

## Near-zone symmetries of Kerr black holes

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**ABSTRACT:** We study the near-zone symmetries of a massless scalar field on four-dimensional black hole backgrounds. We provide a geometric understanding that unifies various recently discovered symmetries as part of an  $SO(4,2)$  group. Of these, a subset are exact symmetries of the static sector and give rise to the ladder symmetries responsible for the vanishing of Love numbers. In the Kerr case, we compare different near-zone approximations in the literature, and focus on the implementation that retains the symmetries of the static limit. We also describe the relation to spin-1 and 2 perturbations.

**KEYWORDS:** Black Holes, Classical Theories of Gravity

**ARXIV EPRINT:** [2203.08832](https://arxiv.org/abs/2203.08832)

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**1 Introduction**

Black hole perturbation theory has a long history dating back to the work of Regge and Wheeler [1] and Zerilli [2]. Interestingly, recent investigations suggest the subject has depths yet to be plumbed. A case in point is a number of symmetries discovered in the past year [3, 4], which shed light on the well-known vanishing of black hole Love numbers (characterizing a black hole’s static, non-dissipative tidal response) [5–14]. In this paper, we present a synthesis of these symmetries, and show how they fit within a larger group containing further symmetries. Some of these are familiar symmetries of the exact dynamics. The rest are approximate symmetries in the low frequency regime. Of these, a subset are exact symmetries of the static sector, and give rise to the ladder symmetries discussed in [4]. To keep the discussion simple, we focus largely on symmetries of a massless scalar, first on a Schwarzschild, then Kerr, background. The connection to spin-1 and 2 perturbations, via a spin ladder, is discussed in the Supplemental Material.

**2 Effective near-zone metric**

We begin by considering the Schwarzschild case. Our starting point is a free massless scalar field  $\phi$  on a fixed 4D Schwarzschild background:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{S^2}^2, \quad f(r) = 1 - \frac{r_s}{r}, \quad (2.1)$$

with  $d\Omega_{S^2}^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$  and where  $r_s \equiv 2GM$  is the Schwarzschild radius. The scalar's action can be written explicitly as

$$S = \frac{1}{2} \int dt dr d\Omega \left[ \frac{r^4}{\Delta} (\partial_t \phi)^2 - \Delta (\partial_r \phi)^2 + \phi \nabla_{\Omega}^2 \phi \right], \quad (2.2)$$

with  $\Delta(r) \equiv r(r - r_s)$  and  $\nabla_{\Omega}^2 \equiv (1/\sin\theta)\partial_{\theta}(\sin\theta\partial_{\theta}) + (1/\sin^2\theta)\partial_{\varphi}^2$ . In frequency space ( $\phi \propto e^{-i\omega t}$ ), we wish to focus on long-wavelength perturbations satisfying  $r_s \ll 1/\omega$ . The behavior of  $\phi$  in the *near-zone* region defined by  $r_s \leq r \ll 1/\omega$  is described by approximating the coefficient in front of the kinetic term as follows:  $r^4/\Delta(r) \simeq r_s^4/\Delta(r)$  [15, 16]. Doing so means in the  $\phi$  equation of motion, the time derivative term  $(r^4/\Delta)\partial_t^2\phi$  is replaced by  $(r_s^4/\Delta)\partial_t^2\phi$ . This has the virtue of preserving the correct singularity as  $r \rightarrow r_s$ , while still accurately capturing the dynamics at larger  $r$ , as long as  $\omega r \ll 1$ .<sup>1</sup>

In this limit, the action (2.2) is the same as that of a massless scalar minimally coupled to an *effective near-zone metric*:

$$ds_{\text{near-zone}}^2 = -\frac{\Delta}{r_s^2} dt^2 + \frac{r_s^2}{\Delta} dr^2 + r_s^2 d\Omega_{S^2}^2. \quad (2.3)$$

In the static limit ( $\omega = 0$ ) the scalar behaves identically on this metric as on the original Schwarzschild background. Nevertheless, it is advantageous to work with the near-zone geometry, both because it allows us to go beyond the strictly static sector, and because it has a richer symmetry structure. In fact, the metric (2.3) is that of  $\text{AdS}_2 \times S^2$ . (The  $(t, r)$  coordinates are a somewhat nonstandard covering of a portion of  $\text{AdS}_2$ , which we describe in the Supplemental Material.) This immediately implies that the near-zone metric (2.3) has 6 Killing vectors (KVs), in contrast to Schwarzschild, which has only 4.

Another advantage of the near-zone metric is that it describes a conformally flat space-time, unlike Schwarzschild. This implies that the metric (2.3) has 9 additional conformal Killing vectors (CKVs). The near-zone metric also has a vanishing Ricci scalar (though not Ricci tensor) because the curvature radii of  $\text{AdS}_2$  and  $S^2$  are identical. This means that the scalar  $\phi$  is effectively conformally coupled, guaranteeing the CKVs generate symmetries of the action in the near zone. We now turn to the study of these symmetries and their physical consequences.

### 3 Near-zone symmetries

The Killing vectors of  $\text{AdS}_2 \times S^2$  in  $(t, r, \theta, \varphi)$  coordinates are

$$T = 2r_s \partial_t, \quad (3.1a)$$

$$L_{\pm} = e^{\pm t/2r_s} (2r_s \partial_r \sqrt{\Delta} \partial_t \mp \sqrt{\Delta} \partial_r), \quad (3.1b)$$

$$J_{23} = \partial_{\varphi}, \quad (3.1c)$$

$$J_{12} = \cos\varphi \partial_{\theta} - \cot\theta \sin\varphi \partial_{\varphi}, \quad (3.1d)$$

$$J_{13} = \sin\varphi \partial_{\theta} + \cot\theta \cos\varphi \partial_{\varphi}. \quad (3.1e)$$

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<sup>1</sup>These criteria do not uniquely fix the near-zone approximation for finite  $\omega$ . Nevertheless, the near-zone approximation of Schwarzschild we use is standard in the literature. The existence of the symmetries that we will discuss below can be viewed as an *a posteriori* motivation for this particular implementation.

The Killing vectors  $L_0 \equiv T$  and  $L_{\pm}$  were first introduced in [17] and coincide with the zero-spin limit of the symmetries discovered for Kerr in [3]. More recently they were encountered in the context of rotating STU supergravity black holes [18].

The near-zone metric (2.3) also possesses 9 conformal Killing vectors:

$$J_{01} = -\frac{2\Delta}{r_s} \cos \theta \partial_r - \frac{\partial_r \Delta}{r_s} \sin \theta \partial_\theta, \quad (3.2a)$$

$$J_{02} = -\cos \varphi \left[ \frac{2\Delta}{r_s} \sin \theta \partial_r + \frac{\partial_r \Delta}{r_s} \left( \frac{\tan \varphi}{\sin \theta} \partial_\varphi - \cos \theta \partial_\theta \right) \right], \quad (3.2b)$$

$$J_{03} = -\sin \varphi \left[ \frac{2\Delta}{r_s} \sin \theta \partial_r - \frac{\partial_r \Delta}{r_s} \left( \frac{\cot \varphi}{\sin \theta} \partial_\varphi + \cos \theta \partial_\theta \right) \right], \quad (3.2c)$$

$$K_{\pm} = e^{\pm t/2r_s} \frac{\sqrt{\Delta}}{r_s} \cos \theta \left( \frac{r_s^3}{\Delta} \partial_t \mp \partial_r \Delta \partial_r \mp 2 \tan \theta \partial_\theta \right), \quad (3.2d)$$

$$M_{\pm} = e^{\pm t/2r_s} \cos \varphi \left[ \frac{r_s^2}{\sqrt{\Delta}} \sin \theta \partial_t \mp \frac{\sqrt{\Delta} \partial_r \Delta \sin \theta}{r_s} \partial_r \pm \frac{2\sqrt{\Delta}}{r_s} \cos \theta \partial_\theta \mp \frac{2\sqrt{\Delta}}{r_s} \frac{\tan \varphi}{\sin \theta} \partial_\varphi \right], \quad (3.2e)$$

$$N_{\pm} = e^{\pm t/2r_s} \sin \varphi \left[ \frac{r_s^2}{\sqrt{\Delta}} \sin \theta \partial_t \mp \frac{\sqrt{\Delta} \partial_r \Delta \sin \theta}{r_s} \partial_r \pm \frac{2\sqrt{\Delta}}{r_s} \cos \theta \partial_\theta \pm \frac{2\sqrt{\Delta}}{r_s} \frac{\cot \varphi}{\sin \theta} \partial_\varphi \right]. \quad (3.2f)$$

Expressing each of the Killing and conformal Killing generators as  $\xi^\mu \partial_\mu$ , the symmetries act on the scalar as

$$\delta \phi = \xi^\mu \partial_\mu \phi + \frac{1}{4} \nabla_\mu \xi^\mu \phi. \quad (3.3)$$

The time translation  $T$  and spatial rotations  $J_{ij}$  ( $i, j = 1, 2, 3$ ) are the familiar symmetries of the exact dynamics. In addition, the symmetry generators  $J_{0i}$  ( $i = 1, 2, 3$ ) have a somewhat privileged status: they generate symmetries of the exact system in the static limit,  $\omega = 0$  [4]; see also [19] for a related discussion. The other (C)KVs do not give rise to exact symmetries in this limit. Each contains a factor of  $e^{\pm t/2r_s}$ , and thus when applied to a static scalar generates a solution with  $\omega = \pm i/2r_s$  (which also means the resulting scalar has an  $|\omega|$  outside the regime of validity of the near-zone approximation). A corollary is that these other (C)KVs are not well-defined in the flat space ( $r_s \rightarrow 0$ ) limit. Nonetheless, these generators can still be used to infer properties of exact static solutions [3]. On the other hand, the generators  $J_{0i}$  have an overall factor of  $1/r_s$  which can be removed without trouble, and thus do have a well-defined flat space limit.

All together, the algebra of the Killing (3.1) and conformal Killing (3.2) symmetries is  $\mathfrak{so}(4, 2)$ , as expected because the metric (2.3) is conformally flat. There are a number of subalgebras of interest. Firstly, the generators  $J_{0i}, J_{ij}$  ( $i, j = 1, 2, 3$ ) form an  $\mathfrak{so}(3, 1)$  subalgebra [4]. In addition, each pair of vectors labeled with the subscripts  $\pm$  in eqs. (3.1) and (3.2) forms a subgroup together with the generator  $T$ . More precisely, denoting  $X = \{L, K, M, N\}$ , we have

$$[T, X_{\pm}] = \pm X_{\pm}, \quad [X_+, X_-] = 2\sigma_X T, \quad (3.4)$$

with  $\sigma_L = -1$  and  $\sigma_K = \sigma_M = \sigma_N = +1$ , giving different  $\mathfrak{sl}(2, \mathbb{R})$  subalgebras. To the best of our knowledge, the consequences of the symmetries  $K_{\pm}, M_{\pm}$ , and  $N_{\pm}$  for perturbations around Schwarzschild have not been explored in the literature.

## 4 Effective Kerr near-zone metric

The Kerr line element in Boyer-Lindquist coordinates is:

$$\begin{aligned}
 ds^2 = & -\frac{\rho^2 - r_s r}{\rho^2} dt^2 - \frac{2ar_s r \sin^2 \theta}{\rho^2} dt d\varphi + \frac{\rho^2}{\Delta} dr^2 \\
 & + \rho^2 d\theta^2 + \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2} \sin^2 \theta d\varphi^2,
 \end{aligned}
 \tag{4.1}$$

where we have defined the quantities

$$\rho^2 \equiv r^2 + a^2 \cos^2 \theta, \quad \Delta \equiv r(r - r_s) + a^2.
 \tag{4.2}$$

The Schwarzschild radius  $r_s$  and the spin parameter  $a$  are related to the outer and inner horizons  $r_{\pm}$ , i.e., the radii where  $\Delta = 0$ , via  $r_{\pm} \equiv r_s/2 \pm \sqrt{(r_s/2)^2 - a^2}$ .

The Klein-Gordon equation on the Kerr background is

$$\begin{aligned}
 \partial_r(\Delta \partial_r \phi) + \nabla_{\Omega}^2 \phi - \frac{a^2}{\Delta} \partial_{\varphi}^2 \phi - \frac{1}{\Delta} [(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta] \partial_t^2 \phi \\
 - \frac{2a}{\Delta} [(r^2 + a^2) - \Delta] \partial_t \partial_{\varphi} \phi = 0.
 \end{aligned}
 \tag{4.3}$$

We define the near-zone region using the same approximation as in the Schwarzschild case. We choose to implement this approximation in Boyer-Lindquist coordinates because they are inertial at infinity. Wherever there are time derivatives, we keep terms that go as  $1/\Delta$  to preserve the singularity as  $r$  approaches the horizon, and set  $r \rightarrow r_+$  in the corresponding numerators. This ensures any corrections are subdominant at the horizon and are small away from it in the low frequency regime:  $\omega a \leq \omega r_+ \leq \omega r \ll 1$ . Thus the near-zone scalar equation is

$$\partial_r(\Delta \partial_r \phi) + \nabla_{\Omega}^2 \phi - \frac{1}{\Delta} [(r_+^2 + a^2) \partial_t + a \partial_{\varphi}]^2 \phi = 0.
 \tag{4.4}$$

Note that, unlike some near-zone approximations of Kerr put forward in the literature, this approximation remains well defined even in the extremal limit  $a \rightarrow r_s/2$ . Expanding in frequency space and spherical harmonics<sup>2</sup>  $\phi = e^{-i\omega t} Y_{\ell m}(\theta, \varphi) R(r)$  (where the  $\ell$  and  $m$  dependence of  $R$  is suppressed), and using  $r_+^2 + a^2 = r_s r_+$ , the near-zone equation reads [15, 16, 21, 22]<sup>3</sup>

$$\partial_r(\Delta \partial_r R) + \left[ \frac{r_s^2 r_+^2}{\Delta} (\omega - m\Omega_+)^2 - \ell(\ell + 1) \right] R = 0,
 \tag{4.5}$$

where for convenience we have introduced  $\Omega_+ \equiv a/(r_s r_+)$ .

<sup>2</sup>For non-static perturbations one should in general use spheroidal harmonics. However, at the order we are working with in the near-zone approximation, it is consistent to decompose the field in terms of spherical harmonics [20].

<sup>3</sup>Note that there is a typo in eqs. (2.11) and (2.12) of [22], where the factor  $r_+^4$  should be replaced by  $r_s^2 r_+^2$ .

It is straightforward to show that eq. (4.4) is the equation of motion for a massless scalar propagating in the following effective near-zone metric:<sup>4</sup>

$$ds_{\text{near-zone}}^2 = -\frac{\Delta - a^2 \sin^2 \theta}{r_s r_+} dt^2 - 2a \sin^2 \theta dt d\varphi + \frac{r_s r_+}{\Delta} dr^2 + r_s r_+ d\Omega_{S^2}^2. \quad (4.6)$$

This metric reduces to (2.3) in the limit  $a \rightarrow 0$ , and moreover is conformally flat, and therefore has the same number of CKVs. A coordinate transformation  $\varphi' = \varphi - (a/r_s r_+)t$  simplifies the metric to

$$ds_{\text{near-zone}}^2 = -\frac{\Delta}{r_s r_+} dt^2 + \frac{r_s r_+}{\Delta} dr^2 + r_s r_+ d\Omega'_{S^2}{}^2. \quad (4.7)$$

In the extremal limit where  $\Delta = (r - r_+)^2 = (r - r_-)^2$ , one can see the  $(t, r)$  subspace is AdS<sub>2</sub> in Poincaré coordinates upon redefining  $r - r_-$  as the new radial coordinate. Interestingly, the extremal near-zone metric coincides with the *near-horizon* limit of the extremal Reissner-Nördstrom solution with  $r_s^2 \rightarrow r_s r_+$ .

Away from the extremal limit, the simplest way to deduce the symmetries is to recognize that the effective near-zone metric for Kerr is in fact equivalent to the near-zone metric for Schwarzschild. To see this, redefine  $t' = (r_*/\sqrt{r_s r_+})t$  and  $r' = \sqrt{r_s r_+}(r - r_-)/r_*$ , with  $r_* \equiv r_+ - r_-$ . The Kerr near-zone metric is then rewritten as:

$$ds_{\text{near-zone}}^2 = -\frac{\tilde{\Delta}}{r_s r_+} dt'^2 + \frac{r_s r_+}{\tilde{\Delta}} dr'^2 + r_s r_+ d\Omega'^2, \quad (4.8)$$

with  $\tilde{\Delta} \equiv r'(r' - \sqrt{r_s r_+}) = (r_s r_+/r_*^2)\Delta$ . This has the same form as the Schwarzschild near-zone metric in (2.3) with  $r_s \rightarrow \sqrt{r_s r_+}$ .

The 15 (conformal) Killing vectors for a spinning black hole in the near-zone can thus be obtained from their Schwarzschild counterparts (3.1)–(3.2) using the coordinate transformations discussed above and replacing  $r_s \rightarrow \sqrt{r_s r_+}$ . For completeness, we report their explicit expressions in the Supplemental Material. There are several  $\mathfrak{sl}(2, \mathbb{R})$  subalgebras, just as in Schwarzschild, taking the same form as in eq. (3.4). Spatial rotations  $J_{ij}$  and boosts  $J_{0i}$  ( $i, j = 1, 2, 3$ ) form an  $\mathfrak{so}(3, 1)$  subalgebra just as before. However, only  $J_{23}$  among the spatial rotations is an exact symmetry of the static sector, while  $J_{12}$  and  $J_{13}$  do not preserve the static nature of field configurations.<sup>5</sup> Physically, this is because the Kerr metric has a preferred direction. Similarly, of the three boosts, only  $J_{01}$  is a symmetry of the exact system in the static limit.<sup>6</sup>

<sup>4</sup>We stress that the near-zone metric (4.6) should not be confused with the near-horizon limit encountered in the context of the extremal Kerr metric (see, e.g., [23]). This near-horizon limit is defined by a rescaling of the radial and time coordinates which keeps the coordinate  $\varphi'$  fixed. As a byproduct, it is suited to study modes with  $\omega \sim m\Omega_+$ , rather than the static regime.

<sup>5</sup>By a static configuration, we mean  $\partial_t \phi = 0$  keeping  $\varphi$  (as well as  $r$  and  $\theta$ ) fixed, as opposed to keeping  $\varphi'$  fixed. This choice is dictated by the fact that  $\varphi$  is an inertial coordinate at infinity.

<sup>6</sup>This is reflected by the fact that only  $J_{01}$  among them has no time dependence when expressed in Boyer-Lindquist coordinates.

## 5 Comparison of different near zones

The effective metric (4.6) captures the near-zone dynamics (4.4) of massless scalar perturbations around a Kerr black hole. For different reasons, various deformations of the near-zone approximation (4.4) have been proposed in the literature. There are in fact many ways of deforming (4.4) at subleading order in  $(r - r_+)/r_+$  [20, 24]. It is instructive to briefly review some of these possibilities and highlight the main differences with (4.4).

One notable example is given in [25]. Supported by the observation that the Cardy formula for a  $\text{CFT}_2$  gives exactly the Bekenstein-Hawking entropy of the Kerr solution, [25] conjectured (see also [23, 26, 27]) that a non-extremal Kerr black hole is dual to a two-dimensional CFT and proposed a near-zone approximation with an  $\text{sl}(2, \mathbb{R})_L \times \text{sl}(2, \mathbb{R})_R$  symmetry. This can be obtained by adding to (4.5) the term  $(\omega r_s(\omega r_s^2 - 2ma)/(r - r_-))R$ . The effect of this term — which is small in the low frequency regime and subleading near the horizon compared to the  $1/\Delta$  term in (4.5)—is to break some of the symmetries while introducing a new  $\text{sl}(2, \mathbb{R})_L \times \text{sl}(2, \mathbb{R})_R$  symmetry (which is not a subalgebra of our  $\text{so}(4, 2)$ ). These  $\text{sl}(2, \mathbb{R})_L \times \text{sl}(2, \mathbb{R})_R$  generators are singular in the Schwarzschild limit (the near zone of [25] is not smoothly connected to the Schwarzschild near zone above in the limit  $a \rightarrow 0$ ) and they are not globally defined, as they do not respect the  $\varphi \rightarrow \varphi + 2\pi$  periodicity.

A different near-zone approximation that overcomes these issues has recently been proposed in [3]. The near zone of [3] differs from the one in eq. (4.4) by the term  $(4\omega\Omega_+m(r-r_+)/(r_+-r_-))R$ . This too breaks some of the symmetries, but keeps an  $\text{sl}(2, \mathbb{R})$  group with generators that are both globally well defined and have a smooth Schwarzschild limit.

All three of these approximations are contained in the one-parameter family of [24], which possesses an  $\text{sl}(2, \mathbb{R})_L \times \text{sl}(2, \mathbb{R})_R$  symmetry except at two special points, one corresponding to (4.4) (where they identified the generators  $T$  (A.4a) and  $L_\pm$  (A.4h)) and the other to [3]. See also [28] for a recent discussion of near-zone approximations and their relations to Killing tensor symmetries.

Each of these approximation schemes has benefits and drawbacks. For our purposes, the main appeal of the approximation (4.4), besides having a smooth Schwarzschild limit and globally well-defined generators, is that its effective metric (4.6) contains in particular the symmetry generator  $J_{01}$  in (A.4b). This is a symmetry of the *exact* dynamics for static field configurations (and is in fact a CKV of an effective 3D metric [4]). Keeping  $J_{01}$  as a symmetry is thus useful for a near-zone approximation intended for low frequency phenomena. The effective metric (4.6) has other special properties: since it is conformally flat, it possesses the maximal number of CKVs (15 in  $d = 4$ ), and since its Ricci scalar vanishes, a massless scalar is automatically conformally coupled, so that each of the (C)KVs  $\xi^\mu$  generates a symmetry acting on the scalar as in (3.3). Each gives rise to a conserved current in the standard way:  $j^\mu = T^{\mu\nu}\xi_\nu$ , where  $T^{\mu\nu}$  is the (traceless) energy momentum tensor of the scalar.

## 6 Ladder symmetries and tidal response

Up to an irrelevant constant factor, the CKV associated with  $J_{01}$  in (A.4b) can be written as  $\xi_{J_{01}}^\mu = (0, \Delta \cos \theta, \frac{1}{2} \Delta' \sin \theta, 0)$ , with  $\Delta' \equiv \partial_r \Delta$ . As mentioned above, this CKV is time independent and survives in the zero-frequency limit, recovering the CKV that we showed in [4] to be associated with a ladder structure for static perturbations around Kerr black holes, leading to vanishing static response.<sup>7</sup> Let us now recall how this works and extend the results of [4] to non-zero frequencies.

The CKV  $\xi_{J_{01}}^\mu$  corresponds to a symmetry of the scalar action, which acts on the scalar field  $\phi$  (in real space) as in (3.3). After decomposing  $\phi$  in spherical harmonics as  $\phi = e^{-i\omega t} Y_{\ell m}(\theta, \varphi) R_\ell(r)$ , and extracting a convenient phase factor from  $R_\ell$ ,

$$R_\ell \equiv e^{-\frac{ir_+ r_s}{r_+ - r_-} (\omega - m\Omega_+) \log\left(\frac{r-r_+}{r-r_-}\right)} \psi_\ell, \quad (6.1)$$

the scalar equation (4.5) takes the form

$$\left[ \partial_r \left( \Delta \partial_r - 2ir_s r_+ (\omega - m\Omega_+) \right) - \ell(\ell + 1) \right] \psi_\ell = 0, \quad (6.2)$$

where derivatives act on everything to their right, and the field transformation takes the form

$$\delta \psi_\ell = \mathcal{Q}_\ell D_{\ell-1}^+ \psi_{\ell-1} - \mathcal{Q}_{\ell+1} D_{\ell+1}^- \psi_{\ell+1}. \quad (6.3)$$

We have defined  $\mathcal{Q}_\ell \equiv \sqrt{(\ell^2 - m^2)/(4\ell^2 - 1)}$  and introduced the operators

$$D_\ell^+ \equiv -\Delta \partial_r - \frac{\ell + 1}{2} \Delta' + ir_s r_+ (\omega - m\Omega_+), \quad (6.4a)$$

$$D_\ell^- \equiv \Delta \partial_r - \frac{\ell}{2} \Delta' - ir_s r_+ (\omega - m\Omega_+). \quad (6.4b)$$

The  $D_\ell^\pm$  are *ladder operators* in the sense that  $\delta \psi_{\ell \pm 1} = D_\ell^\pm \psi_\ell$  are solutions to the near-zone equation (6.2) at level  $\ell \pm 1$  if  $\psi_\ell$  solves (6.2) at level  $\ell$ . These operators are useful because they allow us to recursively define an on-shell conserved charge at each  $\ell$  [4]:

$$P_\ell = \alpha_\ell [\Delta \partial_r - 2ir_s r_+ (\omega - m\Omega_+)] D_1^- \cdots D_\ell^- \psi_\ell, \quad (6.5)$$

where  $\alpha_\ell \equiv -2^{2\ell-1} (\ell!)^2 / [(2\ell)! (2\ell + 1)!]$ . These conserved charges allow us to connect the behavior of solutions near the horizon to the behavior at infinity without solving eq. (6.2) explicitly, making it possible to infer the vanishing of static responses from symmetry [4]. Corresponding to each of these charges is an off-shell symmetry of the action, which can be inferred as described in [4] (see also [29] for a similar construction on de Sitter backgrounds).

Evaluating  $P_\ell$  on  $\psi_\ell = D_{\ell-1}^+ \cdots D_0^+ \psi_0$ , where  $\psi_0 = \text{constant}$  — which is the solution with the correct infalling behavior at  $r = r_+$  — yields

$$P_\ell = -iq(r_+ - r_-)^{2\ell+1} \frac{(\ell!)^2}{(2\ell)! (2\ell + 1)!} \prod_{k=1}^{\ell} (k^2 + 4q^2), \quad (6.6)$$

<sup>7</sup>Note that the  $\mathfrak{sl}(2, \mathbb{R})$  involving  $L_\pm$  in the near zone (4.4) includes (A.4a) as a generator instead of  $\partial_t$ , which precludes applying the arguments of [3] directly.



with  $q \equiv r_s r_+ (m\Omega_+ - \omega) / (r_+ - r_-)$ . The conserved charges  $P_\ell$  in (6.6) reproduce the induced multipole moments of a scalar field on a fixed Kerr geometry [30] (see also [15, 16, 31]). In particular, when  $\omega = 0$ , one recovers the scalar's static dissipative response [4, 10–12, 14].

## Acknowledgments

We would like to thank Albert Law and John Stout for helpful discussions. LH is supported by the DOE DE-SC0011941 and a Simons Fellowship in Theoretical Physics. The work of RP is supported in part by the National Science Foundation under Grant No. PHY-1915611. ARS's research was partially supported by funds from the Natural Sciences and Engineering Research Council (NSERC) of Canada. Research at the Perimeter Institute is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI.

## A Supplemental material

### A.1 Near-zone AdS<sub>2</sub> geometry

Here we elaborate on the geometry of the  $(t, r)$  subspace of the near-zone metric (2.3). This metric describes AdS<sub>2</sub> in de Sitter-slice coordinates. In order to see this, we make the coordinate transformation

$$\tau = \frac{t}{2r_s}, \quad \xi = \cosh^{-1} \left( \frac{2r}{r_s} - 1 \right), \quad (\text{A.1})$$

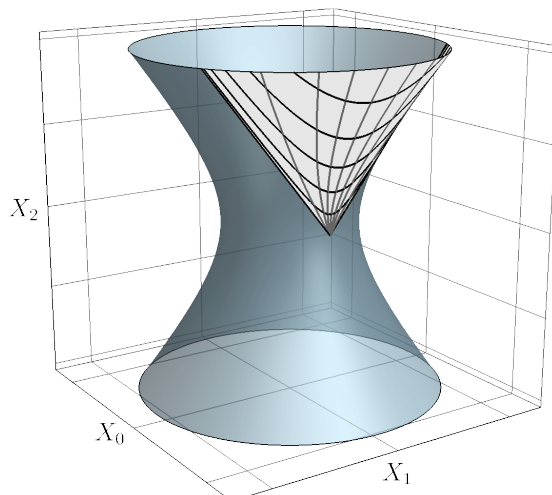
so that the two-dimensional metric  $ds^2 = -\frac{\Delta}{r_s^2} dt^2 + \frac{r_s^2}{\Delta} dr^2$  becomes

$$ds^2 = r_s^2 \left( d\xi^2 - \sinh^2 \xi d\tau^2 \right), \quad (\text{A.2})$$

with  $\tau \in (-\infty, +\infty)$ ,  $\xi \in [0, +\infty)$ . Notice in particular that  $\xi = 0$  corresponds to  $r = r_s$ . This is an AdS<sub>2</sub> metric. To see this explicitly, we note that these coordinates correspond to the embedding

$$X_0 = r_s \cosh \xi, \quad X_1 = r_s \sinh \xi \sinh \tau, \quad X_2 = r_s \sinh \xi \cosh \tau, \quad (\text{A.3})$$

which satisfy  $-X_0^2 - X_1^2 + X_2^2 = -r_s^2$ , and cover the region of this hyperboloid that satisfies  $X_0 \geq r_s, X_2 \geq 0$ . This portion of AdS<sub>2</sub> is depicted in figure 1.



**Figure 1.** Portion of  $\text{AdS}_2$  covered by the de Sitter slice coordinates  $(\tau, \xi)$ , along with lines of constant  $\tau$  and  $\xi$ .

## A.2 (Conformal) Killing vectors of near-zone Kerr geometry

The (conformal) Killing vectors of the near-zone Kerr metric in eq. (4.6) (or, equivalently, eq. (4.7)) are:

$$T = \mathcal{R}\partial_t + \frac{2a}{r_*}\partial_\varphi, \tag{A.4a}$$

$$J_{01} = -\frac{2\Delta}{r_*}\cos\theta\partial_r - \frac{\partial_r\Delta}{r_*}\sin\theta\partial_\theta, \tag{A.4b}$$

$$J_{02} = -\cos\varphi' \left[ \frac{2\Delta}{r_*}\sin\theta\partial_r + \frac{\partial_r\Delta}{r_*} \left( \frac{\tan\varphi'}{\sin\theta}\partial_\varphi - \cos\theta\partial_\theta \right) \right], \tag{A.4c}$$

$$J_{03} = -\sin\varphi' \left[ \frac{2\Delta}{r_*}\sin\theta\partial_r - \frac{\partial_r\Delta}{r_*} \left( \frac{\cot\varphi'}{\sin\theta}\partial_\varphi + \cos\theta\partial_\theta \right) \right], \tag{A.4d}$$

$$J_{12} = \cos\varphi'\partial_\theta - \cot\theta\sin\varphi'\partial_\varphi, \tag{A.4e}$$

$$J_{13} = \sin\varphi'\partial_\theta + \cot\theta\cos\varphi'\partial_\varphi, \tag{A.4f}$$

$$J_{23} = \partial_\varphi, \tag{A.4g}$$

$$L_\pm = e^{\pm t/\mathcal{R}} \left[ \mathcal{R}(\partial_r\sqrt{\Delta})\partial_t \mp \sqrt{\Delta}\partial_r + \frac{2a}{r_*}(\partial_r\sqrt{\Delta})\partial_\varphi \right] \tag{A.4h}$$

$$K_\pm = e^{\pm t/\mathcal{R}} \frac{\sqrt{\Delta}}{r_*} \cos\theta \left( \frac{r_s r_+ r_*}{\Delta} \partial_t \mp \partial_r \Delta \partial_r \mp 2 \tan\theta \partial_\theta + \frac{a r_*}{\Delta} \partial_\varphi \right), \tag{A.4i}$$

$$M_\pm = e^{\pm t/\mathcal{R}} \cos\varphi' \left[ \frac{r_s r_+}{\sqrt{\Delta}} \sin\theta \partial_t \mp \frac{\sqrt{\Delta} \partial_r \Delta \sin\theta}{r_*} \partial_r \pm \frac{2\sqrt{\Delta}}{r_*} \cos\theta \partial_\theta + \left( \frac{a \sin\theta}{\sqrt{\Delta}} \mp \frac{2\sqrt{\Delta}}{r_*} \frac{\tan\varphi'}{\sin\theta} \right) \partial_\varphi \right], \tag{A.4j}$$

$$N_\pm = e^{\pm t/\mathcal{R}} \sin\varphi' \left[ \frac{r_s r_+}{\sqrt{\Delta}} \sin\theta \partial_t \mp \frac{\sqrt{\Delta} \partial_r \Delta \sin\theta}{r_*} \partial_r \pm \frac{2\sqrt{\Delta}}{r_*} \cos\theta \partial_\theta + \left( \frac{a \sin\theta}{\sqrt{\Delta}} \pm \frac{2\sqrt{\Delta}}{r_*} \frac{\cot\varphi'}{\sin\theta} \right) \partial_\varphi \right], \tag{A.4k}$$

where we have defined  $\mathcal{R} = \frac{2r_s r_+}{r_*}$ ,  $\varphi' = \varphi - \frac{a}{r_s r_+} t$  and  $r_* \equiv r_+ - r_-$ . Note that these reduce to (3.1) and (3.2) in the limit  $a \rightarrow 0$ . The generators (A.4) satisfy the  $\text{so}(4, 2)$  algebra. We

can make this explicit by defining  $J_{54} = T$  along with

$$J_{04} = \frac{L_+ - L_-}{2}, \quad J_{05} = \frac{L_+ + L_-}{2}, \quad (\text{A.5a})$$

$$J_{14} = \frac{K_+ - K_-}{2}, \quad J_{15} = \frac{K_+ + K_-}{2}, \quad (\text{A.5b})$$

$$J_{24} = \frac{M_+ - M_-}{2}, \quad J_{25} = \frac{M_+ + M_-}{2}, \quad (\text{A.5c})$$

$$J_{34} = \frac{N_+ - N_-}{2}, \quad J_{35} = \frac{N_+ + N_-}{2}, \quad (\text{A.5d})$$

which then have the  $\text{so}(4, 2)$  commutation relations

$$[J_{AB}, J_{CD}] = \eta_{AD}J_{BC} + \eta_{BC}J_{AD} - \eta_{AC}J_{BD} - \eta_{BD}J_{AC}, \quad (\text{A.6})$$

where  $\eta_{AB} = \text{diag}(-1, 1, 1, 1, 1, -1)$ . A few comments are in order. First, note that only  $J_{01}$  and  $J_{23}$  are time-independent (when expressing all quantities in Boyer-Lindquist coordinates) and are exact symmetries of the static sector [4]. The other generators depend explicitly on time and are not (C)KVs of the effective 3D Kerr metric of [4]. Second, the generators  $L_{\pm}$  differ from the ones introduced in [3] for non-zero values of the spin parameter  $a$ . Interestingly, though, they coincide (up to a rescaling of the time coordinate) in the region close to the horizon defined by  $(r - r_+)/r_+ \ll 1$ . This is a manifestation of the fact that all the near-zone approximations of Kerr put forward in the literature actually coincide in this limit. Note also that some of the generators in (A.4) are manifestly well defined in the extremal limit ( $a \rightarrow r_s/2$ ,  $r_{\star} \rightarrow 0$ ), while others look singular in this limit. This is not a problem because one can consistently recover all the (C)KVs of the metric (4.6) at extremality by multiplying with suitable powers of  $r_{\star}$  and taking linear combinations of the generators (A.4). For instance, in addition to  $J_{ij}$  and  $T$  (after extracting a  $1/r_{\star}$  factor), the other two KVs of (4.6) in the extremal limit are obtained by expanding  $J_{04}$  and the combination  $(T - J_{05})/r_{\star}$  at leading order in  $r_{\star}$ .

### A.3 Ladder in spin and finite frequency

A convenient near-zone approximation that describes the dynamics of particles of generic spin,  $s$ , in the limit  $r_+ \leq r \ll 1/\omega$  is [16, 21]

$$x(x+1)\partial_x^2 R + (s+1)(2x+1)\partial_x R + \left[ -(\ell - s)(\ell + s + 1) + \frac{q^2 + isq(2x+1)}{x(x+1)} \right] R = 0, \quad (\text{A.7})$$

where  $q \equiv r_s r_+ (m\Omega_+ - \omega) / (r_+ - r_-)$  and

$$x \equiv \frac{r - r_+}{r_+ - r_-}. \quad (\text{A.8})$$

It is straightforward to show that eq. (A.7) admits the following set of spin raising and lowering operators:

$$E_s^+ = \left( \partial_r - \frac{ir_s r_+ (\omega - m\Omega_+)}{\Delta} \right) R, \quad E_s^- = \Delta^{-s+1} \left( \partial_r + \frac{ir_s r_+ (\omega - m\Omega_+)}{\Delta} \right) \Delta^s R, \quad (\text{A.9})$$

which generate solutions with spin  $s + 1$  and  $s - 1$  respectively, i.e.,  $R^{(s+1)} = E_s^+ R^{(s)}$  and  $R^{(s-1)} = E_s^- R^{(s)}$ , where  $R^{(s)}$  solves (A.7) with spin  $s$ . The operators (A.9) generalize the Teukolsky-Starobinsky identities [16, 32, 33] in the near-zone regime by connecting solutions with consecutive spin  $s$  and  $s \pm 1$ . These spin raising and lowering operators provide a simple way of extending the results discussed above for spin-0 fields to spin-1 and spin-2 particles described by the Teukolsky equation [4].

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