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Conformal boundaries of Minkowski superspace and their super cuts

Nicolas Boulanger[a], **Yannick Herfray**[b] and **Noémie Parrini**[a,1]

[a] *Physique de l'Univers, Champs et Gravitation, Université de Mons — UMONS, Place du Parc 20, 7000 Mons, Belgium*
[b] *Institut Denis Poisson UMR 7013, Université de Tours, Parc de Grandmont, 37200 Tours, France*

E-mail: nicolas.boulanger@umons.ac.be, yannick.herfray@univ-tours.fr, noemie.parrini@umons.ac.be

Abstract: In this article we carry out a detailed investigation of the geometric nature of the points at infinity of Minkowski superspace. It turns out that there are several sets of points forming the superconformal boundary of Minkowski superspace: on top of a well-behaved super $\mathscr{I}$, we find other sets that we exhibit and study. We also study the intersection of these boundaries with super null cones and explicitly construct the corresponding space of super cuts.

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[1] Research Fellow of the F.R.S.-FNRS (Belgium).

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1 Introduction

Penrose's conformal compactification of spacetimes [1] is an essential tool in general relativity [2–5]. It was introduced as mean to both geometrize and give a global meaning to Bondi-van der Burg-Metzner-Sachs notion of asymptotically flat spacetimes [6, 7]. The very notion of conformal boundary is also at the basis of the AdS/CFT correspondence [8, 9].

In the context of supergeometry it is well known that there exists a notion of superconformal compactification of Minkowski superspace. The construction is detailed in Manin's book [10] and, as we will review, is most naturally realised in terms of supertwistors geometry [11]. The resulting conformally compactified Minkowski superspace $\overline{M}^{4|4\mathcal{N}}_{\mathbb{R}}$ naturally is a homogeneous space[1] for the super conformal group $\mathrm{SU}(2,2|\mathcal{N})$:

$$\overline{M}^{4|4\mathcal{N}}_{\mathbb{R}} = \frac{\mathrm{SU}(2,2|\mathcal{N})}{(\mathbb{R}^* \times \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SU}(\mathcal{N})) \ltimes \mathcal{T}^{4|4\mathcal{N}}}. \tag{1.1}$$

AdS superspace $AdS^{4|4\mathcal{N}}_{\mathbb{R}} \hookrightarrow \overline{M}^{4|4\mathcal{N}}_{\mathbb{R}}$ is recovered by breaking superconformal invariance down to $\mathrm{OSp}(\mathcal{N}|4) \subset \mathrm{SU}(2,2|\mathcal{N})$, the supergroup of isometries of AdS superspace. The action of $\mathrm{OSp}(\mathcal{N}|4)$ on conformally compactified Minowski superspace $\overline{M}^{4|4\mathcal{N}}_{\mathbb{R}}$ is not transitive and resulting orbits include AdS superspace $AdS^{4|4\mathcal{N}}_{\mathbb{R}} \hookrightarrow \overline{M}^{4|4\mathcal{N}}_{\mathbb{R}}$ and its conformal boundary $\overline{M}^{3|2\mathcal{N}}_{\mathbb{R}} \hookrightarrow \overline{M}^{4|4\mathcal{N}}_{\mathbb{R}}$:

$$AdS^{4|4\mathcal{N}}_{\mathbb{R}} \simeq \frac{\mathrm{OSp}(\mathcal{N}|4)}{\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SO}(\mathcal{N})}, \qquad \overline{M}^{3|2\mathcal{N}}_{\mathbb{R}} \simeq \frac{\mathrm{OSp}(\mathcal{N}|4)}{(\mathbb{R}^* \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(\mathcal{N})) \ltimes \mathcal{T}^{3|2\mathcal{N}}}. \tag{1.2}$$

Both supermanifolds naturally fit inside the bigger supermanifold $\overline{M}^{4|4\mathcal{N}}_{\mathbb{R}}$. This superembeding construction, as well as its extended versions, as been studied with a lot of care in [16–18].

Similarly, as has been used in numerous context [19–31], by breaking superconformal invariance down to the super Poincaré group[2] $\mathrm{ISO}(3,1|\mathcal{N}) \subset \mathrm{SU}(2,2|\mathcal{N})$ one obtains an embedding $M^{4|4\mathcal{N}}_{\mathbb{R}} \hookrightarrow \overline{M}^{4|4\mathcal{N}}_{\mathbb{R}}$ of Minkowski superspace. However, it seems that not so much interest has been given in the literature to the investigation of the nature of the boundary "at infinity" of Minkowski superspace. This is however a necessary starting point if, for example, one would like to understand the geometrical nature of the super-BMS group [32–37] or consider the possibility of extending Newman's theory of H-space [38–40] and related asymptotic twistor space [41–43] to the supersymmetric context.

In the first two sections of this work we investigate the asymptotics of Minkowski superspace in details. As a result, we find the following invariant decomposition of conformally compactified Minowski superspace under the action of the super Poincaré group:

$$\overline{M}^{4|4\mathcal{N}}_{\mathbb{R}} = M^{4|4\mathcal{N}}_{\mathbb{R}} \bigsqcup \mathscr{I}^{3|2\mathcal{N}}_{\mathbb{R}} \bigsqcup \mathcal{H}_{\mathbb{R}} \bigsqcup \mathfrak{I}_{\mathbb{R}} \bigsqcup \iota\,. \tag{1.3}$$

[1]Realisations of Minkowski and AdS superspaces as coset spaces is a classical topics in supersymmetry, we refer to the monographs [10, 12–14] for further details and references and to [15] for the classification of all $\mathcal{N} = 1$ kinematical superspaces. In this article, we will focus on the asymptotic boundaries of these models. In this introduction we focus on real Minkowski superspace, however the complexifed superconformal compactifiation appears most naturally in this context. This is of interest because it makes the underlying chirality properties of Minkowski superspace manifest. All these aspects will be discussed in details in the body of the paper.

[2]We refer to eqs. (3.3) and (3.4) for the precise definition of what we call here the super Poincaré group.

The first supermanifold appearing above obviously is Minkowski superspace,

$$M_{\mathbb{R}}^{4|4\mathcal{N}} \simeq \frac{\mathrm{ISO}(1,3|\mathcal{N})}{\mathrm{SL}(2,\mathbb{C})\times \mathrm{SU}(\mathcal{N})}, \tag{1.4}$$

while the four remaining subspaces are four other sets of points left invariant by the action of the super Poincaré group. In the core of this paper we will give explicit parametrisations of each of these spaces "at infinity" of Minkowski superspace. We here briefly sum up the resulting picture: the most well-behaved is super null infinity. It is an homogeneous superspace,[3]

$$\mathscr{I}_{\mathbb{R}}^{3|2\mathcal{N}} \simeq \frac{\mathrm{ISO}(1,3|\mathcal{N})}{(\mathbb{R}^{*}\times \mathrm{ISO}(2)\times \mathrm{SU}(\mathcal{N}))\ltimes \mathcal{T}^{3|2\mathcal{N}}}, \tag{1.5}$$

and as such always is a supermanifold. On the other hand, the set of points that we note $\mathcal{H}$ and $\mathfrak{I}$ only are supermanifolds in the chiral (left or right) complexified setting, in which case they respectively have complex dimensions $(3|2\mathcal{N})$ and $(0|2\mathcal{N})$. For comparison, complexified chiral (left or right) Minkowski and null infinity respectively have complex dimensions $(4|2\mathcal{N})$ and $(3|\mathcal{N})$, thus $\mathcal{H}$ has co-dimension $(1|0)$. However they are never homogeneous spaces: the action of the super Poincaré group happens not to be transitive. If one tries, as we shall explain in the bulk of the paper, to restrict to orbits of the super Poincaré group, the resulting sets turn out not to be supermanifolds. This is indeed a peculiar phenomenon of supergeometries that not all orbits of a supergroup are supermanifolds. The reason is that the stabilizer of a point in the orbit may not be a supergroup. The final invariant set ι is just a point: it stands for both time-like and space-like infinity in the conformal compactification.

To further elucidate the nature of these boundaries, we turn to different but closely related geometrical objects of Minkowski superspace: super null rays. These are natural supermanifolds of dimension $(1|2\mathcal{N})$ which generalise null geodesics [10]: if $(x_0^{AA'}, \theta_0^{IB'})$ is a point in Minkowski superspace then the super-null ray in the direction $[\pi^{A'}] \in \mathbb{C}P^1$ is given by the parametric equations

$$x^{AA'} = x_0^{AA'} + \frac{1}{2}\left(\pi^{A'}\bar{\theta}_{0I}{}^{A}\epsilon^{I} + \theta_0^{IA'}\bar{\pi}^{A}\bar{\epsilon}_I\right) + \epsilon\,\pi^{A'}\bar{\pi}^{A}, \tag{1.6}$$

$$\theta^{IB'} = \theta_0^{IB'} + \epsilon^{I}\pi^{B'}, \tag{1.7}$$

where $(\epsilon, \epsilon^I) \in \mathbb{R}^{(1|0)} \times \mathbb{C}^{0|\mathcal{N}}$ are coordinates along the super null ray. In this way, each point of Minkowski superspace is associated with a Riemann sphere $\mathbb{C}P^1$ of super null rays passing through it. By making use of the fact that super null rays are points in the super ambitwistor space [10, 45, 46], we shall see that super null cones intersect the different boundaries at infinity of Minkowski superspace along preferred super cuts. We will obtain explicit expressions for the resulting space of cuts.

Finally, one can reverse the logic to gain some insight about the boundaries at infinity: in the bosonic geometry, null infinity $\mathscr{I}$ can be understood as a null cone emanating from time-like infinity ι. This interpretation is possible because null cones are conformal invariants. Similarly, in the supergeometric context we can use the fact that super null rays

[3]Even though this realisation of super null infinity is certainly known to experts, we could not find any reference on the subject. See however [44] for the use of null infinity in a superspace context.

are superconformal invariant to interpret points at infinity: it turns out that while super $\mathscr{I}$ genuinely is the super null cone emanating from the point ι, the boundary $\mathcal{H}$ is the union of all super null rays emanating from the purely fermionic points $\mathfrak{I}$. Here $\mathfrak{I}$ should be thought of as the results of translations of ι in fermionic directions.

The paper is organised as follows: to be self-contained and present our notations, we first review how to explicitly realise the superconformal compactification of Minkowski space in terms of supertwistors. We then discuss the different sets of points invariant under the action of the super Poincaré group. Finally, we recall the equivalence between ambitwistors and super null rays and make use of this identification to construct cuts along super null infinity and the other boundaries.

2 Superconformal compactification of Minkowski superspace

In this section we review the well-known superconformal compactification of Minkoswki superspace in terms of supertwistors [10, 11]. We also review the real forms of the complex superconformal group and algebra.

2.1 Supertwistors

2.1.1 Supertwistor space

We take the supertwistor space to be the fundamental representation of the complexified superconformal group $\mathrm{SL}(4|\mathcal{N})$. In coordinates, a supertwistor $Z^{\hat{\alpha}}$ will be represented by an element of $\mathbb{C}^{4|\mathcal{N}}$:

$$Z^{\hat{\alpha}} = \begin{pmatrix} \omega^A \\ \pi_{A'} \\ \theta^I \end{pmatrix}, \tag{2.1}$$

where

$$\omega^A = \begin{pmatrix} \omega^0 \\ \omega^1 \end{pmatrix} \in \mathbb{C}^2_c, \qquad \pi_{A'} = \begin{pmatrix} \pi_{0'} \\ \pi_{1'} \end{pmatrix} \in \mathbb{C}^2_c, \qquad \theta^I = \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^{\mathcal{N}} \end{pmatrix} \in \mathbb{C}^{\mathcal{N}}_a. \tag{2.2}$$

The supergroup $\mathrm{SL}(4|\mathcal{N})$ is parametrised by

$$X^{\hat{\alpha}}{}_{\hat{\beta}} = \begin{pmatrix} M^A{}_B & i\,T^{AB'} & Q^A{}_J \\ -i\,K_{A'B} & -\widetilde{M}_{A'}{}^{B'} & S_{A'J} \\ \widetilde{S}^I{}_B & \widetilde{Q}^{IB'} & R^I{}_J \end{pmatrix}, \tag{2.3}$$

where

$$\begin{pmatrix} M^A{}_B & iT^{AB'} \\ -iK_{A'B} & -\widetilde{M}_{A'}{}^{B'} \end{pmatrix} \in \mathrm{GL}(4,\mathbb{C}_c), \qquad R^I{}_J \in \mathrm{GL}(\mathcal{N},\mathbb{C}_c), \qquad \mathrm{Ber}(X) = 1, \tag{2.4}$$

and $Q^A{}_J$, $\widetilde{Q}^{IB'}$, $S_{A'J}$, and $\widetilde{S}^I{}_B$ are elements of $\mathbb{C}^{2\mathcal{N}}_a$. The supergroup $\mathrm{SL}(4|\mathcal{N})$ acts linearly on the supertwistors:

$$Z^{\hat{\alpha}} \mapsto X^{\hat{\alpha}}{}_{\hat{\beta}}\, Z^{\hat{\beta}}. \tag{2.5}$$

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2.1.2 Real form of the superconformal group

If we restrict to the real case, i.e., if we ask to preserve the hermitian form

$$\overline{Z}^{\hat{\overline{\alpha}}} h_{\overline{\hat{\alpha}}\hat{\beta}} Z^{\hat{\beta}} = \overline{\pi}_A \, \omega^A + \overline{\omega}^{A'} \, \pi_{A'} - \overline{\theta}^J \, \theta^I \delta_{IJ}, \tag{2.6}$$

$$\text{where} \quad h_{\overline{\hat{\alpha}}\hat{\beta}} = \begin{pmatrix} 0 & \mathbb{I}_2 & 0 \\ \mathbb{I}_2 & 0 & 0 \\ 0 & 0 & -\delta_{IJ} \end{pmatrix}, \tag{2.7}$$

then the action of $\mathrm{SL}(4|\mathcal{N})$ reduces to the action of the *real* superconformal group $\mathrm{SU}(2,2|\mathcal{N})$. A generic element x of $\mathfrak{su}(2,2|\mathcal{N})$, the subalgebra of matrices x in $\mathfrak{sl}(4|\mathcal{N})$ such that $hx + x^\dagger h = 0$, can be written as

$$x = \begin{pmatrix} m^A{}_B & it^{AB'} & q^A{}_J \\ -ik_{A'B} & -\overline{m}^{B'}{}_{A'} & \overline{s}_{A'J} \\ s^I{}_B & \overline{q}^{IB'} & r^I{}_J \end{pmatrix} + c\,A, \quad c \in \mathbb{R}_c, \tag{2.8}$$

where $\operatorname{Tr} m = \operatorname{Tr} m^\dagger$, $t^\dagger = t$, $k^\dagger = k$ and $r^\dagger = r$, $\operatorname{Tr} r = 0$. The $\mathcal{N} \times \mathcal{N}$ matrix $(r^I{}_J)$ belongs to the $\mathfrak{su}(\mathcal{N})$ subalgebra while the matrix A is generator of $\mathrm{U}(1)$. In the case $\mathcal{N} = 1$, the matrix A is given in appendix B to which we refer for more details on the superconformal algebra and its presentation. If one denotes by α and β the diagonal elements of the matrix m, the trace constraint on m and $m^\dagger$ implies that $\Im(\alpha + \beta) = 0$, i.e., $\operatorname{Tr} m \in \mathbb{R}_c$. Note that $m^\dagger{}_{A'}{}^{B'} = (m^B{}_A)^* =: \overline{m}^{B'}{}_{A'}$.

2.2 Superconformal compactification

2.2.1 Complex case

Following [10], the model for full (i.e., non-chiral) $\mathcal{N}$ extended compactified *complexified* Minkowski superspace $\overline{M}_{\mathbb{C}}^{4|4\mathcal{N}}$ is taken to be the flag manifold $F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}})$ in the supertwistor space.

There are then two canonical projections from compactified Minkowski superspace to the left/right chiral superspaces:

$$\begin{array}{ccc} & \overline{M}_{\mathbb{C}}^{4|4\mathcal{N}} = F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) & \\ & {}^{\pi_\ell}\swarrow \qquad \searrow^{\pi_r} & \\ \overline{M}_\ell = Gr(2|0, \mathbb{C}^{4|\mathcal{N}}) & & \overline{M}_r = Gr(2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) \end{array} \tag{2.9}$$

explicitly given by

$$\begin{aligned} \pi_\ell : F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) &\longrightarrow Gr(2|0, \mathbb{C}^{4|\mathcal{N}}) \qquad & \pi_r : F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) &\longrightarrow Gr(2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) \\ (P_1, P_2) &\longmapsto P_1 & (P_1, P_2) &\longmapsto P_2. \end{aligned}$$

As we will recall, $\overline{M}_\ell$, $\overline{M}_r$ and $\overline{M}_{\mathbb{C}}^{4|4\mathcal{N}}$ are homogeneous (super)spaces for the complex superconformal group and are respectively of dimensions

$$\dim_{\mathbb{C}} \overline{M}_\ell = \dim_{\mathbb{C}} \overline{M}_r = 4|2\mathcal{N}, \qquad \dim_{\mathbb{C}} \overline{M}_{\mathbb{C}}^{4|4\mathcal{N}} = 4|4\mathcal{N}. \tag{2.10}$$

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We will use grassmanian coordinates $Z^{\hat{\alpha}b}$ with, $\hat{\alpha} \in \{^A, _{A'}, ^I\}$, $b \in \{1,2\}$ to parametrise $Gr(2|0, \mathbb{C}^{4|\mathcal{N}})$ (with similar notations for the other grassmanians). We call elements $Z^{\hat{\alpha}b} \in \overline{M}_\ell$ bi-supertwistors. The elements of $\overline{M}_r$ will be denoted by $Z^{\hat{\alpha}\mathbf{c}}$, with $\mathbf{c} \in \{1, \cdots, \mathcal{N}+2\}$. Making use of the isomorphism

$$Gr(2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) \simeq Gr(2|0, (\mathbb{C}^{4|\mathcal{N}})^*), \tag{2.11}$$

we equivalently represent elements of $\overline{M}_r$ by dual bi-supertwistors $\widetilde{Z}_{\hat{\alpha}}{}^b$. Elements of $\overline{M}_{\mathbb{C}}^{4|4\mathcal{N}} = F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}})$ are then obtained as pairs $(Z^{\hat{\alpha}b}, \widetilde{Z}_{\hat{\alpha}}{}^c)$ of bi-supertwistors and dual bi-supertwistors satisfying $\widetilde{Z}_{\hat{\alpha}}{}^b Z^{\hat{\alpha}c} = 0$. With these notations the projections (2.9) are simply realised as

$$\begin{array}{ccc} & (Z^{\hat{\alpha}b}, \widetilde{Z}_{\hat{\alpha}}{}^c) \in \overline{M}_{\mathbb{C}}^{4|4\mathcal{N}} & \\ \pi_\ell \swarrow & & \searrow \pi_r \\ Z^{\hat{\alpha}b} \in \overline{M}_\ell & & \widetilde{Z}_{\hat{\alpha}}{}^c \in \overline{M}_r \end{array} .$$

The supergroup $\mathrm{SL}(4|\mathcal{N})$ acts on $\overline{M}_\ell$, $\overline{M}_r$ and $\overline{M}_{\mathbb{C}}^{4|4\mathcal{N}}$ via the induced action from $\mathbb{C}^{4|\mathcal{N}}$:

$$\begin{aligned} Z^{\hat{\alpha}b} \in \overline{M}_\ell \quad &\mapsto \quad X^{\hat{\alpha}}{}_{\hat{\beta}} Z^{\hat{\beta}b}, \\ \widetilde{Z}_{\hat{\alpha}}{}^b \in \overline{M}_r \quad &\mapsto \quad \widetilde{Z}_{\hat{\beta}}{}^b (X^{-1})^{\hat{\beta}}{}_{\hat{\alpha}}, \\ (Z^{\hat{\alpha}b}, \widetilde{Z}_{\hat{\alpha}}{}^c) \in \overline{M}_{\mathbb{C}}^{4|4\mathcal{N}} &\mapsto (X^{\hat{\alpha}}{}_{\hat{\beta}} Z^{\hat{\beta}b}, \widetilde{Z}_{\hat{\beta}}{}^c (X^{-1})^{\hat{\beta}}{}_{\hat{\alpha}}). \end{aligned} \tag{2.12}$$

This action is transitive in all three cases, which makes $\overline{M}_{\mathbb{C}}^{4|4\mathcal{N}}$, $\overline{M}_\ell$ and $\overline{M}_r$ all homogeneous spaces for the complex superconformal group. The respective stabilisers are obtained by choosing specific points.

- For $\overline{M}_\ell$, we require (2.3) to stabilise the origin bi-supertwistor

$$\mathrm{Y}^{\hat{\alpha}b} := \begin{bmatrix} 0^{Ab} \\ \delta_{A'}{}^b \\ 0^{Ib} \end{bmatrix}, \tag{2.13}$$

  where the notation $[\cdot]$ in the above definition is used to denote the equivalence class of elements in $Gr(2|0, \mathbb{C}^{4|\mathcal{N}})$, for which a representative is denoted between the square brackets. The stabiliser is the set of supermatrices of $\mathrm{SL}(4, \mathcal{N})$ of the form

$$\mathrm{P}_0^\ell = \begin{pmatrix} M^A{}_B & 0^{AB'} & Q^A{}_J \\ -iK_{A'B} & -\widetilde{M}_{A'}{}^{B'} & S_{A'J} \\ \tilde{S}^I{}_B & 0^{IB'} & R^I{}_J \end{pmatrix}, \tag{2.14}$$

  and forms a parabolic subgroup. Therefore, we have the homogeneous space

$$\overline{M}_\ell \simeq \frac{\mathrm{SL}(4|\mathcal{N})}{\mathrm{P}_0^\ell}. \tag{2.15}$$

- For $\overline{M}_r$, we require (2.3) to stabilise the dual origin bi-supertwistor

$$\widetilde{Y}_{\hat{\alpha}}{}^{b} := \left[\delta_A{}^b \; 0^{A'b} \; 0_I{}^b\right]. \tag{2.16}$$

An equivalent way to find the stabiliser of the dual origin bi-supertwistor is to use the isomorphism (2.11) on the following $(\mathcal{N}+2)$-supertwistor: decomposing the index $\mathbf{c}$ into $(c, \mathtt{C})$ with $c \in \{1,2\}$ and $\mathtt{C} \in \{1, \cdots, \mathcal{N}\}$, we obtain

$$Y^{\hat{\alpha}\mathbf{c}} = \begin{bmatrix} 0^{Ac} & 0^{A\mathtt{C}} \\ \delta_{A'}{}^{c} & 0_{A'}{}^{\mathtt{C}} \\ 0^{Ic} & \delta^{I\mathtt{C}} \end{bmatrix} \tag{2.17}$$

as the dual origin bi-supertwistor. The stabiliser is then the set of supermatrices

$$\mathrm{P}_0^r = \begin{pmatrix} M^A{}_B & 0^{AB'} & 0^A{}_J \\ -iK_{A'B} & -\widetilde{M}_{A'}{}^{B'} & S_{A'J} \\ \tilde{S}^I{}_B & \tilde{Q}^{IB'} & R^I{}_J \end{pmatrix}, \tag{2.18}$$

forming again a parabolic subgroup of $\mathrm{SL}(4|\mathcal{N})$ and yielding the homogeneous space

$$\overline{M}_r \simeq \frac{\mathrm{SL}(4|\mathcal{N})}{\mathrm{P}_0^r}. \tag{2.19}$$

- For $\overline{M}_{\mathbb{C}}^{4|4\mathcal{N}}$, the subgroup preserving the origin $(Y^{\hat{\alpha}b}, \widetilde{Y}_{\hat{\alpha}}{}^{b}) \in \overline{M}_{\mathbb{C}}^{4|4\mathcal{N}}$ is parametrised by

$$\mathrm{P}_0 = \begin{pmatrix} M^A{}_B & 0^{AB'} & 0^A{}_J \\ -iK_{A'B} & -\widetilde{M}_{A'}{}^{B'} & S_{A'J} \\ \tilde{S}^I{}_B & 0^{IB'} & R^I{}_J \end{pmatrix}, \tag{2.20}$$

and therefore

$$\overline{M}_{\mathbb{C}}^{4|4\mathcal{N}} \simeq \frac{\mathrm{SL}(4|\mathcal{N})}{\mathrm{P}_0}. \tag{2.21}$$

2.2.2 Real case

If we now look for the subgroups of (2.14), (2.18) or (2.20) that preserve the hermitian form

$$h_{\bar{\hat{\alpha}}\hat{\beta}} = \begin{pmatrix} 0 & \mathbb{I}_2 & 0 \\ \mathbb{I}_2 & 0 & 0 \\ 0 & 0 & -\delta^I{}_J \end{pmatrix}, \tag{2.22}$$

i.e., the subset of matrices $P^{\hat{\alpha}}{}_{\hat{\beta}}$ such that $\overline{P}^{\bar{\hat{\gamma}}}{}_{\bar{\hat{\beta}}} h_{\bar{\hat{\gamma}}\hat{\delta}} P^{\hat{\delta}}{}_{\hat{\alpha}} = h_{\bar{\hat{\beta}}\hat{\alpha}}$, one finds a unique subgroup given by the set of matrices of the form

$$\mathrm{P}_0^{\mathbb{R}} = \begin{pmatrix} M^A{}_B & 0^{AB'} & 0^A{}_J \\ i\overline{M}_{A'}{}^{C'} k_{C'B} + \frac{1}{2} S_{A'J} (S^\dagger)^J{}_C M^C{}_B & -\overline{M}_{A'}{}^{B'} & S_{A'J} \\ R^I{}_J (S^\dagger)^J{}_A M^A{}_B & 0^{IB'} & R^I{}_J \end{pmatrix}, \tag{2.23}$$

where $R^I{}_J$ is unitary and $k_{C'B}$ is an arbitrary hermitian matrix. Note that if $M = (M^A{}_B)$, we have $M^\dagger{}_{A'}{}^{B'} = \overline{M}^{B'}{}_{A'}$. Therefore we recover the result that chiral and non-chiral superspaces are isomorphic in the real case, and their expression as homogeneous spaces for the real superconformal group is given by

$$\overline{M}^{4|4\mathcal{N}}_{\mathbb{R}} := \overline{M}^{\mathbb{R}}_{\ell} \simeq \overline{M}^{\mathbb{R}}_{r} \simeq \frac{\mathrm{SU}(2,2|\mathcal{N})}{\mathrm{P}^{\mathbb{R}}_0}. \tag{2.24}$$

At the level of algebras, if we now look for the subset of $\mathfrak{su}(2,2|\mathcal{N})$ stabilising (2.13), we get

$$\mathfrak{p}^{\mathbb{R}}_0 = \begin{pmatrix} m^A{}_B & 0^{AB'} & 0^A{}_J \\ -ik_{A'B} & -((m)^\dagger)_{A'}{}^{B'} & \bar{s}_{A'J} \\ s^I{}_B & 0^{IB'} & r^I{}_J \end{pmatrix} + c\,A, \tag{2.25}$$

which is a parabolic subalgebra of $\mathfrak{su}(2,2|\mathcal{N})$.

We say that a point $(Z^{\hat{\alpha}a}, \widetilde{Z}_{\hat{\alpha}}{}^b)$ in $\overline{M}$ is real if the twistors are dual to each others through the metric (2.22), i.e.,

$$\widetilde{Z}_{\hat{\beta}}{}^b = \overline{Z}^{\bar{\hat{\alpha}}b} h_{\bar{\hat{\alpha}}\hat{\beta}}. \tag{2.26}$$

The hermitian structure (2.22) is invertible and we define the inverse $h^{\hat{\alpha}\bar{\hat{\beta}}}$ through $h_{\bar{\hat{\alpha}}\hat{\alpha}}\, h^{\hat{\alpha}\bar{\hat{\gamma}}} = \delta_{\bar{\hat{\alpha}}}{}^{\bar{\hat{\gamma}}}$ or equivalently as

$$\widetilde{Z}_{\hat{\alpha}}\, h^{\hat{\alpha}\bar{\hat{\beta}}}\, \widetilde{Y}_{\bar{\hat{\beta}}} := \overline{Z}^{\bar{\hat{\alpha}}}\, h_{\bar{\hat{\alpha}}\hat{\beta}}\, Y^{\hat{\beta}}, \tag{2.27}$$

where $\widetilde{Y}_{\bar{\hat{\alpha}}} := h_{\bar{\hat{\alpha}}\hat{\beta}}\, Y^{\hat{\beta}}$.

Points of $\overline{M}_\ell$ and $\overline{M}_r$ are real if the corresponding vector spaces are totally null with respect to the metric:

$$\overline{Z}^{\bar{\hat{\alpha}}a} h_{\bar{\hat{\alpha}}\hat{\beta}} Z^{\hat{\beta}b} = 0\,, \qquad \forall a, b, \tag{2.28}$$

$$\widetilde{Z}_{\hat{\alpha}}{}^b\, h^{\alpha\bar{\hat{\beta}}}\, \overline{\widetilde{Z}}_{\bar{\hat{\beta}}}{}^a = 0, \qquad \forall a, b. \tag{2.29}$$

With these definitions, points in (2.24) are real.

3 Invariant subspaces for the super Poincaré group

In this section we break superconformal invariance and discuss the points left at infinity of Minkowski superspace by the action of the Poincaré supergroup.

3.1 Super Poincaré group

We will now break superconformal invariance down to the super Poincaré group by introducing infinity bi-supertwistors

$$I^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \epsilon^{AB} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{I}_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon^{A'B'} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

These bi-supertwistors are simple, $I^{\hat{\alpha}\hat{\beta}} = I_1^{[\hat{\alpha}}I_2^{\hat{\beta}]}$, $I_{\hat{\alpha}\hat{\beta}} = I^1_{[\hat{\alpha}}I^2_{\hat{\beta}]}$, and together define a point $(I^{\hat{\alpha}b}, \tilde{I}_{\hat{\alpha}}{}^b) \in \overline{M}_{\mathbb{C}}^{4|4\mathcal{N}}$ given by

$$I^{\hat{\alpha}b} = \begin{bmatrix} \delta^{Ab} \\ 0_{A'}{}^b \\ 0^{Ib} \end{bmatrix}, \qquad \tilde{I}_{\hat{\alpha}}{}^b = \begin{bmatrix} 0_A{}^b & \delta^{A'b} & 0^{Ib} \end{bmatrix}. \tag{3.2}$$

The subgroup of (2.3) stabilising (3.1) is parametrised by

$$\begin{pmatrix} M^B{}_A & iT^{BA'} & Q^B{}_I \\ 0 & -\widetilde{M}_{B'}{}^{A'} & 0 \\ 0 & \widetilde{Q}^{JA'} & R^J{}_I \end{pmatrix}, \tag{3.3}$$

where now $M^B{}_A$, $\widetilde{M}_{B'}{}^{A'} \in \mathrm{SL}(2, \mathbb{C}_c)$ and $R^J{}_I \in \mathrm{SL}(\mathcal{N}, \mathbb{C}_c)$. This is what we will call the complex super Poincaré group $(\mathrm{ISO}(1,3|\mathcal{N}))_{\mathbb{C}}$, where by an abuse of terminology we include the R-symmetry group $\mathrm{SL}(\mathcal{N}, \mathbb{C}_c)$ in it. The complex super Poincaré group has the structure $(\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(\mathcal{N})) \ltimes \mathcal{T}^{4|4\mathcal{N}}$, where $\mathcal{T}^{4|4\mathcal{N}}$ is the non-Abelian group consisting of the matrices

$$\begin{pmatrix} \delta^B{}_A & iT^{BA'} & Q^B{}_I \\ 0 & \delta_{B'}{}^{A'} & 0 \\ 0 & \widetilde{Q}^{JA'} & \delta^J{}_I \end{pmatrix}. \tag{3.4}$$

If we now impose that the group of matrices (3.3) should preserve the hermitian form (2.6), i.e., that $P^{\hat{\alpha}}{}_{\hat{\beta}}$ should satisfy $\overline{P}^{\bar{\hat{\gamma}}}{}_{\bar{\hat{\beta}}} h_{\bar{\hat{\gamma}}\hat{\delta}} P^{\hat{\delta}}{}_{\hat{\alpha}} = h_{\bar{\hat{\beta}}\hat{\alpha}}$, we find a subgroup given by the set of matrices of the form

$$\mathrm{P}^{\mathbb{R}} = \begin{pmatrix} M^B{}_A & iM^B{}_C t^{CA'} + \frac{1}{2} Q^B{}_I \overline{Q}^{IC'} \overline{M}_{C'}{}^{A'} & Q^B{}_I \\ 0_{B'A} & -\overline{M}_{B'}{}^{A'} & 0_{B'I} \\ 0^J{}_A & -R^J{}_I \overline{Q}^{IB'} \overline{M}_{B'}{}^{A'} & R^J{}_I \end{pmatrix}, \tag{3.5}$$

where $R^I{}_J$ is unitary and $t_{CA'}$ is hermitian. We call this subgroup the real super Poincaré group.

3.2 Minkowski superspace

A point of the superconformal compactification is at infinity whenever $\det(\tilde{I}_{\hat{\alpha}a} Z^{\hat{\alpha}b} = \pi_a{}^b) = 0$, on the contrary points of Minkowski superspace are those satisfying

$$\det(\tilde{I}_{\hat{\alpha}a} Z^{\hat{\alpha}b} = \pi_a{}^b) \neq 0. \tag{3.6}$$

3.2.1 Chiral superspaces

The chiral left Minkowski superspace M_ℓ is defined [10] as the subset of points $Z^{\hat{\alpha}b} \in \overline{M}_\ell$ such that $\det(\tilde{I}_{\hat{\alpha}a} Z^{\hat{\alpha}b} = \pi_a{}^b) \neq 0$. In this chart, the coordinates are such that $\pi_{A'}{}^b$ can be set to the identity, and we write

$$Z^{\hat{\alpha}b} = \begin{bmatrix} iX_+^{Ab} \\ \delta_{A'}{}^b \\ \theta^{Ib} \end{bmatrix}. \tag{3.7}$$

The second line allows to identify A' with b, and $(X_+^{AA'}, \theta^{IA'})$ form the chiral coordinates on M_ℓ. Similarly, we have chiral coordinates $(X_-^{A'A}, \tilde{\theta}^{IA})$ on M_r:

$$\widetilde{Z}_{\hat{\alpha}}{}^b = \left[\delta_A{}^b \; -iX_-^{A'b} \; -\tilde{\theta}_I^b\right]. \tag{3.8}$$

The action of the super Poincaré group (3.3) on M_ℓ is given by

$$\begin{aligned}\begin{pmatrix} M^B{}_A & iT^{BA'} & Q^B{}_I \\ 0 & -\widetilde{M}_{B'}{}^{A'} & 0 \\ 0 & \widetilde{Q}^{JA'} & R^J{}_I \end{pmatrix} \begin{bmatrix} iX_+^{Ab} \\ \delta_{A'}{}^b \\ \theta^{Ib} \end{bmatrix} &= \begin{bmatrix} M^B{}_A iX_+^{Ab} + iT^{Bb} + Q^B{}_I\theta^{Ib} \\ -\widetilde{M}_{B'}{}^b \\ \widetilde{Q}^{Jb} + R^J{}_I\theta^{Ib} \end{bmatrix} \\ &\sim \begin{bmatrix} (\widetilde{M})^b{}_c \cdot (M^B{}_A iX_+^{Ac} + iT^{Bc} + Q^B{}_I\theta^{Ic}) \\ \delta_{B'}{}^b \\ (\widetilde{M})^b{}_c(\widetilde{Q}^{Jc} + R^J{}_I\theta^{Ic}) \end{bmatrix} \in M_\ell. \end{aligned} \tag{3.9}$$

The action is transitive which makes M_ℓ an homogeneous space for the complex super Poincaré group:

$$M_\ell \simeq \frac{(\mathrm{ISO}(1,3|\mathcal{N}))_{\mathbb{C}}}{(\mathrm{SL}(2,\mathbb{C}_c) \times \mathrm{SL}(2,\mathbb{C}_c) \times \mathrm{SL}(\mathcal{N},\mathbb{C}_c)) \ltimes \mathbb{C}^{0|2\mathcal{N}}}. \tag{3.10}$$

The stabiliser is obtained from (3.3) by fixing the point $X_+^{AA'} = 0$, $\theta^{Ib} = 0$ of M_ℓ and this imposes $T^{BA'} = 0$, $\widetilde{Q}^{JA'} = 0$.

For M_r it is obtained in a similar way by asking to fix the point $X_-^{A'A} = 0$, $\theta^{Ib} = 0$ and this imposes instead $T^{BA'} = 0$, $Q^B{}_I = 0$. Therefore, M_ℓ and M_r have the same expression (3.10) as homogeneous spaces, although they are not the same space.

Reality conditions. If we restrict ourselves to the real case, i.e., if we impose on chiral left Minkowski superspace the condition (2.28), we get

$$X_+^{AA'} - \overline{X}_+^{A'A} = -i\,\overline{\theta}^{IA}\,\delta_{IJ}\,\theta^{JA'}. \tag{3.11}$$

As a result of this reality condition, we conclude that there exists a hermitian matrix $x^{AA'}$ such that

$$X_+^{AA'} = x_\ell^{AA'} - \frac{i}{2}\,\overline{\theta}^{IA}\,\delta_{IJ}\,\theta^{JA'}. \tag{3.12}$$

Similarly, when we impose the reality conditions $\overline{Z}_{\bar{\hat{\alpha}}}{}^a\, h^{\bar{\hat{\alpha}}\beta}\, Z_{\hat{\beta}}{}^b = 0$ on chiral right Minkowski superspace, consistently with (2.27), we find that there exists a hermitian matrix $x_r^{AA'}$ such that

$$X_-^{A'A} = x_r^{AA'} + \frac{i}{2}\,\overline{\tilde{\theta}}^{IA}\delta_{IJ}\tilde{\theta}^{JA'}. \tag{3.13}$$

The real Poincaré group (3.5) acts transitively on these real Minkowski superspaces. Accordingly, (3.10) reduces to

$$M_{\mathbb{R}}^{4|4\mathcal{N}} \simeq \frac{\mathrm{ISO}(1,3|\mathcal{N})}{\mathrm{SL}(2,\mathbb{C}_c) \times \mathrm{SU}(\mathcal{N})}. \tag{3.14}$$

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3.2.2 Non-chiral superspace

The non-chiral Minkowski superspace $M_{\mathbb{C}}^{4|4\mathcal{N}}$ is defined via the chiral left and right Minkowski superspaces:

$$M_{\mathbb{C}}^{4|4\mathcal{N}} = \{(Z^{\hat{\alpha}b}, \widetilde{Z}_{\hat{\alpha}b}) \mid Z^{\hat{\alpha}b} \in M_\ell \text{ and } \widetilde{Z}_{\hat{\alpha}b} \in M_r \text{ such that } \widetilde{Z}_{\hat{\alpha}}{}^b Z^{\hat{\alpha}a} = 0\}. \tag{3.15}$$

The flag condition between (3.7) and (3.8) imposes the relation

$$X_+^{AA'} - X_-^{A'A} = -i\tilde{\theta}_I^A \theta^{A'I} \tag{3.16}$$

or, equivalently,

$$X_+^{AA'} = x^{AA'} - \frac{i}{2}\tilde{\theta}_I^A \theta^{A'I}, \qquad X_-^{A'A} = x^{AA'} + \frac{i}{2}\tilde{\theta}_I^A \theta^{A'I}, \tag{3.17}$$

where $(x^{AA'}, \theta^{A'I}, \tilde{\theta}_I^A)$ are coordinates on $M_{\mathbb{C}}^{4|4\mathcal{N}}$. The group (3.3) acts on $M_{\mathbb{C}}^{4|4\mathcal{N}}$ and the stabiliser is the intersection of the stabilisers of the chiral spaces M_ℓ and M_r. One recovers that complex Minkowski superspace is an homogeneous space for the complexified super Poincaré group.

$$M_{\mathbb{C}}^{4|4\mathcal{N}} \simeq \frac{(\mathrm{ISO}(1,3|\mathcal{N}))_{\mathbb{C}}}{\mathrm{SL}(2,\mathbb{C}_c) \times \mathrm{SL}(2,\mathbb{C}_c) \times \mathrm{SL}(\mathcal{N},\mathbb{C}_c)}. \tag{3.18}$$

Reality conditions. The real points are obtained by imposing on the points

$$\{(Z^{\hat{\alpha}b}, \widetilde{Z}_{\hat{\alpha}}{}^b) \mid \widetilde{Z}_{\hat{\alpha}}{}^b Z^{\hat{\alpha}a} = 0\}$$

of complex non-chiral Minkowski space the reality conditions $\widetilde{Z}_{\hat{\alpha}}{}^b = h_{\bar{\hat{\alpha}}\hat{\alpha}} \overline{Z}^{\bar{\hat{\alpha}}b}$, where the complex structure $h_{\bar{\alpha}\beta}$ is given in (2.22). This is equivalent to $X_-^{A'A} = \overline{X_+}^{A'A}$ (i.e., $x^{AA'} = \overline{x}^{A'A}$) and $\tilde{\theta}^{IA} = \overline{\theta}^{IA}$. The corresponding real homogeneous space is again (3.14).

3.2.3 Summary of the coordinates

Keeping in mind that $X_+^{AA'} = x_\ell^{AA'} - \frac{i}{2}\,\overline{\theta}^{IA}\,\delta_{IJ}\,\theta^{JA'}$ and $X_-^{A'A} = x_r^{AA'} + \frac{i}{2}\,\overline{\tilde{\theta}}^{IA}\delta_{IJ}\tilde{\theta}^{JA'}$ the situation can be summarised by the following.

	Complex	Real
Chiral left	$(X_+^{AA'}, \theta^{IA})$	$(x_\ell^{AA'}, \theta^{IA})$ with $x_\ell^{AA'}$ hermitian
(resp. right)	(resp. $(X_-^{AA'}, \tilde{\theta}^{IA'})$)	(resp. $(x_r^{AA'}, \tilde{\theta}^{IA'})$) with $x_r^{AA'}$ hermitian
Non-chiral	$(x^{AA'}, \theta^{IA'}, \tilde{\theta}_I^A)$	$(x^{AA'}, \theta^{IA'})$ (with $\tilde{\theta} = \bar{\theta}$ and $x_\ell^{AA'} = x_r^{AA'} = x^{AA'}$ hermitian)

3.3 Super Null Infinity

We will now discuss points at infinity, i.e., satisfying $\det(\pi_{A'}{}^b) = 0$. Among those, points for which $\pi_{A'}{}^b = 0$ correspond to a supersymmetric version of space/time like infinity which we will here exclude. Under this assumption can always make use of $\mathrm{GL}(2,\mathbb{C})$ invariance in the choice of the generators of plane to write

$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega^A & \hat{\pi}^A \\ \pi_{A'} & 0 \\ \theta^I & \eta^I = 0 \end{bmatrix}. \tag{3.19}$$

The condition $\eta^I = 0$ in the above equation is a Poincaré invariant condition which is part of the definition of chiral left super null infinity $\mathscr{I}_\ell$ (using dual twistors, chiral right super null infinity $\mathscr{I}_r$ can be defined exactly in the same way). The most general situation with $\eta^I \neq 0$ correspond to another invariant subspace which will be investigated in the next section.

3.3.1 Chiral superspaces

Using the remaining $\mathrm{GL}(2,\mathbb{C})$ freedom, we can pose

$$u_+ = -i\omega^A \hat{\pi}_A, \tag{3.20}$$

which can be interpreted as a chiral left coordinate on $\mathscr{I}_\ell$, on which now there are coordinates $(u_+, \hat{\pi}^A, \pi_{A'}, \theta^I) \in \mathbb{C} \times \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{C}^{0|\mathcal{N}}$. Similarly, on $\mathscr{I}_r$, we have coordinates $(u_-, \hat{\tilde{\pi}}^A, \tilde{\pi}_{A'}, \tilde{\theta}^I)$.

The action of the super Poincaré group (3.3) on $\mathscr{I}_\ell$ is given by

$$\begin{pmatrix} M^B{}_A & iT^{BA'} & Q^B{}_I \\ 0 & -\widetilde{M}_{B'}{}^{A'} & 0 \\ 0 & \widetilde{Q}^{JA'} & R^J{}_I \end{pmatrix} \cdot \begin{bmatrix} \omega^A & \hat{\pi}^A \\ \pi_{A'} & 0 \\ \theta^I & 0^I \end{bmatrix} = \begin{bmatrix} M^B{}_A\omega^A + iT^{BA'}\pi_{A'} + Q^B{}_I\theta^I & M^B{}_A\hat{\pi}^A \\ -\widetilde{M}_{B'}{}^{A'}\pi_{A'} & 0 \\ \widetilde{Q}^{JA'}\pi_{A'} + R^J{}_I\theta^I & 0^I \end{bmatrix} \in \mathscr{I}_\ell. \tag{3.21}$$

On the chiral right $\mathscr{I}_r$, it acts by the inverse as described in (2.12).

The action is transitive which makes $\mathscr{I}_\ell$, $\mathscr{I}_r$ both homogeneous spaces for the complex super Poincaré group. The stabiliser, which is constituted of matrices of the form

$$\begin{pmatrix} M^1{}_1 & M^1{}_2 & T^{11} & T^{12} & Q^1{}_I \\ 0 & (M^1{}_1)^{-1} & T^{21} & 0 & Q^2{}_I \\ 0 & 0 & -\widetilde{M}_1{}^1 & 0 & 0_I \\ 0 & 0 & \widetilde{M}_2{}^1 & -(\widetilde{M}_1{}^1)^{-1} & 0_I \\ 0^I & 0^I & \tilde{Q}^{1I} & 0^I & R^J{}_I \end{pmatrix}, \tag{3.22}$$

fixes $M^2{}_2$, $M^2{}_1$, $\widetilde{M}_2{}^2$, $\widetilde{M}_1{}^2$, T^{22} and $\widetilde{Q}^{2I} = 0$ (as compared to a linear combination of Q^{1I} and Q^{2I} for the right space) and parametrises in both cases the subgroup $\mathbb{C}^3 \rtimes (\mathbb{C}^{0|3\mathcal{N}} \rtimes (\mathrm{ISO}(2) \times \mathbb{R}^* \times \mathrm{ISO}(2) \times \mathbb{R}^* \times \mathrm{SL}(\mathcal{N})))$. This can be shown by asking to stabilise the point

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0^I & 0^I \end{bmatrix} \in \mathscr{I}_\ell, \tag{3.23}$$

(respectively the point $\begin{bmatrix} 0 & 1 & 0 & 0 & 0^I \\ 0 & 0 & 0 & 1 & 0^I \end{bmatrix} \in \mathscr{I}_r$).

Therefore, we obtain the expression

$$\mathscr{I}_\ell \simeq \mathscr{I}_r \simeq \frac{(\mathrm{ISO}(1,3|\mathcal{N}))_{\mathbb{C}}}{\mathbb{C}^{3|0} \rtimes (\mathbb{C}^{0|3\mathcal{N}} \rtimes (\mathrm{ISO}(2) \times \mathbb{R}^* \times \mathrm{ISO}(2) \times \mathbb{R}^* \times \mathrm{SL}(\mathcal{N})))}. \tag{3.24}$$

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Reality conditions. By imposing (2.28), we get the following two conditions

$$[\hat{\pi}_A] = [\overline{\pi}_A], \qquad\qquad -2\,\mathrm{Im}(u_+) - \overline{\theta}\theta = 0, \tag{3.25}$$

and so we have local coordinates $(u_\ell, ([\pi^{A'}], [\overline{\pi}^A]), \theta^I) \in \mathbb{R} \times S^2 \times \mathbb{C}^{0|\mathcal{N}}$, where $u_+ = u_\ell - \frac{i}{2}\overline{\theta}^I\theta_I$ (similarly on $\mathscr{I}_r$ we have $(u_r, ([\tilde{\pi}^A], [\overline{\tilde{\pi}}^{A'}]), \tilde{\theta}^I)$ with $u_- = u_r + \frac{i}{2}\overline{\tilde{\theta}}^I\tilde{\theta}_I$).

By intersecting with the subgroup (3.5) that acts transitively on the real points of $\mathscr{I}$, we obtain the expression for the real super null infinity

$$\mathscr{I}_{\mathbb{R}}^{3|2\mathcal{N}} \simeq \frac{\mathrm{ISO}(1,3|\mathcal{N})}{\mathbb{R}^3 \rtimes (\mathbb{R}^{0|2\mathcal{N}} \rtimes (\mathrm{ISO}(2) \times \mathbb{R}^* \times \mathrm{SU}(\mathcal{N})))}. \tag{3.26}$$

3.3.2 Non-chiral superspace

We define the non-chiral super null infinity $\mathscr{I}_{\mathbb{C}}^{3|2\mathcal{N}}$ as the subspace of $\mathscr{I}_\ell \times \mathscr{I}_r$ in $F(2|0,2|\mathcal{N},\mathbb{C}^{4|\mathcal{N}})$. In coordinates, the flag conditions on

$$(Z^{\hat{\alpha}a}, \tilde{Z}^a{}_{\hat{\alpha}}) = \left(\begin{bmatrix} \omega^A & \hat{\pi}^A \\ \pi_{A'} & 0_{A'} \\ \theta^I & 0^I \end{bmatrix}, \begin{bmatrix} \tilde{\pi}_A & \tilde{\omega}^{A'} & -\tilde{\theta}_I \\ 0_A & \hat{\tilde{\pi}}^{A'} & 0_I \end{bmatrix} \right) \in \mathscr{I}_\ell \times \mathscr{I}_r \tag{3.27}$$

are realised by

$$\hat{\tilde{\pi}}^{A'}\,\pi_{A'} = 0, \tag{3.28}$$
$$\tilde{\pi}_A\,\omega^A + \tilde{\omega}^{A'}\,\pi_{A'} - \tilde{\theta}^I\,\theta_I = 0, \tag{3.29}$$
$$\tilde{\pi}_A\,\hat{\pi}^A = 0. \tag{3.30}$$

Let us introduce again the $\mathrm{GL}(2,\mathbb{C})$ invariants

$$iu_+ := \omega^A\,\hat{\pi}_A, \qquad\qquad -iu_- := \tilde{\omega}^{A'}\,\hat{\tilde{\pi}}_{A'}. \tag{3.31}$$

Then the flag conditions (3.29) – (3.30) are solved by

$$[\tilde{\pi}_A] = [\hat{\pi}_A] \qquad\qquad [\hat{\tilde{\pi}}_{A'}] = [\pi_{A'}] \qquad\qquad iu_+ - iu_- - \tilde{\theta}^I\theta_I = 0. \tag{3.32}$$

Setting

$$u_+ = u_l - \frac{i}{2}\,\tilde{\theta}^I\theta_I, \qquad\qquad u_- = u_r + \frac{i}{2}\,\tilde{\theta}^I\theta_I, \tag{3.33}$$

one solves the condition (3.32) with $u_l = u_r = u$. Non-chiral coordinates on $\mathscr{I}_{\mathbb{C}}^{3|2\mathcal{N}}$ are then given by $(u, [\pi_{A'}], [\tilde{\pi}_A], \theta^I, \tilde{\theta}^I)$. The group (3.3) acts on $\mathscr{I}_{\mathbb{C}}^{3|2\mathcal{N}}$ and the stabiliser is the intersection of the stabilisers of the chiral spaces $\mathscr{I}_\ell$ and $\mathscr{I}_r$. One recovers that complex super null infinity is an homogeneous space

$$\mathscr{I}_{\mathbb{C}}^{3|2\mathcal{N}} \simeq \frac{(\mathrm{ISO}(1,3|\mathcal{N}))_{\mathbb{C}}}{\mathbb{C}^{3|0} \rtimes (\mathbb{C}^{0|2\mathcal{N}} \rtimes (\mathrm{ISO}(2) \times \mathbb{R}^* \times \mathrm{ISO}(2) \times \mathbb{R}^* \times \mathrm{SL}(\mathcal{N})))}. \tag{3.34}$$

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Reality conditions. The real points are obtained by imposing on the points $\{(Z^{\hat{\alpha}b}, \widetilde{Z}_{\hat{\alpha}}{}^b)|\ \widetilde{Z}_{\hat{\alpha}}{}^b Z^{\hat{\alpha}a} = 0\}$ of complex non-chiral $\mathscr{I}_{\mathbb{C}}^{3|2\mathcal{N}}$ the reality conditions $\widetilde{Z}_{\hat{\alpha}}{}^b = h_{\bar{\tilde{\alpha}}\hat{\alpha}} \overline{Z}^{\bar{\tilde{\alpha}}b}$, where the complex structure $h_{\bar{\alpha}\beta}$ is given in (2.22). This is equivalent to $\tilde{\pi}_A = \overline{\pi}_A$, $\tilde{\theta}^I = \overline{\theta}^I$ and

$$\overline{iu_+} = \overline{\omega}^{A'}\overline{\hat{\pi}}_{A'} = \tilde{\omega}^{A'}\tilde{\hat{\pi}}_{A'} = iu_-,$$

therefore $\overline{u_+} = u_-$ and this implies $\overline{u} + \frac{i}{2}\overline{\theta}^I\theta_I = u + \frac{i}{2}\overline{\theta}^I\theta_I$, i.e. $u \in \mathbb{R}$.

The corresponding real homogeneous space is again (3.26).

3.3.3 Summary of the coordinates

Keeping in mind the notation $u_+ = u_\ell - \frac{i}{2}\overline{\theta}^I\theta_I$ and $u_- = u_r + \frac{i}{2}\overline{\tilde{\theta}}^I\tilde{\theta}_I$, the coordinates on super null infinity are summarised as follows:

	Complex	Real
Chiral left	$(u_+, [\pi_{A'}], [\hat{\pi}^A], \theta^I)$	$(u_\ell, [\pi_{A'}], \theta^I)$ with $u_\ell \in \mathbb{R}$ and $\hat{\pi}^A = \overline{\pi}^A$
(resp. right)	(resp. $(u_-, [\tilde{\pi}_A], [\tilde{\hat{\pi}}^{A'}], \tilde{\theta}^I)$)	(resp. $(u_r, [\tilde{\pi}_A], \tilde{\theta}^I)$ with $u_r \in \mathbb{R}$ and $\tilde{\hat{\pi}}^{A'} = \overline{\tilde{\pi}}^{A'}$)
Non-chiral	$(u, [\pi_{A'}], [\tilde{\pi}_A], \theta^I, \tilde{\theta}^I)$	$(u, [\pi_{A'}], \theta^I)$ with $u \in \mathbb{R}$, $\tilde{\pi}^A = \overline{\pi}^A$ and $\tilde{\theta}^I = \overline{\theta}^I$

3.4 Other invariants subspaces

3.4.1 Subspace $\mathcal{H}$

This subspace is defined in the chiral left sector as the set of points with $\det(\pi_{A'}{}^b) = 0$, $\pi_{A'}{}^b \neq 0$ but where, as compare to (3.19), η is supposed to be non zero:

$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega^A & \hat{\pi}^A \\ \pi_{A'} & 0 \\ \theta^I & \eta^I \neq 0 \end{bmatrix}. \tag{3.35}$$

The $\mathrm{GL}(2,\mathbb{C})$-invariant quantity

$$iu_+ = \omega^A \hat{\pi}_A, \tag{3.36}$$

can then be interpreted as one of the chiral left coordinates $(u_+, [\pi_{A'}], [\hat{\pi}^A], \theta^I, \eta^I) \in \mathbb{C} \times \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{C}^{0|\mathcal{N}} \times \mathbb{C}^{0|\mathcal{N}}$ on $\mathcal{H}_\ell$. This sub-supermanifold is invariant under super Poincaré. However the action is *not* transitive because the R-symmetry $\mathrm{SL}(\mathcal{N}, \mathbb{C}_c)$ does not act transitively on $\mathbb{C}_a^{\mathcal{N}}$. Let us here discuss the orbits of the particular point

$$Z^{\hat{\alpha}b} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0^I & \eta^I \end{bmatrix} \qquad \text{with} \qquad \eta^I = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.37}$$

For $\mathcal{N} = 1$ all points of $\mathcal{H}_\ell$ lie in such an orbit and $a \in \mathbb{C}_a$ parametrises the different possible orbits; for $\mathcal{N} \geq 2$ there are more complicated orbits of the type $\eta^I = (a, b, \ldots, 0)$ with

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$ab \neq 0$. As we shall see the situation for $\mathcal{N}=1$ is already interesting enough. The action of the super Poincaré group is given by

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & a \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \mapsto \begin{pmatrix} M^1{}_1 & M^1{}_2 & iT^{11'} & iT^{12'} & Q^1{}_1 & \cdots & Q^1{}_N \\ M^2{}_1 & M^2{}_2 & iT^{21'} & iT^{22'} & Q^2{}_1 & \cdots & Q^2{}_N \\ 0 & 0 & -\widetilde{M}_{1'}{}^{1'} & -\widetilde{M}_{1'}{}^{2'} & 0 & \cdots & 0 \\ 0 & 0 & -\widetilde{M}_{2'}{}^{1'} & -\widetilde{M}_{2'}{}^{2'} & 0 & \cdots & 0 \\ 0 & 0 & \widetilde{Q}^{11'} & \widetilde{Q}^{12'} & R^1{}_1 & \cdots & R^1{}_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \widetilde{Q}^{N1'} & \widetilde{Q}^{N2'} & R^N{}_1 & \cdots & R^N{}_N \end{pmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & a \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \tag{3.38}$$

and the subgroup stabilising our preferred point requires the following equations to be satisfied:

$$iT^{22'} = 0, \tag{3.39}$$
$$\widetilde{M}_{1'}{}^{2'} = 0, \tag{3.40}$$
$$\widetilde{Q}^{12'} = iT^{12'}a, \tag{3.41}$$
$$\widetilde{Q}^{J2'} = 0 \qquad \text{if } J > 2, \tag{3.42}$$
$$M^2{}_1 = -Q^2{}_1 a, \tag{3.43}$$
$$M^1{}_1 = R^1{}_1 + \mathfrak{M}^1\, a, \tag{3.44}$$
$$R^J{}_1 = \mathfrak{R}^J\, a \qquad \text{if } J > 2, \tag{3.45}$$

where $\mathfrak{M}^1$ and $\mathfrak{R}^J$ are some unconstrained anti-commuting numbers. Let us now stick to $\mathcal{N}=1$ in order to simplify the discussion: the parameters left unconstrained for the stabiliser are then $M^1{}_2$, $\widetilde{M}^1{}_1$, $\widetilde{M}^1{}_2$, $T^{11'}$, $T^{12'}$, $T^{21'}$, Q^A and $\mathfrak{M}^1$. One might be tempted to conclude that this stabiliser is a supergroup of dimension $\mathbb{C}^{6|3}$ and therefore that the orbits an homogeneous space of dimension $\mathbb{C}^{4|1}$. (It might here be useful for the reader to compare with $\mathscr{I}_l$ which is an homogeneous space (3.24) of dimension $\mathbb{C}^{3|\mathcal{N}}$.) Strictly speaking this would however be an incorrect conclusion: the fact that the "coordinate" $\mathfrak{M}^1$ here only appears as the product "$\mathfrak{M}^1\, a$" means that it is ambiguous and that, following DeWitt [47], it does not define a superchart; the stabiliser is in fact strictly speaking not a supermanifold. If the same analysis had been instead performed at the level of the algebra one would have equivalently concluded that the corresponding stabiliser is a sub supervector space (over Λ_∞) of the super Poincaré algebra *which does not admit a basis*.

We here wish to comment on the peculiar features of supergeometries which are responsible for these phenomena. If one follows DeWitt's approach to supermanifolds (this approach is equivalent to the algebro-geometric approach, see [48] for a proof), then the supervector spaces on which manifolds are modelled are constructed over the supernumbers Λ_∞, which form a graded ring and a graded infinite dimensional vector space over $\mathbb{C}$. Nevertheless, because theses supervector spaces of DeWitt are not over a field but rather over the graded *ring* Λ_∞, they are by definition (graded) modules. In general, the analogy with vector space is dangerous because not all modules over an algebra $\mathcal{A}$ admit a basis. In the case where a (graded) module over $\mathcal{A}$ admit a basis it is said to be free and the parallel with vector spaces is possible: in [49] this analogy between vector spaces and free modules (there called $\mathcal{A}$-vector spaces) has been investigated with great care. This, nevertheless, has certain important limits. The main limit, that turns out to be crucial for our analysis, is the notion of subobject. In fact, there is no natural way to induce a basis on a submodule, once a basis for the free graded $\mathcal{A}$-module is given, due to the counter-intuitive fact that submodules of a free module do not have to be free. One therefore [49] restricts the notion of a graded subspace in this new category as being a graded subspace in the usual sense, with the additional restriction that there exists a homogeneous basis for the total space within its equivalence class such that a subset of it forms a basis for the graded subspace in question; these are the sub-$\mathcal{A}$-vector spaces of [49]. This being introduced, supermanifolds and sub-supermanifold are respectively modelled on Λ_∞-vector spaces and sub-Λ_∞-vector spaces. However, one should realise that, contrary to the ordinary case, one cannot guarantee that through every point of a supermanifold passes a sub-supermanifold of lower dimension [49].

Note however that, despite the fact that these orbits are not a supermanifold, this is very easy to describe them explicitly as

$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega^A & \hat{\pi}^A \\ \pi_{A'} & 0 \\ \theta^I & \alpha a \end{bmatrix}. \tag{3.46}$$

Here $a \in \mathbb{C}_a$ is fixed and parametrises different orbits while $\alpha \in \mathbb{C}_c$ is an extra commuting "coordinate" parametrising the would-be extra bosonic dimension of these orbits as compared to $\mathscr{I}_\ell$.

As we shall see reality conditions complicate further the situation since neither the real orbits nor the real subspace of $\mathcal{H}$ are sub-supermanifolds. The non-chiral models have the same problem.

Reality conditions. We are going to impose the following reality conditions:

$$(Z^1)^2 = 0 \Rightarrow \omega^A\bar{\pi}_A + \pi_{A'}\bar{\omega}^{A'} + \delta_{IJ}\theta^I\bar{\theta}^J = 0 \;\; i.e.\; 2\,\mathrm{Re}(\omega^A\bar{\pi}_A) - \delta_{IJ}\bar{\theta}^J\theta^I = 0, \tag{3.47a}$$

$$(Z^2)^2 = 0 \Rightarrow \delta_{IJ}\eta^I\bar{\eta}^J = 0, \tag{3.47b}$$

$$Z^1 \cdot Z^2 = 0 \Rightarrow \bar{\pi}_A\hat{\pi}^A - \bar{\theta}_J\eta^J = 0. \tag{3.47c}$$

The same type of technical problem we met in the context of the stabiliser of $\mathcal{H}$ shows up at this stage: the second equation above does not define a sub-supermanifold and therefore, here again, strictly speaking the real supermanifold $\mathcal{H}$ cannot be defined.

For $\mathcal{N} = 1$, we can solve the reality conditions and deduce a parametrisation of the real $\mathcal{H}$: first, equation (3.47b) becomes $\eta\bar{\eta} = 0$ and is solved as $\eta = e^{i\phi}a$, with $a \in \mathbb{R}_a$ a real anticommuting supernumber and $\phi \in \mathbb{R}_c$ a real commuting supernumber. Then, in order to solve equation (3.47c), we introduce the parametrisation $\eta = \eta_A\hat{\pi}^A$, where the odd variables η_A is chosen to satisfy $\eta_A\bar{\eta}_{A'} = 0$ (this ensures ensure that (3.47b) still holds). Such a decomposition of η is not unique but always exists. In fact, because $\hat{\pi}^A$ has non zero body, at least one of the body of the components in $\{\hat{\pi}^0, \hat{\pi}^1\}$ is non zero. Assume for example that $\hat{\pi}^0$ has non zero body; then we can take $\eta_A = (\eta/\hat{\pi}^0, 0)$. With this choice of parametrisation, the third reality condition (3.47c) reads

$$(\bar{\pi}_A - \bar{\theta}\eta_A)\hat{\pi}^A = 0, \tag{3.48}$$

and, since $\hat{\pi}^A$ has a non vanishing body, can be solved as

$$\hat{\pi}^A = \alpha(\bar{\pi}^A - \bar{\theta}\eta^A), \qquad \alpha \in \mathbb{C}. \tag{3.49}$$

Using complex projective invariance to rescale $Z^{\hat{\alpha}2}$, one gets

$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega^A & \bar{\pi}^A - \bar{\theta}\eta^A \\ \pi_{A'} & 0 \\ \theta & \eta_A(\bar{\pi}^A - \bar{\theta}\eta^A) \end{bmatrix}. \tag{3.50}$$

Finally, in order to solve the first reality condition (3.47a) we introduce $\Theta := \theta + \eta^A\omega_A$ and $iu_+ := \omega^A(\bar{\pi}_A - \bar{\Theta}\eta_A)$. We then have

$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega^A & \bar{\pi}^A - \bar{\Theta}\eta^A \\ \pi_{A'} & 0 \\ \Theta + \omega^A\eta_A & \eta_A(\bar{\pi}^A - \bar{\Theta}\eta^A) \end{bmatrix} = \begin{bmatrix} \omega^A & \bar{\pi}^A - \bar{\Theta}\eta^A \\ \pi_{A'} & 0 \\ \Theta + \omega^A\eta_A & e^{i\phi}a \end{bmatrix}, \tag{3.51}$$

and equation (3.47a) becomes

$$2\,Im(u_+) = \Theta\bar{\Theta}. \tag{3.52}$$

Which is solved as

$$u_+ = u_\ell + \frac{i}{2}\Theta\bar{\Theta}, \quad u_\ell \in \mathbb{R}_c. \tag{3.53}$$

We thus found a parametrisation of $\mathcal{H}$ by $(u_\ell, [\pi_{A'}], e^{i\phi}, \Theta, a) \in \mathbb{R} \times \mathbb{CP}^1 \times S^1 \times \mathbb{C}_a \times \mathbb{R}_a$. Once again, since $e^{i\phi}$ always appears in the form "$e^{i\phi}a$", it would strictly speaking be incorrect to interpret these as coordinates on a supermanifold.

Non-chiral $\mathcal{H}$. In coordinates, the flag conditions on

$$(Z^{\hat{\alpha}a}, \tilde{Z}^a{}_{\hat{\alpha}}) = \left(\begin{bmatrix} \omega^A & \hat{\pi}^A \\ \pi_{A'} & 0_{A'} \\ \theta^I & \eta^I \end{bmatrix}, \begin{bmatrix} \tilde{\pi}_A & \tilde{\omega}^{A'} & -\tilde{\theta}_I \\ 0_A & \tilde{\hat{\pi}}^{A'} & -\tilde{\eta}_I \end{bmatrix} \right) \in \mathcal{H}_\ell \times \mathcal{H}_r \tag{3.54}$$

are realised by

$$\tilde{\eta}^I\,\eta_I = 0, \tag{3.55a}$$
$$\tilde{\hat{\pi}}^{A'}\,\pi_{A'} - \tilde{\eta}_I\theta^I = 0, \tag{3.55b}$$
$$\tilde{\pi}_A\,\omega^A + \tilde{\omega}^{A'}\,\pi_{A'} - \tilde{\theta}^I\,\theta_I = 0, \tag{3.55c}$$
$$\tilde{\pi}_A\,\hat{\pi}^A - \tilde{\theta}^I\,\eta_I = 0. \tag{3.55d}$$

The first condition can be solved by $\tilde{\eta} = \gamma\,\eta$, where $\gamma \in \mathbb{C}$. As in the real case, we use a decomposition $\eta = \eta_A\hat{\pi}^A$. Similarly, we introduce a decomposition $\tilde{\eta} = \tilde{\hat{\pi}}^{A'}\tilde{\eta}_{A'}$ and to keep satisfied the first reality condition we ask $\eta_A \tilde{\eta}_{A'} = 0$; this is always possible by a similar argument as employed in the chiral case. The equation (3.55d) can therefore be written as

$$(\tilde{\pi}_A - \tilde{\theta}\eta_A)\hat{\pi}^A = 0 \tag{3.56}$$

which can be solved, because $\hat{\pi}^A$ has a non vanishing body, as

$$\hat{\pi}^A = \alpha(\tilde{\pi}^A - \tilde{\theta}\eta^A), \quad \alpha \in \mathbb{C}. \tag{3.57}$$

Similarly, the equation (3.55b) can be written

$$\hat{\tilde{\pi}}^{A'}(\pi_{A'} - \tilde{\eta}_{A'}\theta) = 0, \tag{3.58}$$

which gives

$$\hat{\tilde{\pi}}^{A'} = \tilde{\alpha}(\pi^{A'} - \tilde{\eta}^{A'}\theta), \quad \tilde{\alpha} \in \mathbb{C}. \tag{3.59}$$

We have then

$$(Z^{\hat{\alpha}a}, \tilde{Z}^a{}_{\hat{\alpha}}) = \left(\begin{bmatrix} \omega^A & \tilde{\pi}^A - \tilde{\theta}\eta^A \\ \pi_{A'} & 0_{A'} \\ \theta & \eta_A(\tilde{\pi}^A - \tilde{\theta}\eta^A) \end{bmatrix}, \begin{bmatrix} \tilde{\pi}_A & \tilde{\omega}^{A'} & -\tilde{\theta} \\ 0_A & \pi^{A'} - \tilde{\eta}^{A'}\theta & -(\pi^{A'} - \tilde{\eta}^{A'}\theta)\tilde{\eta}_{A'} \end{bmatrix} \right). \tag{3.60}$$

Introducing $\widetilde{\Theta} = \tilde{\theta} + \tilde{\eta}^{A'}\tilde{\omega}_{A'}$ and $\Theta = \theta + \eta^A\omega_A$, we can write

$$(Z^{\hat{\alpha}a}, \tilde{Z}^a{}_{\hat{\alpha}}) = \left(\begin{bmatrix} \omega^A & \tilde{\pi}^A - \widetilde{\Theta}\eta^A \\ \pi_{A'} & 0 \\ \Theta - \eta^A\omega_A & \eta_A(\tilde{\pi}^A - \widetilde{\Theta}\eta^A) \end{bmatrix}, \begin{bmatrix} \tilde{\pi}_A & \tilde{\omega}^{A'} & -\widetilde{\Theta} + \tilde{\eta}^{A'}\tilde{\omega}_{A'} \\ 0_A & (\pi^{A'} - \tilde{\eta}^{A'}\Theta) & -(\pi^{A'} - \tilde{\eta}^{A'}\Theta)\tilde{\eta}_{A'} \end{bmatrix} \right). \tag{3.61}$$

The last condition to be solved, equation (3.55c), can now be expressed as

$$\begin{aligned} &\tilde{\pi}_A\,\omega^A + \tilde{\omega}^{A'}\,\pi_{A'} - \widetilde{\Theta}\,\Theta + \widetilde{\Theta}\eta^A\omega_A + \tilde{\eta}^{A'}\tilde{\omega}_{A'}\Theta = 0 \\ \iff\ &(\tilde{\pi}_A - \widetilde{\Theta}\eta_A)\omega^A + \tilde{\omega}^{A'}(\pi_{A'} - \tilde{\eta}_{A'}\Theta) = \widetilde{\Theta}\,\Theta. \end{aligned} \tag{3.62}$$

If we define the quantities (u_+, u_-) via $iu_+ := (\tilde{\pi}_A - \widetilde{\Theta}\eta_A)\omega^A$ and $-iu_- := \tilde{\omega}^{A'}(\pi_{A'} - \tilde{\eta}_{A'}\Theta)$, then (3.62) is equivalent to

$$iu_+ - iu_- = \widetilde{\Theta}\,\Theta. \tag{3.63}$$

Setting

$$u_+ = u_l - \frac{i}{2}\widetilde{\Theta}\Theta, \qquad\qquad u_- = u_r + \frac{i}{2}\widetilde{\Theta}\Theta, \tag{3.64}$$

one solves the condition (3.63) with $u_l = u_r = u$. A parametrisation of $\mathcal{H}_\ell \times \mathcal{H}_r$ is then given by $(u, [\pi_{A'}], [\tilde{\pi}_A], \gamma, \Theta, \widetilde{\Theta}, \eta)$ where $\gamma \in \mathbb{C}$.

Reality conditions. The real points are obtained by imposing on the points

$$\{(Z^{\hat{\alpha}b}, \widetilde{Z}_{\hat{\alpha}}{}^b) \mid \widetilde{Z}_{\hat{\alpha}}{}^b Z^{\hat{\alpha}a} = 0\}$$

of complex non-chiral $\mathcal{H}_\ell \times \mathcal{H}_r$ the reality conditions $\widetilde{Z}_{\hat{\alpha}}{}^b = h_{\overline{\hat{\alpha}}\hat{\alpha}}\overline{Z}^{\overline{\hat{\alpha}}b}$, where the complex structure $h_{\overline{\alpha}\beta}$ is given in (2.22). This gives $\tilde{\omega}^{A'} = \overline{\omega}^{A'}$, $\tilde{\pi}_A = \overline{\pi}_A$, $\tilde{\tilde{\pi}}^{A'} = \overline{\tilde{\pi}}^{A'}$, $\tilde{\theta}^I = \overline{\theta}^I$, $\tilde{\eta}^I = \overline{\eta}^I$, which implies that $u \in \mathbb{R}$, $\gamma = e^{-2i\phi}$, and $\widetilde{\Theta} = \overline{\Theta}$.

3.4.2 Fermionic subspace $\mathfrak{I}$

The last invariant subspace that we will consider are points of the form

$$Z^{\hat{a}b} = \begin{bmatrix} \omega^{Ab} \\ 0_{A'}{}^{b} \\ \theta^{Ib} \neq 0 \end{bmatrix}, \tag{3.65}$$

i.e. such that $\pi_{A'}{}^{b} = 0$ but $\theta^{Ib} \neq 0$. These points, which we collectively refer to as $\mathfrak{I}$, therefore appear to be a sort of fermionic extension of time/space-like infinity $\iota := \left\{I^{\hat{a}b}\right\}$ — with this particular isolated point here given by $\theta^{Ib} = 0^{Ib}$.

In order for (3.65) to define a plan ω^{Ab} must be invertible, we can therefore always use the $\mathrm{GL}(2,\mathbb{C})$ freedom to set $\omega^{Ab} = \delta^{Ab}$ and we are left with the complex coordinates $\theta^{Ib} \in \mathbb{C}^{0|2\mathcal{N}}$.

The super Poincaré group does not act transitively on this set

$$\begin{pmatrix} M^{B}{}_{A} & iT^{BA'} & Q^{B}{}_{I} \\ 0 & -\widetilde{M}_{B'}{}^{A'} & 0 \\ 0 & \widetilde{Q}^{JA'} & R^{J}{}_{I} \end{pmatrix} \cdot \begin{bmatrix} \delta^{Ab} \\ 0_{A'}{}^{b} \\ \theta^{Ib} \end{bmatrix} = \begin{bmatrix} M^{B}{}_{A}\delta^{Ab} + Q^{B}{}_{I}\theta^{Ib} \\ 0_{B'}{}^{b} \\ R^{J}{}_{I}\theta^{Ib} \end{bmatrix}. \tag{3.66}$$

However, it clearly stabilises it. There are two different case depending on whether θ^{I1} and θ^{I2} are linearly independent or not. Similarly to the case of $\mathcal{H}$, it is not an homogeneous space for the super Poincaré group and the orbits are not supermanifolds.

Reality condition. Denoting $(\theta^{I1}, \theta^{I2}) = (\theta^{I}, \eta^{I})$, we are going to impose the following reality conditions:

$$\begin{aligned} (Z^1)^2 = 0 &\Rightarrow \overline{\theta}^{I}\delta_{IJ}\theta^{J} = 0, \\ (Z^2)^2 = 0 &\Rightarrow \overline{\eta}^{I}\delta_{IJ}\eta^{J} = 0, \\ Z^1 \cdot Z^2 = 0 &\Rightarrow \overline{\theta}_{J}\eta^{J} = 0. \end{aligned}$$

As explained for $\mathcal{H}$, these conditions do not define a submanifold. If $\mathcal{N} = 1$, the first reality conditions impose that θ can be rewritten as $\theta = e^{i\phi}a$, with $a \in \mathbb{R}_a$ and $\phi \in \mathbb{R}_a$. The remaining reality conditions then give that $\eta = e^{i\psi}a$ with $\psi \in \mathbb{R}_c$. A parametrisation of $\mathfrak{I}$ is then given by $(a, e^{i\phi}, e^{i\psi}) \in \mathbb{R}_a \times S^1 \times S^1$.

3.4.3 Non-chiral invariant subspaces at the boundary

As explained in section 2.2.1, elements of $\overline{M}_{\mathbb{C}}^{4|4\mathcal{N}} = F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}})$ are obtained as pairs $(Z^{\hat{a}b}, \widetilde{Z}_{\hat{\alpha}}{}^{c})$ of bi-supertwistors and dual bi-supertwistors satisfying

$$\widetilde{Z}_{\hat{\alpha}}{}^{b} Z^{\hat{\alpha}c} = 0. \tag{3.67}$$

Non-chiral orbits will therefore be pairs of chiral orbits satisfying the condition (3.67).

One can show that the only admissible orbits under this condition are: $\mathscr{I}_{\ell} \times \mathscr{I}_{r}$, $\mathcal{H}_{\ell} \times \mathcal{H}_{r}$, $\iota_{\ell}^{0|2\mathcal{N}} \times \iota_{r}^{0|2\mathcal{N}}$, $\{\iota_{\ell}\} \times \{\iota_{r}\}$ and $\{\iota_{\ell}\} \times \iota_{r}^{0|2\mathcal{N}}$. In the cases where it applies, the stabilisers can be derived by doing the intersection of the chiral ones.

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4 Super cuts

In this section we will investigate how super null cones intersect with the different boundaries that we previously introduced. In particular we will see that $\mathscr{I}$ and $\mathcal{H}$ are generated by null supergeodesics respectively emanating from ι and $\mathfrak{I}$. In order to achieve this we will need the realisation of null supergeodesics in terms of superambitwistors.

4.1 Superambitwistor space

Here we review elements of superambitwistor geometry, see e.g. [10].

The super ambitwistor space $\mathbb{A}$ is defined as the flag manifold $\mathbb{A} := F(1|0,3|\mathcal{N},\mathbb{C}^{4|\mathcal{N}})$. Consequently we have the double fibration picture:

$$\begin{array}{ccc} & F(1|0,2|0,2|\mathcal{N},3|\mathcal{N},\mathbb{C}^{4|\mathcal{N}}) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{A} = F(1|0,3|\mathcal{N},\mathbb{C}^{4|\mathcal{N}}) & & \overline{M}_{\mathbb{C}}^{4|4\mathcal{N}} = F(2|0,2|\mathcal{N},\mathbb{C}^{4|\mathcal{N}}) \end{array} .$$

In practice elements of $\mathbb{A} = F(1|0,3|\mathcal{N},\mathbb{C}^{4|\mathcal{N}})$ are given by pairs $(Z^{\hat{\alpha}},\widetilde{Z}_{\hat{\alpha}})$ of twistors (and dual twistors) satifying $\widetilde{Z}_{\hat{\alpha}}Z^{\hat{\alpha}} = 0$. Similarly, elements of $F(1|0,2|0,2|\mathcal{N},3|\mathcal{N},\mathbb{C}^{4|\mathcal{N}})$ are given by quadriplets $(Z^{\hat{\alpha}a}\pi_a, Z^{\hat{\alpha}a}, \widetilde{Z}_{\hat{\alpha}}{}^a, \widetilde{Z}_{\hat{\alpha}}{}^a\tilde{\pi}_a)$.

$$\begin{array}{ccc} & (Z^{\hat{\alpha}a}\pi_a, Z^{\hat{\alpha}a}, \widetilde{Z}_{\hat{\alpha}}{}^a, \widetilde{Z}_{\hat{\alpha}}{}^a\tilde{\pi}_a) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ (Z^{\hat{\alpha}a}\pi_a, \widetilde{Z}_{\hat{\alpha}}{}^a\tilde{\pi}_a) & & (Z^{\hat{\alpha}a}, \widetilde{Z}_{\hat{\alpha}}{}^a) \end{array} ,$$

from which one clearly sees e.g. that a point $(Z^{\hat{\alpha}a},\widetilde{Z}_{\hat{\alpha}}{}^a)$ in Minkowski superspace gives an embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$ into $\mathbb{A}$: $([\pi_a],[\tilde{\pi}_a]) \mapsto (Z^{\hat{\alpha}a}\pi_a, \widetilde{Z}_{\hat{\alpha}}{}^a\tilde{\pi}_a)$.

4.1.1 Null supergeodesics

In the reverse sense, a point of the superambitwistor space

$$(Z^{\hat{\alpha}},\widetilde{Z}_{\hat{\alpha}}) = \left(\begin{bmatrix} \omega^A \\ \pi_{A'} \\ \theta^I \end{bmatrix}, [\tilde{\pi}_A \quad \tilde{\omega}^{A'} \quad -\tilde{\theta}_I] \right) \quad \text{s.t.} \quad \widetilde{Z}_{\hat{\alpha}}Z^{\hat{\alpha}} = \tilde{\pi}_A\omega^A + \tilde{\omega}^{A'}\pi_{A'} - \tilde{\theta}_I\theta^I = 0, \tag{4.1}$$

defines a null supergeodesic in the (conformal compactification of) complexified Minkowski space. To see what these are, let us look what the condition is for one point of Minkowski to be on this null supergeodesic. For that, one works in the local chart defining Minkowski superspace:

$$(Z^{\hat{\alpha}b},\widetilde{Z}_{\hat{\alpha}}{}^b) = \left(\begin{bmatrix} iX_+^{AB'} \\ \delta_{A'}{}^{B'} \\ \theta^{IB'} \end{bmatrix}, [\delta_A{}^B \quad -iX_-^{A'B} \quad -\tilde{\theta}_I{}^B] \right) \tag{4.2}$$

with

$$X_+^{AB'} = x^{AB'} - \frac{i}{2}\tilde{\theta}_I{}^A\theta^{IB'}, \qquad X_-^{A'B} = x^{A'B} + \frac{i}{2}\tilde{\theta}_I{}^B\theta^{IA'} .$$

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This spacetime point is part of the null geodesics defined by the ambitwistor (4.1) if and only if it is in the image of this point by the double fibration:

$$Z^{\hat{\alpha}} = Z^{\hat{\alpha}b}\pi_b, \qquad \widetilde{Z}_{\hat{\alpha}} = \widetilde{Z}_{\hat{\alpha}}{}^b\tilde{\pi}_b, \tag{4.3}$$

i.e. if the point satisfies

$$\omega^A = iX_+^{AB'}\pi_{B'}, \qquad \theta^I = \theta^{IB'}\pi'_B, \tag{4.4}$$

$$\tilde{\omega}^{B'} = -iX_-^{AB'}\tilde{\pi}_A, \qquad \tilde{\theta}_I = \tilde{\theta}_I{}^B\tilde{\pi}_B. \tag{4.5}$$

Solutions to these equations can be parametrised by $(\epsilon, \epsilon^I, \tilde{\epsilon}^I) \in \mathbb{C}_c \times \mathbb{C}_a^{\mathcal{N}} \times \mathbb{C}_a^{\mathcal{N}}$ as follows:

$$iX_+^{BB'} = iX_0^{BB'} + \tilde{\theta}_{0I}{}^B\epsilon^I\pi^{B'} + \left(i\epsilon - \frac{1}{2}\epsilon^I\tilde{\epsilon}_I\right)\pi^{B'}\tilde{\pi}^B, \qquad \theta^{IB'} = \theta_0^{IB'} + \epsilon^I\pi^{B'}, \tag{4.6}$$

$$iX_-^{B'B} = i\tilde{X}_0^{BB'} + \theta_0^{IB'}\tilde{\epsilon}_I\tilde{\pi}^B + \left(i\epsilon + \frac{1}{2}\epsilon^I\tilde{\epsilon}_I\right)\pi^{B'}\tilde{\pi}^B, \qquad \tilde{\theta}_I{}^B = \tilde{\theta}_{0I}{}^B + \tilde{\epsilon}_I\tilde{\pi}^B, \tag{4.7}$$

where $(X_0^{AA'}, \tilde{X}_0^{A'A}, \theta_0^{IA'}, \tilde{\theta}_{0I}{}^A)$ is any solution of the equations. Equivalently

$$x^{AA'} = x_0^{AA'} - \frac{i}{2}\left(\pi^{A'}\tilde{\theta}_{0I}{}^A\epsilon^I + \theta_0^{IA'}\tilde{\pi}^A\tilde{\epsilon}_I\right) + \epsilon\pi^{A'}\tilde{\pi}^A, \tag{4.8}$$

$$\theta^{IB'} = \theta_0^{IB'} + \epsilon^I\pi^{B'}, \tag{4.9}$$

$$\tilde{\theta}_I{}^B = \tilde{\theta}_{0I}{}^B + \tilde{\epsilon}_I\tilde{\pi}^B. \tag{4.10}$$

where $X_0^{AB'} = x_0^{AB'} - \frac{i}{2}\tilde{\theta}_{0I}{}^A\theta_0^{IB'}$, $\tilde{X}_0^{B'A} = x_0^{AB'} + \frac{i}{2}\tilde{\theta}_{0I}{}^A\theta_0^{IB'}$.

Keeping all the other parameters fixed in the above equations, $(\epsilon, \epsilon^I, \tilde{\epsilon}^I) \in \mathbb{C}_c \times \mathbb{C}_a^{\mathcal{N}} \times \mathbb{C}_a^{\mathcal{N}}$ parametrise the super null geodesic emanating from $(Z_0{}^{\hat{\alpha}a}, \widetilde{Z}_{0\hat{\alpha}}{}^a)$ and in the direction $(\pi^a, \tilde{\pi}^a)$. Varying $(\pi^a, \tilde{\pi}^a)$ allows to span the whole supernull cone emanating from $(Z_0{}^{\hat{\alpha}a}, \widetilde{Z}_{0\hat{\alpha}}{}^a)$.

4.2 Cuts along super $\mathscr{I}$

4.2.1 $\mathscr{I}$ as the union of super null lines emanating from infinity

We are here interested in the super null cone emanating from (3.2)

$$\{\iota\} = \left(I^{\hat{\alpha}b} = \begin{bmatrix}\delta^{Ab}\\ 0\\ 0\end{bmatrix}, \tilde{I}_{\hat{\alpha}}{}^b = \begin{bmatrix}0 & \delta^{A'b} & 0\end{bmatrix}\right). \tag{4.11}$$

Each of the genenerators of this super null cone corresponds to a superambitwistor

$$(I^{\hat{\alpha}b}\tilde{\lambda}_b, \tilde{I}_{\hat{\alpha}}{}^b\lambda_b). \tag{4.12}$$

At infinity, these null cones intersect several different orbits of the super Poincaré group.

The intersection of the super null cone emanating from $\{\iota\}$ with super $\mathscr{I}$ is composed of points $(Z^{\alpha b}, \widetilde{Z}_\alpha{}^b)$ of the form:

$$Z^{\hat{\alpha}b} = \begin{bmatrix}\omega^A & \tilde{\lambda}^A\\ \lambda_{A'} & 0\\ \theta^I & 0\end{bmatrix}, \qquad \widetilde{Z}_{\hat{\alpha}}{}^b = \begin{bmatrix}\tilde{\lambda}_A & \tilde{\omega}^{A'} & -\tilde{\theta}_I\\ 0 & \lambda^{A'} & 0\end{bmatrix} \tag{4.13}$$

with $\tilde{\lambda}_A\omega^A + \tilde{\omega}^{A'}\lambda_{A'} - \tilde{\theta}^I\theta_I = 0$.

Introducing $iu_+ = \tilde{\lambda}_A \omega^A$ and $-iu_- = \tilde{\omega}^{A'}\lambda_{A'}$, this last equation can be solved as

$$u_+ = u + \frac{i}{2}\,\tilde{\theta}^I\theta_I, \qquad\qquad u_- = u - \frac{i}{2}\,\tilde{\theta}^I\theta_I, \tag{4.14}$$

with $u \in \mathbb{R}$. These points all lie in $\mathscr{I}$ and any point in $\mathscr{I}$ can be obtained in this way: in the coordinate system $(u, [\lambda_{A'}], [\tilde{\lambda}_A], \theta^I, \tilde{\theta}^I)$ the corresponding super null line is obtained by varying u, θ^I, $\tilde{\theta}^I$ while keeping $[\lambda_{A'}]$ and $[\tilde{\lambda}_A]$ fixed.

4.2.2 Cuts at $\mathscr{I}$ emanating from a point at finite distance

We consider the super null cone emanating from a point at finite distance

$$W^{\hat{\alpha}b} = \begin{bmatrix} iX_+^{AB'} \\ \delta_{A'}{}^{B'} \\ \theta^{IB'} \end{bmatrix}, \qquad\qquad \widetilde{W}_{\hat{\alpha}b} = \begin{bmatrix} \delta_A{}^B & -iX_-^{A'B} & -\tilde{\theta}^{IB} \end{bmatrix} \tag{4.15}$$

with $X_\pm^{B'C} = x^{B'C} \mp \frac{i}{2}\,\tilde{\theta}_I{}^C\theta^{IB'}$. Varying $(\lambda_b, \tilde{\lambda}_b)$, the super ambitwistors

$$(W^{\hat{\alpha}b}\lambda_b, \widetilde{W}_{\hat{\alpha}}{}^b\tilde{\lambda}_b) \tag{4.16}$$

then correspond to the super null cone passing through this point. The intersection of this super null cone with $\mathscr{I}$ is the cut

$$(\lambda^a, \tilde{\lambda}^a) \qquad \mapsto \qquad \left(\begin{bmatrix} iX_+^{AB'}\lambda_{B'} & \tilde{\lambda}^A \\ \lambda_{A'} & 0 \\ \theta^{IB'}\lambda_{B'} & 0 \end{bmatrix}, \begin{bmatrix} \tilde{\lambda}_A & -iX_-^{A'B}\tilde{\lambda}_B & -\tilde{\theta}^{IB}\tilde{\lambda}_B \\ 0 & \lambda^{A'} & 0 \end{bmatrix} \right). \tag{4.17}$$

Non-chiral representation of the cuts. In the (complex) *non-chiral* coordinate system $(u, [\pi_{A'}], [\tilde{\pi}_A], \theta^I, \tilde{\theta}^I)$ for $\mathscr{I}$, the cuts of $\mathscr{I}$ by super null cone are:

$$(\lambda^a, \tilde{\lambda}^a) \qquad \mapsto \qquad \left(x^{AA'}\lambda_{A'}\tilde{\lambda}_A, [\lambda_{A'}], [\tilde{\lambda}_A], \theta^{IB'}\lambda_{B'}, \tilde{\theta}^{IB}\tilde{\lambda}_B \right). \tag{4.18}$$

Chiral representation of the cuts. In the (complex) *chiral* left coordinate system $(u_+, [\pi_{A'}], [\hat{\pi}_A], \theta^I)$ for $\mathscr{I}$, the cuts of $\mathscr{I}$ by super null cone are:

$$(\lambda^a, \tilde{\lambda}^a) \qquad \mapsto \qquad \left(X_+^{AA'}\lambda_{A'}\tilde{\lambda}_A, [\lambda_{A'}], [\tilde{\lambda}_A], \theta^{IB'}\lambda_{B'} \right). \tag{4.19}$$

4.3 Cuts along $\mathcal{H}$

4.3.1 $\mathcal{H}$ as the union of super null lines emanating from $\mathfrak{I}$

Here we discuss the intersection with $\mathcal{H}$ of a super null cone emanating from a point of $\mathfrak{I}$ which has the form

$$W^{\hat{\alpha}b} = \begin{bmatrix} \delta^{Ab} \\ 0 \\ \eta^{Ib} \end{bmatrix}, \qquad\qquad \widetilde{W}_{\hat{\alpha}}{}^b = \begin{bmatrix} 0 & \delta^{A'b} & -\tilde{\eta}^{Ib} \end{bmatrix}, \tag{4.20}$$

with $\tilde{\eta}^{IB'}\eta_I{}^B = 0$. This intersection is given by points of the form

$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega^A & \tilde{\lambda}^A \\ \tilde{\eta}_{IA'}\theta^I + \tilde{\alpha}\lambda_{A'} & 0 \\ \theta^I & \eta^{IB}\tilde{\lambda}_B \end{bmatrix}, \qquad \tilde{Z}_{\hat{\alpha}b} = \begin{bmatrix} \tilde{\theta}^I\eta_{IA} + \alpha\tilde{\lambda}_A & \tilde{\omega}^{A'} & -\tilde{\theta}^I \\ 0 & \lambda^{A'} & -\tilde{\eta}_{IB'}\lambda^{B'} \end{bmatrix}. \tag{4.21}$$

with the constraint $\alpha\tilde{\lambda}_A\omega^A + \tilde{\alpha}\lambda_{A'}\tilde{\omega}^{A'} = (\tilde{\theta}^I - \tilde{\eta}^I_A\tilde{\omega}^{A'})(\theta_I - \eta_{IA}\omega^A)$.

Introducing $iu_+ = \alpha\tilde{\lambda}_A\omega^A$, $-iu_- = \tilde{\alpha}\lambda_{A'}\tilde{\omega}^{A'}$, and $\Theta^I = \theta^I - \eta^I{}_A\omega^A$, $\widetilde{\Theta}^I = \tilde{\theta}^I - \tilde{\eta}^I{}_{A'}\tilde{\omega}^{A'}$, the above constraint equation can be solved by $iu_+ - iu_- = \widetilde{\Theta}_I\Theta^I$, i.e.,

$$u_+ = u - \frac{i}{2}\widetilde{\Theta}_I\Theta^I, \qquad u_- = u + \frac{i}{2}\widetilde{\Theta}_I\Theta^I, \qquad u \in \mathbb{C}. \tag{4.22}$$

First, these points (4.21) all lie in $\mathcal{H}$: in the coordinate system $(u, [\lambda_{A'}], [\tilde{\lambda}_A], \Theta^I, \widetilde{\Theta}^I, \eta^I, \tilde{\eta}^I)$ on $\mathcal{H}$ the corresponding super null line is obtained by varying u, Θ^I, $\widetilde{\Theta}^I$ while keeping $\eta^I = \eta^{IB}\tilde{\lambda}_B$, $\tilde{\eta}^I = \tilde{\eta}^{IB'}\lambda_{B'}$, $[\tilde{\lambda}_A]$ and $[\lambda_{A'}]$ fixed.

Second, any point in $\mathcal{H}$ can be obtained in this way, since we showed in 3.4.1 that η^I and $\tilde{\eta}_I$ can always be written in a factorised form $\eta^{IB}\tilde{\lambda}_B$ and $\tilde{\eta}_{IB'}\lambda^{B'}$, respectively.

4.3.2 Cuts at $\mathcal{H}$ emanating from a point at finite distance

In this section we consider the super null cone emanating from a point at finite distance (4.15).

The intersection of this super null cone with $\mathcal{H}$ is given by points of the form

$$Z^{\hat{\alpha}a} = \begin{bmatrix} iX_+^{AA'}\lambda_{A'} & \tilde{\lambda}^B(\delta_B^A - \eta_I^A\tilde{\theta}_B^I) \\ \lambda_{A'} & 0 \\ \theta^{IA'}\lambda_{A'} & \tilde{\lambda}^B(\eta_{IB} - \eta_{IC}\eta_J^C\tilde{\theta}_B^J) \end{bmatrix}, \tag{4.23}$$

$$\widetilde{Z}^b{}_{\hat{\alpha}} = \begin{bmatrix} \tilde{\lambda}_A & -iX_-^{A'B}\tilde{\lambda}_B & -\tilde{\theta}_I{}^B\tilde{\lambda}_B \\ 0 & \lambda^{B'}(\delta_{B'}^{A'} + \tilde{\eta}_I^{A'}\theta_{B'}^I) & -\lambda^{B'}(\tilde{\eta}_{B'}^I + \tilde{\eta}_{C'}^I\tilde{\eta}_J^{C'}\theta_{B'}^J) \end{bmatrix}, \tag{4.24}$$

where $\tilde{\eta}^I{}_A\eta_{IA'} = 0$ and $X_\pm^{B'C} = x^{B'C} \mp \frac{i}{2}\tilde{\theta}_I{}^C\theta^{IB'}$. In terms of the variables $\Theta^I, \widetilde{\Theta}_I$ defined in section 3.4.1, here taking the values $\Theta^I = \theta^{IA'}\lambda_{A'} - i\eta_A^I X_+^{AB'}\lambda_{B'}$, $\widetilde{\Theta}_I = \tilde{\theta}_I^A\tilde{\lambda}_A + i\tilde{\eta}_{A'I}X_-^{A'B}\tilde{\lambda}_B$, we have

$$Z^{\hat{\alpha}a} = \begin{bmatrix} iX_+^{AA'}\lambda_{A'} & \tilde{\lambda}^A - \widetilde{\Theta}^I\eta_I^A \\ \lambda_{A'} & 0 \\ \Theta^I - i\eta_A^I X_+^{AB'}\lambda_{B'} & \eta_{IA}(\tilde{\lambda}^A - \widetilde{\Theta}^J\eta_J^A) \end{bmatrix}, \quad \widetilde{Z}^b{}_{\hat{\alpha}} = \begin{bmatrix} \tilde{\lambda}_A & -iX_-^{A'B}\tilde{\lambda}_B & -\widetilde{\Theta}_I + iX_-^{A'B}\tilde{\eta}_{A'I}\tilde{\lambda}_B \\ 0 & \lambda^{A'} - \tilde{\eta}_I^{A'}\Theta^I & -(\lambda^{B'} - \tilde{\eta}_J^{B'}\Theta^J)\tilde{\eta}_{B'}^I \end{bmatrix}. \tag{4.25}$$

Non-chiral representation of the cuts. In the complex non-chiral coordinate system $(u, [\pi_{A'}], [\tilde{\pi}_A], \Theta^I, \widetilde{\Theta}^I, \eta^I, \tilde{\eta}^I)$ on $\mathcal{H}_\ell \times \mathcal{H}_r$, such that $\tilde{\eta}_I\,\eta^I = 0$, the cuts of $\mathcal{H}_\ell \times \mathcal{H}_r$ by super null cone are:

$$(\lambda^{A'}, \tilde{\lambda}^A, \eta_A^I, \tilde{\eta}_{A'}^I) \mapsto \tag{4.26}$$
$$\left(u,\ [\lambda_{A'}],\ [\tilde{\lambda}_A], [\theta^{IB'} - i\eta_A^I(x^{AB'} - \frac{i}{2}\tilde{\theta}_J^A\theta^{JB'})]\lambda_{B'}, \tilde{\lambda}_B[\tilde{\theta}_I^B + i(x^{BA'} + \frac{i}{2}\tilde{\theta}_J^B\theta^{JA'})\tilde{\eta}_{A'I}], \eta^I, \tilde{\eta}_I\right),$$

where $\eta^I = \eta_A^I(\tilde{\lambda}^A - \widetilde{\Theta}^J\eta_J^A)$, $\tilde{\eta}_I = (\lambda^{B'} - \tilde{\eta}_J^{B'}\Theta^J)\tilde{\eta}_{IB'}$, and

$$u = x^{AB'}\tilde{\lambda}_A\lambda_{B'} + \frac{1}{2}\eta_{IA}x^{AB'}\lambda_{B'}\tilde{\theta}^I + \frac{1}{2}\theta^I\tilde{\lambda}_B x^{BA'}\tilde{\eta}_{A'I} + \frac{i}{4}(\eta_{IA}\tilde{\theta}_J^A)(\tilde{\theta}^I\theta^J) - \frac{i}{4}(\theta_J^{A'}\tilde{\eta}_{A'}^I)(\tilde{\theta}^J\theta^I). \tag{4.27}$$

Chiral representation of the cuts. In the complex *chiral* left coordinate system $(u_+, [\pi_{A'}], [\hat{\pi}^A], \Theta^I, \eta^I)$ for $\mathcal{H}_\ell$, the "cuts" of $\mathcal{H}_\ell$ by super null cone are:

$$(\lambda^a, \tilde{\lambda}^a, , \eta_A^I) \mapsto \tag{4.28}$$
$$\left(X_+^{AB'}\lambda_{B'}\tilde{\lambda}_C(\delta_A^C - \tilde{\theta}_A^I\eta_I^C),\ [\lambda_{A'}], [(\delta_C^A + \tilde{\theta}_I{}^A\eta_C^I)\tilde{\lambda}^C],\ (\theta^{IB'} - i\eta_A^I X_+^{AB'})\lambda_{B'},\ \eta_A^I\tilde{\lambda}^A\right).$$

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A Notations and conventions

The notation $[x]$ is used to express that x is a choice of representative in an equivalence class. A matrix N of $sl(2,\mathbb{C})$ will be denoted $N = (N^A{}_B)$. The complex conjugated matrix $\overline{N}$ will be denoted $\overline{N} = (\overline{N}^{A'}{}_{B'})$. We recall that the Pauli matrices $\vec{\sigma}$ are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

In our convention, the four matrices $\sigma_a = (\mathbb{I}, \vec{\sigma})$ carry the indices $\sigma_a^{AA'}$ while the four matrices $\tilde{\sigma}^a = (\mathbb{I}, -\vec{\sigma})$ carry the indices $\tilde{\sigma}^a_{A'A} = \epsilon_{A'B'}\epsilon_{AB}\sigma^{aBB'}$. We use conventions whereby the invariant $sl(2,\mathbb{C})$ matrices are

$$(\epsilon_{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (\epsilon^{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.2}$$

such that

$$\epsilon^{AB}\,\epsilon_{CB} = \delta^A{}_C. \tag{A.3}$$

This corresponds to the following convention for lowering the indices of a (left) spinor ω^A:

$$\omega_A = \omega^B\,\epsilon_{BA} \quad \Leftrightarrow \quad \omega^A = \epsilon^{AB}\,\omega_B. \tag{A.4}$$

The same convention applies for lowering and raising the indices of a right spinor $\pi^{A'}$ using the invariant matrices $(\epsilon^{A'B'})$ and $(\epsilon_{A'B'})$, numerically equal to the matrices $(\epsilon^{AB}) = (\epsilon_{AB})$.

Given that, we have

$$(\sigma_a)^{AA'} = (\mathbb{I}, \sigma_i)^{AA'}, \qquad (\tilde{\sigma}_a)_{A'A} = (\mathbb{I}, -\sigma_i)_{A'A}, \tag{A.5}$$

$$(\sigma_{ab})^A{}_B = -\frac{1}{4}(\sigma_a\tilde{\sigma}_b - \sigma_b\tilde{\sigma}_a)^A{}_B, \qquad (\tilde{\sigma}_{ab})_{A'}{}^{B'} = -\frac{1}{4}(\tilde{\sigma}_a\sigma_b - \tilde{\sigma}_b\sigma_a)_{A'}{}^{B'}, \tag{A.6}$$

and then,

$$\gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}. \tag{A.7}$$

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B The superconformal algebra

In this appendix we choose a basis for the superconformal algebra in terms of matrices and compute the (anti)commutation relations between the generators of this basis. We stick as much as possible to the conventions of [13].

- Translations: we can decompose $i\,t^{AB'} = -i\,b^a (P_a)^{AB'}$ for certain coefficents $b^a \in \mathbb{R}_c$ and bosonic generators P_a. We fix $(P_a)^{AB'} = -\frac{1}{4}(\sigma_a)^{AB'}$. The matrix of coefficients $t^{AB'}$ is indeed hermitian with this choice.

- Dilations and Lorentz transformations: we can decompose $m^A{}_B = \delta\, D^A{}_B + \frac{i}{2}\lambda_{ab}(J^{ab})^A{}_B$, with $\delta\,,\, \lambda_{ab} \in \mathbb{R}_c$. We fix $D^A{}_B = \frac{1}{2}\delta^A{}_B$ and $(J_{ab})^A{}_B = -i(\sigma_{ab})^A{}_B$.

- Special conformal transformations: we can decompose $-ik_{A'B} = -if^a(K_a)_{A'B}$, where $f^a \in \mathbb{R}_c$. We fix $(K_a)_{A'B} = \frac{1}{4}(\tilde{\sigma}_a)_{A'B}$.

- Supercharges: we can decompose $q^A{}_I = i\,\xi_{BJ}(Q^{BJ})^A{}_I$ where $(Q^{BJ})^A{}_I = \epsilon^{AB}\,\delta^J{}_I$. Similarly, $\overline{q}^{IA'} = i\,\overline{\xi}^J_{B'}\,(\overline{Q}^{B'}_J)^{IA'}$ where we have $(\overline{Q}^{B'}_J)^{IA'} = \epsilon^{A'B'}\,\delta^I_J$. These generators Q and $\overline{Q}$ are odd.

- Super special conformal transformations: we decompose $s^I{}_B = i\,\eta_A{}^J(S^A{}_J)^I{}_B$ where the matrix elements $(S^A{}_J)^I{}_B = \delta^A_B\,\delta^I_J$ and $\eta \in \mathbb{R}_a$. Similarly, $\overline{s}_{A'J} = i\,\overline{\eta}_{B'I}\,(\overline{S}^{B'I})_{A'J}$ where $(\overline{S}^{B'I})_{A'J} = \delta^{B'}_{A'}\,\delta^I_J$.

- R-symmetries: the $\mathcal{N}^2 - 1$ generators of $\mathrm{SU}(\mathcal{N})$ can be written in terms of the $\mathcal{N}\times\mathcal{N}$ matrices $E_{I,J}$ such that $(E_{I,J})^K{}_L = \delta^K{}_I\,\delta_{JL}$ with matrix product $E_{I,J}E_{K,L} = \delta_{JK}E_{I,L}$. In order to give a basis of traceless hermitian matrices in the defining representation of $\mathfrak{su}(\mathcal{N})$, we take the union $\mathcal{T}_D \cup \mathcal{T}_S \cup \mathcal{T}_A$ where

$$\begin{aligned}
\mathcal{T}_D &= \{E_{I,I} - E_{I+1,I+1} | I \in \{1, \dots \mathcal{N}-1\}\},\\
\mathcal{T}_A &= \{i\,(E_{I,J} - E_{J,I}) | 1 \leqslant I < J \leqslant \mathcal{N}\},\\
\mathcal{T}_S &= \{E_{I,J} + E_{I,J} | 1 \leqslant I < J \leqslant \mathcal{N}\}.
\end{aligned}$$

  In the first nontrivial case where $\mathcal{N} = 2$, this basis reproduces the three Pauli matrices. The set $\mathcal{T}_D$ is made of diagonal matrices, while the sets $\mathcal{T}_A$ and $\mathcal{T}_S$ are made of antisymmetric and symmetric matrices, respectively. The former set provides a basis of $\mathfrak{so}(\mathcal{N}) \subset \mathfrak{su}(\mathcal{N})$.

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We then fix $\mathcal{N}=1$ and provide the following matrix realisation of the superconformal algebra:

$$P_a = \left(\begin{array}{c|c} -\frac{1}{8}(\mathbb{I}_4+\gamma_5)\gamma_a & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline \begin{matrix}0&0&0&0\end{matrix} & 0 \end{array}\right) = \left(\begin{array}{cc|c} 0_2 & -\frac{1}{4}\,(\sigma_a)^{AB'} & \begin{matrix}0\\0\end{matrix} \\ 0_2 & 0_2 & \begin{matrix}0\\0\end{matrix} \\ \hline \begin{matrix}0&0\end{matrix} & \begin{matrix}0&0\end{matrix} & 0 \end{array}\right),$$

$$D = \left(\begin{array}{cccc|c} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right) = \frac{1}{2}\left(\begin{array}{c|c} \gamma_5 & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline \begin{matrix}0&0&0&0\end{matrix} & 0 \end{array}\right),$$

$$J_{ab} = \left(\begin{array}{c|c} -i\Sigma_{ab} & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline \begin{matrix}0&0&0&0\end{matrix} & 0 \end{array}\right) = \left(\begin{array}{cc|c} -i(\sigma_{ab})^A{}_B & 0_2 & \begin{matrix}0\\0\end{matrix} \\ 0_2 & -i(\tilde{\sigma}_{ab})_{A'}{}^{B'} & \begin{matrix}0\\0\end{matrix} \\ \hline \begin{matrix}0&0\end{matrix} & \begin{matrix}0&0\end{matrix} & 0 \end{array}\right),$$

$$K_a = \left(\begin{array}{c|c} -\frac{1}{8}(\mathbb{I}_4-\gamma_5)\gamma_a & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline \begin{matrix}0&0&0&0\end{matrix} & 0 \end{array}\right) = \left(\begin{array}{cc|c} 0_2 & 0_2 & \begin{matrix}0\\0\end{matrix} \\ -\frac{1}{4}(\tilde{\sigma}_a)_{A'B} & 0_2 & \begin{matrix}0\\0\end{matrix} \\ \hline \begin{matrix}0&0\end{matrix} & \begin{matrix}0&0\end{matrix} & 0 \end{array}\right),$$

$$Q^1 = \left(\begin{array}{c|c} 0_4 & \begin{matrix}0\\-1\\0\\0\end{matrix} \\ \hline \begin{matrix}0&0&0&0\end{matrix} & 0 \end{array}\right), \quad Q^2 = \left(\begin{array}{c|c} 0_4 & \begin{matrix}1\\0\\0\\0\end{matrix} \\ \hline \begin{matrix}0&0&0&0\end{matrix} & 0 \end{array}\right),$$

$$\overline{Q}^{1'} = \left(\begin{array}{c|c} 0_4 & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline \begin{matrix}0&0&0&-1\end{matrix} & 0 \end{array}\right), \quad \overline{Q}^{2'} = \left(\begin{array}{c|c} 0_4 & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline \begin{matrix}0&0&1&0\end{matrix} & 0 \end{array}\right),$$

$$S^1 = \left(\begin{array}{c|c} 0_4 & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline \begin{matrix}1&0&0&0\end{matrix} & 0 \end{array}\right), \quad S^2 = \left(\begin{array}{c|c} 0_4 & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline \begin{matrix}0&1&0&0\end{matrix} & 0 \end{array}\right),$$

$$\overline{S}^{1'} = \left(\begin{array}{c|c} 0_4 & \begin{matrix}0\\0\\1\\0\end{matrix} \\ \hline \begin{matrix}0&0&0&0\end{matrix} & 0 \end{array}\right), \quad \overline{S}^{2'} = \left(\begin{array}{c|c} 0_4 & \begin{matrix}0\\0\\0\\1\end{matrix} \\ \hline \begin{matrix}0&0&0&0\end{matrix} & 0 \end{array}\right), \quad A = \frac{2i}{3}\left(\begin{array}{c|c} \frac{1}{4}\,\mathbb{I}_4 & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline \begin{matrix}0&0&0&0\end{matrix} & 1 \end{array}\right).$$

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With this choice of basis, we obtain the following (anti)commutation relations between the (super)generators:

$$
\begin{aligned}
&[P_a, P_b] = 0\,, && [J_{ab}, P_c] = i\,\eta_{ac}\,P_b - i\,\eta_{bc}\,P_a\,,\\
&[J_{ab}, J_{cd}] = i\,\eta_{ac}\,J_{bd} - i\,\eta_{ad}\,J_{bc} + i\,\eta_{bd}\,J_{ac} - i\,\eta_{bc}\,J_{ad}\,,\\
&[D, P_a] = P_a\,,\\
&[D, K_a] = -K_a\,, && [P_a, K_b] = -\frac{1}{8}\,(J_{ab} + i\,\eta_{ab}\,D)\,,\\
&[D, J_{ab}] = 0\,, && [J_{ab}, K_c] = i\eta_{ac}\,K_b - i\eta_{bc}\,K_a\,,\\
&[J_{ab}, Q^A] = i(\sigma_{ab})^A{}_B\,Q^B\,, && [J_{ab}, \overline{Q}^{A'}] = -i(\tilde{\sigma}_{ab})_{B'}{}^{A'}\,\overline{Q}^{B'}\,,\\
&[P_a, Q_\alpha] = 0\,, && [P_a, \overline{Q}^{\alpha'}] = 0\,,\\
&\{Q^A, Q^B\} = 0\,, && \{\overline{Q}^{A'}, \overline{Q}^{B'}\} = 0\,,\\
&\{Q^A, \overline{Q}^{A'}\} = 2\,(\sigma^a)^{AA'}\,P_a\,,\\
&[D, Q^A] = \frac{1}{2}\,Q^A\,, && [D, \overline{Q}^{A'}] = \frac{1}{2}\,\overline{Q}^{A'}\,,\\
&[K_a, Q^A] = -\frac{1}{4}\,(\sigma_a)^{AA'}\,\overline{S}_{A'}\,, && [K_a, \overline{Q}^{A'}] = \frac{1}{4}\,(\sigma_a)^{AA'}\,S_A\,,\\
&[J_{ab}, S^A] = i(\sigma_{ab})^A{}_B\,S^B\,, && [J_{ab}, \overline{S}^{A'}] = -i(\tilde{\sigma}_{ab})_{B'}{}^{A'}\,\overline{S}^{B'}\,,\\
&\{S^A, \overline{S}^{A'}\} = 2\,(\sigma^a)^{AA'}\,K_a\,,\\
&[P_a, S^A] = -\,\frac{1}{4}\,(\sigma_a)^{AA'}\,\overline{Q}_{A'}\,, && [P_a, \overline{S}_{A'}] = -\,\frac{1}{4}\,(\tilde{\sigma}_a)_{A'A}\,Q^A\,,\\
&[D, S^A] = -\frac{1}{2}\,S^A\,, && [D, \overline{S}^{A'}] = -\frac{1}{2}\,\overline{S}^{A'}\,,\\
&\{S^A, Q_B\} = \frac{i}{2}\,(\sigma^{ab})^A{}_B\,J_{ab} + \frac{3i}{2}\,\delta^A_B\,A - \frac{1}{2}\,\delta^A_B\,D\,,\\
&\{\overline{S}^{A'}, \overline{Q}_{B'}\} = \frac{i}{2}\,(\tilde{\sigma}^{ab})_{B'}{}^{A'}\,J_{ab} + \frac{3i}{2}\,\delta^{A'}_{B'}\,A + \frac{1}{2}\,\delta^{A'}_{B'}\,D\,,\\
&[A, Q^A] = -\frac{i}{2}\,Q^A\,, && [A, \overline{Q}^{A'}] = \frac{i}{2}\,\overline{Q}^{A'}\,,\\
&[A, S^A] = \frac{i}{2}\,S^A\,, && [A, \overline{S}^{A'}] = -\frac{i}{2}\,\overline{S}^{A'}\,.
\end{aligned}
$$

As a vector space, the superalgebra $\mathfrak{su}(2,2|1)$ can be decomposed through a gradation associated with the eigenvalues of the adjoint action of the dilation operator:

$$
\begin{array}{rcccccccccc}
\mathfrak{su}(2,2|1) & = & \mathfrak{g}_{-1} & \oplus & \mathfrak{g}_{-\frac{1}{2}} & \oplus & \mathfrak{g}_0 & \oplus & \mathfrak{g}_{\frac{1}{2}} & \oplus & \mathfrak{g}_1,\\
\text{Generators:} & & K & & S & & J, A, D & & Q & & P.
\end{array}
$$

C Superconformal boundary of AdS superspace

As we recall in section 2, (complexified) conformally compact Minkowski superspace $\overline{M}^{4|4\mathcal{N}}_{\mathbb{C}}$ is obtained as the flag supermanifold $F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}})$ and is isomorphic to the homogeneous space (2.21) for the (complexified) superconformal group $\mathrm{SL}(4|\mathcal{N})$. The Lorentzian reality

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$$
\begin{array}{ccc}
\mathrm{SL}(4|\mathcal{N},\mathbb{C}) & \xrightarrow{\text{reality condition}} & \mathrm{SU}(2,2|\mathcal{N}) \\
\Big\downarrow \Lambda < 0 & & \Big\downarrow \Lambda < 0 \\
\mathrm{Sp}(4|\mathcal{N},\mathbb{C}) & \xrightarrow{\text{reality condition}} & \mathrm{OSp}(\mathcal{N}|4)
\end{array}
$$

Figure 1. Summary of the group reductions, horizontal arrows correspond to imposing Lorentzian reality condition while vertical arrows correspond to breaking of superconformal invariance.

condition is obtained by introducing a hermitian metric $h_{\bar{\hat{\alpha}}\hat{\beta}}$ and restricting to totally null planes; real Lorentzian conformally compact Minkowski superspace $\overline{M}^{4|4\mathcal{N}}_{\mathbb{R}}$ is then isomorphic to the homogeneous space (2.24) for the (real Lorentzian) superconformal group $\mathrm{SU}(2,2|\mathcal{N})$.

Complexified AdS superspace $AdS^{4|4\mathcal{N}}_{\mathbb{C}} \subset \overline{M}^{4|4\mathcal{N}}_{\mathbb{C}}$ and its real Lorentzian counterpart $AdS^{4|4\mathcal{N}}_{\mathbb{R}} \subset \overline{M}^{4|4\mathcal{N}}_{\mathbb{R}}$ are then obtained by reducing the superconformal group to the respective group of isometries $\mathrm{Sp}(4|\mathcal{N},\mathbb{C})$ and $\mathrm{OSp}(\mathcal{N}|4)$ (see figure 1). We here would like to briefly recall the corresponding realizations in terms of Grasmmanians of $\mathbb{C}^{4|4\mathcal{N}}$: in order to break superconformal invariance one introduces the infinity supertwistors (see e.g. [50, 51]),

$$
I^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \epsilon^{AB} & 0 & 0 \\ 0 & \Lambda\epsilon_{A'B'} & 0 \\ 0 & 0 & \sqrt{|\Lambda|}\delta_{IJ} \end{pmatrix}, \qquad I_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \Lambda\epsilon_{AB} & 0 & 0 \\ 0 & \epsilon^{A'B'} & 0 \\ 0 & 0 & \sqrt{|\Lambda|}\delta^{IJ} \end{pmatrix}, \tag{C.1}
$$

that satisfy $I^{\hat{\alpha}\hat{\gamma}} I_{\hat{\gamma}\hat{\beta}} = |\Lambda| \delta^{\hat{\alpha}}{}_{\hat{\beta}}$ and coincide with (3.1) in the limit where Λ vanishes. For $\Lambda \neq 0$, these are skew-symmetric and invertible; as such these define a symplectic structure on $\mathbb{C}^{4|4\mathcal{N}}$. The requirement to preserve this symplectic form then realises the left hand side reduction of supergroup in figure 1. (Complexified) conformally compact Minkowski superspace $\overline{M}^{4|4\mathcal{N}}_{\mathbb{C}} = F(2|0,2|\mathcal{N},\mathbb{C}^{4|\mathcal{N}})$ then decompose according to the following: if $(Z^{\hat{\alpha}a}, \tilde{Z}_{\hat{\alpha}}{}^{a})$ is a point of $\overline{M}^{4|4\mathcal{N}}_{\mathbb{C}}$ one can restrict the symplectic form (C.1) to the corresponding plane of $F(2|0,2|\mathcal{N},\mathbb{C}^{4|\mathcal{N}})$:

$$
\sigma\epsilon^{ab} := I_{\hat{\alpha}\hat{\beta}} Z^{\hat{\alpha}a} Z^{\hat{\beta}b}, \qquad \tilde{\sigma}\epsilon_{ab} := I^{\hat{\alpha}\hat{\beta}} \tilde{Z}_{\hat{\alpha}}{}^{a} \tilde{Z}_{\hat{\beta}}{}^{b}.
$$

Because of projective invariance the actual values of σ and $\tilde{\sigma}$ are irrelevant; the only alternative to consider is whether or not they vanish, i.e., whether or not the induced skew symmetric bitwistors are invertible. Points of $AdS^{4|4\mathcal{N}}_{\mathbb{C}}$ corresponds to the situation where both induced symplectic structure are non degenerate while points of the superconformal boundary $\overline{M}^{3|3\mathcal{N}}$ correspond to planes which are Lagrangian submanifolds:

$$
(Z^{\hat{\alpha}a}, \tilde{Z}_{\hat{\alpha}}{}^{a}) \in AdS^{4|4\mathcal{N}}_{\mathbb{C}} \qquad \Leftrightarrow \qquad \sigma, \tilde{\sigma} \in \mathrm{GL}(1|0), \tag{C.2}
$$

$$
(Z^{\hat{\alpha}a}, \tilde{Z}_{\hat{\alpha}}{}^{a}) \in \overline{M}^{3|3\mathcal{N}}_{\mathbb{C}} \qquad \Leftrightarrow \qquad \sigma = \tilde{\sigma} = 0. \tag{C.3}
$$

One then obtains the real Lorentzian AdS superspace and boundary by imposing the reality condition $\tilde{Z}_{\hat{\alpha}}{}^{a} = \overline{Z}^{\bar{\hat{\beta}}a} h_{\bar{\hat{\beta}}\hat{\alpha}}$. It is here useful to introduce the map $J^{\hat{\alpha}}{}_{\bar{\hat{\beta}}} := I^{\hat{\alpha}\hat{\gamma}} h_{\bar{\hat{\beta}}\hat{\gamma}}$. It

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satisfies $J^{\hat{\alpha}}{}_{\bar{\hat{\gamma}}}\overline{J}^{\bar{\hat{\gamma}}}{}_{\hat{\beta}} = |\Lambda|\delta^{\hat{\alpha}}{}_{\hat{\beta}}$ and plays the role of a real structure on $\mathbb{C}^{4|4\mathcal{N}}$. Real twistors then correspond to the eigenspace

$$Z^{\hat{\alpha}} \in \mathbb{R}^{4|4\mathcal{N}} \qquad\qquad \Leftrightarrow \qquad\qquad Z^{\hat{\alpha}} = \frac{1}{\sqrt{|\Lambda|}} J^{\hat{\alpha}}{}_{\bar{\hat{\beta}}}\overline{Z}^{\bar{\hat{\beta}}}.$$

With this definition, one obtains another characterisation of the real Lorentzian superconformal boundary $\overline{M}^{3|3\mathcal{N}}_{\mathbb{R}}$ as planes in $\mathbb{C}^{4|\mathcal{N}}$ which are both real and Lagrangian (see e.g. [29]). Finally, since $\mathrm{OSp}(\mathcal{N}|4,\mathbb{R})$ is the subgroup of $\mathrm{Sp}(4|\mathcal{N},\mathbb{C})$ stabilizing the real structure $J^{\hat{\alpha}}{}_{\bar{\hat{\beta}}}$, it naturally acts on both $AdS^{4|4\mathcal{N}}_{\mathbb{R}}$ and $\overline{M}^{3|3\mathcal{N}}_{\mathbb{R}}$. One can prove that this action is transitive and that one recovers in this way the isomorphisms (1.2).

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