrightarrow{\partial}_{\Phi^A} \Gamma_k \overleftarrow{\partial}_{\Phi^B},$$
$$(\Gamma''^{-1}_k)^{AC} \cdot (\Gamma''_k)_{CB} = \delta^A_B \tag{6.65}$$

are used.

The existence of the background mST identity for functional $\Gamma_k[\Phi, \mathcal{B}, \Phi^*]$ does not lead to a solution of the gauge-dependence problem in the FRG approach at least for any finite value of ir parameter k. The case when $k \to 0$ requires special studies of the gauge-dependence problem of the background flow equation. The background flow equation can be formulated for the extended background effective average action $\Gamma_k[\Phi, \mathcal{B}, \Phi^*]$ or for the background effective average action $\Gamma_k[\Phi, \mathcal{B}]$. In what follows, we study the background flow equation for functional $\Gamma_k[\Phi, \mathcal{B}]$ for two reasons. First, this functional is under scrutiny of the FRG community, and second, being invariant under the background gauge transformations the functional remains gauge dependent even on shell. In turn, it shows once again that gauge-invariance and gauge-dependence properties in gauge theories should be considered as independent ones.

The background flow equation for the functional $Z_k[J, \mathcal{B}]$,

$$\partial_k Z_k[J, \mathcal{B}] = \frac{i}{\hbar} \partial_k S_k[-i\hbar \overrightarrow{\partial}_J, \mathcal{B}] Z_k[J, \mathcal{B}], \tag{6.66}$$

and the corresponding equation for the functional $W_k[J, \mathcal{B}]$,

$$\partial_k W_k[J, \mathcal{B}] = \partial_k S_k[\overrightarrow{\partial}_J W_k - i\hbar \overrightarrow{\partial}_J, \mathcal{B}] \cdot 1, \tag{6.67}$$

follow from (6.43). The background effective average action,

$$\Gamma_k[\Phi, \mathcal{B}] = W_k[J, \mathcal{B}] - J_A \Phi^A,$$
$$\Phi^A = \overrightarrow{\partial}_{J_A} W_k[J, \mathcal{B}], \Gamma_k[\Phi, \mathcal{B}] \overleftarrow{\partial}_{\Phi^A} = -J_A, \tag{6.68}$$

satisfies the background flow equation

$$\partial_k \Gamma_k[\Phi, \mathcal{B}] = \partial_k S_k[\hat{\Phi}] \cdot 1, \tag{6.69}$$

where the functional differential operators $\hat{\Phi}^A$ are defined in the form of (4.40) with the functional $\Gamma_k[\Phi, \mathcal{B}]$.

Derivation of the equation describing the gauge dependence of background flow equations (6.67), (6.68), and (6.69) is similar to that used in Sec. V.C. The results read

$$\delta \partial_k Z_k[J, \mathcal{B}] = \left(\frac{i}{\hbar}\right)^2 \partial_k S_k[-i\hbar \overrightarrow{\partial}_J, \mathcal{B}] \delta\Psi_{,A}[-i\hbar \overrightarrow{\partial}_J, \mathcal{B}] R^A(-i\hbar \overrightarrow{\partial}_J, \mathcal{B}) Z_k[J, \Phi^*], \tag{6.70}$$

$$\delta \partial_k W_k[J, \mathcal{B}] = \partial_k S_k[\overrightarrow{\partial}_J W_k - i\hbar \overrightarrow{\partial}_J, \mathcal{B}] \delta\Psi_{,A}[\overrightarrow{\partial}_J W_k - i\hbar \overrightarrow{\partial}_J, \mathcal{B}] R^A(\overrightarrow{\partial}_J W_k - i\hbar \overrightarrow{\partial}_J, \mathcal{B}) \cdot 1, \tag{6.71}$$

$$\delta \partial_k \Gamma_k[\Phi, \mathcal{B}] = \partial_k S_k[\hat{\Phi}, \mathcal{B}] \delta\Psi_{,A}[\hat{\Phi}, \mathcal{B}] R^A(\hat{\Phi}, \mathcal{B}) \cdot 1. \tag{6.72}$$

At any finite value of ir parameter k, the background flow equations (6.67), (6.68), and (6.69) are gauge dependent (6.70), (6.71), and (6.72). At the fixed point, the gauge dependence does not disappear for same reasons which were given in the end of Sec. V.C.

We see that application of the background field method does not help to solve the gauge-dependence problem in the FRG because the BRST symmetry remains broken [86].

VII. DISCUSSION

In the paper, the basic properties of gauge theories in the framework of the FP method, BV formalism, and FRG approach have been analyzed. It is known that the FP and BV quantizations are characterized by the BRST symmetry which governs gauge independence of S-matrix elements. In turn, the BRST symmetry is broken in the FRG approach with all negative consequences for physical interpretation of results. One of the goals of this work was to study the gauge dependence of the effective average action as a solution of the flow equation. For the first time, the equation describing the gauge dependence of the flow equation has been explicitly derived. The gauge dependence of flow equation at any finite value of the ir parameter k was found. As for the limit $k \to 0$, there is a strong motivation given in the paper (see Sec. V C) about the gauge dependence of the effective average action at the fixed point. Quite recently, this point of view has been supported by explicit calculations of some mass parameters in gravity theories at the fixed points [73].

Despite of above feature, it was shown that the FP method, the BV formalism, and the FRG approach can be provided with the ST identity, the Ward identity, and the mST identity, respectively. It was stressed that the existence of these identities is a direct consequence of gauge invariance of the initial classical action of the gauge theory under consideration. Presentation of these identities is essentially simplified by using both the extended generating functionals of the Green's functions and the BRST transformations.

It was proven that using the background field method the background gauge invariance of the effective action within the FP and FRG quantization procedures can be achieved in nonlinear gauges. The gauge-dependence problem within the FP and FRG quantizations in the framework of BFM was studied. Application of the BFM in the case of the FRG approach did not help in solving the problem of gauge dependence of the *S* matrix. Arguments allowing us to state the impossibility of gauge independence of physical results obtained within the FRG approach were given.

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