Published for SISSA by Springer

Received: July 12, 2021
Accepted: September 9, 2021
Published: September 28, 2021

Conformal symmetries for extremal black holes with general asymptotic scalars in STU supergravity

M. Cvetič,[a,b] C.N. Pope[c,d] and A. Saha[c]

[a] *Department of Physics and Astronomy, and Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, U.S.A.*

[b] *Center for Applied Mathematics and Theoretical Physics, University of Maribor, SI2000 Maribor, Slovenia*

[c] *George P. & Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M University, College Station, TX 77843, U.S.A.*

[d] *DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 OWA, U.K.*

E-mail: cvetic@physics.upenn.edu, pope@physics.tamu.edu, aritrasaha@physics.tamu.edu

Abstract: We present a construction of the most general BPS black holes of STU supergravity ($\mathcal{N} = 2$ supersymmetric $D = 4$ supergravity coupled to three vector supermultiplets) with arbitrary asymptotic values of the scalar fields. These solutions are obtained by acting with a subset of the global symmetry generators on STU BPS black holes with zero values of the asymptotic scalars, both in the U-duality and the heterotic frame. The solutions are parameterized by fourteen parameters: four electric and four magnetic charges, and the asymptotic values of the six scalar fields. We also present BPS black hole solutions of a consistently truncated STU supergravity, which are parameterized by two electric and two magnetic charges and two scalar fields. These latter solutions are significantly simplified, and are very suitable for further explicit studies. We also explore a conformal inversion symmetry of the Couch-Torrence type, which maps any member of the fourteen-parameter family of BPS black holes to another member of the family. Furthermore, these solutions are expected to be valuable in the studies of various swampland conjectures in the moduli space of string compactifications.

Keywords: Black Holes, Extended Supersymmetry, Supergravity Models, Black Holes in String Theory

ArXiv ePrint: 2102.02826

https://doi.org/10.1007/JHEP09(2021)188

Contents

1 Introduction

Many intriguing non-perturbative aspects of string theory and M-theory have been brought to light by studying the black hole and higher p-brane solutions (see, for example, [1, 2]). Especially important in this context are the supersymmetric BPS solutions, which are expected to be protected in the face of stringy corrections to the leading-order effective action. Thus by studying the BPS solutions in the low-energy supergravity limit one can expect to gain insights that could remain relevant in the full theory.

The full four-dimensional supergravity theories resulting from the dimensional reduction of the ten-dimensional heterotic or type II superstring are quite complicated, with many field strengths and scalar fields in their bosonic sectors. For many purposes, however, it suffices to focus on the black hole solutions residing within a truncation of the theories to the so-called STU supergravity, which comprises $\mathcal{N}=2$ supergravity coupled to three vector supermultiplets. Solutions within the STU theory can be rotated using the global symmetries of the full heterotic or type II theories to "fill out" solution sets of the larger theories.

There are different ways to present the STU supergravity, which are related by dualisations of one or more of the four gauge field strengths. These different presentations of the theory arise naturally in different contexts. For example, one of the duality complexions, which we refer to as the heterotic formulation, is the one that arises naturally when one performs a toroidal reduction from the ten-dimensional heterotic theory and then makes a consistent truncation to the STU supergravity. A different duality complexion, which we refer to as the U-duality formulation, arises if one makes a consistent reduction of eleven-dimensional supergravity on the 7-sphere, truncates further to the subsector of fields that are invariant under the U(1) maximal torus of the SO(8) isometry group of the 7-sphere, and then turns off the gauge coupling by sending the radius of the sphere to infinity. This formulation of STU supergravity is related to the heterotic formulation by a dualisation of two out of the four field strengths. There is also an intermediate duality complexion which corresponds to dualising any one of the four fields in the U-duality formulation. We refer to this as the 3+1 formulation of the theory.

As a preliminary to subsequent calculations, one of the purposes of this paper is to obtain fully explicit relations between the fields in the three above-mentioned formulations of the STU supergravity theory. These will then be utilised later in the paper, when we construct explicit expressions for the most general static BPS black hole solutions in the STU supergravity theory. The most general such solutions are characterised by a total of eight charges, corresponding to four electric and four magnetic charges carried by the four gauge field strengths. One can use the compact $\mathrm{U}(1)^3$ subgroup of the $\mathrm{SL}(2,\mathbb{R})^3$ global symmetry of the STU theory to rotate these solutions to ones involving only 5 independent charges, but for our purposes it is more useful to present the solutions in the more symmetrical 8-charge characterisation. In fact, we obtain this symmetrical form by using the $\mathrm{U}(1)^3$ symmetry in the opposite direction, to go from a 5-charge to an 8-charge expression. This procedure was originally partially implemented in [3]; for later purposes we also wish to generalise the solutions by allowing for arbitrary asymptotic values of the six scalar fields of the STU supergravity. This can be done by acting with the remaining six generators of the coset $\mathrm{SL}(2,\mathbb{R})^3/\mathrm{U}(1)^3$ within the global symmetry group.

One of our reasons for constructing the most general static BPS black hole solutions explicitly is to study in detail a phenomenon that was first noticed by Couch and Torrence in 1984 in the case of the extremal Reissner-Nordström black hole [4]. This solution, which is a special case [5] of the static BPS black holes of STU supergravity in which all four electric charges are set equal and the magnetic charges are all set to zero, exhibits a conformal inversion symmetry as follows. Writing the extreme Reissner-Nordström metric as

$$ds^2 = -\Big(1+\frac{Q}{r}\Big)^{-2} dt^2 + \Big(1+\frac{Q}{r}\Big)^2 (dr^2 + r^2\, d\Omega_2^2)\,, \tag{1.1}$$

where $d\Omega_2^2$ is the metric on the unit 2-sphere, the horizon is located at $r=0$. Defined an inverted coordinate $\hat{r}$ and a conformally-related metric $d\hat{s}^2$ by

$$\hat{r} = \frac{Q^2}{r}\,, \qquad d\hat{s}^2 = \frac{Q^2}{r^2}\,ds^2\,, \tag{1.2}$$

one finds that metric $d\hat{s}^2$ takes exactly the same form as the original metric ds^2, written now in terms of the inverted coordinate:

$$d\hat{s}^2 = -\Big(1+\frac{Q}{\hat{r}}\Big)^{-2} dt^2 + \Big(1+\frac{Q}{\hat{r}}\Big)^2 (d\hat{r}^2 + \hat{r}^2\, d\Omega_2^2)\,. \tag{1.3}$$

The Couch-Torrence symmetry therefore maps the near-horizon region to the asymptotic region near infinity, and vice versa. This symmetry has been employed in more recent times to related conserved quantities on the horizon to conserved quantities at infinity. (See, for example, [6–8].)

Recently, generalisations of the Couch-Torrence symmetry have been studied for the static black hole solutions of STU supergravity. In [8] it was shown that the set of solutions carrying four electric charges and no magnetic charges maps into itself under a similar conformal inversion. The mapping is no longer a symmetry, in that a black hole with charges (Q_1, Q_2, Q_3, Q_4) maps into a different member of the 4-charge family, with different charges $(\hat{Q}_1, \hat{Q}_2, \hat{Q}_3, \hat{Q}_4)$, related by $\hat{Q}_i = Q_i^{-1}\,(Q_1\, Q_2\, Q_3\, Q_4)^{1/2}$ [8]. More recently, we studied the problem for the general 8-charge static BPS black holes [9]. Our focus was on the metrics where the asymptotic values of the scalar fields were taken to vanish. We established that any such 8-charge black hole will certainly be related to other members of this family under a conformal inversion, although it does not seem to be possible any longer to present an explicit mapping of the charges in the way that could be done for the 4-charge solutions.

A slightly different approach was developed in [11], which could be applied not only to the static BPS black holes of STU supergravity, but more generally in Einstein-Maxwell-scalar theories of type E_7 [11]. An important aspect of this approach is that it is applied in the context of the general class of static BPS black holes including the parameters associated with the asymptotic values of the scalar fields. We exhibit this description of the conformal inversion mapping in detail for the STU supergravity case, where it allows an explicit mapping of the general 14-parameter family of solutions (comprising 8 charges plus 6 asymptotic scalar values). At the price of the conformal inversion always entailing a mapping from one set of asymptotic scalar values to a different set of values, the approach allows one to give fully explicit transformations under the inversion.

The organisation of this paper is as follows. In section 2, we discuss how the four-dimensional STU supergravity can be obtained by Kaluza-Klein reduction from higher dimensions, where the internal space is a torus. In particular, it suffices for our purposes to start from the pure $\mathcal{N} = 2$ non-chiral supergravity in six dimensions (which itself can be obtained from a toroidal reduction of supergravity in ten dimensions), and then perform a reduction on a 2-torus to four dimensions. In addition to discussing the resulting formulation of STU supergravity, known sometimes as the heterotic formulation, we also carry out explicitly the dualisation of two of the four gauge fields. This results in the formulation known as the U-duality basis. This is the form in which the theory would arise if one began with the gauged $\mathcal{N}=8$ supergravity coming from the reduction of eleven-dimensional supergravity on S^7, truncated this to $\mathcal{N}=2$ supergravity plus three vector supermultiplets, and then took the ungauged limit. We also discuss the STU supergravity in a third duality complex, which we refer to as the 3+1 formulation, where just one of the field strengths of the heterotic formulation is dualised. We shall make use of all three of these formulations of STU supergravity, and the explicit relations between them, in the remainder of the paper.

In section 3, we consider the general extremal BPS static black hole solutions of STU supergravity in the U-duality formulation. These solutions carry 8 charges in total, comprising electric and magnetic charges for each of the four gauge fields. These are really equivalent to solutions with just 5 independent charges, since the $\mathrm{U}(1)^3$ subgroup of the global $\mathrm{SL}(2,\mathbb{R})^3$ symmetry group of the STU theory allows 3 of the 8 charges to be transformed away (while maintaining the vanishing of the asymptotic values of the scalar fields at infinity). In fact the 8-charge solutions were constructed in [3], in the heterotic formulation, by starting from a 5-charge "seed solution" and then acting with the $\mathrm{U}(1)^3$ transformations. The analogous process was employed recently in [9] to construct the 8-charge solution in the U-duality formulation of STU supergravity. The new feature that we implement now in section 3 is to allow also for arbitrary asymptotic values of the six scalar fields in the 8-charge black hole solutions.[1] This can be achieved by employing the action of the remaining six symmetries in the coset $\mathrm{SL}(2,\mathbb{R})^3/\mathrm{U}(1)^3$.

In section 4, we carry out the same steps of generating the general static BPS black hole solutions in the heterotic formulation. Again, we first "fill out" the 5-charge seed solution to the full set of 8 charges by acting with the $\mathrm{U}(1)^3$ subgroup of the global symmetry group. We highlight similarities and also differences with the description in the U-duality formulation.

In section 5 we describe the consistent truncation of the STU supergravity to a supersymmetric theory whose bosonic sector comprises the metric, two gauge fields and two scalar fields (a dilaton and an axion). The theory is considerably simpler to work with than the full STU supergravity, and as we show, the general static BPS black hole solutions, characterised by four charges and asymptotic values for the two scalar fields, are very much simpler.

In section 6, we consider the static BPS black holes in the 3+1 formulation, using the description of the theory in terms of the Kähler geometry of the scalar manifold. We then use this to investigate the behaviour of the black hole metrics under an inversion of the radial coordinate, which, together with a conformal rescaling, maps the horizon to infinity and vice versa. In this description of the conformal inversion, which was discussed previously in [11], one can show how any given member of the general 14-parameter family of static BPS black holes (characterised by the 4 electric charges, 4 magnetic charges and 6 asymptotic scalar values) is mapped by the conformal inversion to another member of the 14-parameter family. In this description, unlike one considered previously in [8, 9], the

[1]Partial results including the scalars were obtained in [3].

JHEP09(2021)188

asymptotic values of the scalar fields are always different in the original and the conformally-inverted metrics.

Section 7 contains a summary of our conclusions, and also further discussion. Some details of the Kaluza-Klein reduction from six to four dimensions are relegated to appendix A.

2 The STU supergravity theory

In this section, we shall provide explicit expressions for the bosonic sector of four-dimensional ungauged STU supergravity in the three different formulations we shall be using in this paper, and the explicit relations between them. Specifically, we shall consider what may be called the heterotic formulation, which is the way the theory arises if one starts from ten-dimensional supergravity and toroidally reduces on T^6, together with appropriate truncations. The truncated theory has $\mathcal{N}=2$ supersymmetry, and comprises $\mathcal{N}=2$ supergravity coupled to three vector multiplets. The theory was presented in this form in [12]. We shall also consider the formulation of STU supergravity that one would obtain by reducing eleven-dimensional supergravity on S^7, performing an appropriate truncation from $\mathcal{N}=8$ to $\mathcal{N}=2$ supersymmetry, and also turning off the gauge coupling constant in the four-dimensional theory. This formulation of STU supergravity is sometimes referred to as the U-duality invariant formulation. The two formulations are related in four dimensions by performing an appropriate dualisation of two of the four gauge fields. We shall also consider STU supergravity in an intermediate formulation which we refer to as the 3+1 formulation. In this case, instead of dualising two field strengths in the heterotic formulation only one is dualised.

2.1 STU supergravity from six dimensions

In the heterotic formulation one can in fact conveniently describe the STU theory by first reducing from ten dimensions to six dimensions on T^4 and truncating to pure $\mathcal{N}=2$ non-chiral supergravity. The STU theory is then obtained by reducing this on T^2 with no further truncation. (Except, of course, the usual Kaluza-Klein truncation in which only the singlets under the $\mathrm{U}(1)^2$ isometry of the T^2 are retained.) The bosonic sector of the non-chiral supergravity in six-dimensional supergravity is described by the Lagrangian

$$\mathcal{L}_6 = \hat{R}\,\hat{*}\mathbb{1} - \frac{1}{2}\hat{*}d\hat{\phi}\wedge d\hat{\phi} - \frac{1}{2}e^{-\sqrt{2}\,\hat{\phi}}\,\hat{*}\hat{H}_{(3)}\wedge \hat{H}_{(3)}\,, \tag{2.1}$$

where $\hat{H}_{(3)} = d\hat{B}$. The reduction down to four dimensions is described in detail in appendix A. After dualising the 2-form potential for the field $H_{(3)}$ to an axion χ_1, the resulting Lagrangian for the bosonic sector of STU supergravity is given by (A.14):

$$\begin{aligned}\mathcal{L} = {} & R*\mathbb{1} - \frac{1}{2}\sum_{i=1,3}(*d\varphi_i\wedge d\varphi_i + e^{2\varphi_i}*d\chi_i\wedge d\chi_i) - \frac{1}{2}*d\tilde{\varphi}_2\wedge d\tilde{\varphi}_2 - \frac{1}{2}e^{2\tilde{\varphi}_2}*d\tilde{\chi}_2\wedge d\tilde{\chi}_2 \\ & -\frac{1}{2}e^{-\varphi_1}\Big[e^{\tilde{\varphi}_2-\varphi_3}*\widetilde{\overline{F}}_1\wedge\widetilde{\overline{F}}_1 + e^{-\tilde{\varphi}_2+\varphi_3}*\widetilde{\overline{F}}_2\wedge\widetilde{\overline{F}}_2 \\ & +e^{-\tilde{\varphi}_2-\varphi_3}*\overline{F}^3\wedge\overline{F}^3 + e^{\tilde{\varphi}_2+\varphi_3}*\overline{F}^4\wedge\overline{F}^4\Big] + \chi_1\,(\widetilde{F}_1\wedge\widetilde{F}_2 + F^3\wedge F^4)\,,\end{aligned} \tag{2.2}$$

where $\widetilde{F}_1 = d\widetilde{A}_1$, $\widetilde{F}_2 = d\widetilde{A}_2$, $F^3 = dA^3$ and $F^4 = dA^4$ are the "raw" field strengths, and

$$\begin{aligned} \overline{\widetilde{F}}_1 &= \widetilde{F}_1 - \widetilde{\chi}_2\, F^3\,, & \overline{\widetilde{F}}_2 &= \widetilde{F}_2 - \chi_3\, F^3\,, \\ \overline{F}^3 &= F^3\,, & \overline{F}^4 &= F^4 + \chi_3\, \widetilde{F}_1 + \widetilde{\chi}_2\, \widetilde{F}_2 - \chi_2\, \chi_3\, F^3\,, \end{aligned} \tag{2.3}$$

with overbars, denote the "dressed" field strengths that appear in the kinetic terms in (2.2). The gauge fields numbered 1 and 2 are denoted with tildes here; later on, we shall dualise these fields in order to obtain the STU supergravity theory in the formalism in which the $\mathrm{SL}(2,\mathbb{R})^3$ symmetry is manifest.

2.2 Heterotic formulation of STU supergravity

In this formulation the raw field strengths $(\widetilde{F}_1, \widetilde{F}_2, F^3, F^4)$ are organised into the column vector

$$\boldsymbol{\mathcal{F}} \equiv \begin{pmatrix} \mathcal{F}^1 \\ \mathcal{F}^2 \\ \mathcal{F}^3 \\ \mathcal{F}^4 \end{pmatrix} = \begin{pmatrix} F^3 \\ \widetilde{F}_1 \\ F^4 \\ \widetilde{F}_2 \end{pmatrix}\,, \tag{2.4}$$

and the (φ_1, χ_1) dilaton/axion pair are redefined as

$$\Phi = \frac{1}{2}\varphi_1\,, \qquad \Psi = \chi_1\,. \tag{2.5}$$

The Lagrangian (2.2) can then be written as

$$\begin{aligned} \mathcal{L} = &\; R *\mathbb{1} + \frac{1}{8}\mathrm{Tr}(*dM \wedge LdML) - 2{*d\Phi} \wedge d\Phi - \frac{1}{2}e^{4\Phi} *d\Psi \wedge d\Psi \\ &- \frac{1}{2}e^{-2\Phi}\,(LML)_{ij} *\mathcal{F}^i \wedge \mathcal{F}^j + \frac{1}{2}\Psi\, L_{ij}\, \mathcal{F}^i \wedge \mathcal{F}^j\,. \end{aligned} \tag{2.6}$$

where

$$L = \sigma_1 \otimes \mathbb{I}_2 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}\,. \tag{2.7}$$

The scalar matrix M can be read off by comparing the kinetic terms for the gauge fields $\mathcal{F}^i$ in (2.6) with the kinetic terms in (2.2). It is straightforward to see that M can then be written as

$$M = \begin{pmatrix} \mathbb{G}^{-1} & -\mathbb{G}^{-1}\,\mathbb{B} \\ \mathbb{B}\,\mathbb{G}^{-1} & \mathbb{G} - \mathbb{B}\,\mathbb{G}^{-1}\,\mathbb{B} \end{pmatrix}\,, \tag{2.8}$$

where

$$\mathbb{G} = e^{-\varphi_3} \begin{pmatrix} e^{-\tilde{\varphi}_2} + \widetilde{\chi}_2^2\, e^{\tilde{\varphi}_2} & -\widetilde{\chi}_2\, e^{\tilde{\varphi}_2} \\ -\widetilde{\chi}_2\, e^{\tilde{\varphi}_2} & e^{\tilde{\varphi}_2} \end{pmatrix}\,, \qquad \mathbb{B} = \begin{pmatrix} 0 & -\chi_3 \\ \chi_3 & 0 \end{pmatrix}\,. \tag{2.9}$$

One can recognise $\mathbb{G}$ as being associated with the internal metric on the 2-torus in the Kaluza-Klein reduction (A.1), (i.e. $ds_2^2 = \mathbb{G}_{ij}\, dz^i\, dz^j$ is the metric enclosed in square brackets in (A.1), in the 2-torus directions), and $\mathbb{B}$ as being associated with the internal component $A_{(0)12}$ of the 2-form potential in (A.2).

The four-dimensional Lagrangian (2.6) is in the form that was obtained in [3]. It is invariant under the $\mathrm{O}(2,2)$ transformations (the T-duality of the 2-torus compactification):

$$M \longrightarrow \Omega M \Omega^T\,, \qquad \mathcal{A}^i \longrightarrow \Omega^i{}_j\,\mathcal{A}^j\,, \tag{2.10}$$

where $g_{\mu\nu}$ and S are inert, and $\mathcal{A}^i$ are the potentials for the field strengths $\mathcal{F}^i$; that is, $\mathcal{F}^i = d\mathcal{A}^i$. The transformation matrix $\Omega \in \mathrm{O}(2,2)$ preserves the $\mathrm{O}(2,2)$-invariant matrix L:

$$\Omega^T L \Omega = L\,, \qquad L = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}\,. \tag{2.11}$$

L is in fact the metric tensor in the $(2,2)$-signature flat space on which the $\mathrm{O}(2,2)$ transformations Ω act. L has components L_{ij} with $L_{13} = L_{31} = L_{24} = L_{42} = 0$ and the remainder being zero. The inverse metric L^{-1} has components L^{ij}, and for these too $L^{13} = L^{31} = L^{24} = L^{42} = 0$ with the remainder being zero. Note that one also has

$$L\,M\,L = M^{-1}\,. \tag{2.12}$$

sloppyNote that M has components M^{ij} and its inverse M^{-1} has components $(M^{-1})_{ij} = L_{ik}\,L_{j\ell}\,M^{k\ell}$.

The equations of motion and Bianchi identities are in addition invariant under the $\mathrm{SL}(2,\mathbb{R})$ transformations (electromagnetic S-duality):

$$S \longrightarrow \frac{aS+b}{cS+d}\,, \qquad \mathcal{F}^i \to (c\Psi + d)\,\mathcal{F}^i + c\,e^{-2\Phi}\,(ML)_{ij} *\mathcal{F}^j\,, \tag{2.13}$$

where $g_{\mu\nu}$ and the scalar matrix M are inert.

The equations of motion for the electromagnetic fields that follow from (2.6) are

$$d\mathcal{G}_i = 0\,, \qquad \text{where} \qquad \mathcal{G}_i \equiv -e^{-2\Phi}\,(LML)_{ij} *\mathcal{F}^j + \Psi\,L_{ij}\,\mathcal{F}^j\,, \tag{2.14}$$

which implies that the canonical electric charges $\vec{\alpha}$ will be given by[2]

$$\alpha_i = \frac{1}{4\pi}\int \mathcal{G}_i\,. \tag{2.15}$$

The magnetic charges $\vec{p}$ are given by

$$p^i = \frac{1}{4\pi}\int \mathcal{F}^i\,. \tag{2.16}$$

It is convenient for later purposes to define also magnetic charges $\vec{\beta}$ with a lowered $\mathrm{O}(2,2)$ index:

$$\beta_i = L_{ij}\,p^j\,. \tag{2.17}$$

These will be referred to later as the "canonical" magnetic charges.[3] If the asymptotic values of the scalars are taken to be zero, one has $S_\infty = \mathrm{i}$ and $M_\infty = \mathbb{I}_4$.

[2]We have changed the overall sign of the definition of the dual field strength $\mathcal{G}_i$ in (2.14), relative to the one in [3]. This is for consistency with our conventions for the Hodge dualisation of differential forms (which is made explicit in eq. (3.65)), and our conventions in the rest of the paper.

[3]In toroidally compactified heterotic string theory the canonical electric charges $\vec{\alpha}$ are quantised and span an even self-dual lattice, subject to the constraint: $\vec{\alpha}^T L\vec{\alpha} = -2, 0, 2, \cdots$. For BPS-saturated configurations one further requires $\vec{\alpha}^T L\vec{\alpha} > 0$. By S-duality the same conditions are satisfied for quantised magnetic charges $\vec{\beta}$, along with the condition that when $\vec{\alpha} \propto \vec{\beta}$ with magnetic and electric charge vector components being co-prime integers [13].

2.3 The U-duality formulation of STU supergravity

We arrive at this formulation by dualising the gauge fields $\widetilde{A}_1$ and $\widetilde{A}_2$ appearing in the Lagrangian (2.2). To do this, we employ the standard procedure of introducing dual potentials A^1 and A^2 as Lagrange multipliers, and adding the terms $A^1 \wedge d\widetilde{F}_1 + A^2 \wedge d\widetilde{F}_2$ to the Lagrangian (A.14). Up to total derivatives, this is equivalent to adding

$$\mathcal{L}_{LM} = F^1 \wedge \widetilde{F}_1 + F^2 \wedge \widetilde{F}_2 \,, \tag{2.18}$$

where $F^1 = dA^1$ and $F^2 = dA^2$ are the raw dualised field strengths. Varying the total Lagrangian with respect to $\widetilde{F}_1$ and $\widetilde{F}_2$, now treated as independent fields, shows that $\widetilde{F}_1$ and $\widetilde{F}_2$ satisfy

$$\begin{aligned} F^1 + \chi_1\, \widetilde{F}_2 &= e^{-\varphi_1+\tilde{\varphi}_2}\,(e^{-\varphi_3} *\widetilde{\overline{F}}_1 + \chi_3\, e^{\varphi_3} *\overline{F}^4)\,, \\ F^2 + \chi_1\, \widetilde{F}_1 &= e^{-\varphi_1+\varphi_3}\,(e^{-\tilde{\varphi}_2} *\widetilde{\overline{F}}_2 + \widetilde{\chi}_2\, e^{\tilde{\varphi}_2} *\overline{F}^4)\,, \end{aligned} \tag{2.19}$$

and hence, using (2.3),

$$\begin{aligned} F^1 + \chi_1\, \widetilde{\overline{F}}_2 + \chi_1\,\chi_3\, \overline{F}^3 &= e^{-\varphi_1+\tilde{\varphi}_2}\,(e^{-\varphi_3} *\widetilde{\overline{F}}_1 + \chi_3\, e^{\varphi_3} *\overline{F}^4)\,, \\ F^2 + \chi_1\, \widetilde{\overline{F}}_1 + \chi_1\, \widetilde{\chi}_2\, \overline{F}^3 &= e^{-\varphi_1+\varphi_3}\,(e^{-\tilde{\varphi}_2} *\widetilde{\overline{F}}_2 + \widetilde{\chi}_2\, e^{\tilde{\varphi}_2} *\overline{F}^4)\,, \end{aligned} \tag{2.20}$$

These two equations, together with their Hodge duals, can be solved algebraically for $\widetilde{\overline{F}}_1$ and $\widetilde{\overline{F}}_2$, with the results expressed, using (2.3) again, in terms of (F^1, F^2, F^3, F^4) and $(*F^1, *F^2, *F^3, *F^4)$. Since the expressions are a little unwieldy, we shall not present them here.

Having solved for $\widetilde{\overline{F}}_1$ and $\widetilde{\overline{F}}_2$ these may be substituted back into the Lagrangian (2.2) plus (2.18), to give the theory in the dualised form, with the gauge fields (A^1, A^2, A^3, A^4). After one final step, in which the $\tilde{\varphi}_2$ and $\widetilde{\chi}_2$ scalars are subjected to the discrete involutive $\mathrm{SL}(2,\mathbb{R})$ transformation

$$\tilde{\tau}_2 = -\frac{1}{\tau_2}\,, \qquad \text{where} \quad \tilde{\tau}_2 = \widetilde{\chi}_2 + i\, e^{-\tilde{\varphi}_2}\,, \qquad \tau_2 = \chi_2 + i\, e^{-\varphi_2}\,, \tag{2.21}$$

one finds that the Lagrangian is precisely equal to the one described in appendix B of [15] and appendix A of [9]. (The dualisation to go from the heterotic to the U-duality formulation has also been discussed in [14].) The kinetic terms involving the field strengths given, as in eq. (A.2) of [9], by

$$\mathcal{L}_F = -\frac{1}{2} f^R_{AB} *F^A \wedge F^B - \frac{1}{2} f^I_{AB}\, F^A \wedge F^B\,, \tag{2.22}$$

where f^R_{AB} and f^I_{AB} are the real and imaginary parts of the scalar matrix given in eq. (A.9) of [9]. This is written in the formalism and notation described in Freedman and Van Proeyen [16], in which, defining

$$G_A = -f^R_{AB} *F^B - f^I_{AB}\, F^B\,, \tag{2.23}$$

the field equations $dG_A = 0$ and Bianchi identities $dF^A = 0$ are invariant under the transformations

$$\begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix} \longrightarrow \mathcal{S} \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix}, \qquad \mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.24}$$

where the constant 4×4 matrices A, B, C and D obey

$$A^T C - C^T A = 0\,, \qquad B^T D - D^T B = 0\,, \qquad A^T D - C^T B = \mathbb{I}_4\,, \tag{2.25}$$

and we have defined $\mathbf{F} = (F^1, F^2, F^3, F^4)^T$ and $\mathbf{G} = (G_1, G_2, G_3, G_4)^T$. The equations (2.25) are precisely the conditions for the matrix $\mathcal{S}$ to be an element of $\mathrm{Sp}(8, \mathbb{R})$, obeying [16]

$$S^T \Omega S = \Omega\,, \qquad \text{where} \qquad \Omega = \begin{pmatrix} 0 & \mathbb{I}_4 \\ -\mathbb{I}_4 & 0 \end{pmatrix}. \tag{2.26}$$

The scalar field Lagrangian

$$\mathcal{L}_{\rm scal} = -\frac{1}{2} \sum_{i=1}^{3} ({*d\varphi_i} \wedge d\varphi_i + e^{2\varphi_i}\, {*d\chi_i} \wedge d\chi_i) \tag{2.27}$$

is invariant under $\mathrm{SL}(2, \mathbb{R})^3$, which is a subgroup of $\mathrm{Sp}(8, \mathbb{R})$, and so in this formulation the STU supergravity theory has a manifest $\mathrm{SL}(2, \mathbb{R})^3$ symmetry, at the level of the equations of motion. The explicit forms of the A, B, C and D matrices corresponding to the $\mathrm{SL}(2, \mathbb{R})^3$ subgroup of $\mathrm{Sp}(8, \mathbb{R})$ are given in [9].

The conserved electric and magnetic charges in the U-duality formalism are given by

$$Q_A = \frac{1}{4\pi} \int G_A\,, \qquad P^A = \frac{1}{4\pi} \int F^A\,. \tag{2.28}$$

If we define the charge vectors $\mathbf{P} = (P^1, P^2, P^3, P^4)^T$ and $\mathbf{Q} = (Q_1, Q_2, Q_3, Q_4)$ then the action of $\mathrm{SL}(2, \mathbb{R})^3$ on the charges takes the same form as for the fields and their duals in (2.24), namely

$$\begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix} \longrightarrow \mathcal{S} \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}, \tag{2.29}$$

where again the 4×4 matrices A, B, C and D that form $\mathcal{S}$ are restricted to the $\mathrm{SL}(2, \mathbb{R})^3$ subgroup of $\mathrm{Sp}(8, \mathbb{R})$, as given in [9].

There is in fact a simpler way to present the action of the $\mathrm{SL}(2, \mathbb{R})^3$ global symmetries on the charges, by introducing the charge tensor $\gamma_{aa'a''}$, where a, a' and a'' are doublet indices of the three $\mathrm{SL}(2, \mathbb{R})$ factors in the symmetry group (see, for example, [12]). If we make the assignments

$$\begin{aligned} \gamma_{000} &= -P^1\,, & \gamma_{011} &= P^2\,, & \gamma_{101} &= P^3\,, & \gamma_{110} &= P^4\,, \\ \gamma_{111} &= -Q_1\,, & \gamma_{100} &= Q_2\,, & \gamma_{010} &= Q_3\,, & \gamma_{001} &= Q_4\,, \end{aligned} \tag{2.30}$$

then the $\mathrm{SL}(2, \mathbb{R})^3$ transformation of the charges is given by

$$\gamma_{aa'a''} \longrightarrow (S_1)_a{}^b\, (S_2)_{a'}{}^{b'}\, (S_3)_{a''}{}^{b''}\, \gamma_{bb'b''}\,, \tag{2.31}$$

JHEP09(2021)188

where

$$S_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \qquad a_i\, d_i - b_i\, c_i = 1 \quad \text{for each } i\,. \tag{2.32}$$

This gives exactly the same $\mathrm{SL}(2,\mathbb{R})^3$ transformation as in appendix A of [9]. It furthermore demonstrates that $\gamma_{aa'a''}$ is covariant with respect to $\mathrm{SL}(2,\mathbb{R})^3$ transformations.

One can, of course, conveniently write the $\mathrm{SL}(2,\mathbb{R})^3$ transformation (2.24) in the analogous way, by defining the field strength tensor

$$\begin{aligned} \Phi_{000} &= -F^1\,, & \Phi_{011} &= F^2\,, & \Phi_{101} &= F^3\,, & \Phi_{110} &= F^4\,, \\ \Phi_{111} &= -G_1\,, & \Phi_{100} &= G_2\,, & \Phi_{010} &= G_3\,, & \Phi_{001} &= G_4\,, \end{aligned} \tag{2.33}$$

which transforms according to

$$\Phi_{aa'a''} \longrightarrow (S_1)_a{}^b\,(S_2)_{a'}{}^{b'}\,(S_3)_{a''}{}^{b''}\,\Phi_{bb'b''}\,, \tag{2.34}$$

Furthermore, if the dilaton/axion pairs of scalar fields (φ_1,χ_1), (φ_2,χ_2) and (φ_3,χ_3) are assembled into the matrices

$$M_i = \begin{pmatrix} e^{\varphi_i} & -\chi_i\, e^{\varphi_i} \\ -\chi_i\, e^{\varphi_i} & e^{-\varphi_i} + \chi_i^2\, e^{\varphi_i} \end{pmatrix}, \qquad \text{for} \qquad i = 1,2,3\,, \tag{2.35}$$

whose components are $(M_1)^{ab}$, $(M_2)^{a'b'}$ and $(M_3)^{a''b''}$ respectively, then the scalars transform under the $\mathrm{SL}(2,\mathbb{R})$ factors according to

$$M_i \longrightarrow M_i' = (S_i^T)^{-1}\, M_i\, S_i^{-1}\,, \tag{2.36}$$

(with the i'th scalars transforming only under the i'th $\mathrm{SL}(2,\mathbb{R})$ group).

2.4 Mapping between the heterotic and the U-duality formulations

As we saw in section 2.2, the scalar fields in the heterotic formulation comprise the dilaton/axion pairs (Φ,Ψ), (φ_2,χ_2) and (φ_3,χ_3). These come from the Kaluza-Klein reduction from six dimensions, as described in appendix A, with $\Phi = \frac{1}{2}\varphi_1$ and $\Psi = \chi_1$. The pairs (φ_2,χ_2) and (φ_3,χ_3) are packaged into the $\mathrm{O}(2,2)/\mathrm{U}(1)^2$ scalar coset matrix M given by (2.8) and (2.9), while (Φ,Ψ) parameterise the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ scalar coset.

In the U-duality formulation the scalars comprise the same set (φ_1,χ_1), (φ_2,χ_2) and (φ_3,χ_3), except that now the (φ_2,χ_2) pair are subjected to the involution $\tau_2 \to -1/\tau_2$, where $\tau_2 = \chi_2 + i e^{-\varphi_2}$, as given in (2.21). Thus, in total, the mapping of fields from those of the heterotic formulation and to those of the U-duality formulation is as follows:

$$\begin{aligned} &\Phi \longrightarrow \frac{1}{2}\varphi_1\,, \qquad \Psi \longrightarrow \chi_1\,, && \varphi_3 \longrightarrow \varphi_3\,, \qquad \chi_3 \longrightarrow \chi_3\,, \\ &e^{\tilde\varphi_2} \longrightarrow (1+\chi_2^2\, e^{2\varphi_2})\, e^{-\varphi_2}\,, && \tilde\chi_2 \longrightarrow -\chi_2\, e^{2\varphi_2}\,(1+\chi_2^2\, e^{2\varphi_2})^{-1}\,. \end{aligned} \tag{2.37}$$

The relation between the electric and magnetic charges in the two formulations follows from the relations between the field strengths, which we discussed earlier. The dual field strengths $\mathcal{G}_i$ in the heterotic formulation, given by (2.14), can be seen, after making use of the equations (2.20) to express the fields $\widetilde{F}_1$ and $\widetilde{F}_2$ in (2.4) in terms of the dual fields F^1 and F^2, to be related to the fields F^A and dual fields G_A of the U-duality formulation by

$$\mathcal{G}_1 = G_3\Big|_{\tau_2\to -1/\tau_2}\,, \qquad \mathcal{G}_2 = -F^1\,, \qquad \mathcal{G}_3 = G_4\Big|_{\tau_2\to -1/\tau_2}\,, \qquad \mathcal{G}_4 = -F^2\,, \tag{2.38}$$

where in addition to the involution of (φ_2, χ_2) indicated here, we also make the replacements $\Phi \to \frac{1}{2}\varphi_1$ and $\Psi \to \chi_1$, as given in (2.37). Similarly, one finds that the fields $\mathcal{F}^i$ of the heterotic formulation are related to F^A and G_A by

$$\mathcal{F}^1 = F^3\,, \qquad \mathcal{F}^2 = G_1\Big|_{\tau_2\to -1/\tau_2}\,, \qquad \mathcal{F}^3 = F^4\,, \qquad \mathcal{F}^4 = G_2\Big|_{\tau_2\to -1/\tau_2}\,. \tag{2.39}$$

From the definitions (2.15) and (2.16) for the electric and magnetic charges α_i and p^i in the heterotic formulation, and the definitions (2.28) for the electric and magnetic charges Q_A and P^A in the U-duality formulation we therefore have

$$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (Q_3, -P^1, Q_4, -P^2) \tag{2.40}$$

and

$$\vec{p} = (p^1, p^2, p^3, p^4) = (P^3, Q_1, P^4, Q_2)\,. \tag{2.41}$$

It is also useful to record the mapping from the magnetic charges with lowered index in the heterotic formulation, $\beta_i = L_{ij}\, p^j$, for which we therefore have

$$\vec{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4) = (P^4, Q_2, P^3, Q_1)\,. \tag{2.42}$$

2.5 STU supergravity in the $3+1$ formulation

There is a third choice of duality frame for STU supergravity that is useful for some purposes. In this frame, which we refer to as the "$3+1$ formalism," the starting point is again the STU supergravity Lagrangian given in eqs. (2.2) and (2.3). We then dualise just the $\widetilde{F}_2$ field. in this formalism, therefore, the simple 4-charge BPS solution would carry 3 electric charges and 1 magnetic (or vice versa).

To perform the dualisation of $\widetilde{F}_2$, we add a Lagrange multiplier term $\mathcal{L}_{LM} = F^2 \wedge \widetilde{F}_2$, where $F^2 = dA^2$ is the dualised field, and then solve the algebraic equation of motion obtained by varying F^2. This allows us to solve for the dressed field $\widetilde{\widetilde{F}}_2 = \widetilde{F}_2 - \chi_3\, F^3$, finding

$$\widetilde{\widetilde{F}}_2 = -\frac{1}{(1+\widetilde{\chi}_2^2\, e^{2\widetilde{\varphi}_2})}\Big[e^{\varphi_1+\widetilde{\varphi}_2+\varphi_3}\,({*F^2} + \chi_1 {*\widetilde{F}_1}) + \widetilde{\chi}_2\, e^{2\widetilde{\varphi}_2}\,(F^4 + \chi_3\, \widetilde{F}_1)\Big]\,. \tag{2.43}$$

Substituting back into the original Lagrangian plus the Lagrange multiplier term, we obtain the bosonic STU supergravity Lagrangian written in the $3+1$ formulation. It is convenient to define the new (φ_2, χ_2) scalars related to $(\widetilde{\varphi}_2, \widetilde{\chi}_2)$ by the involution $\widetilde{\tau}_2 = -1/\tau_2$, exactly as in (2.37), and the three "dressed" field strengths:

$$\hat{F}^2 = F^2 + \chi_1\, \widetilde{F}_1\,, \qquad \hat{F}^3 = F^3 + \chi_2\, \widetilde{F}_1\,, \qquad \hat{F}^4 = F^4 + \chi_3\, \widetilde{F}_1\,. \tag{2.44}$$

JHEP09(2021)188

In terms of these, the $3+1$ STU Lagrangian takes the form[4]

$$\begin{aligned}\mathcal{L} = &\ R{*\mathbb{1}} - \frac{1}{2}\sum_{i=1}^{3}({*d\varphi_i}\wedge d\varphi_i + e^{2\varphi_i}\,{*d\chi_i}\wedge d\chi_i) - \frac{1}{2}e^{-\varphi_1-\varphi_2-\varphi_3}\,{*\widetilde{F}_1}\wedge\widetilde{F}_1\\ &-\frac{1}{2}e^{\varphi_1-\varphi_2-\varphi_3}\,{*\hat{F}^2}\wedge\hat{F}^2 - \frac{1}{2}e^{-\varphi_1+\varphi_2-\varphi_3}\,{*\hat{F}^3}\wedge\hat{F}^3 - \frac{1}{2}e^{-\varphi_1-\varphi_2+\varphi_3}\,{*\hat{F}^4}\wedge\hat{F}^4\\ &+\chi_1\,F^3\wedge F^4 + \chi_2\,F^2\wedge F^4 + \chi_3\,F^2\wedge F^3\\ &+(\chi_2\,\chi_3\,F^2 + \chi_1\,\chi_3\,F^3 + \chi_1\,\chi_2\,F^4)\wedge\widetilde{F}_1 + \chi_1\,\chi_2\,\chi_3\,\widetilde{F}_1\wedge\widetilde{F}_1\,. \end{aligned}\tag{2.45}$$

Note that $\widetilde{F}_1$, F^2, F^3 and F^4 are the bare field strengths:

$$\widetilde{F}_1 = d\widetilde{A}_1\,,\qquad F^2 = dA^2\,,\qquad F^3 = dA^3\,,\qquad F^4 = dA^4\,. \tag{2.46}$$

Note also that the scalar fields here are exactly the same as the ones in the U-duality formulation. In this $3+1$ formulation there is a permutation symmetry among the sets of fields

$$(\varphi_1,\chi_1,F^2)\,,\qquad (\varphi_2,\chi_2,F^3)\,,\qquad (\varphi_3,\chi_3,F^4)\,. \tag{2.47}$$

We shall return to a discussion of the 3+1 formulation of STU supergravity in section 6.

3 General extremal BPS static black holes in the U-duality formulation

The most general extremal BPS static black hole solution in STU supergravity will carry 8 independent charges, since each of the four field strengths can carry an electric charge and a magnetic charge. Since the theory has an $\mathrm{SL}(2,\mathbb{R})^3$ global symmetry, this can be used in order to map one solution into another solution that is equivalent under the group action. The 3-parameter compact subgroup $\mathrm{U}(1)^3$ leaves the asymptotic values of the six scalar fields unchanged, and this means that it suffices to consider an $8-3=5$ parameter "seed" solution, with only 5 independent charges, in order to fill out the full 8-parameter family by acting with $\mathrm{U}(1)^3$. If ones starts with a 5-charge seed solution in which the dilatonic and axionic scalars vanish asymptotically at infinity, then the 8-charge solutions obtained in this way will all have vanishing asymptotic scalars. One can also then choose to fill out the solution set further by then acting with the remaining 6-parameter coset $\mathrm{SL}(2,\mathbb{R})^3/\mathrm{U}(1)^3$ transformations, thereby giving arbitrary asymptotic values to the six scalar fields.

In the subsections below, we shall present the general 8-charge static BPS black hole solutions in the U-duality formulation, both for vanishing values of the asymptotic scalar fields and for arbitrary values of the asymptotic scalar fields.

3.1 The general 8-charge static BPS metric

The general 8-charge solution for the case of vanishing asymptotic scalars was constructed in this formulation in [9]. The starting point was the solution with 5 independent charges

[4] A dualisation to obtain the STU theory in the 3+1 formulation can also be found in [17].

that was constructed in [3], after translating it into the U-duality formulation. The five independent charges could be taken to be[5]

$$\mathbf{Q} = (\bar{Q}_1, \bar{Q}_2, \bar{Q}_3, \bar{Q}_4)\,, \qquad \mathbf{P} = (\bar{p}, -\bar{p}, 0, 0)\,. \tag{3.1}$$

(We are placing bars on the charges in the 5-charge seed solution.) For this solution, the metric is given by [3]

$$ds^2 = -\frac{r^2}{\sqrt{V}}\,dt^2 + \frac{\sqrt{V}}{r^2}\,(dr^2 + r^2 d\Omega^2)\,, \tag{3.2}$$

$$V = r^4 + \alpha\, r^3 + \beta\, r^2 + \gamma\, r + \Delta\,, \tag{3.3}$$

where

$$\begin{aligned} \alpha &= \sum_i \bar{Q}_i\,, \qquad \beta = \sum_{i<j} \bar{Q}_i \bar{Q}_j - \bar{p}^2\,, \qquad \gamma = \sum_{i<j<k} \bar{Q}_i \bar{Q}_j \bar{Q}_k - \bar{p}^2\,(\bar{Q}_1 + \bar{Q}_2)\,, \\ \Delta &= \prod_i \bar{Q}_i - \frac{1}{4}\bar{p}^2\,(\bar{Q}_1 + \bar{Q}_2)^2\,. \end{aligned} \tag{3.4}$$

This was filled out to give the general 8-charge solution by acting with the $\mathrm{U}(1)^3$ subgroup of the $\mathrm{SL}(2,\mathbb{R})^3$ global symmetry of the STU supergravity in [9]. It was shown that the constants α, β, γ and Δ are then given by

$$\begin{aligned} \alpha^2 &= \Big(\sum_A Q_A\Big)^2 + \Big(\sum_A P^A\Big)^2\,, \\ \beta &= \sum_{A<B} (Q_A Q_B + P^A P^B)\,, \\ \alpha\gamma &= 4\Delta + \frac{1}{2}\beta^2 - \frac{1}{2}\sum_{A<B}\Big((P^A P^B)^2 + (Q_A Q_B)^2 + P^{AB}\,Q_{AB}\Big) - 3\prod_A P^A - 3\prod_A Q_A\,, \\ \Delta &= \prod_A Q_A + \prod_A P^A + \frac{1}{2}\sum_{A<B} P^A Q_A P^B Q_B - \frac{1}{4}\sum_i (P^A)^2\,(Q_A)^2\,, \end{aligned} \tag{3.5}$$

where $P^{AB} \equiv P^A P^B + (\prod_C P^C)/(P^A P^B)$ and $Q_{AB} \equiv Q_A Q_B + (\prod_C Q_C)/(Q_A Q_B)$ (so $P^{12} = P^1 P^2 + P^3 P^4$, etc.). Stated equivalently, the expressions for α^2, β, $\alpha\gamma$ and Δ given in (3.5) are the unique polynomials of degrees 2, 2, 4 and 4 in the charges, respectively, that are invariant under $\mathrm{U}(1)^3$ and that reduce to those following from (3.4) under the 5-charge specialisation (3.1). Note that Δ is actually invariant under the entire $\mathrm{SL}(2,\mathbb{R})^3$ symmetry group; it is the usual quartic charge invariant of STU supergravity.

The solution described above has 8 independent charges, with the scalar fields (φ_i, χ_i) going to zero at infinity. In order to generalise the solution to include arbitrary values for the six asymptotic scalars, we need only act with the symmetry generators in the six-dimensional coset $\mathrm{SL}(2,\mathbb{R})^3/\mathrm{U}(1)^3$. This is most easily implemented by using the charge-tensor formulation discussed at the end of section 2.3. We proceed in two stages, the first

[5] In [9] the magnetic charges were actually taken to be $\mathbf{P} = (0, 0, p, -p)$, but we are making the equally valid choice in (3.1) here, for consistency with the choice (4.1) in the heterotic formulation, given the mapping (2.41).

being to re-express the quantities α, β, γ and Δ in (3.5) in terms of the charge tensor $\gamma_{aa'a''}$. To do this, we first introduce the SL$(2,\mathbb{R})$-invariant antisymmetric tensors ϵ^{ab}, $\epsilon^{a'b'}$ and $\epsilon^{a''b''}$, with $\epsilon^{01} =$, etc. These obey

$$(S_1)_a{}^c\,(S_1)_b{}^d\,\epsilon^{ab} = \epsilon^{cd}\,, \qquad \text{etc.} \tag{3.6}$$

We also note that since α, β and γ are invariant only under the U$(1)^3$ subgroup but not the full $(SL2,\mathbb{R})^3$, we may in addition employ the Kronecker deltas δ^{ab}, $\delta^{a'b'}$ and $\delta^{a''b''}$ in their construction, since δ is invariant under the U(1) subgroup of SL$(2,\mathbb{R})$. It is straightforward to see that we may now rewrite α, β, γ and Δ in (3.5) as follows:

$$\alpha^2 = \delta^{ab}\,\delta^{a'b'}\,\delta^{a''b''}\,\gamma_{aa'a''}\,\gamma_{bb'b''} + 2\beta\,, \tag{3.7}$$
$$\beta = -\frac{1}{2}(\delta^{ab}\,\epsilon^{a'b'}\,\epsilon^{a''b''} + \epsilon^{ab}\,\delta^{a'b'}\,\epsilon^{a''b''} + \epsilon^{ab}\,\epsilon^{a'b'}\,\delta^{a''b''})\,\gamma_{aa'a''}\,\gamma_{bb'b''}\,, \tag{3.8}$$
$$\gamma = \frac{1}{\alpha}\Big[\Delta + \frac{1}{2}\beta^2 - Y\Big]\,, \tag{3.9}$$
$$\Delta = \frac{1}{8}\epsilon^{ac}\,\epsilon^{a'b'}\,\epsilon^{a''b''}\,\epsilon^{bd}\,\epsilon^{c'd'}\,\epsilon^{c''d''}\,\gamma_{aa'a''}\,\gamma_{bb'b''}\,\gamma_{cc'c''}\,\gamma_{dd'd''}\,, \tag{3.10}$$

where

$$\begin{aligned} Y = \frac{1}{8}\Big(&\delta^{ac}\,\epsilon^{a'b'}\,\epsilon^{a''b''}\,\delta^{bd}\,\epsilon^{c'd'}\,\epsilon^{c''d''} + \delta^{a'c'}\,\epsilon^{ab}\,\epsilon^{a''b''}\,\delta^{b'd'}\,\epsilon^{cd}\,\epsilon^{c''d''} \\ &+\delta^{a''c''}\,\epsilon^{a'b'}\,\epsilon^{ab}\,\delta^{b''d''}\,\epsilon^{c'd'}\,\epsilon^{cd}\Big)\,\gamma_{aa'a''}\,\gamma_{bb'b''}\,\gamma_{cc'c''}\,\gamma_{dd'd''}\,. \end{aligned} \tag{3.11}$$

Note that, as expected, Δ in (3.10) is written without the use of the Kronecker deltas, since it is invariant under the entire SL$(2,\mathbb{R})^3$ symmetry.

The final step is to introduce the non-zero asymptotic values for the scalar fields. As stated above, this could be accomplished by acting on the expressions (3.7)–(3.10) with the coset symmetries SL$(2,\mathbb{R})^3$/U$(1)^3$. In fact a simpler way, with the added advantage that it will directly express α, β, γ and Δ in terms of the asymptotic values of the scalars (φ_i, χ_i), is to note that the only thing that will cause the expressions to change under the action of SL$(2,\mathbb{R})^3$/U$(1)^3$ is the fact that the Kronecker deltas are used in the construction of α, β and γ. If we replace all the Kronecker deltas in the expressions for α, β and γ by the corresponding scalar matrices (2.35), evaluated at infinity,

$$\delta^{ab} \longrightarrow \overline{M}_1^{ab}\,, \qquad \delta^{a'b'} \longrightarrow \overline{M}_2^{a'b'}\,, \qquad \delta^{a''b''} \longrightarrow \overline{M}_3^{a''b''}\,, \tag{3.12}$$

where the bars on the $\overline{M}_i$ indicate that the scalars in (2.35) are evaluated at infinity, then the expressions for these quantities will be invariant under the full SL$(2,\mathbb{R})^3$ symmetry. Thus we see that we shall now have

$$\begin{aligned} \alpha^2 &= \overline{M}_1^{ab}\,\overline{M}_2^{a'b'}\,\overline{M}_3^{a''b''}\,\gamma_{aa'a''}\,\gamma_{bb'b''} + 2\beta\,, \\ \beta &= -\frac{1}{2}(\overline{M}_1^{ab}\,\epsilon^{a'b'}\,\epsilon^{a''b''} + \epsilon^{ab}\,\overline{M}_2^{a'b'}\,\epsilon^{a''b''} + \epsilon^{ab}\,\epsilon^{a'b'}\,\overline{M}_3^{a''b''})\,\gamma_{aa'a''}\,\gamma_{bb'b''}\,, \\ \gamma &= \frac{1}{\alpha}\Big[\Delta + \frac{1}{2}\beta^2 - Y\Big]\,, \end{aligned} \tag{3.13}$$

where now

$$Y = \frac{1}{8}\Big(\overline{M}_1^{ac}\,\epsilon^{a'b'}\,\epsilon^{a''b''}\,\overline{M}_1^{bd}\,\epsilon^{c'd'}\,\epsilon^{c''d''} + \overline{M}_2^{a'c'}\,\epsilon^{ab}\,\epsilon^{a''b''}\,\overline{M}_2^{b'd'}\,\epsilon^{cd}\,\epsilon^{c''d''}$$
$$+\overline{M}_3^{a''c''}\,\epsilon^{a'b'}\,\epsilon^{ab}\,\overline{M}_3^{b''d''}\,\epsilon^{c'd'}\,\epsilon^{cd}\Big)\,\gamma_{aa'a''}\,\gamma_{bb'b''}\,\gamma_{cc'c''}\,\gamma_{dd'd''}\,. \tag{3.14}$$

Δ is unchanged, and is still given by (3.10).

3.2 Scalar fields in the U-duality formulation

The dilatons and axions in the 5-charge static black solutions in the heterotic formulation can be read off from equations (40), (41) and (42) of [3],[6] together with eq. (2.9). Mapping to the charges of U-duality formulation using (2.41) and (2.40), we have

$$e^{2\Phi} = \frac{(r+\overline{Q}_1)(r+\overline{Q}_2)}{\sqrt{V}}\,,\qquad \Psi = -\frac{\bar{p}\,(\overline{Q}_1-\overline{Q}_2)}{2(r+\overline{Q}_1)(r+\overline{Q}_2)}\,, \tag{3.15}$$
$$e^{\tilde{\varphi}_2} = \frac{(r+\overline{Q}_2)(r+\overline{Q}_4)}{\sqrt{V}}\,,\qquad \tilde{\chi}_2 = \frac{\bar{p}\,[r+\frac{1}{2}(\overline{Q}_1+\overline{Q}_2)]}{(r+\overline{Q}_2)(r+\overline{Q}_4)}\,, \tag{3.16}$$
$$e^{\varphi_3} = \frac{(r+\overline{Q}_1)(r+\overline{Q}_4)}{\sqrt{V}}\,,\qquad \chi_3 = -\frac{\bar{p}\,[r+\frac{1}{2}(\overline{Q}_1+\overline{Q}_2)]}{(r+\overline{Q}_1)(r+\overline{Q}_4)}\,. \tag{3.17}$$

After mapping into the scalar variables of the U-duality formulation, using eq. (2.37), we therefore have

$$e^{\varphi_1} = \frac{(r+\overline{Q}_1)(r+\overline{Q}_2)}{\sqrt{V}}\,,\qquad \chi_1 = -\frac{\bar{p}\,(\overline{Q}_1-\overline{Q}_2)}{2(r+\overline{Q}_1)(r+\overline{Q}_2)}\,, \tag{3.18}$$
$$e^{\varphi_2} = \frac{(r+\overline{Q}_1)(r+\overline{Q}_3)}{\sqrt{V}}\,,\qquad \chi_2 = -\frac{\bar{p}\,[r+\frac{1}{2}(\overline{Q}_1+\overline{Q}_2)]}{(r+\overline{Q}_1)(r+\overline{Q}_3)}\,, \tag{3.19}$$
$$e^{\varphi_3} = \frac{(r+\overline{Q}_1)(r+\overline{Q}_4)}{\sqrt{V}}\,,\qquad \chi_3 = -\frac{\bar{p}\,[r+\frac{1}{2}(\overline{Q}_1+\overline{Q}_2)]}{(r+\overline{Q}_1)(r+\overline{Q}_4)}\,. \tag{3.20}$$

To find the expressions for the scalar fields in the general 8-charge solution, we follow the same strategy that we used previously for the metric. Namely, we take the expressions above for the scalar fields in the 5-charge seed solution, and then fill these out to 8-charge solutions by acting with the $\mathrm{U}(1)^3 \in \mathrm{SL}(2,\mathbb{R})^3$ global symmetry transformations. A new feature that arises here is that the scalar fields, unlike the metric, themselves transform under the global symmetries, and so these transformations must be included also in the calculation. To be precise, the dilaton/axion pair (φ_1,χ_1) transforms under $\mathrm{SL}(2,\mathbb{R})_1$ but is inert under $\mathrm{SL}(2,\mathbb{R})_2$ and $\mathrm{SL}(2,\mathbb{R})_3$. Analogous statements apply to (φ_2,χ_2) and to (φ_3,χ_3).

As in our construction of the metric functions for the 8-charge solution, we find it convenient to first obtain the scalar solutions for the case where the asymptotic values of

[6]The labelling of the torus coordinates is opposite in [3] to the labelling we are using in this paper, with our coordinates (z^1,z^2) in appendix A being equal to (y^2,y^1) in [3]. This means that the torus metric and 2-form components G_{11}, G_{22}, G_{12} and B_{12} in eqs. (42) in [3] should be interpreted as our $\mathbb{G}_{22}$, $\mathbb{G}_{11}$, $\mathbb{G}_{12}$ and $-\mathbb{B}_{12}$ respectively (see eqs. (2.8) and (2.9)).

JHEP09(2021)188

the dilatons and axions are all zero. A very simple replacement at the final stage of the calculation allows the introduction of arbitrary values for the asymptotic scalars.

The process of promoting the 5-charge seed solution to a full 8-charge solution proceeds in the same we that we described in [9]. Because we have made a small adjustment in the conventions in this paper, in order to allow a direct mapping to equivalent the results in the heterotic formulation, we need first to record the explicit results for the $\mathrm{U}(1)^3$ rotations that yield the 8-charge solution. Specifically, it was convenient in this paper to change the 5-charge configuration in the U-duality formalism from the one specified in eq. (B.4) of [9] to the one specified in eq. (3.1) of this paper. (That is, $\mathbf{P} = (p, -p, 0, 0)$ rather than $\mathbf{P} = (0, 0, p, -p)$.) This modifies the expressions that were found in [9] when solving for the 5 charges and the three U(1) angles θ_i in terms of the 8 generic charges Q_i and P^i. Thus we solve the 8 equations contained in

$$\bar{\gamma}_{aa'a''} = (U_1)_a{}^b\, (U_2)_{a'}{}^{b'}\, (U_3)_{a''}{}^{b''}\, \gamma_{bb'b''}\,, \tag{3.21}$$

where $\bar{\gamma}_{aa'a''}$ is the charge tensor for the 5-charge seed configuration in (3.1); $\gamma_{aa'a''}$ is the charge tensor of the generic 8-charge configuration, and U_i denote the three U(1) matrices with θ_1, θ_2 and θ_3 as parameters:

$$U_i = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}\,. \tag{3.22}$$

(See eq. (2.30) for a definition of $\gamma_{aa'a''}$.) Eq. (3.21) thus yields expressions for the three angles θ_i and the 5 charges of the seed solution, expressed in terms of the 8 arbitrary charges of the generic 8-charge solution. We find

$$\tan\theta_+ = \frac{(P^1+P^2)-(Q_1+Q_2)\tan\theta_1}{(Q_3+Q_4)+(P^3+P^4)\tan\theta_1}\,, \qquad \tan\theta_- = \frac{(P^3-P^4)+(Q_3-Q_4)\tan\theta_1}{(Q_1-Q_2)-(P^1-P^2)\tan\theta_1}\,, \tag{3.23}$$

$$\tan 2\theta_1 = \frac{2(P^3+P^4)(Q_3+Q_4)-2(P^1+P^2)(Q_1+Q_2)}{(P^1+P^2+P^3+P^4)(P^1+P^2-P^3-P^4)-(Q_1+Q_2+Q_3+Q_4)(Q_1+Q_2-Q_3-Q_4)}\,,$$

where $\theta_\pm = \theta_2 \pm \theta_3$, and

$$\bar{p} = -\bar{\gamma}_{000}\,, \quad \bar{Q}_1 = -\bar{\gamma}_{111}\,, \quad \bar{Q}_2 = \bar{\gamma}_{100}\,, \quad \bar{Q}_3 = \bar{\gamma}_{010}\,, \quad \bar{Q}_4 = \bar{\gamma}_{001}\,, \tag{3.24}$$

where we have placed bars on the 5 seed charges $\bar{Q}_i$ and $\bar{p}$.

To begin, let us consider the dilaton/axion pair (φ_1, χ_1). The first step is to promote the right-hand sides of eqs. (3.18) to 8-charge expressions, by using eqs. (3.24) together with (3.24). We shall write the expressions in (3.18) as

$$e^{-\varphi_1} = \frac{\sqrt{V(r)}}{D(r)} \qquad \chi_1 = \frac{\delta}{D(r)}\,, \tag{3.25}$$

where

$$D(r) = r^2 + \alpha_D\, r + \beta_D\,, \tag{3.26}$$

$$\alpha_D^2 = (\bar{Q}_1 + \bar{Q}_2)^2\,, \qquad \beta_D = \bar{Q}_1\, \bar{Q}_2\,, \qquad \delta = \frac{1}{2}\bar{p}\,(\bar{Q}_1 - \bar{Q}_2)\,. \tag{3.27}$$

JHEP09(2021)188

We first promote these 5-charge expressions to 7-charge expressions, by acting with the $U(1)_2$ and $U(1)_3$ rotations, while keeping $\theta_1 = 0$. From (3.24), this latter requirement means the 7-charge restriction implies the Q_A and P^A must obey

$$(P^1 + P^2)(Q_1 + Q_2) - (P^3 + P^4)(Q_3 + Q_4) = 0\,. \tag{3.28}$$

This leads straightforwardly to the 7-charge augmentations

$$\begin{aligned}\alpha^2_{D(7)} &= (Q_1 + Q_2)^2 + (P^3 + P^4)^2\,, \qquad \beta_{D(7)} = Q_1\, Q_2 + P^3\, P^4\,,\\ \delta_{(7)} &= \frac{1}{4}[(P^1 - P^2)(Q_1 - Q_2) - (P^3 - P^4)(Q_3 - Q_4)]\,.\end{aligned} \tag{3.29}$$

The augmentation from 7 to 8 charges, relaxing the constraint (3.28), is achieved by further rotating the quantities in (3.29) under the remaining $U(1)_1$ transformation (now with θ_2 and θ_3 set to zero). It is helpful at this point to establish a general result for the $U(1)_1$ rotation of a general 2-index symmetric tensor W_{ab}; this obeys the transformation

$$W_{ab} \longrightarrow \widetilde{W}_{ab} = (S_1)_a{}^c\, (S_1)_b{}^d\, W_{cd}\,, \tag{3.30}$$

where S_1 is the $U(1)_1 \in SL(2,\mathbb{R})_1$ matrix

$$S_1 = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}\,. \tag{3.31}$$

If we define the $U(1)_2 \times U(1)_3$ -invariant symmetric tensor

$$Z_{ab} = (\epsilon^{a'b'}\, \epsilon^{a''b''} - \delta^{a'b'}\, \delta^{a''b''})\, \gamma_{aa'a''}\, \gamma_{bb'b''}\,, \tag{3.32}$$

where the charge tensor $\gamma_{aa'a''}$ is defined in (2.30), then the expression for $\tan 2\theta_1$ in eq. (3.24) can be written in the compact form

$$\tan 2\theta_1 = \frac{2Z_{01}}{Z_{00} - Z_{11}}\,. \tag{3.33}$$

From this we find

$$\cos 2\theta_1 = \frac{Z_{00} - Z_{11}}{\Xi}\,, \qquad \sin 2\theta_1 = \frac{2Z_{01}}{\Xi}\,, \tag{3.34}$$

where Ξ is the $U(1)^3$-invariant quantity defined by

$$\Xi^2 = Z_{ab}\, Z_{cd}\, (\delta^{ac}\, \delta^{bd} - \epsilon^{ac}\, \epsilon^{bd})\,. \tag{3.35}$$

It then follows straightforwardly from (3.30) that

$$\begin{aligned}\widetilde{W}_{00} &= \frac{1}{2}\Big[\delta^{ab}\, W_{ab} + \frac{G(W)}{\Xi}\Big]\,, \qquad \widetilde{W}_{11} = \frac{1}{2}\Big[\delta^{ab}\, W_{ab} - \frac{G(W)}{\Xi}\Big]\,,\\ \widetilde{W}_{01} &= \frac{1}{\Xi}\, Z_{ab}\, W_{cd}\, \delta^{ac}\, \epsilon^{bd}\,,\end{aligned} \tag{3.36}$$

where

$$G(W) \equiv W_{ab}\, Z_{cd}\, (\delta^{ac}\, \delta^{bd} - \epsilon^{ac}\, \epsilon^{bd})\,. \tag{3.37}$$

(Note from (3.35) that $G(Z) = \Xi^2$.)

JHEP09(2021)188

To proceed with the augmentation of the 7-charge expressions (3.29), we note that if we define the two $\mathrm{U}(1)_2 \times \mathrm{U}(1)_3$ -invariant tensors

$$X_{ab} = \epsilon^{a'b'}\,\epsilon^{a''b''}\,\gamma_{aa'a''}\,\gamma_{bb'b''}\,, \qquad Y_{ab} = \delta^{a'b'}\,\delta^{a''b''}\,\gamma_{aa'a''}\,\gamma_{bb'b''}\,, \tag{3.38}$$

where the charge tensor $\gamma_{aa'a''}$ is defined in eq. (2.30), then the 7-charge expressions in (3.29) can be written as

$$\alpha^2_{D(7)} = -\widetilde{Z}_{11}\,, \qquad \beta_{D(7)} = -\frac{1}{2}\widetilde{X}_{11}\,, \qquad \delta_{(7)} = \frac{1}{4}(\widetilde{X}_{01} + \widetilde{Y}_{01})\,. \tag{3.39}$$

Using the $\mathrm{U}(1)_1$ transformations (3.36),[7] we therefore obtain the 8-charge expressions

$$\begin{aligned} \alpha_D^2 &= -\frac{1}{2}(\delta^{ab}\,Z_{ab} - \Xi)\,, \\ \beta_D &= -\frac{1}{4}\left[\delta^{ab}\,X_{ab} - \frac{G(X)}{\Xi}\right], \\ \delta &= -\frac{1}{4\Xi}\,(X_{ab} + Y_{ab})\,Z_{cd}\,\delta^{ac}\,\epsilon^{bd}\,, \end{aligned} \tag{3.40}$$

and $G(X)$ is given by substituting $W_{ab} = X_{ab}$ in (3.37). These expressions are all manifestly invariant under $\mathrm{U}(1)^3$. It is useful to note that α_D can be written in terms of α (given in (3.7)) and Ξ as

$$\alpha_D = \frac{\alpha^2 + \Xi}{2\alpha}\,. \tag{3.41}$$

It is also worth noting that if we define the $\mathrm{U}(1)^3$ invariants

$$\begin{aligned} Z_1 &= -\delta^{ab}\,\epsilon^{a'b'}\,\epsilon^{a''b''}\,\gamma_{aa'a''}\,\gamma_{bb'b''}\,, \\ Z_2 &= -\epsilon^{ab}\,\delta^{a'b'}\,\epsilon^{a''b''}\,\gamma_{aa'a''}\,\gamma_{bb'b''}\,, \\ Z_3 &= -\epsilon^{ab}\,\epsilon^{a'b'}\,\delta^{a''b''}\,\gamma_{aa'a''}\,\gamma_{bb'b''}\,, \\ Z_0 &= \delta^{ab}\,\delta^{a'b'}\,\delta^{a''b''}\,\gamma_{aa'a''}\,\gamma_{bb'b''}\,, \end{aligned} \tag{3.42}$$

then Ξ^2, defined in (3.35), can be written in the factorised form

$$\Xi^2 = (Z_0 + Z_1 + Z_2 + Z_3)(Z_0 + Z_1 - Z_2 - Z_3)\,. \tag{3.43}$$

It remains to implement the $\mathrm{U}(1)^3$ transformations that augment the 5 charges to 8 charges on the scalar fields. Since we are looking specifically at φ_1 and χ_1, which transform only under $\mathrm{U}(1)_1$, this means that we just have to make the replacement

$$\chi_1 + i\,e^{-\varphi_1} \longrightarrow \frac{(\chi_1 + i\,e^{-\varphi_1})\,\cos\theta_1 + \sin\theta_1}{\cos\theta_1 - (\chi_1 + i\,e^{-\varphi_1})\,\sin\theta_1}\,. \tag{3.44}$$

In other words, the scalars φ_1 and χ_1 will be given by

$$e^{\varphi_1} = \frac{\mathcal{D}\,D(r)}{\sqrt{V(r)}}\,, \qquad \chi_1 = \frac{\mathcal{C}}{\mathcal{D}}\,, \tag{3.45}$$

[7] The transformation (3.36) is viewed here as the *inverse*, taking us from the general 8-charge (untilded) configuration to the 7-charge (tilded) configuration.

where

$$
\begin{aligned}
\mathcal{C} &= \frac{\delta}{D(r)}\,\cos 2\theta_1 - \frac{1}{2}\Big[\frac{V(r)+\delta^2}{D(r)^2} - 1\Big]\,\sin 2\theta_1\,,\\
\mathcal{D} &= \frac{1}{2}\Big[1 + \frac{V(r)+\delta^2}{D(r)^2}\Big] - \frac{\delta}{D(r)}\,\sin 2\theta_1 + \frac{1}{2}\Big[1 - \frac{V(r)+\delta^2}{D(r)^2}\Big]\,\cos 2\theta_1\,,
\end{aligned}
\tag{3.46}
$$

where $V(r)$ in (3.2) is calculated using the 8-charge expressions (3.7)–(3.10), the quantities δ, α_D and β_D appearing in $D(r)$ are given by the 8-charge expressions in (3.40), and $\cos 2\theta_1$ and $\sin 2\theta_1$ are given by (3.34).

The expressions for φ_1 and χ_1 are in fact considerably simpler than is immediately evident from (3.45) and (3.46). This is because the quadratic function $D(r)$ is actually a divisor of the quartic function $V(r)+\delta^2$ that appears in (3.46). In fact we can show that

$$
V(r)+\delta^2 = D(r)\,\widetilde{D}(r)\,,\qquad \widetilde{D}(r) = r^2 + \tilde{\alpha}_D\,r + \tilde{\beta}_D\,,
\tag{3.47}
$$

where the coefficients in the quadratic function $\widetilde{D}(r)$ are given by

$$
\tilde{\alpha}_D^2 = -\frac{1}{2}(\delta^{ab}\,Z_{ab} + \Xi)\,,\qquad \tilde{\beta}_D = -\frac{1}{4}\Big[\delta^{ab}\,X_{ab} + \frac{G(X)}{\Xi}\Big]\,.
\tag{3.48}
$$

(Note that $\tilde{\alpha}_D$ and $\tilde{\beta}_D$ are closely related to the coefficients α_D and β_D in the function $D(r) = r^2 + \alpha_D\,r + \beta_D$ given in (3.40).) Thus we see that φ_1 and χ_1 are given by

$$
e^{\varphi_1} = \frac{r^2 + b_1\,r + b_0}{\sqrt{V(r)}}\,,\qquad \chi_1 = \frac{c_1\,r + c_0}{r^2 + b_1\,r + b_0}\,,
\tag{3.49}
$$

where

$$
\begin{aligned}
b_1 &= \frac{\alpha_D + \tilde{\alpha}_D}{2} + \frac{(\alpha_D - \tilde{\alpha}_D)(Z_{00} - Z_{11})}{2\Xi}\,,\\
b_0 &= \frac{\beta_D + \tilde{\beta}_D}{2} - \frac{2\delta\,Z_{01}}{\Xi} + \frac{(\beta_D - \tilde{\beta}_D)(Z_{00} - Z_{11})}{2\Xi}\,,\\
c_1 &= \frac{(\alpha_D - \tilde{\alpha}_D)\,Z_{01}}{\Xi}\,,\qquad c_0 = \frac{\delta\,(Z_{00} - Z_{11}) + (\beta_D - \tilde{\beta}_D)\,Z_{01}}{\Xi}\,,
\end{aligned}
\tag{3.50}
$$

Using our previous expressions for the various quantities appearing in (3.50), we find that in terms of the eight charges of the general static BPS black holes,

$$
\begin{aligned}
b_1 &= \frac{1}{\alpha}\Big[(P^3 + P^4)\,\sum_A P^A + (Q_1 + Q_2)\,\sum_A Q_A\Big]\,,\\
b_0 &= P^3\,P^4 + Q_1\,Q_2\,,\\
c_1 &= \frac{1}{\alpha}\Big[(P^1 + P^2)(Q_1 + Q_2) - (P^3 + P^4)(Q_3 + Q_4)\Big]\,,\\
c_0 &= \frac{1}{2}(P^1\,Q_1 + P^2\,Q_2 - P^3\,Q_3 - P^4\,Q_4)\,,
\end{aligned}
\tag{3.51}
$$

where as usual $\alpha = \sqrt{(\sum_A P^A)^2 + (\sum_A Q_A)^2}$.

Finally, it remains to introduce non-vanishing asymptotic values for the scalar fields. As before when we discussed the metric, this is accomplished by making $\mathrm{SL}(2,\mathbb{R})^3/\mathrm{U}(1)^3$ coset transformations on the charges, and now also on the scalar fields. Thus we introduce an asymptotic coset vielbein $\bar{\mathcal{V}}_i$ for each of the $\mathrm{SL}(2,\mathbb{R})_i/\mathrm{U}(1)_i$ coset factors, with

$$\bar{\mathcal{V}}_i = \begin{pmatrix} e^{\frac{1}{2}\bar{\varphi}_i} & -\bar{\chi}_i\, e^{\frac{1}{2}\bar{\varphi}_i} \\ 0 & e^{-\frac{1}{2}\bar{\varphi}_i} \end{pmatrix}, \tag{3.52}$$

where $(\bar{\varphi}_i, \bar{\chi}_i)$ denotes the asymptotic values of the scalar fields. The corresponding transformation of the charge tensor is therefore

$$\gamma_{aa'a''} \longrightarrow \bar{\mathcal{V}}^b_{1\,a}\, \bar{\mathcal{V}}^{b'}_{2\,a'}\, \bar{\mathcal{V}}^{b''}_{3\,a''}\, \gamma_{bb'b''}\,. \tag{3.53}$$

The effect of this is that all $\mathrm{U}(1)^3$ invariants, such as α, β, γ, α_D, β_D, δ, $\tilde{\alpha}_D$, $\tilde{\beta}_D$ will receive modification, namely that each Kronecker delta in the expressions (3.7)–(3.9), (3.32), (3.42), (3.50), etc., will be replaced by the corresponding asymptotic scalar matrix

$$\overline{M}_i = \bar{\mathcal{V}}_i^T\, \bar{\mathcal{V}}_i\,, \tag{3.54}$$

as in eq. (3.12). Additionally, the components of the tensor Z_{ab} appearing in the expressions (3.50) will be transformed by making the replacement

$$Z_{ab} \longrightarrow \bar{\mathcal{V}}^c_{1\,a}\, \bar{\mathcal{V}}^d_{1\,b}\, Z_{cd}\,. \tag{3.55}$$

As already observed, the quantity Δ receives no modification because its construction in (3.10) uses only the $\mathrm{SL}(2,\mathbb{R})$-invariant epsilon tensors, and no Kronecker deltas.

It is worth recording that even when the asymptotic scalars are non-zero, the quantity Ξ, defined now by (3.35) with all Kronecker deltas replaced by the asymptotic scalar M matrices as in (3.12), is again simply factorisable, as

$$\Xi = \alpha\, \tilde{\alpha}\,, \tag{3.56}$$

where α is given in (3.13) and $\tilde{\alpha}$ is the closely related quantity given by

$$\begin{aligned} \tilde{\alpha}^2 \;=\; \Big(&\overline{M}_1^{ab}\, \overline{M}_2^{a'b'}\, \overline{M}_3^{a''b''}\, \gamma_{aa'a''}\, \gamma_{bb'b''} - \overline{M}_1^{ab}\, \epsilon^{a'b'}\, \epsilon^{a''b''} \\ &+ \epsilon^{ab}\, \overline{M}_2^{a'b'}\, \epsilon^{a''b''} + \epsilon^{ab}\, \epsilon^{a'b'}\, \overline{M}_3^{a''b''}\Big)\, \gamma_{aa'a''}\, \gamma_{bb'b''}\,. \end{aligned} \tag{3.57}$$

In other words Ξ^2, which is a quartic multinomial in the charges with coefficients that are multinomial in the $\bar{\chi}_i$ and $e^{\bar{\varphi}_i}$ asymptotic scalar values, factorises as the product of the two quadratic multinomials α^2 and $\tilde{\alpha}^2$.

It is also useful to note that the quantities α_D and $\tilde{\alpha}_D$, given by (3.40) and (3.48) after the replacements (3.12), are related to α and $\tilde{\alpha}$ by

$$\alpha_D + \tilde{\alpha}_D = \alpha\,, \qquad \alpha_D - \tilde{\alpha}_D = \tilde{\alpha}\,. \tag{3.58}$$

Thus after introducing the asymptotic scalar values the coefficients b_1 and c_1 in eqs. (3.50) are particularly simple, and are given by

$$b_1 = \frac{1}{2\alpha}\Big[\alpha^2 + \widetilde{Z}_{00} - \widetilde{Z}_{11}\Big]\,, \qquad c_1 = \frac{1}{\alpha}\, \widetilde{Z}_{01}\,, \tag{3.59}$$

where $\widetilde{Z}_{ab} = \overline{\mathcal{V}}^c_{1a}\,\overline{\mathcal{V}}^d_{1b}\,Z_{cd}$ and all Kronecker deltas involved in the construction of α and Z_{ab} are as usual replaced by M matrices according to (3.12).

The action of the coset transformations on the scalar fields themselves will be given by sending

$$M_i \longrightarrow \overline{\mathcal{V}}_i^T\, M_i\, \overline{\mathcal{V}}_i\,, \qquad \text{for} \quad i = 1, 2, 3\,, \tag{3.60}$$

where the M_i scalar matrices are given in (2.35). These transformations amount to

$$e^{-\varphi_i} \longrightarrow e^{-\varphi_i - \bar{\varphi}_i}\,, \qquad \chi_i \longrightarrow \bar{\chi}_i + \chi_i\, e^{-\bar{\varphi}_i}\,. \tag{3.61}$$

The U(1) rotation angle θ_1 will be modified by the coset transformations, as dictated by substituting the $\gamma_{aa'a''}$ transformations (3.53) into the expression (3.32) for Z_{ab}, and then substituting these transformed Z_{ab} components into (3.33). However, since we have already re-expressed the expressions in (3.45) and (3.46) for φ_1 and χ_1 in the simpler forms given by (3.49) and (3.50), we no longer need to implement the coset transformation on θ_1 explicitly.

The procedure we have described above provides specifically the expressions for the dilaton/axion pair (φ_1, χ_1) in the general 8-charge static BPS black hole solutions. One could repeat the discussion, starting from the same 5-charge seed solution, to augment the expressions (3.19) and (3.20) for the (φ_2, χ_2) and (φ_3, χ_3) dilaton/axion pairs. In fact a simpler way of arriving at the same result is to exploit the triality symmetry of the U-duality formulation of STU supergravity. The precise statement of this triality can be seen from the definition of the $\mathrm{SL}(2,\mathbb{R})^3$ charge tensor $\gamma_{aa'a''}$ in (2.30). The first $\mathrm{SL}(2,\mathbb{R})$ index, a, is associated with the (φ_1, χ_1) pair, and the charges (P^2, Q_2); the a' index with (φ_2, χ_2) and the charges (P^3, Q_3), and the a'' index with (φ_3, χ_3) and the charges (P^4, Q_4). Thus from the expressions we have obtained for (φ_1, χ_1), we just have to permute the labellings according to this triality correspondence, in order to obtain the expressions for (φ_2, χ_2) and (φ_3, χ_3).

Acting with the $\mathbb{Z}_2 \in$ triality symmetry

$$(\varphi_1, \chi_1; P^2, Q_2) \longleftrightarrow (\varphi_2, \chi_2; P^3, Q_3) \tag{3.62}$$

on the results obtained above for the φ_1 and χ_1 scalars will give the expressions for φ_2 and χ_2. Acting instead with the $\mathbb{Z}_2 \in$ triality symmetry

$$(\varphi_1, \chi_1; P^2, Q_2) \longleftrightarrow (\varphi_3, \chi_3; P^4, Q_4) \tag{3.63}$$

will give the expressions for φ_3 and χ_3.

If we specialise for simplicity to the case with vanishing asymptotic values for the scalar fields, acting with the triality transformations (3.62) or (3.63) on the constants in (3.51) will map (3.49) into expressions for φ_2 and χ_2, or φ_3 and χ_3, respectively.

As a check on the calculations in this section, we can take the expressions of the φ_i and χ_i scalars and make the 5-charge specialisation given by eq. (3.1). Doing this, it is straightforward to see that we do indeed recover the expressions given in eqs. (3.18), (3.19) and (3.20). Expressed conversely, this shows that if we were instead to act with the U-duality transformations in order to elevate the 5-charge expressions for (φ_2, χ_2) and (φ_3, χ_3)

to general 8-charge expressions, we would indeed obtain the results that follow by making the triality transformations on the general 8-charge expressions for (φ_1, χ_1) that we have constructed.

3.3 Gauge fields for the 8-charge BPS black holes

Having obtained the expressions for the metric and the scalar fields, the form of the gauge fields in the general 8-charge static BPS black hole solutions follow straightforwardly from the gauge fields equations of motion.

In the U-duality formulation, we find

$$F^A = P^A \sin\theta \, d\theta \wedge d\varphi + \frac{1}{\sqrt{V}} \left((f^R)^{-1}\right)^{AB} (Q_B + f^I_{BC}\, P^C)\, dt \wedge dr\,, \tag{3.64}$$

which is consistent with the definitions of the electric and magnetic charges in (2.28).

Note that our conventions for Hodge dualisation are such that in the metric (3.2) we have

$$* \,(\sin\theta \, d\theta \wedge d\varphi) = \frac{1}{\sqrt{V}}\, dt \wedge dr\,, \qquad *(dt \wedge dr) = -\sqrt{V}\, \sin\theta \, d\theta \wedge d\varphi\,. \tag{3.65}$$

4 General 8-charge static BPS black holes in the heterotic formulation

4.1 8-charge static BPS black hole metric

The starting point for writing the general 8-charge static BPS black holes with arbitrary asymptotic values for the scalar fields is again the metric (3.2). The charges in the seed solution with five independent charges and with vanishing asymptotic scalars take the form

$$\vec{\alpha}_0 = (q_1, -\bar{q}, q_3, \bar{q})\,, \qquad \vec{p}_0 = (0, p^2, 0, p^4) \tag{4.1}$$

in the heterotic basis [3]. (The superscripts 2 and 4 refer to the indexed labelling of charges p^i. See section 2.2 for the notation.) The quantity $\bar{q}$ parameterises the introduction of the fifth, independent, charge) The coefficients α, β, γ and Δ appearing in the metric function $V(r)$ are given by [3]

$$\begin{aligned}
\alpha &= q_1 + q_3 + p^2 + p^4\,,\\
\beta &= q_1\, q_3 - \bar{q}^2 + p^2 p^4 + (q_1 + q_3)(p^2 + p^4)\,,\\
\gamma &= (q_1 q_3 - \bar{q}^2)(p^2 + p^4) + p^2 p^4\,(q_1 + q_3)\,,\\
\Delta &= (q_1 q_3)(p^2 p^4) - \tfrac{1}{4}\bar{q}^2\,(p^2 + p^4)^2\,.
\end{aligned} \tag{4.2}$$

(Here $\bar{q}^2$ means just the square of $\bar{q}$.)

The general solutions with 8 charges and non-vanishing asymptotic scalars can be filled out by acting with the global $\mathrm{O}(2,2) \times \mathrm{SL}(2,\mathbb{R})$ symmetry of the heterotic formulation, where $\mathrm{O}(2,2)$ is the T-duality symmetry from the 2-torus and $\mathrm{SL}(2,\mathbb{R})$ is the electric/magnetic S-duality symmetry. We proceed in three stages; first, acting with the $\mathrm{U}(1)^2$ subgroup of $\mathrm{O}(2,2)$ to augment the 5-charge solution to 7 charges; then acting with the $\mathrm{U}(1)$ subgroup of $\mathrm{SL}(2,\mathbb{R})$ to augment further to 8 charges; and finally, acting with the remaining $\mathrm{O}(2,2)/\mathrm{U}(1)^2$ and $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ cosets in order to introduce the non-vanishing asymptotic scalars.

Acting with $\mathbf{U(1)^2 \in O(2,2)}$. The 5-charge starting point can be filled out to 7 independent charges by acting with the $\mathrm{U}(1)^2$ compact subgroup of $\mathrm{O}(2,2)$. Since α, β and γ should be invariant under $\mathrm{U}(1)^2$ but not under the remaining $\mathrm{O}(2,2)/\mathrm{U}(1)^2$ coset action (that is, when the asymptotic scalars vanish), the 2 additional charges can be introduced by writing α, β and γ as $\mathrm{U}(1)^2$-invariant expressions in $\vec{\alpha}$ and $\vec{p}$ that reduce to (4.2) under the 5-charge specialisation (4.1). In fact, rather than using the magnetic charges $\vec{p}$ with components p^i we shall instead find it convenient to lower the index and work with the magnetic charges $\vec{\beta}$, which have the components $\beta_i = L_{ij}\, p^j$, as defined in (2.17). The available "building blocks" are the matrix L defined in (2.11), which is fully $\mathrm{O}(2,2)$ invariant, and the 4×4 identity matrix $\mathbb{I}_4$, which is invariant only under the $\mathrm{U}(1)^2$ subgroup.[8] We can also construct the more general expression for Δ along similar lines. Since Δ is in fact invariant under the full $\mathrm{O}(2,2)$ symmetry its expression will not involve the use of $\mathbb{I}_4$.

Defining the matrices

$$\nu_{\pm} = I_4 \pm L^{-1}\,, \tag{4.4}$$

one can straightforwardly establish that α, β, γ and Δ may be written for the 7-charge solutions in $\mathrm{U}(1)^2$-invariant terms as follows:

$$\begin{aligned}
\alpha^2 &= \vec{\alpha}^{\,T} \nu_+ \vec{\alpha} + \vec{\beta}^{\,T} \nu_+ \vec{\beta} + 2\bar{\Sigma}\,, && (4.5)\\
2\beta &= \vec{\alpha}^{\,T} L^{-1} \vec{\alpha} + \vec{\beta}^{\,T} L^{-1} \vec{\beta} + 2\bar{\Sigma}\,, && (4.6)\\
2\alpha\gamma &= (\vec{\alpha}^{\,T} L^{-1} \vec{\alpha} + \vec{\beta}^{\,T} L^{-1} \vec{\beta})\bar{\Sigma} \\
&\quad + (\vec{\alpha}^{\,T} L^{-1} \vec{\alpha})(\vec{\beta}^{\,T} \nu_+ \vec{\beta}) + (\vec{\beta}^{\,T} L^{-1} \vec{\beta})(\vec{\alpha}^{\,T} \nu_+ \vec{\alpha})\,, && (4.7)\\
4\Delta &= (\vec{\alpha}^{\,T} L^{-1} \vec{\alpha})(\vec{\beta}^{\,T} L^{-1} \vec{\beta}) - (\vec{\alpha}^{\,T} L^{-1} \vec{\beta})^2\,, && (4.8)
\end{aligned}$$

where

$$\bar{\Sigma}^2 = (\vec{\alpha}^{\,T} \nu_+ \vec{\alpha})(\vec{\beta}^{\,T} \nu_+ \vec{\beta})\,. \tag{4.9}$$

These expressions reduce to those in (4.2) under the 5-charge specialisation (4.1). Note also that in the original 5-charge configuration (4.1), the charges satisfy the constraint

$$\vec{\alpha}^{\,T} \nu_+ \vec{\beta} = 0\,, \tag{4.10}$$

and that this continues to be true after filling out to 7 charges, as we have done by acting with $\mathrm{U}(1)^2$.

[8] Actually things are a bit more complicated. There are two further matrices that are invariant under the $\mathrm{U}(1)^2$ subgroup, namely K_1 and K_2 given by

$$K_1 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \tag{4.3}$$

where σ_2 is the usual Pauli matrix. Note that K_1 and K_2 are antisymmetric. It turns out that by using $K = K_1 + K_2$, one can write the quantity Σ^2 that appears later in (4.15) as a manifest perfect square. It then gets dressed up with asymptotic scalars once these are turned on. See later in this section for a discussion of this point.

Acting with $\mathrm{U}(1) \in \mathrm{SL}(2,\mathbb{R})$. The constraint (4.10) is removed once the 8th and final charge is introduced. This is achieved by acting with the U(1) subgroup of the remaining $\mathrm{SL}(2,\mathbb{R})$ S-duality symmetry. Its action on $\vec{\alpha}$ and $\vec{\beta}$ is

$$\vec{\alpha} \to \vec{\alpha}\,\cos\psi - \vec{\beta}\,\sin\psi\,, \qquad \vec{\beta} \to \vec{\alpha}\,\sin\psi + \vec{\beta}\,\cos\psi\,, \tag{4.11}$$

where

$$\tan 2\psi = -\frac{2(\vec{\alpha}^{\,T}\nu_+\vec{\beta}\,)}{(\vec{\alpha}^{\,T}\nu_+\vec{\alpha}\,) - (\vec{\beta}^{\,T}\nu_+\vec{\beta}\,)}\,. \tag{4.12}$$

Thus we have

$$\cos 2\psi = \frac{1}{\Xi}\,(\vec{\alpha}^{\,T}\nu_+\vec{\alpha} - \vec{\beta}^{\,T}\nu_+\vec{\beta}\,)\,, \qquad \sin 2\psi = -\frac{2}{\Xi}\,\vec{\alpha}^{\,T}\nu_+\vec{\beta}\,, \tag{4.13}$$

where

$$\Xi^2 = (\vec{\alpha}^{\,T}\nu_+\vec{\alpha} + \vec{\beta}^{\,T}\nu_+\vec{\beta}\,)^2 - 4\Sigma^2\,, \tag{4.14}$$
$$\Sigma^2 = (\vec{\alpha}^{\,T}\nu_+\vec{\alpha}\,)(\vec{\beta}^{\,T}\nu_+\vec{\beta}\,) - (\vec{\alpha}^{\,T}\nu_+\vec{\beta}\,)^2\,. \tag{4.15}$$

We now apply the transformation (4.11), with $\cos 2\psi$ and $\sin 2\psi$ given by (4.13), to the expressions (4.5)–(4.8) in order to add in the 8th charge. It is useful to note that for any symmetric 4×4 matrix X we shall therefore have

$$\begin{aligned}
\vec{\alpha}^{\,T}X\,\vec{\alpha} &\longrightarrow \frac{1}{2}(\vec{\alpha}^{\,T}X\,\vec{\alpha} + \vec{\beta}^{\,T}X\,\vec{\beta}\,) + \frac{G(X)}{2\Xi}\,,\\
\vec{\beta}^{\,T}X\,\vec{\beta} &\longrightarrow \frac{1}{2}(\vec{\alpha}^{\,T}X\,\vec{\alpha} + \vec{\beta}^{\,T}X\,\vec{\beta}\,) - \frac{G(X)}{2\Xi}\,,\\
\vec{\alpha}^{\,T}X\,\vec{\beta} &\longrightarrow \frac{1}{\Xi}\Big[(\vec{\alpha}^{\,T}X\,\vec{\beta}\,)(\vec{\alpha}^{\,T}\nu_+\vec{\alpha} - \vec{\beta}^{\,T}\nu_+\vec{\beta}\,) - (\vec{\alpha}^{\,T}\nu_+\vec{\beta}\,)(\vec{\alpha}^{\,T}X\,\vec{\alpha} - \vec{\beta}^{\,T}X\,\vec{\beta}\,)\Big]\,,
\end{aligned} \tag{4.16}$$

where

$$G(X) = (\vec{\alpha}^{\,T}X\,\vec{\alpha} - \vec{\beta}^{\,T}X\,\vec{\beta}\,)(\vec{\alpha}^{\,T}\nu_+\vec{\alpha} - \vec{\beta}^{\,T}\nu_+\vec{\beta}\,) + 4(\vec{\alpha}^{\,T}X\,\vec{\beta}\,)(\vec{\alpha}^{\,T}\nu_+\vec{\beta}\,)\,. \tag{4.17}$$

Note that $G(\nu_+) = \Xi^2$.

Applying these results in (4.5)–(4.8), we therefore find that the general 8-charge expressions of α, β, γ and Δ are given by

$$\alpha^2 = \vec{\alpha}^{\,T}\nu_+\vec{\alpha} + \vec{\beta}^{\,T}\nu_+\vec{\beta} + 2\Sigma\,, \tag{4.18}$$
$$2\beta = \vec{\alpha}^{\,T}L^{-1}\vec{\alpha} + \vec{\beta}^{\,T}L^{-1}\vec{\beta} + 2\Sigma\,, \tag{4.19}$$
$$\begin{aligned}
2\alpha\gamma = {}& (\vec{\alpha}^{\,T}L^{-1}\vec{\alpha} + \vec{\beta}^{\,T}L^{-1}\vec{\beta}\,)\,\Sigma - 2(\vec{\alpha}^{\,T}L^{-1}\vec{\beta}\,)(\vec{\alpha}^{\,T}\nu_+\vec{\beta}\,)\\
&+(\vec{\alpha}^{\,T}L^{-1}\vec{\alpha}\,)(\vec{\beta}^{\,T}\nu_+\vec{\beta}\,) + (\vec{\beta}^{\,T}L^{-1}\vec{\beta}\,)(\vec{\alpha}^{\,T}\nu_+\vec{\alpha}\,)\,,
\end{aligned} \tag{4.20}$$
$$4\Delta = (\vec{\alpha}^{\,T}L^{-1}\vec{\alpha}\,)(\vec{\beta}^{\,T}L^{-1}\vec{\beta}\,) - (\vec{\alpha}^{\,T}L^{-1}\vec{\beta}\,)^2\,. \tag{4.21}$$

We may now observe that these 8-charge expressions may be written in a more compact notation by introducing the $\mathrm{SL}(2,\mathbb{R})$-valued 8-charge vector $\vec{v}^{\,a}$, where

$$\vec{v}^{\,1} = \vec{\alpha}\,, \qquad \vec{v}^{\,2} = \vec{\beta}\,. \tag{4.22}$$

We also introduce the $SL(2,\mathbb{R})$-invariant antisymmetric tensor ε_{ab}, with $\varepsilon_{12}=1$, and the Kronecker delta δ_{ab}, which is invariant only under the $U(1)$ subgroup of $SL(2,\mathbb{R})$. Using these, the quantities α, β, γ and Δ in (4.18)–(4.21) may be rewritten as

$$\begin{aligned}
\alpha^2 &= \delta_{ab}\,\vec{v}^{\,a\,T}\nu_+\,\vec{v}^{\,b} + 2\Sigma\,, && (4.23)\\
2\beta &= \delta_{ab}\,\vec{v}^{\,a\,T}L^{-1}\,\vec{v}^{\,b} + 2\Sigma\,, && (4.24)\\
2\alpha\gamma &= \delta_{ab}\,(\vec{v}^{\,a\,T}L^{-1}\,\vec{v}^{\,b})\,\Sigma + \varepsilon_{ac}\,\varepsilon_{bd}\,(\vec{v}^{\,a\,T}L^{-1}\,\vec{v}^{\,b})(\vec{v}^{\,c\,T}\nu_+\,\vec{v}^{\,d})\,, && (4.25)\\
8\Delta &= \varepsilon_{ac}\,\varepsilon_{bd}\,(\vec{v}^{\,a\,T}L^{-1}\,\vec{v}^{\,b})(\vec{v}^{\,c\,T}L^{-1}\,\vec{v}^{\,d})\,, && (4.26)
\end{aligned}$$

where

$$\Sigma^2 = \frac{1}{2}\varepsilon_{ac}\,\varepsilon_{bd}\,(\vec{v}^{\,a\,T}\nu_+\,\vec{v}^{\,b})(\vec{v}^{\,c\,T}\nu_+\,\vec{v}^{\,d})\,. \qquad (4.27)$$

Introducing the asymptotic scalars. This final step is achieved by acting with the cosets $O(2,2)/U(1)^2$ and $SL(2,\mathbb{R})/U(1)$, appropriately parameterised in terms of the asymptotic values of the scalar fields. This can be done by using a vielbein formulation for both the $O(2,2)/U(1)^2$ scalar coset matrix and the $SL(2,\mathbb{R})/U(1)$ scalar coset matrix.

For $O(2,2)/U(1)^2$, the scalar matrix M given in (2.8) can be written as

$$M = \mathcal{V}^T\,\mathcal{V}\,, \qquad \mathcal{V} = \begin{pmatrix} E^{-1} & -E^{-1}\,\mathbb{B} \\ 0 & E^T \end{pmatrix}, \qquad (4.28)$$

where E is the zweibein for the internal 2-torus metric $\mathbb{G}$. Note that $\mathcal{V}^T L^{-1}\,\mathcal{V} = L^{-1}$. In terms of indices we have

$$M^{ij} = \delta^{k\ell}\,\mathcal{V}_k{}^i\,\mathcal{V}_\ell{}^j\,, \qquad L^{ij} = L^{k\ell}\,\mathcal{V}_k{}^i\,\mathcal{V}_\ell{}^j\,. \qquad (4.29)$$

We shall then make the $O(2,2)/U(1)^2$ transformation $\boldsymbol{\mathcal{F}} \to (\bar{\mathcal{V}}^T)^{-1}\,\boldsymbol{\mathcal{F}}$ on the gauge fields, which implies the transformations

$$\vec{\alpha} \longrightarrow \bar{\mathcal{V}}\,\vec{\alpha}\,, \qquad \vec{\beta} \longrightarrow \bar{\mathcal{V}}\,\vec{\beta}\,, \qquad (4.30)$$

on the charges, where $\bar{\mathcal{V}}$ denotes the asymptotic value of the scalar vielbein $\mathcal{V}$. Since $\mathcal{V}^T L^{-1}\,\mathcal{V} = L$ and $\mathcal{V}^T\,\mathcal{V} = M$, this means that the only change in the expressions (4.18)–(4.27) will be that the matrix I_4 in $\nu_+ = I_4 + L^{-1}$ will change to $\overline{M}$, and so

$$\nu_+ \longrightarrow \mu_+\,, \qquad \mu_+ = \overline{M} + L^{-1}\,. \qquad (4.31)$$

In a similar way, the scalars Φ and Ψ enter the $SL(2,\mathbb{R})/U(1)$ scalar coset in the form

$$\mathcal{N} = e^{2\Phi}\begin{pmatrix} 1 & -\Psi \\ -\Psi & \Psi^2 + e^{-4\Phi} \end{pmatrix}, \qquad (4.32)$$

and this can be written in terms of a vielbein $U = \mathcal{U}$ as

$$\mathcal{N} = \mathcal{U}^T\,\mathcal{U}\,, \qquad \mathcal{U} = \begin{pmatrix} e^{\Phi} & -\Psi\,e^{\Phi} \\ 0 & e^{-\Phi} \end{pmatrix}. \qquad (4.33)$$

The components $\mathcal{U}^a{}_b$ of this matrix obey the $\mathrm{SL}(2,\mathbb{R})$ invariance condition

$$\mathcal{U}^c{}_a \mathcal{U}^d{}_b \, \varepsilon_{cd} = \varepsilon_{ab} \,, \tag{4.34}$$

while, contracted instead with δ_{cd}, we have

$$\mathcal{U}^c{}_a \mathcal{U}^d{}_b \, \delta_{cd} = \mathcal{N}_{ab} \,, \tag{4.35}$$

where $\mathcal{N}_{ab}$ are the components of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ scalar matrix (4.32). Thus if we transform the electric and magnetic charges under the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ transformation

$$\vec{v}^{\,a} \longrightarrow \overline{\mathcal{U}}^a{}_b \, \vec{v}^{\,b} \,, \tag{4.36}$$

where $\overline{\mathcal{U}}^a{}_b$ is the scalar vielbein (4.33) with the scalars set equal to their asymptotic values, then the only change in the expressions (4.23)–(4.26) and (4.27) will be that the Kronecker delta δ_{ab} will be replaced by the corresponding $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ scalar matrix $\overline{\mathcal{N}}_{ab}$, where the scalars in $\mathcal{N}_{ab}$ defined in (4.32) are set equal to their asymptotic values.

Pulling together the threads of the previous discussion, if we act with $\mathrm{O}(2,2)/\mathrm{U}(1)^2$ and $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset elements to introduce asymptotic values for all the scalar fields, the final expressions for the quantities α, β, γ and Δ given previously in (4.23)–(4.26) will be given by

$$\alpha^2 = \overline{\mathcal{N}}_{ab} \, \vec{v}^{\,a\,T} \mu_+ \, \vec{v}^{\,b} + \Sigma \,, \tag{4.37}$$

$$\beta = \frac{1}{2} \overline{\mathcal{N}}_{ab} \, \vec{v}^{\,a\,T} L^{-1} \, \vec{v}^{\,b} + \frac{1}{2}\Sigma \,, \tag{4.38}$$

$$\gamma = \frac{1}{2\alpha} \Big[\overline{\mathcal{N}}_{ab} \, (\vec{v}^{\,a\,T} L^{-1} \, \vec{v}^{\,b}) \, \Sigma + \varepsilon_{ac} \, \varepsilon_{bd} \, (\vec{v}^{\,a\,T} L^{-1} \, \vec{v}^{\,b})(\vec{v}^{\,c\,T} \mu_+ \, \vec{v}^{\,d}) \Big] \,, \tag{4.39}$$

$$\Delta = \frac{1}{8} \varepsilon_{ac} \, \varepsilon_{bd} \, (\vec{v}^{\,a\,T} L^{-1} \, \vec{v}^{\,b})(\vec{v}^{\,c\,T} L^{-1} \, \vec{v}^{\,d}) \,, \tag{4.40}$$

where

$$\Sigma^2 = \frac{1}{2} \varepsilon_{ac} \, \varepsilon_{bd} \, (\vec{v}^{\,a\,T} \mu_+ \, \vec{v}^{\,b})(\vec{v}^{\,c\,T} \mu_+ \, \vec{v}^{\,d}) \,. \tag{4.41}$$

As was first observed in [3], the quartic invariant Δ does not depend on the values of the asymptotic scalars.

Having obtained these general expressions in the heterotic formulation for the coefficients α, β, γ and Δ for static BPS black holes with eight independent charges and arbitrary asymptotic values for the six scalar fields, it is instructive to compare them with the analogous expressions in the U-duality formulation, which we obtained in (3.13) and (3.10). After making use of the mappings (2.42) and (2.40) between the charges $\vec{\alpha}$ and $\vec{\beta}$ of the heterotic formulation and the charges (P^i, Q_i) of the U-duality formulation, and also the mapping (2.37) between the scalar fields in the two formulations, it is straightforward to verify that the two sets of expressions for α, β, γ and Δ agree.

There is, however, one respect in which the two sets of expressions for the constants α, β and γ ostensibly differ in the two formulations. In the U-duality formulation, the $\mathrm{SL}(2,\mathbb{R})^3$-invariant expressions in (3.13) have the feature that α^2, β and $\alpha\gamma$ are manifestly polynomial in the eight charges. By contrast, in the heterotic formulation the corresponding

$O(2,2)\times SL(2,\mathbb{R})$-invariant expressions (4.37)–(4.39) for α^2, β and $\alpha\gamma$ are not *manifestly* polynomial in the eight charges, because they are written using Σ, which is defined via the expression (4.41) for Σ^2. In fact one finds that Σ^2 defined in (4.41) turns out to be a perfect square when one evaluates it, and so Σ is indeed a quadratic polynomial in the charges. However, one cannot write Σ itself as a manifestly $O(2,2)\times SL(2,\mathbb{R})$-invariant quadratic polynomial in the charges and asymptotic scalars. The explanation for this phenomenon turns out to be related to the observation made earlier, in footnote 8, namely that there exists a three-dimensional vector space of matrices that are invariant under the $U(1)\times U(1)$ subgroup of $O(2,2)$ while being non-invariant under the full $O(2,2)$ group.

As we proceeded through the steps described above, we first promoted the 5-charge expressions for α, β, γ and Δ to 7-charge expressions. At that stage, we could have used the $U(1)^2$-invariant matrix (4.3)

$$K = K_1 + K_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \tag{4.42}$$

in order to write $\bar{\Sigma}$ in (4.9) directly, unsquared, as

$$\bar{\Sigma} = -\vec{\alpha}^{\,T} K\,\vec{\beta}\,, \tag{4.43}$$

as may readily be verified. This 7-charge expression can then be augmented to an 8-charge expression by making the replacement (4.11), with ψ given by (4.12). Because the matrix K is antisymmetric, this replacement leaves (4.43) unchanged, and so in the 8-charge expressions for α, β, γ and Δ in (4.18)–(4.21), the expression for Σ following from (4.15) can instead be replaced directly by the unsquared expression

$$\Sigma = -\vec{\alpha}^{\,T} K\,\vec{\beta}\,. \tag{4.44}$$

Finally, when the asymptotic scalars are introduced by acting with the $O(2,2)/U(1)^2$ and $SL(2,\mathbb{R})/U(1)$ coset matrices (4.28) and (4.33) as we did previously, the expression (4.44) for Σ will become

$$\Sigma = \frac{1}{2}\epsilon_{ab}\,\vec{v}^{\,a\,T}\,\overline{P}\,\vec{v}^{\,b}\,, \tag{4.45}$$

where

$$\overline{P} = \overline{\mathcal{V}}^T K\,\overline{\mathcal{V}}\,. \tag{4.46}$$

Thus we have

$$\Sigma = -K^{k\ell}\,\overline{\mathcal{V}}_k{}^i\,\overline{\mathcal{V}}_\ell{}^j\,\alpha_i\,\beta_j = -(\overline{\mathcal{V}}_1{}^i + \overline{\mathcal{V}}_3{}^i)(\overline{\mathcal{V}}_2{}^j + \overline{\mathcal{V}}_4{}^j)(\alpha_i\,\beta_j - \alpha_j\,\beta_i)\,. \tag{4.47}$$

4.2 Scalar fields in the heterotic formulation

The derivation of the expressions for the scalar fields in the heterotic formulation proceeds in an analogous fashion. Here, we shall just present the results for the (Φ, Ψ) dilaton/axion

JHEP09(2021)188

pair, associated with the S-duality in the heterotic formalism. They are given in the 5-charge special case (4.1) by

$$e^{-2\Phi} = \frac{V(r)^{\frac{1}{2}}}{D(r)}\,, \qquad \Psi = \frac{\delta}{D(r)}\,, \tag{4.48}$$

where

$$D(r) = r^2 + \alpha_D\, r + \beta_D\,, \tag{4.49}$$

with

$$\alpha_D = p^2 + p^4\,, \qquad \beta_D = p^2\, p^4\,, \qquad \delta = \frac{1}{2} q\,(p^2 - p^4)\,. \tag{4.50}$$

As in the corresponding derivation in the U-duality formulation, we shall assume the asymptotic values of the scalar fields are all zero until the final stage in the calculation.

The augmentation to a 7-charge solution is accomplished by acting with the $\mathrm{O}(2)\times\mathrm{O}(2)$ subgroup of the $\mathrm{O}(2,2)$ T-duality, and this results in expressions for α_D^2, β_D and δ that are $\mathrm{O}(2)\times\mathrm{O}(2)$ invariant:

$$\begin{aligned}
\alpha^2_{D(7)} &= (p^1+p^3)^2 + (p^2+p^4)^2 = \vec{\beta}^{\,T}\nu_+\,\vec{\beta}\,,\\
\beta_{D(7)} &= p^1\,p^3 + p^2\,p^4 = \frac{1}{2}\vec{\beta}^{\,T} L^{-1}\,\vec{\beta}\,,\\
\delta_{(7)} &= -\frac{1}{4}[(p^1-p^3)(q_1-q_3) + (p^2-p^4)(q_2-q_4)] = \frac{1}{2}\vec{\alpha}^{\,T}L^{-1}\,\vec{\beta}\,.
\end{aligned} \tag{4.51}$$

These quantities are further augmented to 8-charge expressions by acting with the remaining $\mathrm{U}(1) \in \mathrm{SL}(2,\mathbb{R})$ symmetry, with angle ψ given by (4.12). Thus the right-hand-most expressions in (4.51) are replaced according to the rules (4.16), leading to the 8-charge expressions

$$\begin{aligned}
\alpha_D^2 &= \frac{1}{2}\Big(\vec{\alpha}^{\,T}\nu_+\,\vec{\alpha} + \vec{\beta}^{\,T}\nu_+\,\vec{\beta} - \Xi\Big)\,,\\
\beta_D &= \frac{1}{4}\Big(\vec{\alpha}^{\,T}L^{-1}\,\vec{\alpha} + \vec{\beta}^{\,T}L^{-1}\,\vec{\beta} - \frac{\sigma}{\Xi}\Big)\,,\\
\delta &= \frac{1}{2\Xi}\Big[(\vec{\alpha}^{\,T}\nu_+\,\vec{\beta})(\vec{\alpha}^{\,T}L^{-1}\,\vec{\alpha} - \vec{\beta}^{\,T}L^{-1}\,\vec{\beta}) - (\vec{\alpha}^{\,T}L^{-1}\,\vec{\beta})(\vec{\alpha}^{\,T}\nu_+\,\vec{\alpha} - \vec{\beta}^{\,T}\nu_+\,\vec{\beta})\Big]\,,
\end{aligned} \tag{4.52}$$

where Ξ is given by (4.14) and (4.15), and

$$\sigma = G(L^{-1}) = (\vec{\alpha}^{\,T}L^{-1}\,\vec{\alpha} - \vec{\beta}^{\,T}L^{-1}\,\vec{\beta})(\vec{\alpha}^{\,T}\nu_+\,\vec{\alpha} - \vec{\beta}^{\,T}\nu_+\,\vec{\beta}) + 4(\vec{\alpha}^{\,T}L^{-1}\,\vec{\beta})(\vec{\alpha}^{\,T}\nu_+\,\vec{\beta})\,. \tag{4.53}$$

These expressions can be written more compactly in terms of the 8-component charge vector $\vec{v}^{\,a}$ defined in (4.22), giving

$$\begin{aligned}
\alpha_D^2 &= \frac{1}{2}\Big(\delta_{ab}\,\vec{v}^{\,a\,T}\nu_+\,\vec{v}^{\,b} - \Xi\Big)\,,\\
\beta_D &= \frac{1}{4}\Big(\delta_{ab}\,\vec{v}^{\,a\,T}L^{-1}\,\vec{v}^{\,b} - \frac{\sigma}{\Xi}\Big)\,,\\
\delta &= -\frac{1}{2\Xi}\,\delta_{ac}\,\varepsilon_{bd}\,(\vec{v}^{\,a\,T}L^{-1}\,\vec{v}^{\,b})(\vec{v}^{\,c\,T}\nu_+\,\vec{v}^{\,d})\,,
\end{aligned} \tag{4.54}$$

with Ξ and σ given by

$$\begin{aligned}\Xi^2 &= (\delta_{ab}\,\vec{v}^{\,a\,T}\nu_+\,\vec{v}^{\,b})^2 - 4\Sigma^2\,,\\ &= (\delta_{ab}\,\delta_{cd} - 2\varepsilon_{ac}\,\varepsilon_{bd})\,(\vec{v}^{\,a\,T}\nu_+\,\vec{v}^{\,b})(\vec{v}^{\,c\,T}\nu_+\,\vec{v}^{\,d})\,,\\ \sigma &= (\delta_{ab}\,\delta_{cd} - 2\varepsilon_{ac}\,\varepsilon_{bd})(\vec{v}^{\,a\,T}\nu_+\,\vec{v}^{\,b})(\vec{v}^{\,c\,T}L^{-1}\,\vec{v}^{\,d})\,.\end{aligned} \tag{4.55}$$

Thus α_D^2, β_D and δ are all written for the general 8-charge configuration, in forms that are manifestly invariant under the $\mathrm{U}(1)^3$ subgroup of the full $\mathrm{O}(2,2)\times\mathrm{SL}(2,\mathbb{R})$ duality group.

As in the analogous earlier discussion in the U-duality formulation, here the dilaton/axion pair (Φ,Ψ) transforms under the $\mathrm{U}(1)$ subgroup of the $\mathrm{SL}(2,\mathbb{R})$ S-duality group, and so to obtain the expressions for Φ and Ψ after the augmentation from the 5-charge seed solution to the general 8-charge solution, we should transform these too, according to (2.13) with $a=d=\cos\psi$ and $b=-c=\sin\psi$, where the U(1) angle is given by (4.12). Thus, analogously to (3.45) the dilaton and axion will then be given by

$$e^{-2\Phi} = \frac{\sqrt{V(r)}}{\mathcal{D}\,D(r)}\,,\qquad \Psi = \frac{\mathcal{C}}{\mathcal{D}}\,, \tag{4.56}$$

where

$$\begin{aligned}\mathcal{C} &= \frac{\delta}{D(r)}\,\cos 2\psi - \frac{1}{2}\Big[\frac{V(r)+\delta^2}{D(r)^2} - 1\Big]\,\sin 2\psi\,,\\ \mathcal{D} &= \frac{1}{2}\Big[1 + \frac{V(r)+\delta^2}{D(r)^2}\Big] - \frac{\delta}{D(r)}\,\sin 2\psi + \frac{1}{2}\Big[1 - \frac{V(r)+\delta^2}{D(r)^2}\Big]\,\cos 2\psi\,.\end{aligned} \tag{4.57}$$

The function $V(r)$ is given by the expression in (3.2) with the coefficients α, β, γ and Δ given by (4.23)–(4.26); the function $D(r)$ is given by (3.26) with the coefficients given by (4.54); the coefficient δ is given also in (4.54); and the expressions for $\cos 2\psi$ and $\sin 2\psi$ are given, as in (4.13), by

$$\cos 2\psi = \frac{1}{\Xi}\,(\delta_{a1}\,\delta_{b1} - \delta_{a2}\,\delta_{b2})\,\vec{v}^{\,a\,T}\nu_+\,\vec{v}^{\,b}\,,\qquad \sin 2\psi = \frac{2}{\Xi}\,\delta_{a1}\,\delta_{b2}\,\vec{v}^{\,a\,T}\nu_+\,\vec{v}^{\,b}\,. \tag{4.58}$$

(Of course these last expressions should not be invariant under the U(1) subgroup of the $\mathrm{SL}(2,\mathbb{R})$ S-duality, since ψ is the U(1) angle of the final rotation that introduced the 8th charge.)

Finally, the process of turning on non-vanishing asymptotic values for the scalar fields can be accomplished by means of transformations under the $[\mathrm{O}(2,2)\times\mathrm{SL}(2,\mathbb{R})]/\mathrm{U}(1)^3$ coset, as in section 4.1. This means that in all the expressions above one makes the replacements

$$\nu_+ \longrightarrow \mu_+\,,\qquad \delta_{ab} \longrightarrow \overline{\mathcal{N}}_{ab}\,, \tag{4.59}$$

where μ_+ is defined in eq. (4.31) and $\overline{\mathcal{N}}_{ab}$ is obtained by setting Φ and Ψ to their asymptotic values $\overline{\Phi}$ and $\overline{\Psi}$ in (4.32). Note that δ_{a1} and δ_{a2} in (4.58) will be replaced by $\overline{\mathcal{N}}_{a1}$ and $\overline{\mathcal{N}}_{a2}$ also. Finally, the dilaton/axion pair (Φ,Ψ) will transform under the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset also, undergoing the replacements

$$e^{-2\Phi} \longrightarrow e^{-2\Phi-2\overline{\Phi}}\,,\qquad \Psi \longrightarrow \overline{\Psi} + \Psi\,e^{-2\overline{\Phi}}\,. \tag{4.60}$$

4.3 Gauge fields for the 8-charge BPS black holes

Having obtained the expressions for the metric and the scalar fields, the form of the gauge fields in the general 8-charge static BPS black hole solutions follow straightforwardly from the gauge fields equations of motion.

In the heterotic formulation we find

$$\mathcal{F}^i = p^i \sin\theta\, d\theta \wedge d\varphi + \frac{e^{2\Phi}}{\sqrt{V}} M^{ij} (\alpha_j - \Psi\, \beta_j)\, dt \wedge dr\,. \tag{4.61}$$

As may be verified, these expressions are consistent with the definitions of the electric and magnetic charges in eqs. (2.15) and (2.16).

5 Truncation to pairwise-equal charges

The STU supergravity theory admits a consistent truncation in which the four gauge fields are set equal in pairs, and at the same time two of the dilaton/axion pairs are set to zero. Because of the triality symmetry in the U-duality formulation, the three possible ways of equating pairs of gauge fields are equivalent. We shall choose the pairing

$$A^1 = A^2\,, \qquad A^3 = A^4 \tag{5.1}$$

in the theory described in section 2.3. It is straightforward to see that this truncation is consistent provided that at the same time one sets

$$\varphi_2 = \varphi_3 = 0\,, \qquad \chi_2 = \chi_3 = 0\,. \tag{5.2}$$

The entire bosonic Lagrangian then reduces to

$$\begin{aligned}\mathcal{L}_U &= R\,{*1\!\!1} - \frac{1}{2}{*d\varphi_1} \wedge d\varphi_1 - \frac{1}{2} e^{2\varphi_1}\, {*d\chi_1} \wedge d\chi_1 \qquad\qquad (5.3)\\ &\quad - \frac{e^{2\varphi_1}}{1+\chi_1^2\, e^{2\varphi_1}} \Big[e^{-\varphi_1}\, {*F^1} \wedge F^1 + \chi_1\, F^1 \wedge F^1\Big] - e^{-\varphi_1}\, {*F^3} \wedge F^3 + \chi_1\, F^3 \wedge F^3\,.\end{aligned}$$

Note that this Lagrangian is invariant under the discrete symmetry

$$A^1 \longrightarrow A^3\,, \qquad A^3 \longrightarrow A^1\,, \qquad \tau_1 \longrightarrow -\frac{1}{\tau_1}\,, \tag{5.4}$$

where as usual $\tau_1 = \chi_1 + \mathrm{i}\, e^{-\varphi_1}$, that is,

$$e^{-\varphi_1} \longleftrightarrow \frac{e^{\varphi_1}}{(1+\chi_1^2\, e^{2\varphi_1})}\,, \qquad \chi_1 \longleftrightarrow -\frac{\chi_1\, e^{2\varphi_1}}{(1+\chi_1^2\, e^{2\varphi_1})}\,. \tag{5.5}$$

In the heterotic formulation, as can be seen from (2.38) and (2.39), the corresponding truncation amounts to setting

$$\mathcal{A}^1 = \mathcal{A}^3\,, \qquad \mathcal{A}^2 = \mathcal{A}^4\,, \tag{5.6}$$

together, again, with

$$\varphi_2 = \varphi_3 = 0\,, \qquad \chi_2 = \chi_3 = 0\,. \tag{5.7}$$

The bosonic Lagrangian in the heterotic formulation, described in section 2.2, therefore reduces to

$$\begin{aligned}\mathcal{L}_H = & \; R\,{*\mathbb{1}} - 2{*d\Phi}\wedge d\Phi - \frac{1}{2}e^{4\Phi}\,{*d\Psi}\wedge d\Psi \\ & -e^{-2\Phi}\left({*\mathcal{F}^1}\wedge\mathcal{F}^1 + {*\mathcal{F}^2}\wedge\mathcal{F}^2\right) + \Psi\left(\mathcal{F}^1\wedge\mathcal{F}^1 + \mathcal{F}^2\wedge\mathcal{F}^2\right).\end{aligned} \tag{5.8}$$

The general static BPS black hole solutions in the truncated theory are obtained by setting the charges associated with the corresponding pairs of equated fields to be equal also. Thus in the U-duality formulation we set

$$P^1 = P^2\,, \qquad Q_1 = Q_2\,, \qquad P^3 = P^4\,, \qquad Q_3 = Q_3\,. \tag{5.9}$$

In the heterotic formulation, this corresponds to setting

$$p^1 = p^3\,, \qquad q_1 = q_3\,, \qquad p^2 = p^4\,, \qquad q_2 = q_4\,. \tag{5.10}$$

Not only is the STU theory greatly simplified in the pairwise-equal truncation of the fields, as we saw above, but also the form of the black hole solutions becomes considerably simpler. In particular, the quartic metric function $V(r)$ (3.2) now becomes a perfect square, The metric function V in eq. (3.2) also becomes a perfect square:

$$V(r) = \left(r^2 + \frac{1}{2}\alpha\,r + \Delta^{1/2}\right)^2, \tag{5.11}$$

with α and Δ now given by

$$\alpha = 2\sqrt{(P^1+P^3)^2 + (Q_1+Q_3)^2}\,, \qquad \Delta = (P^1\,P^3 + Q_1\,Q_3)^2\,. \tag{5.12}$$

(For simplicity, we first consider the case where the asymptotic values of the scalar fields are taken to be zero here.)

The scalar fields (φ_1, χ_1) themselves can be read off from eqs. (3.49) and (3.50), together with the triality-related expressions for (φ_2, χ_2) and (φ_3, χ_3) as detailed in (3.62) and (3.63), after making the pairwise-equal specialisation (5.9). These latter expressions reproduce the vanishing of the (φ_2, χ_2) and (φ_3, χ_3) as in (5.7), and the former give (3.49)

$$\begin{aligned} e^{\varphi_1} &= \frac{r^2 + b_1\,r + b_0}{r^2 + d_1\,r + d_0}\,, & \chi_1 &= \frac{c_1\,r + c_0}{r^2 + b_1\,r + b_0}\,, \\ b_1 &= \frac{4}{\alpha}\left[P^3\,(P^1+P^3) + Q_1\,(Q_1+Q_3)\right], & b_0 &= (P^3)^2 + (Q_1)^2\,, \\ c_1 &= \frac{4}{\alpha}\,(P^1\,Q_1 - P^3\,Q_3)\,, & c_0 &= P^1\,Q_1 - P^3\,Q_3\,, \\ d_1 &= \frac{1}{2}\alpha\,, & d_0 &= P^1\,P^3 + Q_1\,Q_3\,. \end{aligned} \tag{5.13}$$

It is worth noting that the asymmetry of the expressions for φ_1 and χ_1 with respect to exchanging (P^1, Q_1) and (P^3, Q_3), because of the asymmetry of b_1 and b_0 given in (5.13), is in fact precisely consistent with the exchange symmetry of the pairwise-equal truncation of the STU theory. This symmetry is given in (5.4). One may verify that the scalar fields φ_1 and χ_1 given in (5.13) have the property that

$$\frac{e^{\varphi_1}}{(1+\chi_1^2\, e^{2\varphi_1})} = \frac{r^2+d_1\, r+d_0}{r^2+e_1\, r+e_0}\,, \qquad \frac{\chi_1\, e^{2\varphi_1}}{(1+\chi_1^2\, e^{2\varphi_1})} = -\frac{c_1\, r+c_0}{r^2+e_1\, r+e_0}\,, \tag{5.14}$$

where

$$e_1 = \frac{4}{\alpha}\Big[P^1\,(P^1+P^3)+Q_3\,(Q_1+Q_3)\Big]\,, \qquad e_0 = (P^1)^2+(Q_3)^2\,, \tag{5.15}$$

and so indeed the pairwise-equal solution is compatible with the exchange symmetry given by (5.5).

If the asymptotic values of the scalar fields are taken to be non-vanishing, $\varphi_1 \to \bar\varphi_1$ and $\chi_1 \to \bar\chi_1$, it is straightforward to check using the results in section 3.2 that the constants b_1, b_0, c_1 and c_0 in (5.13) are replaced by

$$\begin{aligned}
b_1 &= \frac{4}{\alpha}\Big\{e^{-\bar\varphi_1}\,[(P^3)^2+Q_1^2]+P^1\,P^3+Q_1\,Q_3+\bar\chi_1\,(P^1\,Q_1-P^3\,Q_3)\\
&\quad +\bar\chi_1\,e^{\bar\varphi_1}\Big[P^1\,Q_1-P^3\,Q_3+\frac{1}{2}\bar\chi_1\,[(P^1)^2+(P^3)^2+Q_1^2+Q_3^2]\Big]\Big\}\,,\\
b_0 &= e^{-\bar\varphi_1}\,[(P^3)^2+Q_1^2]+\bar\chi_1\,(P^1\,Q_1-P^3\,Q_3)\\
&\quad +\bar\chi_1\,e^{\bar\varphi_1}\Big[P^1\,Q_1-P^3\,Q_3+\frac{1}{2}\bar\chi_1\,[(P^1)^2+(P^3)^2+Q_1^2+Q_3^2]\Big]\,,\\
c_1 &= \frac{4}{\alpha}\Big[P^1\,Q_1-P^3\,Q_3+\bar\chi_1\,e^{\bar\varphi_1}\,[(P^1)^2+Q_3^2]\Big]\,,\\
c_0 &= P^1\,Q^1-P^3\,Q_3+\bar\chi_1\,e^{\bar\varphi_1}\,[(P^1)^2+Q_3^2]\,.
\end{aligned} \tag{5.16}$$

Note that $b_1 = \frac{4}{\alpha}\,(b_0+P^1\,P^3+Q_1\,Q_3)$, and $c_1 = \frac{4}{\alpha}\,c_0$. The quantity α is now given by

$$\begin{aligned}
\alpha^2 &= 4e^{\bar\varphi_1}\Big[(P^1+\bar\chi_1\,Q_1)^2+(Q_3-\bar\chi_1\,P^3)^2\Big]+8(P^1\,P^3+Q_1\,Q_3)\\
&\quad +4e^{-\bar\varphi_1}\,[(P^3)^2+Q_1^2]\,.
\end{aligned} \tag{5.17}$$

Finally, the scalar fields φ_1 and χ_1 themselves should be transformed, as in eq. (3.61).

6 Symmetry transformations and asymptotic scalars

In this section, we shall address a conformal transformation for BPS black holes, studied in [11], in the context of STU theory. In order to study the conformal transformation, we need to go outside the approach of the rest of this paper, and adopt the pre-potential formalism studied in [18, 19]. In [19], the authors discuss various stationary solutions of $D = 4$, $\mathcal{N} = 2$ supergravity, among which the one with pre-potential

$$F(X^I) = -\frac{X^1 X^2 X^3}{X^0} \tag{6.1}$$

is of interest to us. The resulting theory, as we shall see later, corresponds to STU supergravity in the 3+1 formulation that we discussed in section 2.5.

Let us consider a Kähler manifold with Kähler potential given by

$$K = -\log[\,\mathrm{i}\,(\bar{X}^I W_I - X^I \bar{W}_I)] \tag{6.2}$$

where X^I and W_I,[9] are related to the "holomorphic section" of the underlying Kähler manifold via the usual definition

$$\begin{pmatrix} X^I \\ W_I \end{pmatrix} = e^{-\frac{K}{2}} \begin{pmatrix} L^I \\ M_I \end{pmatrix} \tag{6.3}$$

where $\begin{pmatrix} L^I \\ M_I \end{pmatrix}$ constitute the "holomorphic section". The four quantities, W_I, are evaluated from the pre-potential mentioned in (6.1) using the definition: $W_I = \frac{\partial F}{\partial X^I}$. Physical scalars z^A, $A = 1, 2, 3$, which parameterise the Kähler manifold, are given in terms of X^I functions as follows

$$z^A = \frac{X^A}{X^0}, \quad A = 1, 2, 3. \tag{6.4}$$

It can be shown that the metric function of the 8-charge static BPS black hole should be related to the Kähler potential (since the metric has to be duality invariant, it should be related to a similar duality invariant quantity in special geometry), and X^I, W_I and their complex conjugates should be given by harmonic functions. These are made to ensure that the bosonic solutions are supersymmetric (i.e. solutions which render the supersymmetry variations $\delta\psi_\mu$ and $\delta\lambda^A$ equal to zero). We refer to [19] for detailed results.

$$\begin{aligned}
e^{-2U} &= e^{-K} = \mathrm{i}\,(\bar{X}^I W_I - X^I \bar{W}_I)\,, \\
\mathrm{i}\,(X^I - \bar{X}^I) &= \widetilde{H}^I, \quad \text{where} \quad \widetilde{H}^I = \tilde{h}^I + \frac{p^I}{r}, \\
\mathrm{i}\,(W_I - \bar{W}_I) &= H_I, \quad \text{where} \quad H_I = h_I + \frac{q_I}{r}.
\end{aligned} \tag{6.5}$$

Here $\tilde{h}^I$ and h_I are integration constants, while q_I and p^I are the electric and magnetic charges. Using (6.4) and (6.5), the metric and scalar fields are expressed in terms of the harmonic functions below,

$$\begin{aligned}
z^A &= \frac{\left(2\widetilde{H}^A H_A - \widetilde{H}^I H_I\right) - \mathrm{i}\,e^{-2U}}{\left(d_{ABC}\widetilde{H}^B\widetilde{H}^C + 2\widetilde{H}^0 H_A\right)}\,, \qquad A = 1, 2, 3\,, \\
e^{-4U} &= \frac{1}{r^4} V(r) \\
&= -(\widetilde{H}^I H_I)^2 + (d_{ABC}\widetilde{H}^B\widetilde{H}^C d^{ADE} H_D H_E) \\
&\quad + 4\,\widetilde{H}^0 H_1 H_2 H_3 - 4\,H_0\,\widetilde{H}^1\widetilde{H}^2\widetilde{H}^3\,.
\end{aligned} \tag{6.6}$$

d_{ABC} is the "symmetrised ϵ tensor," $d_{ABC} = |\epsilon_{ABC}|$, so

$$d_{123} = 1 = d_{213} = (\text{any permutation of 1,2,3 indices})\,. \tag{6.7}$$

[9] W_I is denoted as F_I in most of the literature dealing with supergravity and special geometry, however we choose not to use the notation F_I to avoid confusion with field strength.

Towards the end of this section we shall relate the charges, scalar fields and the metric mentioned in the above equation to those in the U-duality formulation, via the 3+1 formulation discussed in section 2.5.

Not all the eight constants h_I and $\tilde{h}^I$ are independent; they are constrained by two conditions. Since the metric is asymptotically flat, we have the constraint

$$\begin{aligned} 1 &= -(\tilde{h}^I h_I)^2 + (d_{ABC}\, \tilde{h}^B \tilde{h}^C d^{ADE}\, h_D h_E) \\ &\quad + 4\, \tilde{h}^0 h_1 h_2 h_3 - 4\, h_0\, \tilde{h}^1 \tilde{h}^2 \tilde{h}^3 \end{aligned} \tag{6.8}$$

The other constraint is [18]

$$\tilde{h}^I\, q_I - h_I\, p^I = 0\,, \tag{6.9}$$

which arises from a more fundamental requirement of special geometry:

$$\langle V, \mathcal{D}_A V\rangle = 0, \quad \text{where} \quad V = \begin{pmatrix} L^I \\ M_I \end{pmatrix}, \qquad \mathcal{D}_A \equiv \partial_A + \frac{1}{2}(\partial_A K)\,, \tag{6.10}$$

where $\partial_A = \partial/\partial z^A$. The angular bracket denotes the inner product with respect to the symplectic metric. We again refer to [18, 19] for details. Thus only six out of the eight integration constants $\tilde{h}^I$ and h_I are independent.

We now have all the necessary ingredients for discussing the conformal inversion of the static 8-charge BPS metric. Since the metric is in general is parameterised by eight charges and the asymptotic values of the six scalar fields, the conformal inversion transforms the charges and the asymptotic values of the scalars into a new set of charges and asymptotic scalar values in the conformally rescaled metric (see, for example, section 4 in [9]). Although it is relatively easy to find the relations between constants α, β, γ and Δ appearing in the metric function $V(r)$ in section 3.1 and those in the conformally rescaled metric, it is quite difficult to find the relation between original and the transformed charges and scalars. It is nonetheless possible to circumvent this difficulty by viewing the metric as being parameterised by q_I, p^I, h_I, and $\tilde{h}^I$. It should again be emphasised that out of these sixteen parameters, only fourteen (i.e. eight charges and six out of the eight h_I and $\tilde{h}^I$ constants) are independent, thus giving us the same number of parameters as we have when the metric is written in terms of the eight charges and the asymptotic values of the six scalars. Following [11], if we implement the transformation

$$\begin{pmatrix} h_I \\ \tilde{h}^I \end{pmatrix} \to \begin{pmatrix} \hat{h}_I \\ \hat{\tilde{h}}^I \end{pmatrix} = \Delta^{-\frac{1}{4}} \begin{pmatrix} q_I \\ p^I \end{pmatrix}, \qquad \begin{pmatrix} q_I \\ p^I \end{pmatrix} \to \begin{pmatrix} \hat{q}_I \\ \hat{p}^I \end{pmatrix} = \Delta^{\frac{1}{4}} \begin{pmatrix} h_I \\ \tilde{h}^I \end{pmatrix}, \tag{6.11}$$

together with $r \to \hat{r} = \sqrt{\Delta}/r$, it is easy to check that we obtain the correct conformally rescaled metric, namely

$$\begin{aligned} V(r) &= \frac{\Delta}{\hat{r}^4} \hat{V}(\hat{r}), \qquad ds^2 = \frac{\sqrt{\Delta}}{\hat{r}^2} \hat{ds}^2 \\ \hat{ds}^2 &= -\frac{\hat{r}^2}{\sqrt{\hat{V}(\hat{r})}}\, dt^2 + \frac{\sqrt{\hat{V}(\hat{r})}}{\hat{r}^2}\,(d\hat{r}^2 + \hat{r}^2 d\Omega^2)\,, \end{aligned} \tag{6.12}$$

where $\hat{V}(\hat{r})$ is obtained from the second equation in (6.6) by replacing $\widetilde{H}^I$ and H_I by $\hat{\widetilde{H}}^I$ and $\hat{H}_I$, with $\hat{\widetilde{H}}^I = \hat{\tilde{h}}^I + \hat{p}^I/\hat{r}$, etc.[10]

It is now instructive to see what happens to the asymptotic values of the scalar fields under these transformations. Under the transformations in (6.11),

$$\begin{pmatrix} \hat{\widetilde{H}}^I \\ \hat{H}_I \end{pmatrix} = r\,\Delta^{-\frac{1}{4}} \begin{pmatrix} \widetilde{H}^I \\ H_I \end{pmatrix}, \qquad e^{-2\hat{U}} = r^2 \Delta^{-\frac{1}{2}} e^{-2U}\,. \tag{6.13}$$

According to (6.6), this implies for scalars that

$$\hat{z}^A\left(\hat{r}, \hat{q}_I, \hat{p}^I, \hat{\tilde{h}}^I, \hat{h}_I\right) = z^A\left(r, q_I, p^I, \tilde{h}^I, h_I\right). \tag{6.14}$$

Thus the functional form of the scalar fields remain unchanged under the conformal transformation. This implies that

$$\lim_{r\to\infty} z^A = \frac{(2\tilde{h}^A h_A - \tilde{h}^I h_I) - \mathrm{i}}{d_{ABC}\,\tilde{h}^B \tilde{h}^C + 2\,\tilde{h}^0 h_A} = \lim_{\hat{r}\to 0} \hat{z}^A = \frac{2\,\hat{p}^A \hat{q}_A - \hat{p}^I \hat{q}_I - \mathrm{i}\,\sqrt{\hat{V}(\hat{p}^I, \hat{q}_I)}}{d_{ABC}\,\hat{p}^B \hat{p}^C + 2\,\hat{p}^0 \hat{q}_A}\,, \tag{6.15}$$

where $\hat{V}(\hat{p}^I, \hat{q}_I)$ is the metric function at the horizon after inversion, which can be obtained by the replacements $\widetilde{H}^I \to \hat{p}^I$ and $H_I \to \hat{q}_I$ in the lower equation in (6.6). In other words, the above equation implies that the asymptotic values of the scalar fields in the original metric map to the values of the fields at the horizon of the transformed metric.

It is important to distinguish this property of the scalar fields undergoing conformal inversion from that considered in [9]. In [9], the asymptotic values of the scalar fields were taken to be zero in the static black hole solution of STU supergravity, and it was mandated that after the conformal inversion the metric should again be an 8-charge solution with vanishing asymptotic values for the scalar fields. By contrast, in the present discussion, if we start with zero asymptotic values for the scalar fields, we will end up with the scalar fields becoming *zero at the horizon of the inverted metric*, but at spatial infinity the *scalar fields will be non-zero*. Therefore, the type of transformation considered in (6.11) clearly does not satisfy the requirements of the conformal inversion discussed in [9], which involved mapping any member of the restricted 8-parameter class of charged black holes with vanishing asymptotic scalars to another member of this 8-parameter class. Instead it provides an alternative way of implementing the inversion, describing a mapping from any member of the 14-parameter general class of black hole solutions to another member of the class.

Although we have in principle demonstrated the effect that the conformal inversion has on the metric, the charges, the constants $\tilde{h}^I$, h_I, and the scalars, the discussion might appear abstract to some degree since no real connection has been made between a particular theory and the particular pre-potential considered in (6.1). To establish that connection,

[10]In [11] and in [9] the quantities after the conformal inversion were denoted using tildes. Here, we are instead using hats to denote the conformally-inverted quantities, reserving tildes for the functions $\widetilde{H}^I = \tilde{h}^I + p^I/r$ in the notation of [19].

we need to look at two equations which arise as a consequence of holomorphicity of the section (L^I, M_I) of the underlying Kähler manifold, namely,

$$W_I = \mathcal{N}_{IJ} X^J , \qquad \mathcal{D}_A W_I = \overline{\mathcal{N}}_{IJ}\, \mathcal{D}_A X^J \,, \tag{6.16}$$

where $\mathcal{N}_{IJ}$ is the complex matrix describing the coupling of the scalars to the field strengths (see below). Here $\mathcal{D}_A = \partial_A + (\partial_A K)$ is the appropriate covariant derivative with the Kähler connection. Note that the change of the coefficient of the $(\partial_A K)$ term from $\frac{1}{2}$ in eq. (6.10) to 1 here is because the different weights of the differentiands with respect to K, as can be seen in eq. (6.3).

Since we know the relation between X^A and the scalars z^A via (6.4), it is possible to solve the $(4+12) = 16$ conditions coming from (6.16) to determine $\mathcal{N}_{IJ}$ in terms of z^A. The gauge field part of the Lagrangian in the 3+1 formulation considered in section 2.5 has this structure, and if we choose

$$z^A = -\chi_A - ie^{-\varphi_A} \tag{6.17}$$

then it is straightforward to see that the Lagrangian in (2.45) becomes

$$\mathcal{L} = R + \mathcal{L}(\varphi,\chi) - \frac{1}{2}\Im(N_{IJ}) * F^I \wedge F^J + \frac{1}{2}\Re(N_{IJ}) F^I \wedge F^J \,, \tag{6.18}$$

where scalar kinetic terms are given by

$$\mathcal{L}(\varphi,\chi) = -2 g_{A\bar{B}} * dz^A \wedge d\bar{z}^{\bar{B}} = -\frac{1}{2}\left(\sum_{i=1}^{3} *d\varphi_i \wedge d\varphi_i + e^{2\varphi_i} * d\chi_i \wedge d\chi_i\right) , \tag{6.19}$$

with $g_{A\bar{B}} = \partial_A \partial_{\bar{B}} K$.

Finally, the metric considered in (6.6) as a general function of $\widetilde{H}^I$ and H_I can be shown to be equal to the static 8-charge metric in the U-duality formulation upon relating the charges in this section to the charges of U-duality frame via the mapping[11]

$$\begin{aligned} p^A &= \frac{P^A}{\sqrt{2}}, \qquad q_A = \frac{Q_A}{\sqrt{2}}, \qquad A = 1,2,3. \\ p^0 &= \frac{Q^4}{\sqrt{2}}, \qquad q_0 = -\frac{P^4}{\sqrt{2}} \,. \end{aligned} \tag{6.20}$$

Note that it would also be natural, when mapping from the (3+1) formulation of STU supergravity used in [19] to the U-duality formulation, to relabel the constants $\tilde{h}^0$ and h_0 in the same way as p^0 and q_0 are relabelled in (6.20):

$$\tilde{h}^0 = h_4 \,, \qquad h_0 = -\tilde{h}^4 \,. \tag{6.21}$$

Thus the functions $\widetilde{H}^I$ and H_I with $I = 0,1,2,3$ would also now be defined instead for $I = 1,2,3,4$, with

$$\widetilde{H}^I = \tilde{h}^I + \frac{P^I}{\sqrt{2}\, r} \,, \qquad H_I = h_I + \frac{Q_I}{\sqrt{2}\, r} \,, \qquad I = 1,2,3,4 \,. \tag{6.22}$$

[11] The field strengths $F_{(BLS)}$ in [19] are normalised differently from ours, with $F_{(BLS)} = 1/(2\sqrt{2})\, F$. The Lagrangian (6.18) is written in terms of our convention for the field strength normalisation. Comparison of the expressions for the fields in the black hole solutions in [19] with our expressions leads to the relations in eq. (6.20).

Although we have given a way to write the metric in terms of the constants $\tilde{h}^I$, h_I and the charges, and shown how these should transform under conformal inversion, it should be emphasised that the quantities $\tilde{h}^I$ and h_I are not all physical and independent. In order to evaluate them in terms of physical charges and asymptotic scalar values, we need to use the first equation in (6.6). Keeping in mind the equation (6.4), we could write six equations for $\tilde{h}^I$ and h_I:

$$
\begin{aligned}
0 &= 2\tilde{h}^A h_A - \tilde{h}^I h_I + \left(d_{ABC}\tilde{h}^B\tilde{h}^C + 2\tilde{h}^0 h_A\right)\bar{\chi}_A, && A = 1,2,3. \\
f_A &= d_{ABC}\tilde{h}^B\tilde{h}^C + 2\tilde{h}^0 h_A, && A = 1,2,3\,,
\end{aligned} \tag{6.23}
$$

where $\bar{\chi}_A$ and $f_A \equiv e^{\bar{\varphi}_A}$ are the asymptotic values of the scalars. These six equations, along with (6.8) and (6.9), enable us to solve for the eight constants, giving

$$
\begin{aligned}
\tilde{h}^0 &= a\,, \quad \tilde{h}^1 = -a\,\bar{\chi}_1 + \frac{b}{f_1}\,, \quad \tilde{h}^2 = -a\,\bar{\chi}_2 + \frac{b}{f_2}\,, \quad \tilde{h}^3 = -a\,\bar{\chi}_3 + \frac{b}{f_3} \\
h_0 &= -a\left(\bar{\chi}_1\bar{\chi}_2\bar{\chi}_3 - \frac{\bar{\chi}_1}{f_2 f_3} - \frac{\bar{\chi}_2}{f_1 f_3} - \frac{\bar{\chi}_3}{f_1 f_2}\right) - b\left(\frac{1}{f_1 f_2 f_3} + \frac{\bar{\chi}_2\bar{\chi}_3}{f_1} + \frac{\bar{\chi}_1\bar{\chi}_3}{f_2} + \frac{\bar{\chi}_1\bar{\chi}_2}{f_3}\right), \\
h_1 &= a\left(\frac{1}{f_2 f_3} + \bar{\chi}_2\bar{\chi}_3\right) + b\left(\frac{\bar{\chi}_2}{f_2} + \frac{\bar{\chi}_3}{f_3}\right), \\
h_2 &= a\left(\frac{1}{f_1 f_3} + \bar{\chi}_1\bar{\chi}_3\right) + b\left(\frac{\bar{\chi}_1}{f_1} + \frac{\bar{\chi}_3}{f_3}\right), \\
h_3 &= a\left(\frac{1}{f_2 f_1} + \bar{\chi}_2\bar{\chi}_1\right) + b\left(\frac{\bar{\chi}_2}{f_2} + \frac{\bar{\chi}_1}{f_1}\right),
\end{aligned} \tag{6.24}
$$

where a and b are given by

$$
a = \frac{\sqrt{f_1 f_2 f_3}}{\sqrt{2}}\frac{D_q}{\sqrt{D_q^2 + D_p^2}}\,, \qquad b = \frac{\sqrt{f_1 f_2 f_3}}{\sqrt{2}}\frac{D_p}{\sqrt{D_q^2 + D_p^2}} \tag{6.25}
$$

and

$$
\begin{aligned}
D_q &= q_4 + f_1 f_2\left(q_1 - \bar{\chi}_2\,p^3 - \bar{\chi}_3\,p^2 - \bar{\chi}_2\,\bar{\chi}_3\,q_4\right) + f_1 f_3\left(q_2 - \bar{\chi}_1\,p^3 - \bar{\chi}_3\,p^1 - \bar{\chi}_1\,\bar{\chi}_3\,q_4\right) \\
&\quad + f_1 f_2\left(q_3 - \bar{\chi}_1\,p^2 - \bar{\chi}_2\,p^1 - \bar{\chi}_1\,\bar{\chi}_2\,q_4\right), \\
D_p &= f_1 f_2 f_3\left(p^4 + \bar{\chi}_1\,q_1 + \bar{\chi}_2\,q_2 + \bar{\chi}_3\,q_3 - \bar{\chi}_2\,\bar{\chi}_3\,p^1 - \bar{\chi}_1\,\bar{\chi}_3\,p^2 - \bar{\chi}_1\,\bar{\chi}_2\,p^3 - \bar{\chi}_1\,\bar{\chi}_2\,\bar{\chi}_3\,q_4\right) \\
&\quad + f_1\left(p^1 + \bar{\chi}_1\,q_4\right) + f_2\left(p^2 + \bar{\chi}_2\,q_4\right) + f_3\left(p^3 + \bar{\chi}_3\,q_4\right).
\end{aligned} \tag{6.26}
$$

Note that in the special case where the asymptotic values of the scalar fields φ_i and χ_i are taken to vanish, the $\tilde{h}^I$ and h_I constants become simply

$$
\begin{aligned}
\tilde{h}^1 = \tilde{h}^2 = \tilde{h}^3 = \tilde{h}^4 &= \frac{\sum_I P^I}{\sqrt{2}\sqrt{(\sum_J P^J)^2 + (\sum_J (Q_J)^2}}\,, \\
h_1 = h_2 = h_3 = h_4 &= \frac{\sum_I Q_I}{\sqrt{2}\sqrt{(\sum_J P^J)^2 + (\sum_J (Q_J)^2}}\,,
\end{aligned} \tag{6.27}
$$

(after the change to the U-duality notation in (6.21)).

Plugging the solutions (6.24) into the metric in (6.6), along with the mapping (6.20), one can show that the metric is indeed equal to the static metric of the U-duality formulation (it is most easily done when the asymptotic scalars are set to zero). These voluminous but symmetric equations enable one to find the eight constants $\tilde{h}^I$ and h_I in terms of the eight charges and the asymptotic values of the six scalars. One can think of this as an "initial value" assignment for the eight $\tilde{h}^I$ and h_I constants, prior to the inversion. One could then formulate the inversion problem entirely in terms of these constants and charges, forgetting about the scalars. It is always possible, after inversion, to re-express: I) the transformed charges in terms of original charges and scalars using (6.11) and the aforementioned equations, and II) the transformed scalars in terms of the original charges and asymptotic values of the original scalars using (6.6), after expressing (6.6) in terms of "hatted" quantities. The latter, as we have already shown, turn out to be equal to the original set of scalars.

7 Concluding remarks and outlook

Studies of BPS black holes in four-dimensional ungauged supergravity theories, such as the extremal STU black holes of the $\mathcal{N} = 2$ supergravity theory coupled to three vector supermultiplets, have been a subject of intense research ever since their discovery [3, 5]. The study of their properties attracted extensive efforts over years, with recent ones focusing on their enhanced symmetries, such as Aretakis and Newman-Penrose charges [8, 20], and Couch-Torrence-type symmetries [9]. It is important to note that these features of extremal STU black holes stem from and are closely related to those of extremal Reissner-Nordström and Kerr-Newman black holes of Maxwell-Einstein gravity, these are a special case of STU black holes. The STU BPS black holes are fourteen-parameter solutions, specified by four electric and four magnetic charges, *and* by the asymptotic values of the six scalar fields.

In the past, most analyses focused on the BPS black holes where the asymptotic values of scalar fields were taken to be their canonical zero values. The explicit form of such black holes was constructed in [3], in the heterotic formulation, and the analogous process was employed recently in [9] to construct the eight-charge solutions in the U-duality formulation of STU supergravity. In [10], all attractor flows for BPS and non-BPS black holes were described in full generality in the STU symplectic frame.

One of the main purposes of the present paper was to derive systematically the full explicit solutions, both in the U-duality and the heterotic frames, in sections 3 and 4 respectively.[12] The solutions are specified by coefficients in the functions giving the metric and scalar fields that are expressed in terms of manifestly covariant U-duality or heterotic-duality quantities. In section 5 we also presented BPS black hole solutions of the consistently truncated STU supergravity obtained by equating the four electromagnetic fields in pairs and at the same time setting four of the six scalar fields to zero. This supergravity theory, with two gauge fields and one complex (axion-dilaton) scalar field, and its black

[12] In [3] explicit results for the mass and the horizon area of these black holes in the heterotic frame were obtained. This analysis showed that the horizon area, and thus the Bekenstein entropy, is *independent* of the asymptotic values of the scalar fields.

JHEP09(2021)188

hole solutions, are significantly simplified, and are well suited to further explicit studies. In section 6 we made use of some of the general results we obtained in order to discuss in detail a conformal inversion symmetry of the BPS black hole solutions.

The new results we have obtained immediately lend themselves to study further symmetry structures of these generalised solutions, such as generalisations of the conformal inversion symmetries of the Couch-Torrence type. For that purpose, in section 6, we consider the eight-charge static BPS black holes [19], formulated in the description of the theory in terms of the Kähler geometry of the scalar manifold [16]. In this formulation the BPS black hole solutions are expressed in terms of eight harmonic functions, subject to two constraints [18] and thus, again, the solution is specified in terms of fourteen parameters. We employed a transformation, discussed in [11], by acting with an inversion of the radial coordinate, which, together with a conformal rescaling of the metric, maps the horizon to infinity and vice versa. As we also obtained the explicit map of the black hole solutions in this formulation to those in the U-duality frame, we can show how any member of the general fourteen-parameter family of static BPS black holes (characterised by the eight charges and six asymptotic scalar values) is mapped by this conformal inversion to another member of the family. This type of conformal inversion is different from, although closely related to, the one considered in [8, 9], where the asymptotic values of scalar fields remained unchanged (actually, always set to zero) under the inversion.

The conformal inversion symmetries of extremal black holes have been used previously to relate the conserved charges on the future horizon to conserved Newman-Penrose charges at infinity. The conserved charges on the horizon have been shown to imply the existence of certain kinds of instability, associated with growing modes of scalar or gravitational perturbations at large time v on the future horizon (for a discussion of some of these issues, see, for example, [7, 21, 22]). On the face of it such instabilities might seem to be at odds with the fact that the black hole backgrounds under consideration are extremal, obeying BPS conditions. However, it should be recalled that in a theory of gravity, as opposed to the situation in a flat-space Yang-Mills/Higgs theory, one cannot in general turn a surface integral at infinity (related to the mass) into an integral over positive-definite invariants in the interior. Indeed, counter-examples have been discussed in the literature, such as the instability even of Minkowski spacetime to the formation of black holes from ingoing finite-energy gravitational radiation. Some discussion of related issues can be found, for example, in [7, 23].

Building on technical advances presented in this paper, we foresee a number of important future research directions. First of all, black hole solutions of the STU supergravity have in principle an immediate generalisation to black holes of maximally supersymmetric ungauged supergravity theories, i.e. the $\mathcal{N} = 8$ and the $\mathcal{N} = 4$ supergravities of the toroidally compactified effective Type II and heterotic superstring theories, respectively. In particular, we expect that the BPS solution of $\mathcal{N} = 4$ supergravity with $\mathrm{O}(6, 22)$ T-duality symmetry and $\mathrm{SL}(2, \mathbb{R})$ S-duality symmetry (at the level of the equations of motion) can be obtained in a straightforward way by acting on the STU black hole with a subset of the global symmetry generators in the heterotic frame. The final solution will then be parameterised by 28 electric and 28 magnetic charges, and the asymptotic values of the 134 scalar fields in the coset of $\mathrm{O}(6,22)/[\mathrm{O}(6)\times\mathrm{O}(22)]\times\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$. It is expected that the final result should represent a straightforward generalisation of the resulting section 4, with charge vectors $\vec{\alpha}$ and $\vec{\beta}$ characterising the 28 electric and 28 magnetic charges, respectively, the matrix L now denoting the $\mathrm{O}(6,22)$ invariant matrix, and the matrix μ_+ parameterising the corresponding asymptotic values of 132 scalar fields in the $\mathrm{O}(6,22)/[\mathrm{O}(6)\times\mathrm{O}(22)]$ coset. (Note again, that in this case the mass and the horizon area were obtained in [3].) On the other hand, $\mathcal{N}=8$ supergravity has an $E_{7,7}$ U-duality symmetry, and the general BPS black hole solution can in principle be generated by acting on the STU black hole solution with a subset of $E_{7,7}$ generators. It would be of great interest to obtain the explicit form of the full BPS black hole solution in terms of manifestly $E_{7,7}$-covariant coefficients.

Another important application is to focus on black hole properties as a function of the asymptotic values of the scalar fields, which in effective string theory specify the moduli of the corresponding string compactification. The explicit expressions obtained in this paper would allow us to employ these black holes for studies of various so-called swampland conjectures [24–26]. In particular, we plan to explore the implications of these black holes for the swampland distance conjecture [25], and its connection to the weak gravity conjecture [26]. (The swampland distance conjecture argues that as one moves toward the boundary of moduli space, there appears an infinite tower of states with masses approaching zero exponentially as a function of the traversed distance. The weak gravity conjecture argues that in any consistent quantum gravity there must exist a particle whose charge-to-mass ratio equals or exceeds the extremality bound for black hole solutions of that theory.) For recent work, tying together the two conjectures via special examples of BPS black holes, see [27] and references therein. We would also like to emphasise that since the BPS black hole solutions presented in this paper account for both electric and magnetic charges, they would allow for probing non-perturbative effects in the moduli space of string compactifications, such as the appearance of light/massless dyonic BPS states in the middle of moduli space, where they could signify, for example, the appearance of enhanced gauge symmetry and/or supersymmetry (cf., [28]).

Acknowledgments

We are grateful to Gary Gibbons for discussions. The work of M.C. is supported in part by the DOE (HEP) Award DE-SC0013528, the Fay R. and Eugene L. Langberg Endowed Chair (M.C.) and the Slovenian Research Agency (ARRS No. P1-0306). The work of C.N.P. is supported in part by DOE grant DE-FG02-13ER42020.

A Kaluza-Klein T^2 reduction from six dimensions

Using the notation and conventions of [29, 30], the T^2 reduction of the six-dimensional Lagrangian (2.1) is accomplished by means of the reduction ansätze

$$d\hat{s}_6^2 = e^{\frac{1}{2}(\varphi_1+\varphi_3)}\,ds_4^2 + e^{-\frac{1}{2}(\varphi_1+\varphi_3)}\left[e^{-\tilde{\varphi}_2}\,(h^1)^2 + e^{\tilde{\varphi}_2}\,(h^2)^2\right], \tag{A.1}$$

$$\hat{B}_{(2)} = B_{(2)} + A_{(1)1}\wedge dz^1 + A_{(1)2}\wedge dz^2 - A_{(0)12}\,dz^1\wedge dz^2\,, \tag{A.2}$$

$$\hat{\phi} = \frac{1}{\sqrt{2}}\,(\varphi_1-\varphi_3)\,, \tag{A.3}$$

JHEP09(2021)188

where

$$h^1 = dz^1 + \mathcal{A}^1_{(1)} + \mathcal{A}^1_{(0)2}\, dz^2\,, \qquad h^2 = dz^2 + \mathcal{A}^2_{(1)}\,, \tag{A.4}$$

and (z^1, z^2) are the coordinates on the internal 2-torus. The six-dimensional 3-form field strength is then given by

$$\hat{H}_{(3)} = d\hat{B}_{(2)} = H_{(3)} + F_{(2)1} \wedge h^1 + F_{(2)2} \wedge h^2 - F_{(1)12}\, h^1 \wedge h^2\,, \tag{A.5}$$

with the four-dimensional field strengths $H_{(3)}$, $F_{(2)1}$, $F_{(2)2}$ and $F_{(1)12}$ being read off by substituting (A.2) into (A.5). It turns out to be advantageous to make redefinitions of certain of the potentials in order to obtain the four-dimensional theory in a parameterisation in which the axionic scalars $\mathcal{A}^1_{(0)2}$ from the metric and $A_{(0)12}$ from the $\hat{B}_{(2)}$ field occur without derivatives in their couplings to the vector fields. Thus we define primed potentials as follows

$$A_{(1)1} = A'_{(1)1} + \mathcal{A}^1_{(0)2}\,, \qquad A_{(1)2} = A'_{(1)2} + A_{(0)12}\, \mathcal{A}^1_{(1)}\,, \qquad \mathcal{A}^1_{(1)} = \mathcal{A}^1_{(1)}{}' + \mathcal{A}^1_{(0)2}\, \mathcal{A}^2_{(1)}\,. \tag{A.6}$$

We also define a primed potential $B'_{(2)}$ for the four-dimensional 3-form $H_{(3)}$ via

$$B_{(2)} = B'_{(2)} + A_{(0)12}\, \mathcal{A}^{1'}_1 \wedge \mathcal{A}^2_{(1)}\,. \tag{A.7}$$

The four-dimensional dressed field strengths are now given by

$$\begin{aligned}
\mathcal{F}^1_{(2)} &= d\mathcal{A}^1_{(1)}{}' + \mathcal{A}^1_{(0)2}\, d\mathcal{A}^2_{(1)}\,, \qquad \mathcal{F}^2_{(2)} = d\mathcal{A}^2_{(1)}\,,\\
F_{(2)1} &= dA'_{(1)1} - A_{(0)12}\, d\mathcal{A}^2_{(1)}\,,\\
F_{(2)2} &= dA'_{(1)2} + A_{(0)12}\, d\mathcal{A}^1_{(1)}{}' - \mathcal{A}^1_{(0)2}\, dA'_{(1)1} + \mathcal{A}^1_{(0)2}\, A_{(0)12}\, d\mathcal{A}^2_{(1)}\,,\\
H_{(3)} &= dB'_{(2)} - dA'_{(1)1} \wedge \mathcal{A}^1_{(1)}{}' - dA'_{(1)2} \wedge \mathcal{A}^2_{(1)}\,.
\end{aligned} \tag{A.8}$$

In this form, after then dualising the 2-form potential $B_{(2)}$ to an axion, the bosonic STU supergravity Lagrangian was presented in [31].

In this paper we shall denote the four gauge potentials by $(\widetilde{A}_1, \widetilde{A}_2, A^3, A^4)$, with

$$\widetilde{A}_1 = \mathcal{A}^1_{(1)}{}'\,, \qquad \widetilde{A}_2 = A'_{(1)1}\,, \qquad A^3 = \mathcal{A}^2_{(1)}\,, \qquad A^4 = A'_{(1)2}\,. \tag{A.9}$$

We also define the axions $\widetilde{\chi}_2$ and χ_3 by

$$\widetilde{\chi}_2 = -\mathcal{A}^1_{(0)2}\,, \qquad \chi_3 = A_{(0)12}\,. \tag{A.10}$$

(The tildes are placed on the potentials $\widetilde{A}_1$ and $\widetilde{A}_2$ to signify that these are the fields that we shall dualise when passing to the U-duality formulation of the STU supergravity. The tildes on $\widetilde{\chi}_2$, and on $\tilde{\varphi}_2$ earlier, are used because we are reserving the symbols χ_2 and φ_2 for redefined fields we shall be using later.) The four-dimensional Lagrangian is then given by

$$\begin{aligned}
\mathcal{L} = &\; R\,{*\mathbb{1}} - \frac{1}{2}{*d\varphi_1} \wedge d\varphi_1 - \frac{1}{2}e^{2\varphi_1}\, {*d\chi_1} \wedge d\chi_1 - \frac{1}{2}{*d\tilde{\varphi}_2} \wedge d\tilde{\varphi}_2 - \frac{1}{2}e^{2\tilde{\varphi}_2}\, {*d\widetilde{\chi}_2} \wedge d\widetilde{\chi}_2\\
&-\frac{1}{2}{*d\varphi_1} \wedge d\varphi_1 - \frac{1}{2}e^{-2\varphi_1}\, {*dH_{(3)}} \wedge dH_{(3)}\\
&-\frac{1}{2}e^{-\varphi_1}\Big[e^{\tilde{\varphi}_2-\varphi_3}\, {*\overline{\widetilde{F}}_1} \wedge \overline{\widetilde{F}}_1 + e^{-\tilde{\varphi}_2+\varphi_3}\, {*\overline{\widetilde{F}}_2} \wedge \overline{\widetilde{F}}_2\\
&+e^{-\tilde{\varphi}_2-\varphi_3}\, {*\overline{F}^3} \wedge \overline{F}^3 + e^{\tilde{\varphi}_2+\varphi_3}\, {*\overline{F}^4} \wedge \overline{F}^4\Big]\,,
\end{aligned} \tag{A.11}$$

where $\widetilde{F}_1 = d\widetilde{A}_1$, $\widetilde{F}_2 = d\widetilde{A}_2$, $F^3 = dA^3$ and $F^4 = dA^4$ are the "raw" field strengths,

$$\begin{aligned} \widetilde{\widetilde{F}}_1 &= \widetilde{F}_1 - \widetilde{\chi}_2\, F^3\,, & \widetilde{\widetilde{F}}_2 &= \widetilde{F}_2 - \chi_3\, F^3\,, \\ \overline{F}^3 &= F^3\,, & \overline{F}^4 &= F^4 + \chi_3\,\widetilde{F}_1 + \chi_2\,\widetilde{F}_2 - \widetilde{\chi}_2\,\chi_3\, F^3\,, \end{aligned} \tag{A.12}$$

are the "dressed" field strengths appearing in the kinetic terms in (A.11), and

$$H_{(3)} = dB'_{(2)} - \widetilde{A}_1 \wedge d\widetilde{A}_2 - A^3 \wedge dA^4\,. \tag{A.13}$$

Finally, we may dualise the 2-form potential $B'_{(2)}$ to an axion χ_1 by adding a Lagrangian multiplier term $\chi_1\,(dH_{(3)} + \widetilde{F}_1 \wedge \widetilde{F}_2 + F^3 \wedge F^4)$ to (A.11), treating $H_{(3)}$ as an independent field, and substituting its equation of motion $H_{(3)} = -e^{2\varphi_1} *d\chi_1$ back into the Lagrangian. This gives the bosonic STU supergravity Lagrangian in the form

$$\begin{aligned} \mathcal{L} = R*\mathbb{1} &- \frac{1}{2}\sum_{i=1,3}(*d\varphi_i \wedge d\varphi_i + e^{2\varphi_i} *d\chi_i \wedge d\chi_i) - \frac{1}{2}*d\widetilde{\varphi}_2 \wedge d\widetilde{\varphi}_2 - \frac{1}{2}e^{2\widetilde{\varphi}_2} *d\widetilde{\chi}_2 \wedge d\widetilde{\chi}_2 \\ &-\frac{1}{2}e^{-\varphi_1}\Big[e^{\widetilde{\varphi}_2-\varphi_3} *\widetilde{\widetilde{F}}_1 \wedge \widetilde{\widetilde{F}}_1 + e^{-\widetilde{\varphi}_2+\varphi_3} *\widetilde{\widetilde{F}}_2 \wedge \widetilde{\widetilde{F}}_2 \\ &+e^{-\widetilde{\varphi}_2-\varphi_3} *\overline{F}^3 \wedge \overline{F}^3 + e^{\widetilde{\varphi}_2+\varphi_3} *\overline{F}^4 \wedge \overline{F}^4\Big] + \chi_1\,(\widetilde{F}_1 \wedge \widetilde{F}_2 + F^3 \wedge F^4)\,. \end{aligned} \tag{A.14}$$

References

[1] C.M. Hull and P.K. Townsend, *Unity of superstring dualities*, Nucl. Phys. B **438** (1995) 109 [hep-th/9410167] [INSPIRE].

[2] E. Witten, *Some comments on string dynamics*, in *STRINGS 95: Future Perspectives in String Theory*, Los Angeles U.S.A. (1995), pg. 501 [hep-th/9507121] [INSPIRE].

[3] M. Cvetič and A.A. Tseytlin, *Solitonic strings and BPS saturated dyonic black holes*, Phys. Rev. D **53** (1996) 5619 [*Erratum ibid.* **55** (1997) 3907] [hep-th/9512031] [INSPIRE].

[4] W.E. Couch and R.J. Torrence, *Conformal invariance under spatial inversion of extreme Riessner-Nordström black holes*, Gen. Rel. Grav. **16** (1984) 789.

[5] M. Cvetič and D. Youm, *Dyonic BPS saturated black holes of heterotic string on a six torus*, Phys. Rev. D **53** (1996) 584 [hep-th/9507090] [INSPIRE].

[6] P. Bizon and H. Friedrich, *A remark about wave equations on the extreme Reissner-Nordström black hole exterior*, Class. Quant. Grav. **30** (2013) 065001 [arXiv:1212.0729] [INSPIRE].

[7] J. Lucietti, K. Murata, H.S. Reall and N. Tanahashi, *On the horizon instability of an extreme Reissner-Nordström black hole*, JHEP **03** (2013) 035 [arXiv:1212.2557] [INSPIRE].

[8] H. Godazgar, M. Godazgar and C.N. Pope, *Aretakis Charges and Asymptotic Null Infinity*, Phys. Rev. D **96** (2017) 084055 [arXiv:1707.09804] [INSPIRE].

[9] M. Cvetič, C.N. Pope and A. Saha, *Generalized Couch-Torrence symmetry for rotating extremal black holes in maximal supergravity*, *Phys. Rev. D* **102** (2020) 086007 [arXiv:2008.04944] [INSPIRE].

[10] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, *stu Black Holes Unveiled*, *Entropy* **10** (2008) 507 [arXiv:0807.3503] [INSPIRE].

[11] L. Borsten, M.J. Duff and A. Marrani, *Freudenthal duality and conformal isometries of extremal black holes*, arXiv:1812.10076 [INSPIRE].

[12] M.J. Duff, J.T. Liu and J. Rahmfeld, *Four-dimensional string-string-string triality*, *Nucl. Phys. B* **459** (1996) 125 [hep-th/9508094] [INSPIRE].

[13] A. Sen, *Strong-weak coupling duality in four-dimensional string theory*, *Int. J. Mod. Phys. A* **9** (1994) 3707 [hep-th/9402002] [INSPIRE].

[14] D.D.K. Chow and G. Compère, *Dyonic AdS black holes in maximal gauged supergravity*, *Phys. Rev. D* **89** (2014) 065003 [arXiv:1311.1204] [INSPIRE].

[15] M. Cvetič et al., *Embedding AdS black holes in ten-dimensions and eleven-dimensions*, *Nucl. Phys. B* **558** (1999) 96 [hep-th/9903214] [INSPIRE].

[16] D.Z. Freedman and A. Van Proeyen, *Supergravity*, Cambridge University Press, Cambridge U.K. (2012).

[17] D.D.K. Chow and G. Compère, *Seed for general rotating non-extremal black holes of $\mathcal{N}=8$ supergravity*, *Class. Quant. Grav.* **31** (2014) 022001 [arXiv:1310.1925] [INSPIRE].

[18] W.A. Sabra, *Black holes in $N=2$ supergravity theories and harmonic functions*, *Nucl. Phys. B* **510** (1998) 247 [hep-th/9704147] [INSPIRE].

[19] K. Behrndt, D. Lüst and W.A. Sabra, *Stationary solutions of $N=2$ supergravity*, *Nucl. Phys. B* **510** (1998) 264 [hep-th/9705169] [INSPIRE].

[20] M. Cvetič, C.N. Pope, A. Saha and A. Satz, *Gaussian Null Coordinates for Rotating Charged Black Holes and Conserved Charges*, *Phys. Rev. D* **101** (2020) 066025 [arXiv:2001.04495] [INSPIRE].

[21] S. Aretakis, *Horizon Instability of Extremal Black Holes*, *Adv. Theor. Math. Phys.* **19** (2015) 507 [arXiv:1206.6598] [INSPIRE].

[22] J. Lucietti and H.S. Reall, *Gravitational instability of an extreme Kerr black hole*, *Phys. Rev. D* **86** (2012) 104030 [arXiv:1208.1437] [INSPIRE].

[23] G.W. Gibbons, S.A. Hartnoll and A. Ishibashi, *On the stability of naked singularities*, *Prog. Theor. Phys.* **113** (2005) 963 [hep-th/0409307] [INSPIRE].

[24] C. Vafa, *The String landscape and the swampland*, hep-th/0509212 [INSPIRE].

[25] H. Ooguri and C. Vafa, *On the Geometry of the String Landscape and the Swampland*, *Nucl. Phys. B* **766** (2007) 21 [hep-th/0605264] [INSPIRE].

[26] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, *The String landscape, black holes and gravity as the weakest force*, *JHEP* **06** (2007) 060 [hep-th/0601001] [INSPIRE].

[27] N. Gendler and I. Valenzuela, *Merging the weak gravity and distance conjectures using BPS extremal black holes*, *JHEP* **01** (2021) 176 [arXiv:2004.10768] [INSPIRE].

JHEP09(2021)188

[28] M. Cvetič and D. Youm, *Singular BPS saturated states and enhanced symmetries of four-dimensional $N = 4$ supersymmetric string vacua*, Phys. Lett. B **359** (1995) 87 [hep-th/9507160] [INSPIRE].

[29] H. Lü and C.N. Pope, *p-brane solitons in maximal supergravities*, Nucl. Phys. B **465** (1996) 127 [hep-th/9512012] [INSPIRE].

[30] E. Cremmer, B. Julia, H. Lü and C.N. Pope, *Dualization of dualities. 1*, Nucl. Phys. B **523** (1998) 73 [hep-th/9710119] [INSPIRE].

[31] Z.W. Chong, M. Cvetič, H. Lü and C.N. Pope, *Charged rotating black holes in four-dimensional gauged and ungauged supergravities*, Nucl. Phys. B **717** (2005) 246 [hep-th/0411045] [INSPIRE].