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Horizon states and the sign of their index in $\mathcal{N} = 4$ dyons

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Abstract: Classical single centered solutions of 1/4 BPS dyons in $\mathcal{N} = 4$ theories are usually constructed in duality frames which contain non-trivial hair degrees of freedom localized outside the horizon. These modes are in addition to the fermionic zero modes associated with broken supersymmetry. Identifying and removing the hair from the 1/4 BPS index allows us to isolate the degrees of freedom associated with the horizon. The spherical symmetry of the horizon then ensures that index of the horizon states has to be positive. We verify that this is indeed the case for the canonical example of dyons in type IIB theory on $K3\times T^2$ and prove this property holds for a class of states. We generalise this observation to all CHL orbifolds, this involves identifying the hair and isolating the horizon degrees of freedom. We then identify the horizon states for 1/4 BPS dyons in $\mathcal{N} = 4$ models obtained by freely acting $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds of type IIB theory compactified on T^6 and observe that the index is again positive for single centred black holes. This observation coupled with the fact the 1/4 BPS index of single centred solutions without removal of the hair violates positivity indicates that there exists no duality frame in these models without non-trivial hair.

Keywords: Black Holes in String Theory, Gauge-gravity correspondence, Superstring Vacua

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Contents

1 Introduction

Counting microscopic degrees of freedom for extremal black holes in string theory is a useful probe into aspects of quantum gravity [1]. For supersymmetric black holes, one should in principle be able identify the degrees of freedom both from the macroscopic solution as well as count them from the microscopic description of these black holes. The 1/4 BPS dyonic black holes in $\mathcal{N} = 4$ theory is a system which has been extensively studied in this context, see [2, 3] for reviews. The identification of the degrees of freedom is complicated by the fact that classical solutions of black holes are multi-centered and usually they contain hair degrees of freedom localized outside the horizon [4–6]. The microscopic analysis counts all these configurations together. Let us make this precise, let $d_{\rm micro}(\vec{q})$ be the degeneracy or in the case of the extremal supersymmetric black holes the appropriate supersymmetric index evaluated from the microscopic description of a BPS state with charge $\vec{q}$. Similarly let $d_{\rm macro}(\vec{q})$ be the corresponding macroscopic index. Then

$$d_{\rm macro}(\vec{q}) = \sum_n \sum_{\substack{\{\vec{q}_i\},\vec{q}_{\rm hair} \\ (\sum_{i=1}^n \vec{q}_i)+\vec{q}_{\rm hair}=\vec{q}}} \left(\prod_{i=1}^n d_{\rm hor}(\vec{q}_i) \right) d_{\rm hair}(\vec{q}_{\rm hair}; \{\vec{q}_i\}) \,. \tag{1.1}$$

Each term on the right hand side of (1.1) is the contribution to the index of say, the n-centered black hole configuration. $d_{\rm hor}(\vec{q}_i)$ is the contribution to the index from the horizon degrees of freedom with charge q_i and $d_{\rm hair}(\vec{q}_{\rm hair}; \{\vec{q}_i\})$ is the index of the hair carrying total charge $\vec{q}_{hair}$ of a n-centered black hole whose horizons carry charges $\vec{q}_1, \cdots \vec{q}_n$. We expect

$$d_{\rm macro}(\vec{q}) = d_{\rm micro}(\vec{q}). \tag{1.2}$$

It would simplify matters if we can restrict our attention to single centred black hole configurations. Then (1.1) indicates that we would need to identify the hair to isolate the horizon degrees of freedom. Since we are dealing with 1/4 BPS states in $\mathcal{N} = 4$ theories, which break 12 supersymmetries, the degeneracy $d(\vec{q})$ will refer to the index

$$B_6 = \frac{1}{6!}{\rm Tr}((2J)^6(-1)^{2J}), \tag{1.3}$$

where J is the component of the angular momentum in say, the 3 direction. The factorized form of the Hilbert space corresponding to the hair degrees of freedom and the horizon degrees of freedom follows from the fact the these are well separated due the presence of an infinite throat [4].

The utility of identifying the horizon degrees of freedom lies in the fact that the horizon is spherically symmetric and therefore carries zero momentum $J = 0$. The index taken over the horizon states reduces to $(-1)^{2J}d_{\rm hor} = d_{\rm hor}$, where $d_{\rm hor}$ is the total number of states associated with the horizon. Therefore the index of the horizon states must be a positive number. This leads to an important check on the microscopic counting and the equality (1.2). Once one determines the hair degrees of freedom for a given macroscopic black hole and factors them out of the index, what must remain is a positive number which counts the index of the horizon states. This argument clearly relies on what are the hair degrees of freedom and this in turn depends on the duality frame of the macroscopic solution. This prediction was tested in [7] with the assumption that there exists a frame in which the only hair degrees of freedom are the fermionic zero modes associated with the broken supersymmetry generators. For black holes in $\mathcal{N} = 8$ there is evidence towards this fact in [8, 9]. These authors worked in a frame in which the black hole configuration reduced to a system of only D-branes and showed the only hair degrees of freedom were the fermionic zero modes and the BPS configuration indeed had zero angular momentum. However such a frame has not yet been shown to exist for black holes in $\mathcal{N} = 4$ theory.

Given this situation, one way of proceeding is to evaluate the partition functions corresponding to the hair degrees of freedom and isolate the horizon degrees of freedom in a given frame. This has already been done in [5, 6], for 1/4 BPS dyons in the type IIB frame, but a test of positivity of the index for the resulting horizon degrees of freedom has not been done. We perform this analysis in this paper and indeed demonstrate that the $d_{\rm hor}$ is indeed positive. This is quite remarkable as we will see, since factorizing the hair degrees of freedom naively seems to introduce terms with negative contributions to the index. We adapt the proof of [10] for configurations with magnetic charge $P^2 = 2$ and demonstrate that the index is positive. We then extend this observation to all the CHL models and to other orbifolds associated with Mathieu moonshine introduced in [11, 12].

In [11, 13] it was observed that for $\mathcal{N} = 4$ models obtained by freely acting $\mathbb{Z}_2, \mathbb{Z}_3$ orbifolds of type IIB on T^6, the index for single centered configurations after factorising

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the sign due to the fermionic zero modes did not obey the expectation $d_{\rm hor}$ is positive.[1] But as the above discussion shows, a possible reason for this could be that the assumption that there exists a frame in which the fermionic zero modes are the only hair degrees of freedom might not be true. Therefore we re-examine this question in this paper. Following the same procedure used in the CHL models we isolate the hair degrees of freedom in the type IIB frame. Then on examining the sign of the index for single centered black holes we observe that $d_{\rm hor}$ is positive.

The organisation of the paper is as follows. In the section 2 we briefly review the statements about the hair and the partition function for the horizon degrees of freedom for the 1/4 BPS dyonic black hole in type IIB compactified on $K3 \times T^2$. We then generalise this to all the CHL orbifolds as well as other orbifolds associated with Mathieu Moonshine. Finally we construct the partition function for the horizon states for the toroidal models obtained by freely acting $\mathbb{Z}_2, \mathbb{Z}_3$ orbifolds of type IIB on T^6. In section 3 we perform a consistency check on the $d_{\rm hor}$ obtained. This check relies on the fact that the 5-dimensional BMPV black hole has the same near horizon geometry [5, 6]. Therefore $d_{\rm hor}$ for the BMPV black hole should agree with that of the 1/4 BPS dyon. We show that this is indeed the case for all the examples. Finally in section 4, armed with the $d_{\rm hor}$ for all the models we study the positivity of the index for single centered black holes for all the models. We have evaluated numerically the indices of horizon states for several charges in all the $\mathcal{N} = 4$ models for which dyon partition functions are known which confirm that the index is positive. We adapt the proof of [10] to show that the index is positive for charge configurations with $P^2 = 2$. Section 5 contains our conclusions.

2 Horizon states for the 1/4 BPS dyon

In this section we construct the partition function for the horizon states for 1/4 BPS dyons in $\mathcal{N} = 4$ compactifications. This is done by identifying the 'hair' degrees of freedom which are localized outside the horizon. Such a partition function for the horizon states was constructed for the canonical $\mathcal{N} = 4$ theory obtained by compactifying type IIB string theory $K3 \times T^2$ in [5, 6] in the type IIB frame. We review this in section and then extend the analysis for other $\mathcal{N} = 4$ models.

The $\mathcal{N} = 4$ compactifications of interest are type IIB theory on $K3 \times T^2/\mathbb{Z}_N$ where $\mathbb{Z}_N$ acts as an automorphism g' on $K3$ along with a shift of $1/N$ units on one of the circles of T^2. The action of g' can be labelled by the 26 conjugacy classes of the Mathieu group M_{23}. The classes pA with $p = 2, 3, 5, 6, 7, 8$ and the class $4B$ are called as Nikulin's automorphism of $K3$. They were first introduced in [14, 15] as models dual to heterotic string theory with $\mathcal{N} = 4$ superysmmetry but with gauge groups with reduced from the maximal rank of 28. All these compactifications admit 1/4 BPS dyons, let (Q, P) be the electric and magnetic charge vector of these dyons, then the 1/4 BPS index B_6 is given by [16–20]

$$-B_6 = \frac{1}{N}(-1)^{Q\cdot P+1} \int_{\mathcal{C}} d\rho d\sigma dv \; e^{-\pi i(N\rho Q^2 + \sigma P^2/N + 2vQ\cdot P)} \frac{1}{\tilde{\Phi}_k(\rho, \sigma, v)}, \tag{2.1}$$

[1] Please see tables 16, 17, 18 reproducing this observation.

where $\mathcal{C}$ is a contour in the complex 3-plane defined by

$$\begin{aligned} \rho_2 &= M_1, & \sigma_2 &= M_2, & v_2 &= -M_3, \\ 0 \le \rho_1 &\le 1, & 0 \le \sigma_1 &\le N, & 0 \le v_1 &\le 1. \end{aligned} \tag{2.2}$$

Here $\rho = \rho_1 + i\rho_2, \sigma = \sigma_1 + i\sigma_2, v = v_1 + iv_2$ and M_1, M_2, M_3 are positive numbers, which are fixed and large and $M_3 \ll M_1, M_2$. The contour in (2.2) implies that we first expand in powers or $e^{2\pi i\rho}, e^{2\pi i\sigma}$ and at the end perform the expansion in $e^{2\pi i v}$.

The Siegel modular form of weight k given by $\tilde{\Phi}_k(\rho,\sigma,v)$ transforming under $\mathrm{Sp}(2,\mathbb{Z})$, or its subgroups for $N > 1$ admits an infinite product representation given by

$$\tilde{\Phi}_k(\rho,\sigma,v) = e^{2\pi i(\rho+\sigma/N+v)} \prod_{r=0}^{N} \prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},\, l\in\mathbb{Z}, \\ j\in\mathbb{Z} \text{ if } k',l\ge 0; \\ j<0 \text{ if } k'=l=0}} (1 - e^{2\pi i(k'\sigma + l\rho + jv)})^{\sum_{s=0}^{N-1} c^{(r,s)}(4k'l - j^2)}. \tag{2.3}$$

The coefficients $c^{(r,s)}$ are determined from the expansion of the twisted elliptic genera for the various order N orbifolds g' of K^3. The twisted elliptic genus of $K3$ is defined by

$$\begin{aligned} F^{(r,s)}(\tau,z) &= \frac{1}{N} \mathrm{Tr}_{RR\;\, g'^r} \left[(-1)^{F_{K3}+\bar{F}_{K3}} g'^s e^{2\pi i z F_{K3}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right], \\ &= \sum_{j\in\mathbb{Z},\, n\in\mathbb{Z}/N} c^{(r,s)}(4n - j^2) e^{2\pi i n\tau + 2\pi i j z}. \qquad 0 \le r, s \le N-1. \end{aligned} \tag{2.4}$$

The trace is performed over the Ramond-Ramond sector of the $\mathcal{N} = (4,4)$ super conformal field theory of $K3$ with $(c,\bar{c}) = (6,6)$, F is the Fermion number and j is the left moving $\mathrm{U}(1)$ charge of the $\mathrm{SU}(2)$ R-symmetry of $K3$. The twisted elliptic genera for the g' corresponding to conjugacy classes of $M_{23} \cup M_{24}$ have been evaluated in [11]. These take the form

$$\begin{aligned} F^{(0,0)}(\tau,z) &= \alpha_{g'}^{(0,0)} A(\tau,z), \\ F^{(r,s)}(\tau,z) &= \alpha_{g'}^{(r,s)} A(\tau,z) + \beta_{g'}^{(r,s)}(\tau) B(\tau,z), \\ & \qquad\qquad r, s \in \{0, 1, \cdots N-1\} \text{ with}(r,s) \neq (0,0), \end{aligned} \tag{2.5}$$

where we can write $A(\tau,z)$ and $B(\tau,z)$ in terms of Jacobi theta functions θ_i and Dedekind eta functions as follows

$$\begin{aligned} A(\tau,z) &= \frac{\theta_2^2(\tau,z)}{\theta_2^2(\tau,0)} + \frac{\theta_3^2(\tau,z)}{\theta_3^2(\tau,0)} + \frac{\theta_4^2(\tau,z)}{\theta_4^2(\tau,0)}, \\ B(\tau,z) &= \frac{\theta_1^2(\tau,z)}{\eta^6(\tau)}. \end{aligned} \tag{2.6}$$

We have used the following notation for all cases:

$$\begin{aligned} \theta_1(\tau,z) &= \sum_n (-1)^{n-1/2} q^{\frac{(n-1/2)^2}{2}} z^{n-1/2}, \quad \theta_2(\tau,z) = \sum_n q^{\frac{(n-1/2)^2}{2}} z^{n-1/2}, \\ \theta_3(\tau,z) &= \sum_n q^{\frac{n^2}{2}} z^n, \quad \theta_4(\tau,z) = \sum_n (-1)^n q^{\frac{n^2}{2}} z^n, \\ \eta(\tau) &= q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \end{aligned}$$

The coefficients $\alpha_{g'}^{(r,s)}$ in (2.5) are numerical constants, while $\beta_{g'}^{(r,s)}(\tau)$ are modular forms that transform under $\Gamma_0(N)$. For g' corresponding to conjugacy classes of M_{23}, they can be read out from appendix E of [11]. For example, in the case of the pA orbifolds with $p = 1, 2, 3, 5, 7$, they are given by [18].

$$\begin{aligned}
F^{(0,0)} &= \frac{8}{N} A(\tau, z), \\
F^{(0,s)} &= \frac{8}{(N+1)N} A(\tau, z) - \frac{2}{N+1} B(\tau, z)\mathcal{E}_N(\tau), \\
F^{(r,rk)} &= \frac{8}{N(N+1)} A(\tau, z) + \frac{2}{N(N+1)} B(\tau, z)\mathcal{E}_N\left(\frac{\tau+k}{N}\right), \\
\mathcal{E}_N(\tau) &= \frac{12i}{\pi(N-1)} \partial_\tau[\ln \eta(\tau) - \ln \eta(N\tau)].
\end{aligned} \tag{2.7}$$

For N composite corresponding to the classes $4B, 6A, 8A$, the strategy for construction of the twisted elliptic genus was first given in [21] and it was worked out explicitly for the $4B$ example.[2] The papers [22–24] contain the twining characters, $F^{(0,s)}$ and [25] also contains the strategy to construct the twisted elliptic genera for other conjugacy classes of M_{23} and a Mathematica code for generating the elliptic genera.

The weight of the Siegel modular form $\tilde{\Phi}(\rho, \sigma, v)$ is given by

$$k = \frac{1}{2} \sum_{s=0}^{N-1} c^{(0,s)}(0). \tag{2.8}$$

For the classes $pA, p = 1, 2, 3, 5, 7, 11$ we have

$$k = \frac{24}{p+1} - 2, \tag{2.9}$$

for $4B, 6A, 8A$ we have $k = 3, 2, 1$ respectively and for $14A, 15A$ $k = 0$.

Finally, as discussed in the introduction the study of horizon states would be much simpler if one could focus on single centered dyons. Such a system would have only one horizon. The choice of the contour chosen in (2.2) together with some kinematic constraints on charges such as (4.3) ensures that we are in the attractor region of the axion-dilaton moduli and the index given by (2.1) is that of single centred dyons [7, 26]. All the indices evaluated in this section paper is done using the contour (2.2).

2.1 The canonical example: $K3 \times T^2$

In the work of [5, 6] the hair modes of the 1/4 BPS dyonic black hole in type IIB theory compactified on $K3 \times T^2$ were constructed. Here we briefly review this construction. These modes were shown to be deformations localized outside the horizon and they preserved supersymmetry. Let us first recall that the dyonic black hole in 4-dimensions is constructed by placing the 5 dimensional BMPV black hole or the rotating D1-D5 system [27] in Taub-Nut space [28]. The Taub-Nut space has the geometry which at the origin is R^4 but at

[2] Suresh Govindarajan informed us that the authors of [21] also explicitly constructed all the sectors of the $6A$ and $8A$ twisted elliptic genera though it was not reported in the paper.

infinity it is $R^3 \times \tilde{S}$. The isometry along S^1 coincides with the angular direction the BMPV rotates. The hair modes arise from the collective modes of the D1-D5 system thought of as an effective string along say the x^5 and the time t directions. Therefore these modes are oscillations of the effective string, they are left moving since they have to preserve supersymmetry[3] After allowing the fermionic zero modes associated with the 12 broken susy generators to saturate $(2J)^6/6!$ in the helicity trace given in (1.3), the non-trivial hair modes consist of

- 4 left moving fermionic modes arising from the deformations of the gravitino giving rise to the contribution
$$Z^{4d:f}_{\text{hair}:1A} = \prod_{l=1}^{\infty}(1-e^{2\pi i l\rho})^4. \tag{2.10}$$
- 3 left moving bosonic modes associated with the oscillation of the effective string in the 3 transverse directions $\mathbb{R}^3$ as Taub-NUT is asymptotically $\mathbb{R}^3 \times \tilde{S}^1$
$$Z^{4d:\perp}_{\text{hair}:1A} = \prod_{l=1}^{\infty}\frac{1}{(1-e^{2\pi i l\rho})^3}. \tag{2.11}$$
- 21 left moving bosonic modes, these arise from the deformations of the 21 anti-self-dual forms of type IIB on K3. These deformations involve 21 scalar functions folded with the 2 form $d\omega_{TN}$ on the Taub-Nut given by
$$\delta H^s = h^s(v)dv \wedge d\omega_{TN}, \qquad v = t + x^5, \quad s = 1, \cdots 21. \tag{2.12}$$
Counting these oscillations we obtain
$$Z^{4d:\text{asd}}_{\text{hair}:1A} = \prod_{l=1}^{\infty}\frac{1}{(1-e^{2\pi i l\rho})^{21}}. \tag{2.13}$$
The 21 anti-self dual forms arise from compactifying the RR 4-form on the 19 anti-self dual 2 form of the $K3$ together with the NS 2-form and the RR 2-form of type IIB.

Note that in the partition function we labelled the chemical potential to count the oscillations by ρ, this is because exciting these left moving momentum modes correspond to exciting the electric charge of the dyon [19]. Now combining these partition functions we obtain

$$\begin{aligned} Z^{4d}_{\text{hair}:1A} &= Z^{4d:f}_{\text{hair}:1A} \times Z^{4d:\perp}_{\text{hair}:1A} \times Z^{4d:\text{asd}}_{\text{hair}:1A} \\ &= \prod_{l=1}^{\infty}(1-e^{2\pi i(l\rho)})^{-20}. \end{aligned} \tag{2.14}$$

The Bosonic hair partition function is given by

$$Z^{\text{bosons}}_{\text{hair}:1A} = Z^{4d:\perp}_{\text{hair}:1A} \times Z^{4d:\text{asd}}_{\text{hair}:1A} = \frac{e^{2\pi i\rho}}{\eta^{24}(\rho)}, \tag{2.15}$$

[3] It is easy to see from the heteroric frame that only left moving oscillations preserve supersymmetry.

this is identical to that of the counting the degeneracy of purely electric states in this model without the zero point energy. This observation will help in the generalizations to CHL models.

To obtain the partition function of horizon states we factor out the hair degrees of freedom resulting in

$$Z_{\rm hor} = \frac{1}{\tilde{\Phi}_{10}(\rho,\sigma,v) Z^{4d}_{\text{hair:1}A}}. \tag{2.16}$$

The index for the horizon states can be then be obtained by extracting the Fourier coefficients using the expression given by

$$d_{\rm hor} = -(-1)^{Q\cdot P} \int_{\mathcal{C}} d\rho d\sigma dv\, e^{-\pi i(\rho Q^2+\sigma P^2+2vQ\cdot P)} \frac{1}{\tilde{\Phi}_{10}(\rho,\sigma,v)} \prod_{l=1}^{\infty}(1-e^{2\pi i(l\rho)})^{20}. \tag{2.17}$$

Here the contour $\mathcal{C}$ is same as that defined in (2.2).

2.2 Orbifolds of $K3 \times T^2$

$2A$ orbfiold. Before we present the analysis for the most general orbifold, let us examine in detail the analysis for the $2A$ orbifold. In this case, the orbifold acts by exchanging 8 pairs of anti-self dual $(1,1)$ forms out of the 19 anti-self dual forms of $K3$ with the $1/2$ shift on S^1 [15]. Note that because of the $1/2$ shift, the natural unit of momentum on S^1 is $N = 2$. With this input we are ready to repeat the analysis for the partition function of the hair modes

- The 4 left moving fermionic modes arising from the deformations of the gravitino give rise to the contribution

$$Z^{4d:f}_{\text{hair:2}A} = \prod_{l=1}^{\infty}(1-e^{4\pi i l\rho})^4. \tag{2.18}$$

  Note that due to the fact that the periodicity is now $\frac{2\pi}{N}$, the unit of momentum is doubled.

- The 3 transverse bosonic deformations along R^3 of the effective string results in

$$Z^{4d:\perp}_{\text{hair:2}A} = \prod_{l=1}^{\infty}\frac{1}{(1-e^{4\pi i l\rho})^3}. \tag{2.19}$$

- The action of the orbifold projects out 8 anti-self dual forms. The analysis for $13 = 11+2$.[4] invariant anti-self dual forms proceeds as before except for the fact that the unit of momentum is 2

$$Z^{4d:\text{asd}}_{\text{hair:2}A}\big|_{\text{invariant}} = \prod_{l=1}^{\infty}\frac{1}{(1-e^{4\pi i l\rho})^{13}}. \tag{2.20}$$

[4]The 2 arises from the anti-self dual component of the RR 2-form and the NS 2-form.

Consider the following boundary conditions of the function $h(s)$ in (2.12) for the 8 projected anti-self dual forms

$$h\left(v + \frac{2\pi}{N}\right) = -h(v), \qquad N = 2. \tag{2.21}$$

These deformation pick up sign when one move by 1/2 unit on S^1. The partition function corresponding to these modes is given by

$$Z^{4d\text{:asd}}_{\text{hair:}2A}|_{\text{twisted}} = \prod_{l=1}^{\infty} \frac{1}{(1 - e^{2\pi i(2l-1)\rho})^8}. \tag{2.22}$$

Note that these modes are twisted for the circle of radius $2\pi/N, N = 2$, they obey anti-periodic boundary conditions. However in supergravity periodicities are measured over the circle of radius 2π and they are periodic for this radius, therefore these modes can be counted as hair modes. Together, the contribution of the anti-self dual forms to the partition function is given by

$$\begin{aligned} Z^{4d\text{:asd}}_{\text{hair:}2A} &= Z^{4d\text{:asd}}_{\text{hair:}2A}|_{\text{invariant}} \times Z^{4d\text{:asd}}_{\text{hair:}2A}|_{\text{twisted}} \\ &= \prod_{l=1}^{\infty} \frac{1}{(1 - e^{4\pi i l\rho})^5} \prod_{l=1}^{\infty} \frac{1}{(1 - e^{2\pi i l\rho})^8}. \end{aligned} \tag{2.23}$$

Now combining all the hair modes we obtain

$$\begin{aligned} Z^{4d}_{\text{hair:}2A} &= Z^{4d:f}_{\text{hair:}2A} \times Z^{4d:\perp}_{\text{hair:}2A} \times Z^{4d\text{:asd}}_{\text{hair:}2A} \\ &= \prod_{l=1}^{\infty} (1 - e^{4\pi i l\rho})^{-4} (1 - e^{2\pi i l\rho})^{-8}. \end{aligned} \tag{2.24}$$

Observe that the partition function of the bosonic hair modes is given by

$$\begin{aligned} Z^{4d:\,b}_{\text{hair:}2A} &= Z^{4d:\perp}_{\text{hair:}2A} \times Z^{4d\text{:asd}}_{\text{hair:}2A} \\ &= \prod_{l=1}^{\infty} (1 - e^{4\pi i l\rho})^{-8} (1 - e^{2\pi i l\rho})^{-8} \\ &= \frac{e^{2\pi i\rho}}{\eta^8(2\rho)\eta^8(\rho)}. \end{aligned} \tag{2.25}$$

This is the partition function of the fundamental string in the $N = 2$ CHL orbifold of the heterotic theory with the zero point energy removed [19, 29].

pA orbifolds $p = 2, 3, 5, 7$. The construction of the hair modes for the case of orbifolds of prime order, the method proceeds as discussed in detail for the $2A$ orbifold. In each case we need to count the number of 2-forms which are left invariant and which pick up phases and evaluate the partition function. The result for the bosonic hair modes is given by

$$Z^{4d:\,b}_{\text{hair:}\,pA} = \prod_{l=1}^{\infty} \frac{1}{(1 - e^{2\pi i\rho N l})^{k+2} (1 - e^{2\pi i l\rho})^{k+2}}. \tag{2.26}$$

N	l	$-b^2$	$\sum_{s=0}^{N-1} e^{-2\pi isl/N} c^{(0,s)}(-b^2)$
p	$N \vert l$	0	$2k = \frac{48}{N+1} - 4$
		-1	2
	$N \nmid l$	0	$k + 2 = \frac{24}{N+1}$
		-1	0

Table 1. Values of $\sum_{s=0}^{N-1} e^{-2\pi isl/N} c^{(0,s)}(-b^2)$ for orbifolds of $K3$ with prime order ($N = p$).

where

$$k = \frac{24}{p+1} - 2. \tag{2.27}$$

Note that this is the partition of the states containing only the electric charges or the fundamental string without the zero point energy [19]. Now including the 4 fermionic deformations we obtain

$$\begin{aligned} Z^{4d}_{\text{hair: }pA} &= \prod_{l=1}^{\infty} (1 - e^{2\pi i(Nl\rho)})^{-(k+2)} (1 - e^{2\pi i(l\rho)})^{-(k+2)} (1 - e^{2\pi i(Nl\rho)})^4 \\ &= \prod_{l=1}^{\infty} (1 - e^{2\pi i(Nl\rho)})^{-2k} \prod_{N \nmid l} (1 - e^{2\pi i(l\rho)})^{-(k+2)}. \end{aligned} \tag{2.28}$$

It is useful to rewrite this expression as follows

$$\begin{aligned} Z^{4d}_{\text{hair: }pA} &= \prod_{l=1}^{\infty} (1 - e^{2\pi i(Nl\rho)})^{-\sum c^{(0,s)}(0)} \prod_{N \nmid l} (1 - e^{2\pi i(l\rho)})^{-\sum e^{-2\pi isl/N} c^{(0,s)}(0)} \\ &= \prod_{l \neq 0} (1 - e^{2\pi i(l\rho)})^{-\sum e^{-2\pi isl/N} c^{(0,s)}(0)}. \end{aligned} \tag{2.29}$$

The sum is on the range of $s = 0$ to $N - 1$ and $N \nmid l$ implies N does not divide l.

The values of $\sum_{s=0}^{N-1} e^{-2\pi isl/N} c^{(0,s)}(-b^2)$ for prime N are listed in table 1.

Orbifolds of composite order: $4B, 6A, 8A$. One can count the hair modes in a similar fashion to the case of orbifolds with prime order. The only difference would arise for the bosonic modes $Z^{\text{bosons}}_{\text{hair}}$, which needs to be replaced by the fundamental string in these theories without the zero point energy. Including the 4 fermionic hairs, we see that the answer can be written in the same form as that seen for orbifolds with prime order

$$Z^{4d}_{\text{hair:CHL}} = \prod_{l=1}^{\infty} (1 - e^{2\pi i(Nl\rho)})^{-\sum c^{(0,s)}(0)} \prod_{N \nmid l} (1 - e^{2\pi i(l\rho)})^{-\sum e^{-2\pi isl/N} c^{(0,s)}(0)}. \tag{2.30}$$

The sum ranges from $s = 0$ to $N - 1$. This can be rewritten as

$$Z^{4d}_{\text{hair:CHL}} = \prod_{l=1}^{\infty} (1 - e^{2\pi i(l\rho)})^{-\sum e^{-2\pi isl/N} c^{(0,s)}(0)}. \tag{2.31}$$

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N	l	$-b^2$	$\sum_{s=0}^{N-1} e^{-2\pi isl/N} c^{(0,s)}(-b^2)$
4	$4\vert l$	0	6
		-1	2
	$2\vert l,\ 4\nmid l$	0	6
	$2\nmid l$	0	4
6	$6\vert l$	0	4
		-1	2
	$2\vert l,\ 6\nmid l$	0	4
	$3\vert l,\ 6\nmid l$	0	4
	$2\nmid l, 3\mid l$	0	2
8	$8\vert l$	0	2
		-1	2
	$2\vert l,\ 4\nmid l$	0	3
	$4\vert l,\ 8\nmid l$	0	4
	$2\nmid l$	0	2

Table 2. Values of $\sum_{s=0}^{N-1} e^{-2\pi isl/N} c^{(0,s)}(-b^2)$ for non-prime CHL orbifolds of $K3$. $\sum_{s=0}^{N-1} e^{-2\pi isl/N} c^{(0,s)}(-1) = 0$ if $N \nmid l$ for any of these cases.

For the geometric CHL orbifolds , we list $\sum_{s=0}^{N-1} e^{-2\pi isl/N} c^{(0,s)}(-b^2)$ for different $N = 4, 6, 8$ in table 2. Using the data from table 2 we obtain

$$\begin{aligned}
Z^{4d}_{\text{hair:4B}} &= \prod_{l=1}^{\infty}(1-e^{2\pi i(4l\rho)})^4(1-e^{2\pi i(4l\rho)})^{-4}(1-e^{2\pi i(2l\rho)})^{-2}(1-e^{2\pi i(l\rho)})^{-4} \qquad (2.32)\\
&= \prod_{l=1}^{\infty}(1-e^{2\pi i(2l\rho)})^{-2}(1-e^{2\pi i(l\rho)})^{-4}\\
Z^{4d}_{\text{hair:6A}} &= \prod_{l=1}^{\infty}(1-e^{2\pi i(6l\rho)})^4(1-e^{2\pi i(6l\rho)})^{-2}(1-e^{2\pi i(2l\rho)})^{-2}(1-e^{2\pi i(3l\rho)})^{-2}(1-e^{2\pi i(l\rho)})^{-2}\\
&= \prod_{l=1}^{\infty}(1-e^{2\pi i(6l\rho)})^2(1-e^{2\pi i(2l\rho)})^{-2}(1-e^{2\pi i(3l\rho)})^{-2}(1-e^{2\pi i(l\rho)})^{-2}\\
Z^{4d}_{\text{hair:8A}} &= \prod_{l=1}^{\infty}(1-e^{2\pi i(8l\rho)})^4(1-e^{2\pi i(8l\rho)})^{-2}(1-e^{2\pi i(2l\rho)})^{-1}(1-e^{2\pi i(4l\rho)})^{-1}(1-e^{2\pi i(l\rho)})^{-2}\\
&= \prod_{l=1}^{\infty}(1-e^{2\pi i(8l\rho)})^2(1-e^{2\pi i(2l\rho)})^{-1}(1-e^{2\pi i(4l\rho)})^{-1}(1-e^{2\pi i(l\rho)})^{-2}.
\end{aligned}$$

Horizon states. We factor out the hair degrees of freedom to obtain the horizon states, this is given by

$$Z^{4d}_{\text{hor:CHL}} = -\frac{1}{\tilde{\Phi}_k(\rho,\sigma,v)}\prod_{l=1}^{\infty}(1-e^{2\pi i(l\rho)})^{\sum_s e^{-2\pi isl/N}c^{(0,s)}(0)}. \qquad (2.33)$$

It is useful to use the product form of $\tilde{\Phi}_k$ given in (2.3) to rewrite the partition function of the horizon states as follows

$$
\begin{aligned}
Z^{4d}_{\text{hor:CHL}} &= -e^{-2\pi i(\rho+\sigma/N+v)} \prod_{r=0}^{N-1} \prod_{\substack{k'\in\mathbb{Z}+r/N, l\in\mathbb{Z},\\ j\in\mathbb{Z}\\ k'>0, l\geq 0}} (1-e^{2\pi i(k'\sigma+l\rho+jv)})^{-\sum_s e^{-2\pi i s l/N} c^{(r,s)}(4k'l-j^2)} \\
&\quad \times \prod_{l=1}^{\infty}(1-e^{2\pi i(Nl\rho+v)})^{-2} \prod_{l=1}^{\infty}(1-e^{2\pi i(Nl\rho-v)})^{-2}(1-e^{-2\pi i v})^{-2} \qquad (2.34)\\
&= -e^{-2\pi i(\rho+\sigma/N)} \prod_{r=0}^{N-1} \prod_{\substack{k'\in\mathbb{Z}+r/N, l\in\mathbb{Z},\\ j\in\mathbb{Z}\\ k'>0, l\geq 0}} (1-e^{2\pi i(k'\sigma+l\rho+jv)})^{-\sum_s e^{-2\pi i s l/N} c^{(r,s)}(4k'l-j^2)} \\
&\quad \times \prod_{l=1}^{\infty}(1-e^{2\pi i(Nl\rho+v)})^{-2} \prod_{l=1}^{\infty}(1-e^{2\pi i(Nl\rho-v)})^{-2}(e^{\pi i v}-e^{-\pi i v})^{-2}.
\end{aligned}
$$

This form of the horizon partition function is useful in the next section. The index for the horizon states is given by

$$
\begin{aligned}
d_{\text{hor:CHL}} &= -(-1)^{Q\cdot P} \int_{\mathcal{C}} d\rho d\sigma dv \; e^{-\pi i(N\rho Q^2+\sigma P^2/N+2vQ\cdot P)} \frac{1}{\tilde{\Phi}_k(\rho,\sigma,v)} \\
&\quad \times \prod_{l=1}^{\infty}(1-e^{2\pi i(l\rho)})^{\sum_s e^{-2\pi i s l/N} c^{(0,s)}(0)}. \qquad (2.35)
\end{aligned}
$$

Non-geometric orbifolds: $\mathbf{11}\boldsymbol{A}, \mathbf{14}\boldsymbol{A}, \mathbf{15}\boldsymbol{A}, \mathbf{23}\boldsymbol{A}$. For completeness we note that we can extend the counting of hair modes to g' orbifolds of $K3$ where g' corresponds all the remaining conjugacy classes of M_{23}. The CHL orbifolds also form a part of these, however the ones discussed in this section are non-geometric. The hair modes in these cases can also be written as:

$$
Z^{4d}_{\text{hair}:g'} = \prod_{l\neq 0}(1-e^{2\pi i(l\rho)})^{-\sum e^{-2\pi i s l/N} c^{(0,s)}(0)}. \qquad (2.36)
$$

To be explicit, we list the of values of $\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c^{(0,s)}(-b^2)$ for different N for $N = 11, 14, 15, 23$ in table 3. Using the results from table 3 we write:

$$
\begin{aligned}
Z^{4d}_{\text{hair}:11A} &= \prod_{l=1}^{\infty}(1-e^{2\pi i(11l\rho)})^4(1-e^{2\pi i(l\rho)})^{-2}(1-e^{2\pi i(11l\rho)})^{-2}, \qquad (2.37)\\
Z^{4d}_{\text{hair}:14A} &= \prod_{l=1}^{\infty}(1-e^{2\pi i(14l\rho)})^4(1-e^{2\pi i(14l\rho)})^{-1} \qquad (2.38)\\
&\quad \times(1-e^{2\pi i(2l\rho)})^{-1}(1-e^{2\pi i(7l\rho)})^{-1}(1-e^{2\pi i(l\rho)})^{-1},\\
Z^{4d}_{\text{hair}:15A} &= \prod_{l=1}^{\infty}(1-e^{2\pi i(15l\rho)})^4(1-e^{2\pi i(15l\rho)})^{-1} \qquad (2.39)\\
&\quad \times(1-e^{2\pi i(3l\rho)})^{-1}(1-e^{2\pi i(5l\rho)})^{-1}(1-e^{2\pi i(l\rho)})^{-1},\\
Z^{4d}_{\text{hair}:23A} &= \prod_{l=1}^{\infty}(1-e^{2\pi i(23l\rho)})^4(1-e^{2\pi i(23l\rho)})^{-1}(1-e^{2\pi i(l\rho)})^{-1}. \qquad (2.40)
\end{aligned}
$$

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N	l	$-b^2$	$\sum_{s=0}^{N-1} e^{-2\pi isl/N} c^{(0,s)}(-b^2)$
11	$11\vert l$	0	0
		-1	2
	$11\nmid l$	0	2
14	$14\vert l$	0	0
		-1	2
	$2\vert l,\ 7\nmid l$	0	2
	$7\vert l,\ 2\nmid l$	0	2
	$2\nmid l, 7\nmid l$	0	1
15	$15\vert l$	0	0
		-1	2
	$3\vert l,\ 5\nmid l$	0	2
	$5\vert l,\ 3\nmid l$	0	2
	$3\nmid l, 5\nmid l$	0	1
23	$23\vert l$	0	-2
		-1	2
	$23\nmid l$	0	1

Table 3. Values of $\sum_{s=0}^{N-1} e^{-2\pi isl/N} c^{(0,s)}(-b^2)$ for non-geometric orbifolds of $K3$ where $g' \in [M_{23}]$. $\sum_{s=0}^{N-1} e^{-2\pi isl/N} c^{(0,s)}(-1) = 0$ if $N \nmid l$ for any of these cases.

The partition function of the horizon states in these models are given by the same expressions as in (2.34) with N replaced by the order of the conjugacy class and the coefficients $c^{(r,s)}$ read out from the respective twisted elliptic genus. Let us conclude by writing the general formula for the horizon states as

$$Z^{4d}_{\text{hor: g}'} = -e^{-2\pi i(\rho+\sigma/N)} \prod_{r=0}^{N-1} \prod_{\substack{k'\in\mathbb{Z}+r/N, l\in\mathbb{Z}, \\ j\in\mathbb{Z} \\ k'>0, l\geq 0}} (1-e^{2\pi i(k'\sigma+l\rho+jv)})^{-\sum_s e^{-2\pi isl/N} c^{(r,s)}(4k'l-j^2)}$$
$$\times \prod_{l=1}^{\infty}(1-e^{2\pi i(Nl\rho+v)})^{-2} \prod_{l=1}^{\infty}(1-e^{2\pi i(Nl\rho-v)})^{-2}(e^{\pi iv}-e^{-\pi iv})^{-2}. \tag{2.41}$$

2.3 Toroidal orbifolds

In this section we construct the hair for $\mathcal{N}=4$ theories obtained by freely acting $\mathbb{Z}_2, \mathbb{Z}_3$ involutions on T^6 [30]. Let us first briefly recall how these are constructed. In the type IIB frame, they are obtained by an inversion of 4 of the co-ordinates of T^4 together with a half shift along one of the S^1. The type IIA description of the theory is that of a freely acting orbifold with the action of $(-1)^{F_L}$ and a 1/2 shift along one of the circles of T^6.[5] A

[5]For details of these descriptions and the dyon configuration refer [18].

similar compactification of order 3 given by a $2\pi/3$ rotation along one 2D plane of T^4 and a $-2\pi/3$ rotation along another plus an $1/3$ shift along one of the circles of T^2 was also discussed in [18]. We call these models $\mathbb{Z}_2$ and $\mathbb{Z}_3$ toroidal orbifolds.

One key property of these models to keep in mind which will be important is that the breaking of the 32 supersymmetries of type IIB to 16 is determined by the size of S^1. This was not the case for the orbifolds of $K3\times T^2$, where supersymmetry was broken by the $K3$. For the toroidal models if the size of S^1 is infinite, the theory effectively behaves as though the theory has 32 supersymmetries. We will use this fact to propose certain fermionic zero modes which were present for the CHL models will become singular at the horizon.

The dyon partition function for the toroidal models is given by [31]:

$$\begin{aligned}\tilde{\Phi}_k(\rho,\sigma,v) &= e^{2\pi i(\rho+v)} \\ &\times \prod_{r=0}^{N-1} \prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},l\in\mathbb{Z},\\ j\in\mathbb{Z}\\ k',l\geq 0,\, j<0\; k'=l=0}} \left(1-e^{2\pi i(k'\sigma+l\rho+jv)}\right)^{\sum_{s=0}^{N-1} e^{2\pi isl/N} c^{r,s}(4k'l-j^2)}.\end{aligned} \tag{2.42}$$

The coefficients $c^{(r,s)}$ are read out from the following twisted elliptic genus for $\mathbb{Z}_2$ orbifold:

$$\begin{aligned}F^{(0,0)} &= 0, \\ F^{(0,1)} &= \frac{8}{3}A(\tau,z) - \frac{4}{3}B(\tau,z)\mathcal{E}_2(\tau), \\ F^{(1,0)} &= \frac{8}{3}A(\tau,z) + \frac{2}{3}B(\tau,z)\mathcal{E}_2\Big(\frac{\tau}{2}\Big), \\ F^{(1,1)} &= \frac{8}{3}A(\tau,z) + \frac{2}{3}B(\tau,z)\mathcal{E}_2\Big(\frac{\tau+1}{2}\Big).\end{aligned} \tag{2.43}$$

The corresponding Siegel form of weight $k=2$ can be written as

$$\tilde{\Phi}_2(\rho,\sigma,v) = \frac{\tilde{\Phi}_6^2(\rho,\sigma,v)}{\tilde{\Phi}_{10}(\rho,\sigma,v)}, \tag{2.44}$$

where $\tilde{\Phi}_6$ is the weight 6 Siegel modular form associated with the order 2 CHL orbifold. For the $\mathbb{Z}_3$ toroidal case the twisted elliptic genus is given by

$$\begin{aligned}F^{(0,0)} &= 0 \\ F^{(0,s)} &= A(\tau,z) - \frac{3}{4}B(\tau,z)\mathcal{E}_3(\tau) \\ F^{(r,rk)} &= A(\tau,z) + \frac{1}{4}B(\tau,z)\mathcal{E}_3\Big(\frac{\tau+k}{3}\Big), \quad r=1,2.\end{aligned} \tag{2.45}$$

The Siegel modular form associated with the $\mathbb{Z}_3$ toroidal orbifold has weight $k=1$ and is given by

$$\tilde{\Phi}_1(\rho,\sigma,v) = \frac{\tilde{\Phi}_4^{3/2}(\rho,\sigma,v)}{\tilde{\Phi}_{10}^{1/2}(\rho,\sigma,v)}, \tag{2.46}$$

where $\tilde{\Phi}_4$ is the weight 4 Siegel modular form associated with the order 3 CHL orbifold.

Let us construct the hair modes and horizon states for these models.

$T^6/\mathbb{Z}_2$ model.

- Just as in the case of the CHL models, we have 4 left moving fermions. This gives rise to

$$Z^{4d:f}_{\text{hair}:T^6/\mathbb{Z}_2} = \prod_{l=1}^{\infty}(1-e^{2\pi i(2l)\rho})^4. \tag{2.47}$$

- The deformations corresponding to the motion of the effective string in the 3 transverse directions of $R^3 \times \tilde{S}^1$ of the Taub-Nut space together with the fluctuations of the anti-self dual forms can be determined easily by examining the partition function of the fundamental string in this theory and removing the zero point energy. This partition function was determined in [18], using this result we obtain[6]

$$Z^{4d\ b}_{\text{hair}:T^6/\mathbb{Z}_2} = \prod_{l=1}^{\infty}\left[(1-e^{2\pi i(2l-1)\rho})^8(1-e^{4\pi i l\rho})^{-8}\right]. \tag{2.48}$$

- Contribution of the zero modes: The quantum mechanics of the bosonic zero modes describing the motion of the D1-D5 system in the Taub-Nut result in the following partition function [19]

$$Z^{4d:\text{ zeromodes}}_{\text{hair}:T^6/\mathbb{Z}_2} = -e^{2\pi i v}(1-e^{2\pi i v})^{-2}. \tag{2.49}$$

  For orbifolds of $K3$, this contribution from the bosonic zero modes was cancelled by the zero modes of 4 fermions from the right moving sector carrying angular momentum $J = \pm\frac{1}{2}$ whose partition function is given by $-(e^{\pi i v} - e^{-\pi i v})^2$ [5]. However for the toroidal model, we propose that these zero modes do not form part of the hair. They are either singular at the horizon or they are not localized outside the horizon. This is possible due to the fact that we are in a theory with 16 supersymmetries is tied to the radius of S^1. Verification of this proposal would involve a detailed study of the zero mode wave functions which we leave for the future. However we will perform consistency checks of this proposal in section 4. by evaluating the index of the horizon states.

Thus the hair modes of the $\mathbb{Z}_2$ toroidal model is given by

$$Z^{4d}_{\text{hair}:T^6/\mathbb{Z}_2} = -(e^{\pi i v} - e^{-\pi i v})^{-2}\prod_{l=1}^{\infty}\left[(1-e^{2\pi i(2l-1)\rho})^8(1-e^{4\pi i l\rho})^{-4}\right]. \tag{2.50}$$

The partition function of the horizon states of this model are given by

$$Z^{4d}_{\text{hor}:T^6/\mathbb{Z}_2} = -\frac{1}{\tilde{\Phi}_2(\rho,\sigma,v)Z^{4d}_{\text{hair}:T^6/\mathbb{Z}_2}}. \tag{2.51}$$

where $\tilde{\Phi}_2(\rho,\sigma,v)$ is given in (2.44) or (2.42).

[6] One can also obtain this by counting the number of invariant 2-forms and the forms which pick up a phase as done in [20].

The toroidal model has another special feature, they admit Wilson lines along T^4 [31], their partition function is given by

$$Z_{\text{Wilson:}T^4/\mathbb{Z}_2} = \prod_{l=1}^{\infty}\left[(1-e^{2\pi i(2l-1)\rho+2\pi i v})^2(1-e^{2\pi i(2l-1)\rho-2\pi i v})^2(1-e^{2\pi i(2l-1)\rho})^{-4}\right]. \tag{2.52}$$

It is possible that the Wilson lines might also be part of the hair modes. In section 4 we will see that including the Wilson lines as hair modes instead of the bosonic zero modes given in (2.49) does not preserve the positivity of the index of the horizon states.

$T^6/\mathbb{Z}_3$ model. Performing the same analysis as done for the $\mathbb{Z}_2$ orbfiold we obtain the following partition function for the hair modes

$$Z^{4d}_{\text{hair:T}^6/\mathbb{Z}_3} = -(e^{\pi i v}-e^{-\pi i v})^{-2}\prod_{l=1}^{\infty}\left[\frac{(1-e^{2\pi i(3l-1)\rho})^3(1-e^{2\pi i(3l-2)\rho})^3}{(1-e^{2\pi i(3l)\rho})^{-2}}\right]. \tag{2.53}$$

The horizon states is given by

$$Z^{4d}_{\text{hor:T}^6/\mathbb{Z}_3} = -\frac{1}{\tilde{\Phi}_1(\rho,\sigma,v)Z^{4d}_{\text{hair:T}^6/\mathbb{Z}_3}}, \tag{2.54}$$

where $\tilde{\Phi}_1$ is given by (2.46) or (2.42). For reference we also provide the partition function of the Wilson lines in this model

$$Z_{\text{Wilson:}T^4/\mathbb{Z}_3} = \prod_{l=1}^{\infty}\left[\frac{(1-e^{2\pi i((3l-1)\rho+v)})(1-e^{2\pi i((3l-2)\rho+v)})(1-e^{2\pi i((3l-1)\rho-v)})(1-e^{2\pi i((3l-2)\rho-v)})}{(1-e^{2\pi i((3l-1)\rho)})^2(1-e^{2\pi i((3l-2)\rho)})^2}\right] \tag{2.55}$$

From the expression for the Wilson lines and the infinite product representation given for $\tilde{\Phi}_k$ given in (2.42) we obtain the following useful expression for the partition function for the horizon modes for both the toroidal orbifolds.

$$\begin{aligned} Z^{4d}_{\text{hor; T}^6/\mathbb{Z}_\text{N}} = {}& e^{-2\pi i\rho}\prod_{r=0}^{N-1}\prod_{\substack{k\in\mathbb{Z}+r/N, l\in\mathbb{Z},\\ j\in\mathbb{Z}\\ k>0, l\geq 0}}(1-e^{2\pi i(k\sigma+l\rho+jv)})^{-\sum_s e^{-2\pi i s l/N}c^{(r,s)}(4kl-j^2)} \\ &\times\prod_{l=1}^{\infty}\left[(1-e^{2\pi i(Nl\rho+v)})^{-2}(1-e^{2\pi i(Nl\rho-v)})^{-2}\right](e^{\pi i v}-e^{-\pi i v})^2\times Z_{\text{Wilson:}T^4/\mathbb{Z}_N}. \end{aligned} \tag{2.56}$$

3 Horizon states for the BMPV black hole

We now examine the BMPV black hole in 5 dimensions, that is the transverse space now does not have the Taub-Nut solution. The main reason for studying the problem in 5

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dimensions is that the near horizon geometry of the BMPV black hole in 5 dimensions is same as the of the 1/4 BPS dyon in 4 dimensions. This implies that the partition function of the horizon states of these 2 systems should be identical. In this section we construct the partition function of the hair and the horizon states for the BMPV black hole in type IIB on $K3 \times S^1/g'$ as well as toroidal orbifolds of T^5. Here g' corresponds to all the conjugacy classes of M_{23}.

3.1 Partition function of BMPV black holes

The partition function for these black holes in the canonical compactification $K3 \times S^1$, was constructed in [5]. The same analysis can be extended to all the CHL models. The partition function receives contributions from the following sectors.

- The bound states of the D1-D5 system, this is given by the elliptic genus of the symmetric product of $K3/g'$. This contribution was evaluated in [19]. It is given by

$$Z^{5d}_{S^N K3/g'} = e^{-2\pi i\sigma/N} \prod_{r=0}^{N-1} \prod_{\substack{k\in\mathbb{Z}+r/N, l\in\mathbb{Z},\\ j\in\mathbb{Z}\\ k>0, l\geq 0}} (1-e^{2\pi i(k\sigma+l\rho+jv)})^{-\sum_s e^{-2\pi i s l/N} c^{(r,s)}(4kl-j^2)}. \tag{3.1}$$

- The centre of mass motion of the D1-D5 system in flat space. The degrees of freedom consist of 4 bosons and 4 fermions. 2 pairs of bosons carry the angular momentum $J = \pm 1$ [5]

$$Z^{5d}_{\text{c.o.m}} = \prod_{l=1}^{\infty} \Big[(1-e^{2\pi i(Nl\rho+v)})^{-2}(1-e^{2\pi i(Nl\rho-v)})^{-2}(1-e^{2\pi i Nl\rho})^4\Big]. \tag{3.2}$$

  Note that the only difference from the canonical model is that the unit of momentum on S^1 is N due to the $1/N$ shift.

- 4 right chiral zero modes which contribute as $(-1)^J e^{2\pi J}$ which contribute in pairs with $J = \pm\frac{1}{2}$

$$Z^{5d}_{\text{zeromodes}} = -(e^{\pi iv} - e^{-\pi iv})^2. \tag{3.3}$$

- A shift of $e^{-2\pi i\rho}$ to ensure to take into account of the difference in the electric charge measured at infinity and the horizon [5].

Combining all the sectors we obtain the following expression for the partition function for BMPV black hole for all orbifolds of $K3 \times S^1$.

$$\begin{aligned} Z^{5d}_{g'} = &-e^{-2\pi i(\rho+\sigma/N)} \prod_{r=0}^{N-1} \prod_{\substack{k\in\mathcal{Z}+r/N, l\in\mathcal{Z},\\ j\in\mathcal{Z}\\ k>0, l\geq 0}} (1-e^{2\pi i(k\sigma+l\rho+jv)})^{-\sum_s e^{-2\pi i s l/N} c^{(r,s)}(4kl-j^2)} \\ &\times \prod_{l=1}^{\infty} \Big[(1-e^{2\pi i(Nl\rho+v)})^{-2}(1-e^{2\pi i(Nl\rho-v)})^{-2}(e^{\pi iv}-e^{-\pi iv})^2(1-e^{2\pi i Nl\rho})^4\Big]. \end{aligned} \tag{3.4}$$

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Here the coefficients $c^{(r,s)}$ have to be read out from the twisted elliptic genus of $K3$ by g' corresponding to the conjugacy classes of M_{23}.

Using the counting of states for the dyon partition function done in [31] we can extend the analysis to the toroidal models. We present the analysis in some detail for the $T^6/\mathbb{Z}_2$ model Here the contributions arise from the following:

- The bound state of the D1-D5 system on the $T^4/\mathbb{Z}_2$ orbifold is given by

$$Z^{5d}_{S^N T^4/\mathbb{Z}_2} = \prod_{r=0}^{N-1} \prod_{\substack{k\in\mathbb{Z}+r/N, l\in\mathbb{Z}, \\ j\in\mathbb{Z} \\ k>0, l\geq 0}} (1-e^{2\pi i(k\sigma+l\rho+jv)})^{-\sum_s e^{-2\pi i s l/N} c^{(r,s)}(4kl-j^2)} \qquad \text{with } N=2. \tag{3.5}$$

  Here the coefficients $c^{(r,s)}$ are read out from the expansion of the functions given in (2.43).

- The contribution of the Wilson lines on $T^4/\mathbb{Z}_2$ which is given by

$$Z^{5d}_{\text{Wilson}:T^4/\mathbb{Z}_2} = \prod_{l=1}^{\infty} \left[(1-e^{2\pi i(2l-1)\rho+2\pi i v})^2 (1-e^{2\pi i(2l-1)\rho-2\pi i v})^2 (1-e^{2\pi i(2l-1)\rho})^{-4}\right]. \tag{3.6}$$

- The partition function corresponding to the centre of mass motion of the D1-D5 system in the transverse space

$$Z^{5d}_{\text{c.o.m}} = \prod_{l=1}^{\infty} \left[(1-e^{2\pi i(Nl\rho+v)})^{-2} (1-e^{2\pi i(Nl\rho-v)})^{-2} (1-e^{2\pi i N l\rho})^4\right], \quad \text{with } N=2. \tag{3.7}$$

- The contribution of the zero modes

$$Z^{5d}_{\text{zeromodes}} = -(e^{\pi i v} - e^{-\pi i v})^2. \tag{3.8}$$

- The shift in the electric charge accounted for by the factor $e^{-2\pi i\rho}$.

Combining all the contributions we obtain

$$\begin{aligned} Z^{5d}_{T^5/\mathbb{Z}_N} = -e^{-2\pi i\rho} &\prod_{\substack{k\in\mathcal{Z}+r/N, l\in\mathcal{Z}, \\ j\in\mathcal{Z} \\ k>0, l\geq 0}} (1-e^{2\pi i(k\sigma+l\rho+jv)})^{-\sum_s e^{-2\pi i s l/N} c^{(r,s)}(4kl-j^2)} \\ &\times \prod_{l=1}^{\infty} \left[(1-e^{2\pi i(Nl\rho+v)})^{-2} (1-e^{2\pi i(Nl\rho-v)})^{-2} (1-e^{2\pi i N l\rho})^4\right] (e^{\pi i v}-e^{-\pi i v})^2 \\ &\times Z_{\text{Wilson}:T^4/\mathbb{Z}_N} \qquad \text{with } N=2. \end{aligned} \tag{3.9}$$

The partition function of the BMPV black hole in the $T^5/\mathbb{Z}_3$ is obtained model is given by the same expression as in (3.9) except that the coefficients $c^{(r,s)}$ must be read out from the functions given in (2.45) and $N\to 3$.

3.2 Orbifolds of $K3 \times S^1$

We now construct the hair modes in 5 dimensions for the $K3 \times S^1/g'$ where the quotient by g' is associated with any conjugacy classes of the Mathieu group M_{23}. The analysis proceeds identical to that done in [6], the only difference being that the unit of momentum on S^1 is N. Here we briefly state the contributions.

- The contribution of the 4 real left moving gravitino deformations of the BMPV black hole[7]

$$Z^{5d;\, f}_{\text{hair: } g'} = \prod_{l=1}^{\infty} (1 - e^{2\pi i l N \rho})^4. \tag{3.10}$$

- The contribution of the 8 real gravitino zero modes among the 12 modes due to broken supersymmetries which carry angular momentum $J = \pm\frac{1}{2}$

$$Z^{5d;\,\text{zero modes}}_{\text{hair: } g'} = (e^{\pi i v} - e^{-\pi i v})^4. \tag{3.11}$$

Combining these contributions we obtain

$$Z^{5d}_{\text{hair: g}'} = (e^{\pi i v} - e^{-\pi i v})^4 \prod_{l=1}^{\infty} (1 - e^{2\pi i l N \rho})^4. \tag{3.12}$$

The partition function for the horizon states is given by

$$Z^{5d}_{\text{hor: } g'} = \frac{Z^{5d}_{g'}}{Z^{5d}_{\text{hair: g}'}}. \tag{3.13}$$

Now comparing the horizon states of the $4d$ dyons from (2.41) and using (3.4) and (3.12) in (3.13) we can easily conclude

$$Z^{5d}_{\text{hor: } g'} = Z^{4d}_{\text{hor: } g'}. \tag{3.14}$$

3.3 Toroidal models

For the toroidal models the contributions of the hair are as follows.

- The contribution of the 4 left moving gravitino modes which result in

$$Z^{5d;\, f}_{\text{hair: } T^5/\mathbb{Z}_N} = \prod_{l=1}^{\infty} (1 - e^{2\pi i l N \rho})^4, \qquad N = 2, 3. \tag{3.15}$$

- The contirbution of the zero modes. As we discussed earlier, supersymmetry in these models is tied to the radius of S^1. We propose that due to this, out of 8 gravitino zero modes arising from broken supersymmetries which has angular momentum $J = \pm\frac{1}{2}$, the wave functions of 4 of them either become singular at the horizon or they not localized outside the horizon. These 4 modes should not be counted as hair modes. Therefore the contribution of the zero modes in these models are given by

$$Z^{5d;\,\text{zero modes}}_{\text{hair: } T^5/\mathbb{Z}_N} = -(e^{\pi i v} - e^{-\pi i v})^2. \tag{3.16}$$

As we will see consistency checks for this proposal will be done in section 4.

[7] The bosonic deformations were shown to be singular at the horizon in [6].

Combining these contributions we obtain

$$Z^{5d}_{\text{hair: T}^5/\mathbb{Z}_\text{N}} = -(e^{\pi i v} - e^{-\pi i v})^2 \prod_{l=1}^{\infty}(1 - e^{2\pi i l N\rho})^4. \tag{3.17}$$

The horizon partition function from the 5d perspective is given by

$$Z^{5d}_{\text{hor:T}^5/\mathbb{Z}_\text{N}} = \frac{Z^{5d}_{T^5/\mathbb{Z}_N}}{Z^{5d}_{\text{hair: T}^5/\mathbb{Z}_\text{N}}}. \tag{3.18}$$

Comparing the 4d horizon partition function given in (2.56) and using (3.9) and (3.17) in (3.18) we see that

$$Z^{5d}_{\text{hor:T}^5/\mathbb{Z}_\text{N}} = Z^{4d}_{\text{hor:T}^6/\mathbb{Z}_\text{N}}. \tag{3.19}$$

4 The sign of the index for horizon states

In this section we will address the main goal of the paper. We observe that the index of horizon states is always positive.

4.1 Canonical example: $K3 \times T^2$

For the un-orbifolded model recall that the hair in 4d is given by

$$Z^{4d}_{\text{hair}:1A} = \prod_{l=1}^{\infty} \frac{1}{(1 - e^{2\pi i l\rho})^{20}}. \tag{4.1}$$

The partition function of the horizon states is obtained by

$$Z_{\text{hor}:1A} = \frac{1}{\tilde{\Phi}_{10}(\rho,\sigma,v) Z^{4d}_{\text{hair}:1A}} = \frac{\prod_{l=1}^{\infty}(1 - e^{2\pi i l\rho})^{20}}{\tilde{\Phi}_{10}(\rho,\sigma,v)}. \tag{4.2}$$

It was observed in [7] that the index $-B_6$ or the Fourier coefficients of $1/\Phi_{10}$ extracted using the contour in (2.2) subject to the kinematic restrictions

$$Q.P \geq 0, \quad Q \cdot P \leq Q^2, \quad Q \cdot P \leq P^2, \quad Q^2, P^2, (Q^2P^2 - (Q.P)^2) > 0 \tag{4.3}$$

were positive. The contour together with the above kinematic constraints ensures that the index counts single centred dyons. Furthermore [10] proved that the index of all single centered dyons with $P^2 = 2, 4$ is positive. These works assumed that there existed a frame in which the fermionic zero modes associated with broken supersymmetries were the only hair. We have seen that the type IIB frame the hair degrees of freedom is given by (4.1). Now naively it seems from the expression for the horizon states in (4.2) there are negative terms introduced due to the factor in the numerator and the observation of positivity seen in [7] and [10] might be violated once the hair in the type IIB frame is factored out. However we will show by adapting the proof of [10] that single centred dyons with $P^2 = 2$ do have positive index. For other values of charges we evaluate the index numerically, our results are presented in table 4. We observe that for single centered dyons the index is indeed positive.

$(Q^2,\ P^2)\quad \backslash\ Q\cdot P$	0	1	2	3	4
(2, 2)	28944	13863	1608	327	0
(2, 4)	761312	406296	72424	6936	− 648
(2, 6)	12324920	6995541	1423152	96619	−13680
(2, 8)	148800072	88006584	19366320	1152216	−164244
(4, 2)	272832	154236	28944	1836	− 648
(4, 4)	12980224	8595680	2665376	406296	25760
(4, 6)	333276712	235492308	85781820	16141380	1423152
(6, 6)	6227822652	4771720755	2158667028	572268361	85781820

Table 4. Index of horizon states for $K3\times T^2$, note that negative numbers have zero or negative values for $Q^2P^2-(Q\cdot P)^2$.

Proof of positivity at $P^2=2$. Let us perform a Fourier expansion of $\frac{1}{\tilde{\Phi}_{10}(\tau,\sigma,z)}$ in terms of Jacobi forms.[8]

$$\frac{1}{\tilde{\Phi}_{10}(q,p,y)}=\sum_{m=-1}^{\infty}\psi_m(\tau,z)p^m,\qquad q=e^{2\pi i\tau}\quad ,p=e^{2\pi i\sigma},\quad y=e^{2\pi iz}. \tag{4.4}$$

$\psi_m(\tau,z)\eta^{24}(\tau)$ is a weak Jacobi form of weight 2 and index m. In [32] it was shown that $\psi_m(\tau,z)$ admits the following decomposition

$$\psi_m(\tau,z)=\psi_m^{\rm P}(\tau,z)+\psi_m^{\rm F}(\tau,z), \tag{4.5}$$

where, $\psi_m^{\rm F}(\tau,z)$ has no poles in z. The polar part is given by an Appell-Lerch sum:

$$\begin{aligned}\psi_m^{\rm P}(\tau,z)&=\frac{p_{24}(m+1)}{\eta^{24}(\tau)}\mathcal{A}_{2,m}(\tau,z),\\ \mathcal{A}_{2,m}(\tau,z)&=\sum_{s\in\mathbb{Z}}\frac{q^{ms^2+s}y^{2ms+1}}{(1-q^sy)^2}.\end{aligned} \tag{4.6}$$

At $P^2=2$ we have $m=1$ and we can write

$$\psi_1^{\rm F}(\tau,z)=-\frac{3}{\Delta}(E_4B(\tau,z)+216\mathcal{H}(\tau,z)). \tag{4.7}$$

We need to show that $\psi_1^h=-\frac{3}{q\prod(1-q^n)^4}(E_4B(\tau,z)+216\mathcal{H}(\tau,z))$ has the positivity property. Here $\mathcal{H}$ is the simplest mock Jacobi form defined by the Hurwitz-Kronecker class numbers $H(n)$

$$\mathcal{H}(\tau,z)=\sum_{\substack{n=0\\ j\in\mathbb{Z}}}^{\infty}H(4n-j^2)q^ny^j. \tag{4.8}$$

[8]We use the variable τ instead of ρ and z in place of v to be consistent with the earlier work [13].

The coefficients $H(n)$ are defined by[9]

$$H(n) = 0 \qquad \text{for } n < 0, \tag{4.9}$$

$$\sum_{n\in\mathbb{Z}} H(n)q^n = -\frac{1}{12} + \frac{1}{3}q^3 + \frac{1}{2}q^4 + q^7 + q^8 + q^{11} + \cdots \tag{4.10}$$

$$\mathcal{H}(\tau, z) = \theta_3(2\tau, 2z)h_0(\tau) + \theta_2(2\tau, 2z)h_1(\tau). \tag{4.11}$$

We can write the weak Jacobi form $B(\tau, z)$ given in (2.6) as:

$$B(\tau, z) = \frac{\theta_1^2(\tau, z)}{\eta^6} = \frac{1}{\eta^6}(\theta_2(2\tau)\theta_3(2\tau, 2z) - \theta_3(2\tau)\theta_2(2\tau, 2z)) \tag{4.12}$$

where, $\theta_2(\tau, z) = \sum_{n\in\mathbb{Z}} q^{\frac{(n+1/2)^2}{2}} y^{n+1/2}$ and $\theta_3(\tau, z) = \sum_{n\in\mathbb{Z}} q^{n^2/2} y^n$ and $y = e^{2\pi i z}$. So we see that even and odd powers of y are separated in ψ_1^F by the two theta functions. With this we can write ψ_1^F and ψ_1^h as follows:

$$\begin{aligned}
\psi_1^F &= \frac{3}{\Delta}\left(\theta_2(2\tau, 2z)\left(\frac{\theta_3(2\tau)}{\eta^6}E_4 - 216h_1(\tau)\right) - \theta_3(2\tau, 2z)\left(\frac{\theta_2(2\tau)}{\eta^6}E_4 + 216h_0(\tau)\right)\right) \\
\psi_1^h &= \frac{3}{\Delta_4}\left(\theta_2(2\tau, 2z)\left(\frac{\theta_3(2\tau)}{\eta^6}E_4 - 216h_1(\tau)\right) - \theta_3(2\tau, 2z)\left(\frac{\theta_2(2\tau)}{\eta^6}E_4 + 216h_0(\tau)\right)\right),
\end{aligned} \tag{4.13}$$

where $\Delta_4 = q\prod_{n=1}^{\infty}(1 - q^n)^4$. The following results are known

1. The Fourier coefficients in $h_0(\tau)$ and $h_1(\tau)$ are positive except for q^0 in $h_0(\tau)$ [10].
2. All Fourier coefficients in the q expansion of $\frac{\theta_2(2\tau)}{\eta^6}$ or $\frac{\theta_3(2\tau)}{\eta^6}$ are positive.
3. $E_4 = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n$, where $\sigma_3(n)$ is given by, $\sum_{d,d|n} d^3$.

Let us observe the expression: $\left(\frac{\theta_2(2\tau)}{\eta^6}E_4 + 216h_0(\tau)\right)$. The only negative Fourier coefficient appears at q^0. Using these observations we can prove the following lemma:

Lemma 1. *For a function $f(q) = -1 + \sum_{n=1}^{\infty} a(n)q^n$ having all positive $a(n)$, the function $\frac{f(q)}{\prod_{n=1}^{\infty}(1-q^n)^k}$ has positive coefficients as long as $a(1) > k$ and $a(n+1) > k$ for all $n \in \mathbb{N}$.*

Proof. We prove this for $\frac{1}{(1-q)^k}$ and then the rest can be similarly proved by using $q \to q^r$ and taking $f_{r+1}(q) = \frac{f_r(q)}{(1-q^{r+1})^k}$. For f_2 the coefficient of q^1 is, $a(1) - k > 0$ and the coefficient of q^N for $N > 1$ is given by,

$$-\binom{N+k-1}{N} + \binom{N+k-2}{N-1}a(1) + \binom{N+k-3}{N-2}a(2)\cdots > k. \qquad \text{Q.E.D.}$$

[9] Explicit formula and detailed properties of $H(n), h_0(\tau), h_1(\tau)$ can be found in [33].

We can write

$$\frac{1}{16}\left(\frac{\theta_2(2\tau)}{\eta^6}E_4+216h_0(\tau)\right)=-1+\sum_{n=1}^{\infty}a(n)q^n. \tag{4.14}$$

Here $a(1)>15\sigma(1)>4$. Hence the removal of hair degrees of freedom ensures positivity of $-B_6$ for the sector $Q\cdot P=$ even when $Q^2\geq 0$.

In the series asociated with $\theta_2(2\tau,2z)$ in equation (4.13) the Fourier coefficient of $q^{n-1/4}$ is bounded from below by,

$$10\sigma_3(n)-9H(4n-1).$$

Its positivity is ensured starting from $n=2$ using the following bounds:

1. $\sigma_3(n)\geq n^3$,
2. $H(n)<n$ [10].

For $n=1$ the positivity still holds as $H(3)=1/3$. So the complete q series expansion of $\left(\frac{\theta_3(2\tau)}{\eta^6}E_4-216h_1(\tau)\right)$ contains no negative Fourier coefficient. This could also be seen from the Fourier expansion of $\left(\frac{\theta_3(2\tau)}{\eta^6}E_4-216h_1(\tau)\right)$,

$$\left(\frac{\theta_3(2\tau)}{\eta^6}E_4-216h_1(\tau)\right)=q^{-1/4}(1+176q+\cdots). \tag{4.15}$$

This ensures the positivity of $-B_6$ for $Q\cdot P=$ odd and hence the coefficients of ψ_1^h obey positivity as expected for $P^2=2$.

4.2 Orbifolds of $K3\times T^2$

For the $2A$ orbifold we extract the index of single centred dyons by using the contour in (2.2) together with the following kinematic constraints on the charges [7]

$$\begin{aligned}
&Q^2>0, \qquad P^2>0, \qquad Q.P\geq 0, \qquad P^2Q^2-(Q\cdot P)^2>0,\\
&2Q^2\geq Q\cdot P, \qquad P^2\geq Q\cdot P, \qquad P^2+2Q^2\geq 3Q\cdot P\,.
\end{aligned} \tag{4.16}$$

The index of the horizon states for the $2A$ orbifold is given in table 5.

The kinematic constraints on the charges for the $3B$ orbifold so that the dyons are single centered are given by

$$\begin{aligned}
&\{Q^2,\ P^2, P^2Q^2-(Q\cdot P)^2\}>0 \qquad Q.P\geq 0,\\
&3Q^2\geq Q\cdot P, \qquad P^2\geq Q\cdot P,\\
&2P^2+3Q^2\geq 5Q\cdot P, \qquad P^2+6Q^2\geq 5Q\cdot P, \qquad 2P^2+6Q^2\geq 7Q\cdot P.
\end{aligned} \tag{4.17}$$

The index for the horizon states is then obtained using contour (2.2) and is listed in table 6.

For an orbifold of order $N>3$ there are infinite set of constraints for the charges to ensure that the index corresponds to single centered dyons [7]. However we see as long as the norms of electric and magnetic charges are positive and $Q\cdot P\geq 0$ together with $Q^2P^2-(Q\cdot P)^2>0$, the index $-B_6$ remains positive for the orbifolds of $K3$ (see the

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$(Q^2,\ P^2) \quad \backslash\ Q \cdot P$	0	1	2	3	4
(1, 2)	580	176	−2	0	0
(1, 4)	5504	1856	32	0	0
(1, 6)	41476	16200	996	52	0
(1, 10)	1293256	589200	63556	2752	−104
(2, 2)	1312	576	48	0	0
(2, 4)	16896	8640	1280	64	0
(3, 2)	9708	4696	580	52	0

Table 5. Index of horizon states for the $2A$ orbifold of $K3$.

$(Q^2,\ P^2) \quad \backslash\ Q \cdot P$	0	1	2	3	4
(2/3, 2)	216	27	0	0	0
(2/3, 4)	1548	342	0	0	0
(2/3, 6)	8532	2430	54	0	0
(4/3, 2)	540	216	0	0	0
(4/3, 4)	5820	2698	136	0	0
(2, 2)	1728	621	54	0	0
(2, 6)	204264	117837	23400	765	0
(2, 8)	1440288	896670	216540	13932	54

Table 6. Index of horizon states for the $3A$ orbifold of $K3$.

$(Q^2,\ P^2) \quad \backslash\ Q \cdot P$	0	1	2	3	4
(1/2, 2)	64	8	0	0	0
(1/2, 4)	288	80	0	0	0
(1/2, 6)	1088	464	24	0	0
(1, 2)	96	48	0	0	0
(1, 4)	464	480	16	0	0
(3/2, 4)	640	1680	160	0	0
(3/2, 6)	3958	11448	2026	38	0
(3/2, 22)	232188670	421276388	228036842	43979890	2695862

Table 7. Index of horizon states for the $4B$ orbifold of $K3$.

$(Q^2,\ P^2) \quad \backslash\ Q \cdot P$	0	1	2	3	4
(2/5, 2)	44	1	0	0	0
(2/5, 4)	220	20	0	0	0
(2/5, 6)	880	125	0	0	0
(4/5, 2)	88	16	0	0	0
(4/5, 4)	560	160	0	0	0
(6/5, 6)	8360	3755	310	0	0
(6/5, 8)	37394	18720	2202	16	0

Table 8. Index of horizon states for the $5A$ orbifold of $K3$.

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$(Q^2,\ P^2)\ \ \backslash\, Q\cdot P$	0	1	2	3	4
(1/3, 2)	24	1	0	0	0
(1/3, 4)	92	12	0	0	0
(1/3, 6)	318	49	0	0	0
(2/3, 2)	44	10	0	0	0
(2/3, 4)	236	68	0	0	0
(1, 4)	564	216	8	0	0
(1, 6)	2702	1201	100	0	0
(1/3, 34)	15836220	6614053	409414	1789	−14

Table 9. Index of horizon states $6A$ orbifold of $K3$.

$(Q^2,\ P^2)\ \ \backslash\, Q\cdot P$	0	1	2	3	4
(2/7, 2)	18	0	0	0	0
(2/7, 4)	72	3	0	0	0
(2/7, 6)	240	18	0	0	0
(4/7, 2)	30	3	0	0	0
(4/7, 4)	150	31	0	0	0
(6/7, 8)	5580	2304	0	0	0
(2/7, 40)	46940778	18696804	1139238	4689	−18

Table 10. Index of horizon states for the $7A$ orbifold of $K3$.

$(Q^2,\ P^2)\ \ \backslash\, Q\cdot P$	0	1	2	3	4
(1/4, 2)	12	0	0	0	0
(1/4, 4)	40	2	0	0	0
(1/4, 6)	124	10	0	0	0
(1/2, 2)	20	2	0	0	0
(1/2, 4)	88	16	0	0	0
(3/4, 4)	176	52	0	0	0
(3/4, 6)	708	248	6	0	0
(1/4, 46)	37469836	15088039	845410	2491	−10

Table 11. Index of horizon states for the $8A$ orbifold of $K3$.

tables 7–15). These orbifolds maybe geometric like that of CHL or even non-geometric where $g' \in [M_{23}]$.

It is interesting to see that the index for horizon states even in non-geometric orbifolds of $K3$ retains positivity of the index in the domain $NQ^2 \geq Q\cdot P, P^2 \geq Q\cdot P$, $Q^2P^2 - (Q\cdot P)^2 > 0$.

$(Q^2,\ P^2)\quad\backslash\ Q\cdot P$	0	1	2	3	4
(2/11, 2)	6	0	0	0	0
(2/11, 4)	18	0	0	0	0
(2/11, 6)	50	1	0	0	0
(4/11, 2)	8	0	0	0	0
(4/11, 4)	32	4	0	0	0
(6/11, 8)	592	172	2	0	0
(6/11, 10)	1568	527	16	0	0

Table 12. Index of horizon states for the 11A orbifold of $K3$.

$(Q^2,\ P^2)\quad\backslash\ Q\cdot P$	0	1	2	3	4
(1/7, 2)	3	0	0	0	0
(1/7, 4)	7	0	0	0	0
(1/7, 6)	18	0	0	0	0
(2/7, 2)	4	0	0	0	0
(2/7, 4)	14	1	0	0	0
(3/7, 8)	163	45	0	0	0
(3/7, 10)	390	116	2	0	0
(4/7, 10)	774	329	14	0	0

Table 13. Index of horizon states for the $14A$ orbifold of $K3$.

$(Q^2,\ P^2)\quad\backslash\ Q\cdot P$	0	1	2	3	4
(2/15, 2)	3	0	0	0	0
(2/15, 4)	6	0	0	0	0
(2/15, 6)	15	0	0	0	0
(4/15, 2)	3	1	0	0	0
(4/15, 4)	10	4	0	0	0
(2/5, 8)	125	31	0	0	0
(2/5, 10)	277	80	1	0	0
(8/15, 10)	527	227	9	0	0

Table 14. Index of horizon states for the $15A$ orbifold of $K3$.

$(Q^2,\ P^2)\quad\backslash\ Q\cdot P$	0	1	2	3	4
(2/23, 2)	1	0	0	0	0
(2/23, 4)	2	0	0	0	0
(2/23, 6)	5	0	0	0	0
(4/23, 2)	14	2	0	0	0
(4/23, 4)	28	4	0	0	0
(6/23, 8)	87	36	4	0	0
(6/23, 10)	144	57	6	0	0

Table 15. Index of horizon states for the $23A$ orbifold of $K3$.

Q^2 \\ P^2	2	4	6	8
1	**-224**	**-1248**	1728	95104
2	1152	18240	233984	2432544
3	**-3392**	**-10320**	542976	12103360
4	-11520	200736	4575744	86712256
5	**-30336**	**-55424**	12914944	412163328
6	83968	1544832	61928448	2013023104
7	**-202560**	**-179022**	175358304	8292093664
8	496512	9480000	638922240	32998944096
9	**-1118496**	**-155232**	1735394112	119618619520
10	2521600	49523328	5364983808	415768863360

Table 16. The index $d(Q, P)$ for the $\mathbb{Z}_2$ toroidal orbifold some low lying values of Q^2, P^2 with $Q \cdot P = 0$.

Q^2 \\ P^2	2	4	6	8
1	96	1968	22528	190047
2	**-256**	840	70912	1127672
3	1376	34656	728256	11046139
4	**-3840**	16632	2497408	61486056
5	13152	343152	13144832	348876305
6	**-33536**	171152	42058240	1603241304
7	92928	2476752	162898624	7016918625
8	**-220672**	1265256	480911872	27503872048
9	540416	14545584	1556561664	102315259287
10	**-1204992**	7558560	4271142656	354800345088

Table 17. The index $d(Q, P)$ for the $\mathbb{Z}_2$ toroidal orbifold some low lying values of Q^2, P^2 with $Q \cdot P = 1$.

4.3 Toroidal orbifolds

In [13] we have seen that positivity of index for single centred dyons was violated for the toroidal models. For completeness we have reproduced some of the indices evaluated in [13] in tables 16, 17, 18.

Positivity of the horizon states for toroidal models. The indices in tables 16, 17, 18 were obtained under the assumption that there exists a frame in which the fermionic zero modes associated with broken supersymmetries are the only hair. In (2.50) and (2.53) we have proposed the partition function for the hair degrees of freedom in the type IIB frame for the $\mathbb{Z}_2, \mathbb{Z}_3$ toroidal orbifolds respectively. We evaluate the indices of horizon states in the following tables (19–24) and observe that they are all positive for single centered dyons.

We now enumerate the consistency checks we have done for the proposal for the hair modes in the $T^6/\mathbb{Z}_2$ toroidal model given in (2.50).

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Q^2 \\ P^2	2	4	6	8
1	0	- 12	- 224	-1248
2	64	2592	43264	491904
3	**- 224**	2432	191168	3805600
4	1152	43392	1440256	30853488
5	**-3392**	33720	5363680	171782688
6	11520	414336	24533248	893029504
7	**-30336**	302400	80281536	3963098880
8	83968	2926080	287831552	16432262672
9	**-202560**	2049968	851816352	62214237440
10	496512	16919712	2627695616	222752294016

Table 18. The index $d(Q, P)$ for the $\mathbb{Z}_2$ toroidal orbifold some low lying values of Q^2, P^2 with $Q \cdot P = 2$.

Q^2 \\ P^2	2	4	6	8
1	832	14816	158848	1283902
2	3840	101008	1425920	14471264
3	14624	556176	10273024	129971582
4	48128	2588336	62037760	971443680
5	143424	10594400	325402624	6254176746
6	394112	39145344	1521266688	35582718576
7	1016080	133122060	6465235840	182481593350
8	2480512	422430736	25355844096	856661245280
9	5786240	1264061344	92844570752	3726638152610
10	12968576	3595680768	320340466176	15170555788976

Table 19. Index of horizon states for the $\mathbb{Z}_2$ orbifold of T^6 for $Q \cdot P = 0$.

Q^2 \\ P^2	2	4	6	8
1	480	9012	98784	811166
2	2496	69328	1001472	10329280
3	9888	403448	7664064	98689790
4	33664	1946480	48074496	766539920
5	102272	8155848	258619232	5063997322
6	286208	30667504	1231379200	29352001136
7	747456	105699406	5306269024	152656500694
8	1847040	339109664	21040306176	724593923536
9	4350816	1024054008	77737446688	3180401982114
10	9841408	2935991504	270248202752	13043376086768

Table 20. Index of horizon states for the $\mathbb{Z}_2$ orbifold of T^6 for $Q \cdot P = 1$.

$Q^2 \ \backslash\backslash\ P^2$	2	4	6	8
1	96	1880	21056	178660
2	640	21312	329728	3577216
3	2992	151056	3115712	42306045
4	11008	813280	22062720	371908656
5	35840	3669600	128569280	2665839255
6	105472	14554120	647882496	16372365048
7	288192	52296704	2913889600	88924896642
8	738560	173535528	11950263808	436628175032
9	1798688	539123792	45385181120	1969579830259
10	4187008	1583791144	161466383616	8262793111120

Table 21. Index of horizon states for the $\mathbb{Z}_2$ orbifold of T^6 for $Q \cdot P = 2$.

$Q^2 \ \backslash\backslash\ P^2$	2	4	6	8
1	0	-12	-224	-1046
2	64	2480	40960	484752
3	320	26590	632544	9430780
4	1408	178096	5723136	106304080
5	5088	916872	38694432	887612004
6	16896	4001712	215960576	6052758272
7	50432	15481304	1047526432	35500683214
8	140352	54572672	4557481728	184959084864
9	365536	178371800	18160058144	874917932484
10	905600	547471520	67260039168	3817189761008

Table 22. Index of horizon states for the $\mathbb{Z}_2$ orbifold of T^6 for $Q \cdot P = 3$. Note that it is only when $Q^2P^2 - (Q \cdot P)^2 < 0$ we observe that the index is negative.

$Q^2 \ \backslash\backslash\ P^2$	2	4	6	8
1	0	0	0	37
2	0	-8	-256	1232
3	16	1900	50880	868435
4	0	17928	757376	16261008
5	96	114160	6613888	176919248
6	512	576016	43399680	1427632608
7	2416	2506512	236442496	9431113673
8	8320	9731384	1124958848	53751377384
9	26592	34532368	4818946176	272969682473
10	75904	113759408	18960610304	1262218427744

Table 23. Index of horizon states for the $\mathbb{Z}_2$ orbifold of T^6 when $Q \cdot P = 4$. Note that only when $Q^2P^2 - (Q \cdot P)^2 < 0$ we observe that the index is negative.

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$(Q^2,\ P^2)\quad \backslash\, Q\cdot P$	0	1	2	3	4
(2/3, 2)	162	90	9	0	0
(2/3, 4)	1944	1134	162	0	0
(2/3, 6)	14598	8748	1149	0	0
(4/3, 2)	540	324	72	0	0
(4/3, 4)	8856	5724	1458	54	0
(2, 2)	1566	1008	243	18	0
(2, 4)	34344	23652	7290	810	0
(2, 6)	402972	286734	98613	13614	249

Table 24. Index of horizon states for the $T^6/\mathbb{Z}_3$ orbifold.

1. If we do not include the zero modes $-e^{2\pi iv}(1-e^{2\pi iv})^{-2}$ as part of the hair partition function in $T^6/\mathbb{Z}_2$, then we observe the violation of positivity in index for $P^2 = 6, Q^2 = 1, Q\cdot P = 2$ and $P^2 = 6, Q^2 = 2, Q\cdot P = 3$. The indices for these dyonic charges are -224 and -256 respectively. These charges are within the kinematic domain defined by (4.16).

2. If we include the contribution of the Wilson lines given in (2.52) as part of the hair partition function and remove the contribution of the zero modes $-e^{2\pi iv}(1-e^{2\pi iv})^{-2}$, we find violations in positivity of the index. This can be observed at $P^2 = 6,\ Q^2 = 1,\ Q\cdot P = 2,\ P^2 = 6,\ Q^2 = 2,\ Q\cdot P = 3,\ P^2 = 4,\ Q^2 = 4, Q\cdot P = 3$, the indices are given by $-64, -64, -4$ respectively.

These two observations show that we certainly need to include the contribution of the zero modes $-e^{2\pi iv}(1-e^{2\pi iv})^{-2}$ as part of the hair partition function which is consistent with our proposal. It would be interesting to prove this by studying the wave function of the gravitino zero modes in the toroidal models.

A very similar analysis holds true for $T^6/\mathbb{Z}_3$. The index of horizon states obtained by considering the proposal given in (2.50) for the hair partition function is positive as shown in the subsequent tables. We have also repeated the consistency checks we mentioned earlier for the $\mathbb{Z}_2$ orbifold in this case with the same conclusions.

5 Conclusions

We have constructed the horizon partition function of the 1/4 BPS dyonic black hole in $\mathcal{N} = 4$ theories obtained by compactifying type IIB on orbifolds of $K3\times T^2$. We then observed that the index of the horizon states of single centred black holes are all positive. We adapted the proof of [10] and showed that the index of the horizon partition function of single centred dyons with $P^2 = 2$ remains positive.

For the toroidal models we propose that the hair modes are given by (2.50) and (2.53). We showed the index of horizon states with this proposal is positive and performed consistency checks. As mentioned earlier it would be interesting to study the wave function of the zero modes of the gravitino in the toroidal models to check the proposal in (2.50) and (2.53). In [13] it was noticed that the index of single centred dyons in these models were not positive when one assumed that the only hair modes are the Fermionic zero modes associated with broken supersymmetry generators. Since hair modes are frame dependent, the observations in this paper indicates that there is possibly no duality frame for these models which contains only the Fermionic zero modes as the hair. It will be interesting to verify this explicitly by an study similar to that done in [8, 9] for the $\mathcal{N} = 8$ theory.

The observation that the index of horizon states in the canonical compactification on $K3 \times T^2$ is positive is worth further study. It should be possible to extend the proof of [10] to higher values of P^2.

Note added: as this work was nearing completion, we became aware of the work done in [34]. The analysis of the hair modes done for the CHL orbifolds of $K3$ in section 2 and 3 overlaps with parts of [34].

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