Spontaneous non-Hermiticity in the (2 + 1)-dimensional Gross-Neveu model

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Using a nonperturbative approach based on the Cornwall-Jackiw-Tomboulis (CJT) effective action $\Gamma(S)$ for composite operators (S is the full fermion propagator), the phase structure of the simplest massless $(2+1)$-dimensional Gross-Neveu model is investigated. We have calculated $\Gamma(S)$ and its stationary (or Dyson-Schwinger) equation in the first order of the bare coupling constant G and have shown that there exist a well-defined dependence of $G \equiv G(\Lambda)$ on the cutoff parameter Λ, such that the Dyson-Schwinger equation is renormalized. It has three different solutions for fermion propagator S corresponding to possible dynamical appearance of three different mass terms in the model. One is a Hermitian, but two others are non-Hermitian and $\mathcal{PT}$ even or odd. It means that two phases with spontaneous non-Hermiticity can be emerged in the system. Moreover, mass spectrum of quasiparticles is real in these non-Hermitian and $\mathcal{PT}$ even/odd phases.

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I. INTRODUCTION

For a long time, it was believed that to describe quantum systems it is necessary to use theories with Hermitian Hamiltonians (or Lagrangians), since in this case the energy spectrum is real. However, in recent decades, it has been understood that there are situations, especially in open physical systems interacting with the environment, which can be effectively considered in the framework of non-Hermitian Hamiltonians (see, e.g., in review [1]). Moreover, it was claimed that if non-Hermitian theories have in addition a space-time reflection symmetry $\mathcal{PT}$, then its energy spectrum is real [2,3], i.e., the Hermitian nature of the Hamiltonian is only a sufficient, but far from necessary, condition for the energy spectrum of the system to be real. This statement is supported by a number of bosonic theories in which non-Hermiticity together with $\mathcal{PT}$ symmetry lead to real mass spectrum.

However, the situation is more involved in fermionic systems. On the one hand, indeed, as the considerations of some $(1+1)$- and $(3+1)$-dimensional (D) and non-Hermitian field theory models with four-fermion interaction show, the $\mathcal{PT}$-symmetry together with non-Hermiticity leads to a real spectrum of mass [4–6]. On the other hand, in the same paper [6] other non-Hermitian and anti-$\mathcal{PT}$-symmetric extensions of the four-fermion models are also presented, in which, nevertheless, a real spectrum of fermion masses is also generated, i.e., in fact $\mathcal{PT}$ symmetry of the model is not a necessary condition for real fermion masses to exist. Thus, the relationship between the phenomena of non-Hermiticity, $\mathcal{PT}$ symmetry and the reality of the energy spectrum in any quantum system remains a far from solved problem and deserves further study. Moreover, it should be noted one more feature of the non-Hermiticity phenomenon, which was observed recently just within the framework of the $(3+1)$-D Nambu–Jona-Lasinio (NJL) model [7] with four-fermion interactions. Namely, in this model the non-Hermiticity can arise spontaneously [8]. (Quite recently, it was noted in Ref. [9] that, perhaps, the phenomenon of spontaneous non-Hermiticity occurs also in some models of Yukawa type.) It means that (i) initial Lagrangian of the massless NJL model is taken to be Hermitian and $\mathcal{PT}$-symmetric, (ii) but, as it was proved in Ref. [8], there exists a ground state corresponding to a dynamical (spontaneous) appearance of the $\mathcal{PT}$-symmetric and non-Hermitian Yukawa-type term in the effective Lagrangian. In addition, quasiparticle excitations of this ground state obey a real mass.

In the present paper, we show that in the simplest $(2+1)$-D massless Gross-Neveu (GN) model (for the first time, it was discussed in Ref. [10]) with four-fermion interaction (initially, its Lagrangian is Hermitian and $\mathcal{PT}$-symmetric[1]) the non-Hermiticity can also arise spontaneously. In contrast to the situation observed in the massless NJL model, in $(2+1)$-D massless GN model (i) both $\mathcal{PT}$- and anti-$\mathcal{PT}$-symmetric non-Hermitian ground states are allowed to be realized. And (ii) it means that mass terms, and not terms with Yukawa interaction, with corresponding symmetry properties are generated in the model Lagrangian.

In this connection, it is necessary to note that recently much attention has been paid to the investigation of $(2+1)$-D field theory models, which can be used to predict and study the condensed matter physical phenomena of planar nature such as quantum Hall effect, high-temperature superconductivity, low-energy graphene physics, etc. To a fairly large extent, these phenomena are usually considered within the framework of models with a four-fermion interaction [11–19]. One of the reasons is that in these models the spontaneous symmetry breaking occurs dynamically, i.e., without taking into account additional scalar Higgs bosons. Moreover, despite the perturbative nonrenormalizability of these $(2+1)$-dimensional models, in the framework of nonperturbative approaches such as large-N technique, etc., they are renormalizable [20]. And just using this nonperturbative $1/N$ approach, spontaneous symmetry breaking and the associated effect of dynamical generation of the fermion mass were investigated in the simplest $(2+1)$-D GN model with four-fermion interaction. In particular, it was shown, e.g., in Ref. [21] that at zero temperature and zero chemical potential (as well as at fixed value of the cutoff parameter Λ) in this $(2+1)$-D GN model a phase with dynamical chiral symmetry breaking occurs only at sufficiently large (positive) values of the bare coupling constant $G \equiv G(\Lambda)$. For a rather weak interaction, the symmetric phase is realized in the model, and it is not an asymptotically free one. (In contrast, the $(1+1)$-D GN model [10] is an asymptotically free and dynamic generation of the fermionic mass occurs there for arbitrary values of bare coupling constant.) Qualitatively the same properties of this $(2+1)$-D GN model one can observe in terms of variational optimized expansion technique [22] and other nonperturbative variational approaches [23], etc.

Unlike the aforementioned papers, we investigate phase structure of the $(2+1)$-D GN model (1) within the framework of another nonperturbative approach based on the effective action for composite operators. Originally, the approach was proposed in the paper by Cornwall-Jackiw-Tomboulis (CJT) [24] when considering mainly a scalar ϕ^4-field model, etc. Then in a series of papers [25–29] the CJT effective action for composite operators method has been extended to (Hermitian) quantum field theory models with fermions. As a result, a nonperturbative method has emerged for calculating various multifermion Green's functions based on functional equations of the Dyson-Schwinger type. Moreover, in this CJT effective action approach it is possible to investigate the possibility of dynamical generation of the fermion mass and chiral symmetry breaking, etc., as it was demonstrated, e.g., in the framework of the $(1+1)$-D GN model in Ref. [27]. And in the last case, i.e., in $(1+1)$-D, the results of the CJT effective action studies of the model are qualitatively the same as in the large-N expansion technique. It is also worth mentioning that the possibility of dynamically generating fermion mass in some non-Hermitian quantum field theory models has been investigated in Refs. [30–33]. Namely, in the first of these papers, the problem is considered within the framework of the $1/N$ expansion in the $(3+1)$-D NJL model (with a complex coupling constant), while in the remaining papers, for this purpose, the approach of the Dyson-Schwinger equation was used in non-Hermitian Yukawa-type models with additional four-fermion interaction term.

In the recent paper [34] we have studied phase structure of the massless $(2+1)$-D GN model also using the CJT effective action method. It turns out that in this case, in contrast to $(1+1)$-D GN model, the CJT approach predicts a much richer phase structure compared to the result obtained with other generally accepted nonperturbative methods, i.e., large-N and optimized expansion techniques, etc. And each of the observed phases is associated with some dynamically generated (Hermitian) mass term of the Lagrangian. So in the present paper, we show that non-Hermiticity can arise spontaneously in the $(2+1)$-D GN model under consideration just in the framework of the CJT composite operator approach. It means that for a certain well-defined behavior of the bare coupling constant, the ground state of the system can be characterized by a dynamically arising non-Hermitian mass term of the Lagrangian, which can be both $\mathcal{PT}$- and anti-$\mathcal{PT}$ symmetric.

The paper is organized as follows. Section II A presents the N-flavor massless $(2+1)$-dimensional Gross-Neveu model symmetric under several discrete transformations, two chiral transformations as well as with respect to spatial $\mathcal{P}$ and time $\mathcal{T}$ reflections. It also clarifies the question of how different fermion-antifermion structures (possible massive terms of the model Lagrangian) are transformed under the influence of $\mathcal{PT}$. In Sec. II B the CJT effective action $\Gamma(S)$ of the composite bilocal and bifermion operator $\bar\psi(x)\psi(y)$ is constructed, which is actually the functional of the full fermionic propagator $S(x,y)$. In real situations, the propagator is a translation invariant solution of the stationary Schwinger-Dyson-type equation of the CJT effective action. In this section, the unrenormalized expression for $\Gamma(S)$ is obtained up to a first order in the bare coupling constant G. Based on this expression, we show in Sec. III that for a some well-defined behavior of the coupling constant $G(\Lambda)$ vs Λ, there exist three different

[1]Its Lagrangian is presented below in Eq. (1).

renormalized, i.e., without ultraviolet divergences, solutions of the Schwinger-Dyson equation for the propagator. One of them corresponds to a phase in which a dynamically Hermitian mass term arises for fermions. The other two solutions correspond to two different phases with dynamically emerging non-Hermitian mass terms. In each of these cases, the non-Hermiticity appears spontaneously in the originally Hermitian model, and it is accompanied by a real spectrum of fermions. Finally, in Sec. IV we show that the spontaneous non-Hermiticity of the model arises only in the chiral limit, i.e., if initially the Lagrangian of the model contains a (Hermitian) nonzero mass term, then non-Hermiticity does not arise.

II. (2 + 1)-DIMENSIONAL GN MODEL AND ITS CJT EFFECTIVE ACTION

A. Model, its symmetries, etc.

We investigate the spontaneous (dynamical) generation of non-Hermitian mass terms in the simplest massless $(2+1)$-dimensional GN model. Its Lagrangian has the following form

$$L = \bar{\psi}_k \gamma^\nu i\partial_\nu \psi_k + \frac{G}{2N}(\bar{\psi}_k \psi_k)^2, \tag{1}$$

where for each $k = 1, \ldots, N$ the field $\psi_k \equiv \psi_k(t,x,y)$ is a (reducible) four-component Dirac spinor [its spinor indices are omitted in Eq. (1)], γ^ν $(\nu = 0,1,2)$ are 4×4 matrices acting in this four-dimensional spinor space (the algebra of these γ-matrices and their particular representation used in the present paper is given in the Appendix, where the matrices γ^3, γ^5 and $\tau = -i\gamma^3\gamma^5$ are also introduced), and the summation over repeated k- and ν-indices is assumed in Eq. (1) and below. The bare coupling constant G has a dimension of $[\text{mass}]^{-1}$. The Lagrangian is invariant under two discrete chiral transformations Γ^5 and Γ^3,

$$\begin{aligned} \Gamma^5:\ & \psi_k(t,x,y) \to \gamma^5\psi_k(t,x,y); \\ & \bar{\psi}_k(t,x,y) \to -\bar{\psi}_k(t,x,y)\gamma^5, \\ \Gamma^3:\ & \psi_k(t,x,y) \to \gamma^3\psi_k(t,x,y); \\ & \bar{\psi}_k(t,x,y) \to -\bar{\psi}_k(t,x,y)\gamma^3. \end{aligned} \tag{2}$$

Moreover, it is symmetric with respect to space parity $\mathcal{P}$, time reversal $\mathcal{T}$ and $\mathcal{PT}$ symmetries, which we now discuss in more detail within the framework of model (1).

In $(2+1)$ dimensions the space reflection, or parity, transformation $\mathcal{P}$ is defined by $(t,x,y)\xrightarrow{\mathcal{P}}(t,-x,y)$.[2] Moreover, we assume that an evident relation $\mathcal{PP} = \mathbf{1}$ is valid. Let us derive the transformation of the spinor fields ψ under $\mathcal{P}$. To find this transformation, we postulate that the Lagrangian L_0 of the free massless spinor fields ψ remains intact under space reflection $\mathcal{P}$, i.e., L_0 equals to $\mathcal{P}L_0\mathcal{P}$, where (below, for the sake of brevity we denote by x and x' the set of coordinates (t,x,y) and $(t,-x,y)$, respectively)

$$\begin{aligned} & L_0 \equiv \bar{\psi}(x)\mathcal{D}\psi(x), \qquad \mathcal{P}L_0\mathcal{P} = \overline{\psi^{\mathcal{P}}}(x')\mathcal{D}'\psi^{\mathcal{P}}(x'), \\ & \mathcal{D} = i\gamma^0\partial_0 + i\gamma^1\partial_1 + i\gamma^2\partial_2, \\ & \mathcal{D}' = \mathcal{PDP} = i\gamma^0\partial_0 - i\gamma^1\partial_1 + i\gamma^2\partial_2, \\ & \psi^{\mathcal{P}}(x') = \mathcal{P}\psi(x)\mathcal{P}, \qquad \overline{\psi^{\mathcal{P}}}(x') = \mathcal{P}\bar{\psi}(x)\mathcal{P}. \end{aligned} \tag{3}$$

It is not very difficult to notice from Eq. (3) that L_0 is invariant under the action of $\mathcal{P}$ only when

$$\begin{aligned} & \psi^{\mathcal{P}}(x') \equiv \psi^{\mathcal{P}}(t,-x,y) = \gamma^5\gamma^1\psi(t,x,y); \quad \text{i.e.} \\ & \overline{\psi^{\mathcal{P}}}(x') \equiv \overline{\psi^{\mathcal{P}}}(t,-x,y) = \bar{\psi}(t,x,y)\gamma^5\gamma^1, \end{aligned} \tag{4}$$

So the parity $\mathcal{P}$ transformation of the fermion fields can be defined by Eq. (4). Now, it is easy to show that the bifermion structure $\bar{\psi}(x)\psi(x)$ is $\mathcal{P}$ invariant. Indeed, it is clear from Eqs. (3) and (4) that

$$\begin{aligned} \bar{\psi}(x)\psi(x) & \xrightarrow{\mathcal{P}} \mathcal{P}\bar{\psi}(x)\mathcal{PP}\psi(x)\mathcal{P} = \overline{\psi^{\mathcal{P}}}(x')\psi^{\mathcal{P}}(x') \\ & = \bar{\psi}(x)\gamma^5\gamma^1\gamma^5\gamma^1\psi(x) = \bar{\psi}(x)\psi(x). \end{aligned} \tag{5}$$

As a result, we conclude that the GN model (1) is $\mathcal{P}$ invariant. In a similar way, one can find the $\mathcal{P}$ transformations of some other Hermitian bispinor forms such as

$$\begin{aligned} \bar{\psi}(x)i\gamma^5\psi(x) & \xrightarrow{\mathcal{P}} -\bar{\psi}(x)i\gamma^5\psi(x), \\ \bar{\psi}(x)i\gamma^3\gamma^5\psi(x) & \xrightarrow{\mathcal{P}} -\bar{\psi}(x)i\gamma^3\gamma^5\psi(x), \\ \bar{\psi}(x)i\gamma^3\psi(x) & \xrightarrow{\mathcal{P}} \bar{\psi}(x)i\gamma^3\psi(x). \end{aligned} \tag{6}$$

Now, let us consider the time reversal $\mathcal{T}$ in the framework of the $(2+1)$-D GN model (1). In the $(2+1)$-dimensional spacetime it is defined as $(t,x,y)\xrightarrow{\mathcal{T}}(-t,x,y)$, i.e., we suppose that $\mathcal{TT} = \mathbf{1}$. To determine how the spinor fields ψ are transformed under this operation in $(2+1)$-D spacetime, we also assume from the very beginning (as in the case of spatial reflection $\mathcal{P}$) that the Lagrangian L_0 (see in Eq. (3)) of free massless fermionic fields ψ remains invariant with respect to $\mathcal{T}$, i.e., $L_0 = \mathcal{T}L_0\mathcal{T}$, where (now, for the sake of brevity we denote below by x and x' the set of coordinates (t,x,y) and $(-t,x,y)$, respectively)

$$\begin{aligned} & \mathcal{T}L_0\mathcal{T} = \overline{\psi^{\mathcal{T}}}(x')\mathcal{D}'\psi^{\mathcal{T}}(x'), \qquad \psi^{\mathcal{T}}(x') = \mathcal{T}\psi(x)\mathcal{T}, \\ & \overline{\psi^{\mathcal{T}}}(x') = \mathcal{T}\bar{\psi}(x)\mathcal{T}, \end{aligned} \tag{7}$$

[2] In $(2+1)$ spacetime dimensions, parity corresponds to inverting only one spatial axis [11,29], since the inversion of both axes is equivalent to rotating the entire space by π.

and in this formula $\mathcal{D}' = \mathcal{T}\mathcal{D}\mathcal{T}$. In the following, it is very important to take into account that time-reversal operation $\mathcal{T}$ (i) changes the sign of the time coordinate, $t \to -t$, and (ii) that it is an antilinear or antiunitary one, which means that its action on any complex number or matrix C transforms it into the complex conjugate C^*, i.e., $\mathcal{T}C\mathcal{T} = C^*$ (for details, see, e.g., in Refs. [3,35,36]). Taking into account these (i) and (ii) properties of the $\mathcal{T}$ transformation, we have

$$\mathcal{D}' = i\gamma^{0*}\partial_0 - i\gamma^{1*}\partial_1 - i\gamma^{2*}\partial_2 = i\gamma^0\partial_0 + i\gamma^1\partial_1 - i\gamma^2\partial_2. \quad (8)$$

In the last equality of Eq. (8) we used the relations $\gamma^{0*} = \gamma^0$, $\gamma^{1*} = -\gamma^1$ and $\gamma^{2*} = \gamma^2$ (see in Appendix). Now, it is rather evident from Eqs. (7) and (8) that L_0 is invariant under the action of $\mathcal{T}$ only when

$$\psi^{\mathcal{T}}(x') \equiv \psi^{\mathcal{T}}(-t, x, y) = \gamma^5\gamma^2\psi(t, x, y); \quad \text{i.e.}$$
$$\overline{\psi^{\mathcal{T}}}(x') \equiv \overline{\psi^{\mathcal{T}}}(-t, x, y) = \bar\psi(t, x, y)\gamma^5\gamma^2. \quad (9)$$

And just the relations (9) can be considered as a time reversal $\mathcal{T}$ transformation of the spinor fields ψ. Now, taking into account Eqs. (9), it is possible to obtain the $\mathcal{T}$ transformations of some Hermitian bispinor forms. In particular,

$$\begin{aligned}
\bar\psi(x)\psi(x) &\xrightarrow{\mathcal{T}} \mathcal{T}\bar\psi(x)\mathcal{T}\mathcal{T}\psi(x)\mathcal{T} = \overline{\psi^{\mathcal{T}}}(x')\psi^{\mathcal{T}}(x') \\
&= \bar\psi(x)\gamma^5\gamma^2\gamma^5\gamma^2\psi(x) = \bar\psi(x)\psi(x), \\
\bar\psi(x)i\gamma^3\psi(x) &\xrightarrow{\mathcal{T}} \mathcal{T}\bar\psi(x)\mathcal{T}\mathcal{T}i\gamma^3\mathcal{T}\mathcal{T}\psi(x)\mathcal{T} \\
&= \overline{\psi^{\mathcal{T}}}(x')(-i)\gamma^3\psi^{\mathcal{T}}(x') \\
&= \bar\psi(x)\gamma^5\gamma^2(-i)\gamma^3\gamma^5\gamma^2\psi(x) = -\bar\psi(x)i\gamma^3\psi(x),
\end{aligned} \quad (10)$$

where we have used the antiunitary property of the $\mathcal{T}$ transformation, $\mathcal{T}i\mathcal{T} = -i$. In a similar way it is possible to find the $\mathcal{T}$ transformations of some other Hermitian bispinor forms,

$$\begin{aligned}
\bar\psi(x)i\gamma^5\psi(x) &\xrightarrow{\mathcal{T}} -\bar\psi(x)i\gamma^5\psi(x), \\
\bar\psi(x)i\gamma^3\gamma^5\psi(x) &\xrightarrow{\mathcal{T}} -\bar\psi(x)i\gamma^3\gamma^5\psi(x).
\end{aligned} \quad (11)$$

As it follows from Eqs. (5), (6), (10), and (11), the Hermitian bispinor structures $\bar\psi(x)\psi(x)$, $\bar\psi(x)i\gamma^5\psi(x)$, and $\bar\psi(x)i\gamma^3\gamma^5\psi(x)$ are $\mathcal{P}\mathcal{T}$ even (invariant), whereas the Hermitian $\bar\psi(x)i\gamma^3\psi(x)$ form is a $\mathcal{P}\mathcal{T}$ odd, i.e., it changes the sign under the action of the $\mathcal{P}\mathcal{T}$ transformation, and due to this reason is called sometimes anti-$\mathcal{P}\mathcal{T}$-symmetric [6].

From the point of view of further consideration, we are also interested in the behavior of such non-Hermitian structures as $\bar\psi(x)\gamma^5\psi(x)$ and $\bar\psi(x)\gamma^3\psi(x)$ with respect to $\mathcal{P}$ and $\mathcal{T}$ transformations. Obviously, under the action of $\mathcal{P}$ they transform in the same way as the corresponding Hermitian forms in Eq. (6). However, when reversing time, we have (below, we use evident relations $\mathcal{T}\gamma^5\mathcal{T} = \gamma^{5*} = -\gamma^5$ and $\mathcal{T}\gamma^3\mathcal{T} = \gamma^3$)

$$\begin{aligned}
\bar\psi(x)\gamma^5\psi(x) &\xrightarrow{\mathcal{T}} \bar\psi(x)\gamma^5\psi(x), \\
\bar\psi(x)\gamma^3\psi(x) &\xrightarrow{\mathcal{T}} \bar\psi(x)\gamma^3\psi(x).
\end{aligned} \quad (12)$$

Hence, $\bar\psi(x)\gamma^5\psi(x)$ is a $\mathcal{P}\mathcal{T}$-odd structure, but $\bar\psi(x)\gamma^3\psi(x)$ is a $\mathcal{P}\mathcal{T}$-even one.

Below, we are going to study, using the CJT composite operator technique, the possibility for dynamical generation of the following mass terms in the model (1)

$$\begin{aligned}
\mathcal{M}_H &= im_5\bar\psi(x)\gamma^5\psi(x) + im_3\bar\psi(x)\gamma^3\psi(x), \\
\mathcal{M}_{NH1} &= im_5\bar\psi(x)\gamma^5\psi(x) + m_3\bar\psi(x)\gamma^3\psi(x), \\
\mathcal{M}_{NH2} &= m_5\bar\psi(x)\gamma^5\psi(x) + im_3\bar\psi(x)\gamma^3\psi(x),
\end{aligned} \quad (13)$$

(note that in this formula and below it is assumed that m_3 and m_5 are real quantities). The mass term $\mathcal{M}_H$ is Hermitian and its dynamical appearance corresponds to spontaneous breaking of some of the above mentioned discrete symmetries of the model. Each of the remaining two mass terms in Eq. (13) is non-Hermitian; therefore, their dynamic appearance corresponds to the spontaneous non-Hermitian nature of the model (in addition to the spontaneous breaking of some of its discrete symmetries). Moreover, the mass term $\mathcal{M}_{NH1}$ is $\mathcal{P}\mathcal{T}$ even (symmetric), but the non-Hermitian mass term $\mathcal{M}_{NH2}$ is $\mathcal{P}\mathcal{T}$ odd, i.e., it changes the sign under this transformation.

B. CJT effective action of the model

Before starting to consider the question of the spontaneous emergence of the non-Hermiticity of the model, it is necessary to add few words about the method for solving the problem, i.e., about the CJT composite operator approach.

Let us define $Z(K)$, the generating functional of the Green's functions of bilocal fermion-antifermion composite operators $\sum_{k=1}^N \bar\psi_k^\alpha(x)\psi_{k\beta}(y)$ in the framework of a $(2+1)$-D GN model (1) (the corresponding technique for theories with four-fermion interaction is elaborated in details, e.g., in Ref. [28])

$$\begin{aligned}
Z(K) \equiv \exp(iNW(K)) \int \mathcal{D}\bar\psi_k\mathcal{D}\psi_k \exp\Bigg(i\Bigg[I(\bar\psi,\psi) \\
+ \int d^3x d^3y \bar\psi_k^\alpha(x)K_\alpha^\beta(x,y)\psi_{k\beta}(y)\Bigg]\Bigg),
\end{aligned} \quad (14)$$

where α, $\beta = 1, 2, 3, 4$ are spinor indices, $K_\alpha^\beta(x,y)$ is a bilocal source of the fermion bilinear composite field $\bar\psi_k^\alpha(x)\psi_{k\beta}(y)$ (recall that in all expressions the summation over repeated indices is assumed).[3] Moreover, $I(\bar{\psi},\psi)=\int L d^3x$, where L is the Lagrangian (1) of a (2 + 1)-dimensional GN model under consideration. It is evident that

$$
\begin{aligned}
I(\bar{\psi},\psi) &= \int d^3x d^3y \bar{\psi}_k^\alpha(x) D_\alpha^\beta(x,y)\psi_{k\beta}(y) + I_{\rm int}(\bar{\psi}_k^\alpha \psi_{k\beta}),\\
D_\alpha^\beta(x,y) &= (\gamma^\nu)_\alpha^\beta i\partial_\nu \delta^3(x-y),\\
I_{\rm int} &= \frac{G}{2N}\int d^3x (\bar{\psi}_k\psi_k)^2\\
&= \frac{G}{2N}\int d^3x d^3t d^3u d^3v \delta^3(x-t)\delta^3(t-u)\\
&\quad\times \delta^3(u-v)\bar{\psi}_k^\alpha(x)\delta_\alpha^\beta \psi_{k\beta}(t)\bar{\psi}_l^\rho(u)\delta_\rho^\xi \psi_{l\xi}(v).
\end{aligned}
\tag{15}
$$

Note that in Eq. (15) and similar expressions below, $\delta^3(x-y)$ denotes the three-dimensional Dirac delta function. There is an alternative expression for $Z(K)=\exp(iNW(K))$,

$$
\begin{aligned}
\exp(iNW(K)) &= \exp\left(iI_{\rm int}\left(-i\frac{\delta}{\delta K}\right)\right)\int \mathcal{D}\bar{\psi}_k \mathcal{D}\psi_k \exp\left(i\int d^3x d^3y \bar{\psi}_k(x)[D(x,y)+K(x,y)]\psi_k(y)\right)\\
&= \exp\left(iI_{\rm int}\left(-i\frac{\delta}{\delta K}\right)\right)[\det(D(x,y)+K(x,y))]^N\\
&= \exp\left(iI_{\rm int}\left(-i\frac{\delta}{\delta K}\right)\right)\exp[N{\rm Tr}\ \ln(D(x,y)+K(x,y))],
\end{aligned}
\tag{16}
$$

where instead of each bilinear form $\bar{\psi}_k^\alpha(s)\psi_{k\beta}(t)$ appearing in $I_{\rm int}$ of the Eq. (15) we use a variational derivative $-i\delta/\delta K_\alpha^\beta(s,t)$. Moreover, the Tr-operation in Eq. (16) means the trace both over spacetime and spinor coordinates. The effective action (or CJT effective action) of the composite bilocal and bispinor operator $\bar{\psi}_k^\alpha(x)\psi_{k\beta}(y)$ is defined as a functional $\Gamma(S)$ of the full fermion propagator $S_\beta^\alpha(x,y)$ by a Legendre transformation of the functional $W(K)$ entering in Eqs. (14) and (16),

$$
\Gamma(S)=W(K)-\int d^3x d^3y S_\beta^\alpha(x,y)K_\alpha^\beta(y,x), \tag{17}
$$

where

$$
S_\beta^\alpha(x,y)=\frac{\delta W(K)}{\delta K_\alpha^\beta(y,x)}. \tag{18}
$$

Taking into account the relation (14), it is clear that $S(x,y)$ is the full fermion propagator at $K(x,y)=0$. Hence, in order to construct the CJT effective action $\Gamma(S)$ of Eq. (17), it is necessary to solve Eq. (18) with respect to K and then to use the obtained expression for K (it is a functional of S) in Eq. (17). It is clear from the definition (17)–(18) that

$$
\begin{aligned}
\frac{\delta\Gamma(S)}{\delta S_\beta^\alpha(x,y)} &= \int d^3u d^3v \frac{\delta W(K)}{\delta K_\nu^\mu(u,v)}\frac{\delta K_\nu^\mu(u,v)}{\delta S_\beta^\alpha(x,y)}\\
&\quad - K_\alpha^\beta(y,x) - \int d^3u d^3v S_\mu^\nu(v,u)\frac{\delta K_\nu^\mu(u,v)}{\delta S_\beta^\alpha(x,y)}.
\end{aligned}
\tag{19}
$$

(In Eq. (19) and below, the Greek letters α,β,μ,ν, etc., also denote the spinor indices, i.e., $\alpha,\ldots\nu,\ldots=1,\ldots,4$.) Now, due to the relation (18), it is easy to see that the first term in Eq. (19) cancels there the last term, so

$$
\frac{\delta\Gamma(S)}{\delta S_\beta^\alpha(x,y)}=-K_\alpha^\beta(y,x). \tag{20}
$$

Hence, in the true GN theory, in which bilocal sources $K_\alpha^\beta(y,x)$ are zero, the full fermion propagator is a solution of the following stationary equation,

$$
\frac{\delta\Gamma(S)}{\delta S_\beta^\alpha(x,y)}=0. \tag{21}
$$

Note that in the nonperturbative CJT approach the stationary/gap equation (21) for fermion propagator $S_\alpha^\beta(x,y)$ is indeed a Schwinger–Dyson equation [28]. Further, in order to simplify the calculations and obtain specific information about the phase structure of the model, we calculate the

[3]We denote a matrix element of an arbitrary matrix (operator) $\hat{A}$ acting in the four dimensional spinor space by the symbol A_β^α, where the upper (low) index $\alpha(\beta)$ is the column (row) number of the matrix $\hat{A}$. In particular, the matrix elements of any γ^μ matrix is denoted by $(\gamma^\mu)_\beta^\alpha$.

effective action (17) up to a first order in the coupling G. In this case (the detailed calculations are given in Appendix B of Ref. [34])

$$\Gamma(S) = -i\mathrm{Tr}\,\ln(-iS^{-1}) + \int d^3x d^3y S^\alpha_\beta(x,y) D^\beta_\alpha(y,x) + \frac{G}{2}\int d^3x [\mathrm{tr}S(x,x)]^2 - \frac{G}{2N}\int d^3x \mathrm{tr}[S(x,x)S(x,x)]. \tag{22}$$

Notice that in Eq. (22) the symbol tr means the trace of an operator over spinor indices only, but Tr is the trace operation both over spacetime coordinates and spinor indices. Moreover, there the operator $D(x,y)$ is introduced in Eq. (15). The stationary equation (21) for the CJT effective action (22) looks like

$$0 = i[S^{-1}]^\beta_\alpha(x,y) + D^\beta_\alpha(x,y) + G\delta^\beta_\alpha \delta^3(x-y)\mathrm{tr}S(x,y) - \frac{G}{N} S^\beta_\alpha(x,y)\delta^3(x-y). \tag{23}$$

Now suppose that $S(x,y)$ is a translationary invariant operator. Then

$$S^\beta_\alpha(x,y) \equiv S^\beta_\alpha(z) = \int \frac{d^3p}{(2\pi)^3} \overline{S^\beta_\alpha}(p) e^{-ipz},$$
$$\overline{S^\beta_\alpha}(p) = \int d^3z S^\beta_\alpha(z) e^{ipz},$$
$$(S^{-1})^\beta_\alpha(x,y) \equiv (S^{-1})^\beta_\alpha(z) = \int \frac{d^3p}{(2\pi)^3} \overline{(S^{-1})^\beta_\alpha}(p) e^{-ipz}, \tag{24}$$

where $z = x - y$ and $\overline{S^\beta_\alpha}(p)$ is a Fourier transformation of $S^\beta_\alpha(z)$. After Fourier transformation, the Eq. (23) takes the form

$$\overline{(S^{-1})^\beta_\alpha}(p) - ip_\nu(\gamma^\nu)^\beta_\alpha = iG\delta^\beta_\alpha \int \frac{d^3q}{(2\pi)^3} \mathrm{tr}\bar{S}(q) - i\frac{G}{N}\int \frac{d^3q}{(2\pi)^3} \overline{S^\beta_\alpha}(q). \tag{25}$$

It is clear from Eq. (25) that in the framework of the four-fermion model (1) the Schwinger-Dyson equation for fermion propagator $\bar{S}(p)$ reads in the first order in G like the Hartree-Fock equation for its self-energy operator $\Sigma(p)$. In particular, the first and second terms on the right-hand side of Eq. (25) are, respectively, the so-called Hartree and Fock contributions to the fermion self energy (for details, see, e.g., the section 4.3.1 in Ref. [37]).

Finally note that both the CJT effective action (22) and its stationary equation (23)–(25), in which G is a bare coupling constant, contain ultraviolet (UV) divergences and need to be renormalized. In the next sections, we will find out at what behavior of the bare coupling constant $G \equiv G(\Lambda)$ vs Λ it is possible to renormalize the stationarity equation (25), the finite solution of which corresponds at $\Lambda \to \infty$ to the dynamical appearance of the mass terms of the form (13) in the Lagrangian.

III. POSSIBILITY FOR DYNAMICAL GENERATION OF THE MASS TERMS (13)

A. Dynamical generation of the Hermitian mass $\mathcal{M}_H$

Let us explore the possibility that the solution of the gap equation (25) has the form

$$\overline{S^{-1}}(p) = i(\hat{p} + i\gamma^5 m_5 + i\gamma^3 m_3), \quad \text{i.e.}$$
$$\bar{S}(p) = -i\frac{\hat{p} + i\gamma^5 m_5 + i\gamma^3 m_3}{p^2 - (m_3^2 + m_5^2)}. \tag{26}$$

It corresponds to a dynamically generated mass term of the form $\mathcal{M}_H = (m_5\bar{\psi}i\gamma^5\psi + m_3\bar{\psi}i\gamma^3\psi)$ in the Lagrangian (1) (the Hermitian matrices $\gamma^{3,5}$ are presented in Appendix). Since we suppose that m_5 and m_3 are some real numbers, this mass term is a Hermitian one. And it is not invariant under each of the discrete transformations (2) or (3) (at nonzero m_3 and m_5). Substituting Eq. (26) into Eq. (25), one can obtain for m_3 and m_5 the following system of gap equations

$$m_3 = \frac{im_3G}{N}\int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 - (m_3^2 + m_5^2)},$$
$$m_5 = \frac{im_5G}{N}\int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 - (m_3^2 + m_5^2)}. \tag{27}$$

After a Wick rotation in Eq. (27) to Euclidean energy-momentum, i.e., $p_0 \to ip_0$, we see that (m_3, m_5) should obey the equation system (in which $p^2 = p_0^2 + p_1^2 + p_2^2$)

$$\frac{m_3}{G} = \frac{m_3}{N}\int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + (m_3^2 + m_5^2)},$$
$$\frac{m_5}{G} = \frac{m_5}{N}\int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + (m_3^2 + m_5^2)}. \tag{28}$$

This system contains UV-divergent integrals, i.e., it is unrenormalized. For its regularization, we use the spherical coordinate system when $\int d^3p f(\sqrt{p_0^2 + p_1^2 + p_2^2}) = 4\pi\int_0^\infty p^2 dp f(p)$ and $p = \sqrt{p_0^2 + p_1^2 + p_2^2}$. Then, cutting the region of integration in the obtained one-dimensional UV-divergent integral by Λ, we have for (m_3, m_5) the *regularized* gap equations

$$\frac{m_3}{G} = \frac{m_3}{2N\pi^2}\int_0^\Lambda \frac{p^2}{p^2+m_3^2+m_5^2}dp,$$
$$\frac{m_5}{G} = \frac{m_5}{2N\pi^2}\int_0^\Lambda \frac{p^2}{p^2+m_3^2+m_5^2}dp. \qquad (29)$$

Notice that at $\Lambda \to \infty$ an integral term in Eqs. (29) has the following asymptotic expansion

$$\int_0^\Lambda \frac{p^2}{p^2+m_3^2+m_5^2}dp = \Lambda - \frac{\pi}{2}\sqrt{m_3^2+m_5^2} + \sqrt{m_3^2+m_5^2}\,\mathcal{O}\left(\frac{\sqrt{m_3^2+m_5^2}}{\Lambda}\right). \qquad (30)$$

Hence, taking into account the expansion (30), the UV divergence can be removed from the gap equations (29) if we require (it is clear from the form of this equation system) the following behavior of the bare coupling constant $G \equiv G(\Lambda)$ vs Λ,

$$\frac{1}{G(\Lambda)} = \frac{1}{2N\pi^2}\left(\Lambda + g\frac{\pi}{2} + g\mathcal{O}\left(\frac{g}{\Lambda}\right)\right), \qquad (31)$$

where g is a finite Λ-independent and renormalization group invariant quantity, and it can also be considered as a new free parameter of the model. Now, comparing Eqs. (31) and (29), we obtain in the limit $\Lambda \to \infty$ the following renormalized, i.e., without UV divergences, gap equations for the masses m_3 and m_5

$$m_3\left(g + \sqrt{m_3^2+m_5^2}\right) = 0,$$
$$m_5\left(g + \sqrt{m_3^2+m_5^2}\right) = 0. \qquad (32)$$

Hence, at $g > 0$ only a trivial solution of the gap equations (32) exists, $m_3 = m_5 = 0$, and all discrete symmetries of the model remain intact. However, at $g < 0$ there are two solutions, (i) $m_3 = 0$, $m_5 = 0$ and (ii) $m_3 = |g|\cos\alpha$, $m_5 = |g|\sin\alpha$ (where $0 \le \alpha \le \pi/2$ is some arbitrary fixed angle), of the system (32) of gap equations. To find which of the solutions of the gap equations, (i) or (ii), is more preferable in this case, it is necessary to consider the so-called CJT effective potential $V(S)$ of the model, which is defined on the basis of the CJT effective action (22) by the following relation [24,26]

$$V(S)\int d^3x \equiv -\Gamma(S)|_{\text{transl.-inv. } S(x,y)}, \qquad (33)$$

where $S(x,y)$ is a translation invariant quantity, i.e., $S(x,y) \equiv S(x-y)$, as assumed in Eq. (24). It is evident that for arbitrary values of the bare coupling constant G the CJT effective potential (33) is UV-divergent unrenormalized quantity. However, if G is constrained by the condition (31), then the UV divergences of $V(S)$ are eliminated, and for fermion propagator of the form (26) it looks at $\Lambda \to \infty$ like (up to unessential m_3- and m_5-independent infinite constant)

$$V(S) \equiv V_H(m_3, m_5) = \frac{1}{6\pi}\left(2(m_3^2+m_5^2)^{3/2} + 3g(m_3^2+m_5^2)\right). \qquad (34)$$

It is clear from Eq. (34) that at $g < 0$ the effective potential V_H takes on the solution (ii) the value $-|g|^3/(6\pi)$, and this quantity is smaller than $V_H(m_3 = 0, m_5 = 0) = 0$. This allows us to conclude that if in the original model (1) the bare coupling constant G behaves vs Λ like in the expression (31) with $g < 0$, then the system undergoes dynamic generation of the $m_3 = |g|\cos\alpha$ and $m_5 = |g|\sin\alpha$ masses, i.e., a phase with spontaneous violation of all discrete symmetries is realized in the model (if $\alpha \ne 0, \pi/2$). But if $\alpha = 0$ then only $\mathcal{PT}$-odd $m_3 = |g|$ mass term is generated, and Γ^3 chiral symmetry (2) is dynamically violated. However, at $\alpha = \pi/2$ only $\mathcal{PT}$-even $m_5 = |g|$ mass term appears dynamically, and in this case chiral Γ^5 symmetry is broken spontaneously. Finally notice that at $g < 0$ in all above mentioned cases, i.e., at arbitrary values of the angle parameter α, the genuine physical fermion mass, which is indeed a pole of the fermion propagator (26), is equal to $M_F = \sqrt{m_3^2+m_5^2} \equiv |g|$.

In terms of dimensionless bare coupling constant $\lambda \equiv \lambda(\Lambda) = \Lambda G(\Lambda)$, where $G(\Lambda)$ is given by Eq. (31), the situation looks as follows. It is clear that in this case for a sufficiently high values of $\Lambda \gg |g|$ both the dimensional bare coupling $G(\Lambda)$ and the dimensionless coupling λ are positive. In addition, it is easy to see that at $\Lambda \to \infty$ the dimensionless bare coupling λ tends to the quantity $\lambda_{\text{crit}} \equiv 2N\pi^2$, which in fact is the UV-stable fixed point of the model (see in Ref. [34]). Then the relation

$$\lambda - \lambda_{\text{crit}} \sim -\frac{2\pi^2 Ng}{\Lambda} \qquad (35)$$

can be obtained. It follows from Eq. (35) that on the positive λ-semiaxis the UV-fixed point λ_{crit} separates the symmetric phase from the one in which fermions are massive. Indeed, if $\lambda > \lambda_{\text{crit}}$ then, as it is clear from Eq. (35), the parameter g must be negative. In this case dynamical generation of the Hermitian mass term $\mathcal{M}_H = (m_5\bar{\psi}i\gamma^5\psi + m_3\bar{\psi}i\gamma^3\psi)$ occurs in the Lagrangian (which indeed corresponds to a physical mass M_F of fermion quasiparticles equal to $|g|$). In contrast, at $\lambda < \lambda_{\text{crit}}$ we have $g > 0$ from Eq. (35) and symmetric phase of the model [see the text below Eq. (32)].

B. Dynamical generation of the non-Hermitian mass terms $\mathcal{M}_{NH1}$ and $\mathcal{M}_{NH2}$

Generation of the $\mathcal{M}_{NH1}$ mass term. First, let us explore, using the CJT approach, the possibility of the dynamic appearance of a non-Hermitian and $\mathcal{PT}$ symmetric mass term $\mathcal{M}_{NH1}$ (13) in the original $(2+1)$-D GN model (1). In this case we should find the solution of the gap equation (25) in the form

$$\overline{S^{-1}}(p) = i(\hat{p} + i\gamma^5 m_5 + \gamma^3 m_3), \quad \text{i.e.} \quad \bar{S}(p) = -i\frac{\hat{p} + i\gamma^5 m_5 + \gamma^3 m_3}{p^2 - (m_5^2 - m_3^2)}, \tag{36}$$

where m_3 and m_5 are real quantities. In addition, we suppose that $m_5^2 \geq m_3^2$. Substituting Eq. (36) into the CJT stationary equation (25), one can obtain for m_3 and m_5 the UV-divergent system of gap equations. In it, one can also perform a Wick rotation and then to integrate over spherical angles. Finally, after cutting off the region of integration by Λ, we obtain for m_3 and m_5 a *regularized* system, which looks like

$$\frac{m_3}{G} = \frac{m_3}{2N\pi^2}\int_0^\Lambda \frac{p^2}{p^2 + m_5^2 - m_3^2}dp, \quad \frac{m_5}{G} = \frac{m_5}{2N\pi^2}\int_0^\Lambda \frac{p^2}{p^2 + m_5^2 - m_3^2}dp. \tag{37}$$

Since in our consideration it is assumed that $m_5^2 \geq m_3^2$, one can apply in Eq. (37) the asymptotic expansion (30) and find that the bare coupling constant G, which behaves vs Λ like in Eq. (31), removes at $\Lambda \to \infty$ all UV-divergences from the gap system (37). And its finite, i.e., *renormalized*, form reads

$$m_3\left(g + \sqrt{m_5^2 - m_3^2}\right) = 0, \quad m_5\left(g + \sqrt{m_5^2 - m_3^2}\right) = 0. \tag{38}$$

There are several solutions of the system (38). To find the more preferable from the energy point of view, we again use the CJT effective potential (33), which, after substituting the expressions (36) and (31) into Eq. (22), takes the form (recall that in our case $m_5^2 - m_3^2 \geq 0$)

$$V(S) \equiv V_{NH1}(m_3, m_5) = \frac{1}{6\pi}\left(2(m_5^2 - m_3^2)^{3/2} + 3g(m_5^2 - m_3^2)\right). \tag{39}$$

Hence, at $g > 0$ its global minimum lies at the point $m_5 = m_3 = 0$, and dynamical mass generation is absent. But at $g < 0$ the global minimum of the $V_{NH1}(m_3, m_5)$ is $-|g|^3/(6\pi)$, and it is achieved at arbitrary (m_3, m_5) point such that $m_5^2 - m_3^2 = g^2$, i.e., when $m_3 = |g|\sinh\beta$ and $m_5 = |g|\cosh\beta$, where $\beta \in \mathbb{R}$. Note that such a structure of the global minimum point of the model appears due to the emergent symmetry of the CJT effective potential (39) with respect to nonunitary transformations

$$\begin{pmatrix} m_5 \\ m_3 \end{pmatrix} \to \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}\begin{pmatrix} m_5 \\ m_3 \end{pmatrix}. \tag{40}$$

The energies of all these ground states at which a non-Hermitian and $\mathcal{PT}$-symmetric fermion mass term $\mathcal{M}_{NH1}$ (13) arises spontaneously in the system are equal to each other and, moreover, coincide with the energy of any vacuum state corresponding to the dynamic appearance of a Hermitian mass term $\mathcal{M}_H$ of fermions in the system (see in the previous subsection). The singularity of the fermion propagator (36) corresponds to the fact that the quasiparticle excitations of each of these non-Hermitian and $\mathcal{PT}$-even ground states of the system have a real mass spectrum, i.e., their masses are real and also equal to the same value $M_F = \sqrt{m_5^2 - m_3^2} \equiv |g|$, which is observed in the case with Hermitian vacuum (see in the Sec. III A).

Generation of the $\mathcal{M}_{NH2}$ mass term. Omitting unnecessary details, it can be shown in exactly the same way that for the same dependence (31) of the bare coupling constant G vs Λ, there exists a nontrivial solution of the renormalized stationary (Dyson-Schwinger) equation (25) of the form

$$\overline{S^{-1}}(p) = i(\hat{p} + \gamma^5 m_5 + i\gamma^3 m_3), \quad \text{i.e.} \quad \bar{S}(p) = -i\frac{\hat{p} + \gamma^5 m_5 + i\gamma^3 m_3}{p^2 - (m_3^2 - m_5^2)}, \tag{41}$$

which corresponds to spontaneous generation at $g < 0$ of the non-Hermitian but $\mathcal{PT}$-odd mass term $\mathcal{M}_{NH2}$ (13) in the model. In this case $m_3 = |g|\cosh\omega$ and $m_5 = |g|\sinh\omega$, where $\omega \in \mathbb{R}$. And in this phase fermion propagator (41) describes, for any real value of ω, the quasiparticles with the same real mass $M_F = \sqrt{m_3^2 - m_5^2} \equiv |g|$.

Conclusions. As a result, we see that at $\lambda > \lambda_{\text{crit}} = 2N\pi^2$, where $\lambda \equiv \Lambda G(\Lambda)$ is the dimensionless coupling constant of the model (see in the last paragraph of the previous subsection III A), there might appear, on the same footing, three different phases in the framework of the Hermitian massless $(2+1)$-D GN model (1). One of them is characterized by dynamical appearance of the Hermitian mass term $\mathcal{M}_H$ in the Lagrangian, i.e., the ground state of the system remains Hermitian. However, in each of the remaining two phases, a non-Hermitian mass term, $\mathcal{M}_{NH1}$ or $\mathcal{M}_{NH2}$, is dynamically generated in the Lagrangian of the model. And in this case non-Hermiticity of the model appears spontaneously, which is accompanied by a real mass spectrum of quasiparticle excitations. In the first case, when $\mathcal{M}_{NH1}$ is generated, the non-Hermitian ground state of the model remains $\mathcal{PT}$ symmetrical, but when $\mathcal{M}_{NH2}$ appears dynamically, the $\mathcal{PT}$-invariance of the model is broken spontaneously. Moreover, we note that the ground states of all these three different phases, both Hermitian and non-Hermitian, have the same energy density equal to $-|g|^3/(6\pi)$, i.e., they are degenerate. The similar result was obtained in Ref. [8] within $(3+1)$-D NJL model, where it was shown that in the chiral limit the uniform non-Hermitian ground state has the same (finite) free energy density as the usual Hermitian ground state.

Finally, two remarks. (i) The phenomenon of spontaneous emergence of non-Hermiticity in the massless $(2+1)$-D GN model (1) is characteristic only for finite values of N and cannot be observed in the framework of the large-N expansion technique. Indeed, it might occur only at $\lambda > \lambda_{\rm crit} = 2N\pi^2$. So at $N \to \infty$ it is absent. (ii) In the $(3+1)$-D NJL model, as it is proved in Ref. [8], the spontaneous non-Hermiticity occurs only in the chiral limit, i.e., if bare quark mass is zero. However, if it is a nonzero quantity, then this effect does not appear. In the next section, we show that in the framework of the $(2+1)$-D GN model (1) with nonzero (Hermitian) bare mass of fermions, non-Hermiticity also cannot arise spontaneously.

IV. THE CASE OF NONZERO HERMITIAN BARE m_5 MASS

Let us study the dynamical symmetry breaking in the $(2+1)$-D GN model (1) when its Lagrangian contains, e.g., a nonzero bare Hermitian chiral mass m_5, i.e., the Lagrangian of the model looks like

$$L = \bar\psi_k\gamma^\nu i\partial_\nu\psi_k + im_{05}\bar\psi_k\gamma^5\psi_k + \frac{G}{2N}(\bar\psi_k\psi_k)^2. \tag{42}$$

In this case it is invariant only under discrete chiral Γ^3 and $\mathcal{PT}$ transformations [other discrete symmetries Γ^5, $\mathcal{P}$, and $\mathcal{T}$ of the massless model (1) are violated explicitly, as it is shown in Sec. II A, by the mass term of the Lagrangian (42)]. In the present section, we are going to consider in the framework of the model (42) the possibility of dynamical generation in it of the m_3-mass term of the form $\kappa m_3\bar\psi\gamma^3\psi$, where κ is equal to 1 or i and m_3 is real. If $\kappa = i$, then both the mass term and the model remain Hermitian, however if $\kappa = 1$ then the ground state of the model corresponds to spontaneous emergence of non-Hermiticity in it. Note that in both cases the discrete Γ^3 symmetry (2) of the model is spontaneously broken down.

The consideration is again performed on the basis of the CJT effective action (22) and its stationary equation (23), in which this time $D(x,y) = [\gamma^\nu i\partial_\nu + im_{05}\gamma^5]\delta^3(x-y)$, i.e., $\bar D(p) = \hat p + im_{05}\gamma^5$. Fourier transformation of this gap equation reads as

$$-i\overline{(S^{-1})^\beta_\alpha}(p) = p_\nu(\gamma^\nu)^\beta_\alpha + im_{05}\gamma^5 + G\delta^\beta_\alpha\int\frac{d^3q}{(2\pi)^3}{\rm tr}\bar S(q) - \frac{G}{N}\int\frac{d^3q}{(2\pi)^3}\overline{S^\beta_\alpha}(q). \tag{43}$$

Here we explore the possibility that the solution of this gap equation has the form

$$\overline{S^{-1}}(p) = i(\hat p + i\gamma^5 m_5 + \kappa\gamma^3 m_3), \quad \text{i.e.} \quad \bar S(p) = -i\frac{\hat p + i\gamma^5 m_5 + \kappa\gamma^3 m_3}{p^2 - (m_5^2 - \kappa^2 m_3^2)}. \tag{44}$$

It corresponds to a dynamically generated mass term of the form $\kappa m_3\bar\psi\gamma^3\psi$ in the Lagrangian (42) (the Hermitian matrices $\gamma^{3,5}$ are presented in Appendix). Moreover, we suppose that for each value of $\kappa = 1, i$ the mass parameters m_5 and m_3 both in Eq. (44) and throughout the consideration are some real numbers. Substituting Eq. (44) into Eq. (43) and taking into account the technical details discussed in previous two sections, one can obtain for m_3 and m_5 the following system of gap equations (in which $p^2 = p_0^2 - p_1^2 - p_2^2$)

$$m_3 = \frac{im_3G}{N}\int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2 - (m_5^2 - \kappa^2 m_3^2)},$$
$$m_5 = m_{05} + \frac{im_5G}{N}\int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2 - (m_5^2 - \kappa^2 m_3^2)}. \tag{45}$$

After a Wick rotation in Eq. (45) to Euclidean energy-momentum, i.e., $p_0 \to ip_0$, it is possible to use there a spherical coordinate system. Then we integrate in the obtained expressions over spherical angles and cut off by Λ the resulting one-dimensional integral. As a result, we see that the set (m_3, m_5) should obey the following *regularized* system

$$\frac{m_3}{G} = \frac{m_3}{N}\int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2 + (m_5^2 - \kappa^2 m_3^2)} = \frac{m_3}{2N\pi^2}\int_0^\Lambda p^2dp\frac{1}{p^2 + (m_5^2 - \kappa^2 m_3^2)},$$
$$\frac{m_5}{G} - \frac{m_{05}}{G} = \frac{m_5}{N}\int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2 + (m_5^2 - \kappa^2 m_3^2)} = \frac{m_5}{2N\pi^2}\int_0^\Lambda p^2dp\frac{1}{p^2 + (m_5^2 - \kappa^2 m_3^2)}. \tag{46}$$

Taking into account in these *regularized* equations the asymptotic expansion (30), we see that the system of equations (46) can be *renormalized* if we demand the same behavior (31) of the bare coupling constant G vs Λ, as well as that $\frac{m_{05}}{G} = \pm\frac{m^2}{4\pi N}$, in addition (here $m \neq 0$). Then (m_3, m_5) should obey the following finite gap system

$$m_3(g+\sqrt{m_5^2-\kappa^2 m_3^2})=0,$$
$$m_5(g+\sqrt{m_5^2-\kappa^2 m_3^2})=\pm m^2. \quad (47)$$

Recall that both parameters g and m are some finite and renormalization group invariant quantities with dimension of [mass]. Since $m \neq 0$ (otherwise the bare mass m_{05} would be equal to zero), the expression in parentheses of Eq. (47) is always nonzero. So the solution of Eq. (47) is such that $m_3=0$, and m_5 should obey the equation

$$m_5(g+|m_5|)=\pm m^2. \quad (48)$$

Hence, we conclude that both for $\kappa=i$ and $\kappa=1$ there no exist solutions of the gap equations (47) with $m_3 \neq 0$. It means that in the case when bare mass $m_{05} \neq 0$, neither Hermitian $im_3\bar{\psi}\gamma^3\psi$ nor non-Hermitian $m_3\bar{\psi}\gamma^3\psi$ mass terms can arise dynamically in the model. In a similar way, it can be shown that if some other Hermitian bare mass term (instead of the chiral m_{05} considered in this section) is present in the Lagrangian (1) (for example, the Dirac $\bar{\psi}\psi$, the Haldane $i\bar{\psi}\gamma^3\gamma^5\psi$-mass terms, etc.), then it is this mass term that is modified in the framework of the CJT composite approach. The dynamic emergence of other mass terms, both Hermitian and non-Hermitian, is impossible.

So the spontaneous emergence of non-Hermiticity in the initially Hermitian (2 + 1)-D GN model is allowed only in the chiral limit, i.e., at zero bare masses.

V. SUMMARY AND CONCLUSIONS

In the present paper we have studied the possibility of the dynamical appearance of both Hermitian and non-Hermitian mass terms in the originally Hermitian massless $(2+1)$-dimensional GN model (1). It is invariant with respect to several discrete transformations, two chiral Γ^3 and Γ^5, space reflection (or parity) $\mathcal{P}$ and time reversal $\mathcal{T}$ (see in Sec. II A). As a consequence, it is $\mathcal{PT}$ symmetric. The problem is investigated, using a nonperturbative approach based on the CJT effective action $\Gamma(S)$ (17) for the composite bifermion operator $\bar{\psi}(x)\psi(y)$. In fact, $\Gamma(S)$ is a functional of a full fermion propagator $S(x,y)$ (see in the Sec. II B). In this case, in order to find the true fermion propagator of the original GN model and to determine what kind of fermionic mass terms, Hermitian $\mathcal{M}_H$ or non-Hermitian $\mathcal{M}_{NH1}$ and $\mathcal{M}_{NH2}$ [see in Eq. (13)], can arise dynamically in the model, it is sufficient to solve the stationary (gap or Dyson-Schwinger) equation (21) for the functional $\Gamma(S)$, which we have calculated up to the first order in the coupling constant G [see Eqs. (22) and (25)].

It turns out that due to the behavior (31) of the bare coupling constant $G(\Lambda)$ vs cutoff parameter Λ, the gap equation (25) is renormalized and it has three different finite solutions. The first one (see in Sec. III A) corresponds to a dynamical generation of the Hermitian mass term $\mathcal{M}_H$ in the model Lagrangian. In this case the phase with spontaneous breaking of all above mentioned discrete symmetries (including the $\mathcal{PT}$ one) is realized in the system. The second finite solution of the Dyson-Schwinger equation (25) corresponds to dynamical appearing of the non-Hermitian mass term $\mathcal{M}_{NH1}$ in the model (see in Sec. III B). In this case the phase with spontaneous emergence of non-Hermiticity is induced in the system, but it is still $\mathcal{PT}$ invariant and the mass spectrum of its quasiparticle excitations is real. Finally, there is a solution of the gap equation (25) that indicates on the possibility of spontaneous realizing in the system another non-Hermitian phase, in which fermionic quasiparticles are described effectively by free Lagrangian with also non-Hermitian but $\mathcal{PT}$-odd mass term $\mathcal{M}_{NH2}$. Note that the ground state in each of these three qualitatively different phases has the same energy density, i.e., the phases can appear spontaneously in the *massless* $(2+1)$-D GN model (1) on the same footing. It means that in the space, filled with one of these degenerated phases, bubbles of the other two phases can be created, i.e.. one can observe in space the mixture (or coexistence) of these three phases. Moreover, as it is noted in Sec. III B, the effect of spontaneous non-Hermiticity can be detected only at finite N, i.e., outside the large-N expansion technique.

We have also shown (see in Sec. IV) that in the *massive* $(2+1)$-D GN model (1), i.e., when one or another nonzero bare Hermitian mass term is added to the Lagrangian (1), the effect of its spontaneous non-Hermiticity is impossible.

It is worth recalling that earlier the effect of spontaneous emergence of non-Hermiticity of a quantum system (accompanied by a real mass spectrum of its quasiparticle excitations) was discovered on the basis of the $(3+1)$-D NJL model [8]. Comparing our results with the results of this paper, we see that despite the large qualitative difference between the $(2+1)$-D GN and $(3+1)$-D NJL models, the effects of their spontaneous non-Hermiticity (with real mass spectrums) have many similar features. Indeed, in the NJL model, this effect is also observed only in the chiral limit, and the phase corresponding to it is $\mathcal{PT}$ symmetric. In addition, the ground state of this non-Hermitian NJL phase has the same free energy density as some $\mathcal{PT}$-invariant phase of the NJL model with Hermitian ground state. However, there are some differences. On the one hand, in the $(2+1)$-D GN model there is one more phase with spontaneous non-Hermiticity, which is $\mathcal{PT}$ odd (nonsymmetric). On the other hand, as shown in [8], in the NJL model in the strong-coupling limit, the spontaneously non-Hermitian ground state generates its inhomogeneity, i.e., in this case the translational invariance of the system is violated. Within the framework of the CJT effective action approach used in our paper, it is impossible to detect such a phase, since from the very beginning the presence of translational invariance is assumed (see in Sec. II B).

We hope that the results of this article can be useful for describing physical phenomena in condensed matter systems having a planar crystal structure, or in thin films, e.g., like graphene. In such situations, it often happens that the elementary excitations of the system are massless. As a result, at low energies and in the continuum limit, physical phenomena in it can be effectively described by *massless* quantum field theory models with four-fermion interactions of the type (1) [13,14,17]. And just in these cases, the effect of spontaneous non-Hermiticity could be manifested.

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APPENDIX: ALGEBRA OF THE γ MATRICES IN THE CASE OF SO(2,1) GROUP

The two-dimensional irreducible representation of the $(2+1)$-dimensional Lorentz group SO(2,1) is realized by the following 2×2 $\tilde{\gamma}$-matrices:

$$\tilde{\gamma}^0 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tilde{\gamma}^1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$
$$\tilde{\gamma}^2 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A1}$$

acting on two-component Dirac spinors. They have the properties:

$$\mathrm{Tr}(\tilde{\gamma}^\mu\tilde{\gamma}^\nu) = 2g^{\mu\nu}; \qquad [\tilde{\gamma}^\mu, \tilde{\gamma}^\nu] = -2i\varepsilon^{\mu\nu\alpha}\tilde{\gamma}_\alpha;$$
$$\tilde{\gamma}^\mu\tilde{\gamma}^\nu = -i\varepsilon^{\mu\nu\alpha}\tilde{\gamma}_\alpha + g^{\mu\nu}, \tag{A2}$$

where $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1,-1,-1)$, $\tilde{\gamma}_\alpha = g_{\alpha\beta}\tilde{\gamma}^\beta$, $\varepsilon^{012} = 1$. There is also the relation:

$$\mathrm{Tr}(\tilde{\gamma}^\mu\tilde{\gamma}^\nu\tilde{\gamma}^\alpha) = -2i\varepsilon^{\mu\nu\alpha}. \tag{A3}$$

Note that the definition of chiral symmetry is slightly unusual in $(2+1)$-dimensions (spin is here a pseudoscalar rather than a (axial) vector). The formal reason is simply that there exists no other 2×2 matrix anticommuting with the Dirac matrices $\tilde{\gamma}^\nu$ which would allow the introduction of a γ^5-matrix in the irreducible representation. The important concept of "chiral" symmetries and their breakdown by mass terms can nevertheless be realized also in the framework of $(2+1)$-dimensional quantum field theories by considering a four-component reducible representation for Dirac fields. In this case the Dirac spinors ψ have the following form:

$$\psi(x) = \begin{pmatrix} \tilde{\psi}_1(x) \\ \tilde{\psi}_2(x) \end{pmatrix}, \tag{A4}$$

with $\tilde{\psi}_1$, $\tilde{\psi}_2$ being two-component spinors. In the reducible four-dimensional spinor representation one deals with $4\times 4\gamma$-matrices: $\gamma^\mu = \mathrm{diag}(\tilde{\gamma}^\mu, -\tilde{\gamma}^\mu)$, where $\tilde{\gamma}^\mu$ are given in (A1) (This particular reducible representation for γ-matrices is used, e.g., in Ref. [29]). One can easily show, that (μ, $\nu = 0$, 1, 2):

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}; \qquad \gamma^\mu\gamma^\nu = \sigma^{\mu\nu} + g^{\mu\nu};$$
$$\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \mathrm{diag}(-i\varepsilon^{\mu\nu\alpha}\tilde{\gamma}_\alpha, -i\varepsilon^{\mu\nu\alpha}\tilde{\gamma}_\alpha). \tag{A5}$$

In addition to the Dirac matrices γ^μ ($\mu = 0$, 1, 2) there exist two other matrices, γ^3 and γ^5, which anticommute with all γ^μ ($\mu = 0$, 1, 2) and with themselves

$$\gamma^3 = \begin{pmatrix} 0, & I \\ I, & 0 \end{pmatrix}, \qquad \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = i\begin{pmatrix} 0, & -I \\ I, & 0 \end{pmatrix},$$
$$\tau = -i\gamma^3\gamma^5 = \begin{pmatrix} I, & 0 \\ 0, & -I \end{pmatrix} \tag{A6}$$

with I being the unit 2×2 matrix.

[1] Y. Ashida, Z. Gong, and M. Ueda, Adv. Phys. **69**, 249 (2021).
[2] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[3] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).
[4] A. Beygi, S. P. Klevansky, and C. M. Bender, Phys. Rev. A **99**, 062117 (2019).
[5] A. Felski, A. Beygi, and S. P. Klevansky, Phys. Rev. D **101**, 116001 (2020).
[6] A. Felski and S. P. Klevansky, Phys. Rev. D **103**, 056007 (2021).
[7] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).
[8] M. N. Chernodub, A. Cortijo, and M. Ruggieri, Phys. Rev. D **104**, 056023 (2021).
[9] N. E. Mavromatos, S. Sarkar, and A. Soto, arXiv:2111.05131.
[10] D. J. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).
[11] G. W. Semenoff and L. C. R. Wijewardhana, Phys. Rev. Lett. **63**, 2633 (1989).
[12] G. W. Semenoff, I. A. Shovkovy, and L. C. R. Wijewardhana, Mod. Phys. Lett. A **13**, 1143 (1998); V. C. Zhukovsky, K. G.

Klimenko, V. V. Khudyakov, and D. Ebert, JETP Lett. **73**, 121 (2001).
[13] V. P. Gusynin, S. G. Sharapov, and J. P. Carbotte, Int. J. Mod. Phys. B **21**, 4611 (2007).
[14] D. Mesterhazy, J. Berges, and L. von Smekal, Phys. Rev. B **86**, 245431 (2012).
[15] A. S. Vshivtsev, B. V. Magnitsky, V. C. Zhukovsky, and K. G. Klimenko, Phys. Part. Nucl. **29**, 523 (1998).
[16] K. G. Klimenko, B. V. Magnitsky, and A. S. Vshivtsev, Nuovo Cim. **107A**, 439 (1994); V. C. Zhukovsky and K. G. Klimenko, Teor. Mat. Fiz. **134**, 289 (2003) [Theor. Math. Phys. **134**, 254 (2003)].
[17] D. Ebert, K. G. Klimenko, P. B. Kolmakov, and V. C. Zhukovsky, Ann. Phys. (Amsterdam) **371**, 254 (2016); D. Ebert and D. Blaschke, Prog. Theor. Exp. Phys. **2019**, 123I01 (2019).
[18] T. Kanazawa, M. Kieburg, and J. J. M. Verbaarschot, J. High Energy Phys. 06 (2020) 015.
[19] Y. M. P. Gomes and R. O. Ramos, Phys. Rev. B **104**, 245111 (2021).
[20] B. Rosenstein, B. J. Warr, and S. H. Park, Phys. Rep. **205**, 59 (1991).
[21] Y. Kikukawa and K. Yamawaki, Phys. Lett. B **234**, 497 (1990); G. Gat, A. Kovner, B. Rosenstein, and B. J. Warr, Phys. Lett. B **240**, 158 (1990); H. J. He, Y. P. Kuang, Q. Wang, and Y. P. Yi, Phys. Rev. D **45**, 4610 (1992); M. Modugno, G. Pettini, and R. Gatto, Phys. Rev. D **57**, 4995 (1998); A. S. Vshivtsev, K. G. Klimenko, and B. V. Magnitsky, Teor. Mat. Fiz. **101**, 391 (1994) [Theor. Math. Phys. **101**, 1436 (1994)].
[22] K. G. Klimenko, Z. Phys. C **50**, 477 (1991); Mod. Phys. Lett. A **09**, 1767 (1994).
[23] J. L. Kneur, M. B. Pinto, R. O. Ramos, and E. Staudt, Phys. Lett. B **657**, 136 (2007); Phys. Rev. D **76**, 045020 (2007); J. L. Kneur, M. B. Pinto, and R. O. Ramos, Phys. Rev. D **88**, 045005 (2013).
[24] J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974).
[25] M. E. Peskin, in *Recent Advances in Field Theory and Statistical Mechanics*, 1982 Les Houches Lectures, edited by J. B. Zuber and R. Stora (North-Holland, Amsterdam, 1984).
[26] R. Casalbuoni, S. De Curtis, D. Dominici, and R. Gatto, Phys. Lett. **140B**, 357 (1984); A. Barducci, R. Casalbuoni, S. De Curtis, D. Dominici, and R. Gatto, Phys. Rev. D **38**, 238 (1988).
[27] N. Dorey and R. D. Kenway, Nucl. Phys. **B333**, 419 (1990).
[28] A. A. Garibli, R. G. Jafarov, and V. E. Rochev, Symmetry **11**, 668 (2019); R. G. Jafarov and V. E. Rochev, Phys. At. Nucl. **76**, 1149 (2013); V. E. Rochev, J. Phys. A **45**, 205401 (2012).
[29] T. W. Appelquist, M. Bowick, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. D **33**, 3704 (1986).
[30] T. Kanazawa, J. High Energy Phys. 03 (2021) 121.
[31] J. Alexandre and N. E. Mavromatos, Phys. Lett. B **807**, 135562 (2020).
[32] J. Alexandre, N. E. Mavromatos, and A. Soto, Nucl. Phys. **B961**, 115212 (2020).
[33] N. E. Mavromatos and A. Soto, Nucl. Phys. **B962**, 115275 (2021).
[34] T. G. Khunjua, K. G. Klimenko, and R. N. Zhokhov, arXiv:2111.04539.
[35] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley Publishing Company, Reading, MA, 1995).
[36] M. N. Chernodub, J. Phys. A **50**, 385001 (2017).
[37] M. Buballa, Phys. Rep. **407**, 205 (2005).