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Yang-Baxter deformation of WZW model based on Lie supergroups: The cases of $GL(1|1)$ and $(C^3 + A)$

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We proceed to generalize the Yang-Baxter (YB) deformation of Wess-Zumino-Witten (WZW) model to the Lie supergroups case. This generalization enables us to utilize various kinds of solutions of the (modified) graded classical Yang-Baxter equation ((m)GCYBE) to classify the YB deformations of WZW models based on the Lie supergroups. We obtain the inequivalent solutions (classical r-matrices) of the (m)GCYBE for the $gl(1|1)$ and $(\mathcal{C}^3 + \mathcal{A})$ Lie superalgebras in the non-standard basis, in such a way that the corresponding automorphism transformations are employed. Then, the YB deformations of the WZW models based on the $GL(1|1)$ and $(C^3 + A)$ Lie supergroups are specified by skew-supersymmetric classical r-matrices satisfying (m)GCYBE. In some cases for both families of deformed models, the metrics remain invariant under the deformation, while the components of B-fields are changed. After checking the conformal invariance of the models up to one-loop order, it is concluded that the $GL(1|1)$ and $(C^3 + A)$ WZW models are conformal theories within the classes of the YB deformations preserving the conformal invariance. However, our results are interesting in themselves, but at a constructive level, may prompt many new insights into (generalized) supergravity solutions.

Contents

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1. Introduction

Klimcik [1–3] proposed the YB σ-model as a systematic way to consider integrable deformations of two-dimensional non-linear σ-models. Then, this systematic procedure refined by Delduc, Magro and Vicedo in [4]. The deformations obtained via this method are called *Yang-Baxter* deformations, due to the central place that the CYBE takes in the construction. The principal chiral models deformed by Klimcik were also generalized by Delduc, Magro and Vicedo in [5,6] for the $AdS_5 \times S^5$ superstring action (see, also [7–9]). In [5], the integrable deformation of the type IIB $AdS_5 \times S^5$ superstring action along with the deformed field equations, Lax connection, and κ-symmetry transformations have been presented. Moreover, one can see the supercoset constructions in the YB deformed $AdS_5 \times S^5$ superstring with the $SU(2,2|4)$ Lie supergroup based on the homogeneous CYBE in [10] (see, also [11]). Actually, the integrable deformations of the $AdS_5 \times S^5$ superstring is an important application of the YB σ-model description. So far in all the works done on the deformation of the superstring action, the attention has been concentrated to the case where the deformations are created by bosonic generators of the Lie supergroup. Unlike these works, in the present work, the deformation is performed on both bosonic and fermionic sectors of the models.

The YB σ-model was then generalized by adding a WZW term. A prescription of the YB deformation of WZW model invented by Delduc, Magro and Vicedo in [12] (see, also [13–15]). In most of the works, the deformations of the YB WZW models have been studied on semisimple or compact Lie groups. Some interesting examples of the deformed YB WZW models were constructed on the Lie groups Nappi-Witten [16], H_4 [17] and $GL(2,\mathbb{R})$ [18] with classical r-matrices satisfying the (m)CYBE. A fundamental fact about them is that all can be considered as unique conformal theories within the class of the YB deformations preserving the conformal invariance.

The goal of the present work is to generalize the YB deformation of WZW model to the Lie supergroups case and present the resulting YB deformed backgrounds for the $GL(1|1)$ and $(C^3 + A)$ Lie supergroups along with inequivalent classical r-matrices satisfying the (m)GCYBE. This generalization would be important from the viewpoint of its applications, because the YB deformed backgrounds on the Lie supergroups have a wider class of the (generalized) supergravity solutions [19] in general rather than the bosonic Lie groups.

This paper is organized as follows. In Section 2, by introducing a useful notation of $\mathbb{Z}_2$-graded vector space we generalize the YB deformation of WZW model to the Lie supergroups case. In Section 3, we find the R-operators and inequivalent r-matrices for the $gl(1|1)$ Lie superalgebra. We furthermore construct the YB deformed backgrounds of the $GL(1|1)$ WZW model in this section. Calculating inequivalent r-matrices for the $(\mathcal{C}^3 + \mathcal{A})$ Lie superalgebra and followed by the YB deformations of the $(C^3 + A)$ WZW model are devoted to Section 4. In Section 5, it is shown that the deformed backgrounds satisfy the one-loop beta function equations which is the most important feature of the obtained models. In this way, we obtain the dilaton fields making the deformed models conformal up to the one-loop order. Some concluding remarks are given in the last section.

2. YB deformation of WZW model based on Lie supergroups and (m)GCYBE

We are now interested in studying the YB deformation of WZW model based on Lie supergroups. The general procedure that we shall apply is a straightforward generalization of the well-known prescription of Delduc, Magro and Vicedo [12]. Thus, in this section, inspired by a prescription invented by authors of Ref. [12], we generalize the YB deformation of WZW model from Lie groups to Lie supergroups. Before setting the model with Lie supergroups, let us recall the properties of $\mathbb{Z}_2$-graded vector space and also the definition of a Lie superalgebra $\mathscr{G}$ [20]. A super vector space V is a $\mathbb{Z}_2$-graded vector space, i.e., a vector space over a field $\mathbb{K}$ with a given decomposition of subspaces of grade 0 and grade 1, $V = V_0 \oplus V_1$. The parity of a nonzero homogeneous element, denoted by $|x|$, is 0 (even) or 1 (odd)[1] according to whether it is in V_0 or V_1, namely, $|x| = 0$ for any $x \in V_0$, while for any $x \in V_1$ we have $|x| = 1$. A Lie superalgebra $\mathscr{G}$ is a $\mathbb{Z}_2$-graded vector space, thus admitting the decomposition $\mathscr{G} = \mathscr{G}_B \oplus \mathscr{G}_F$, equipped with a bilinear superbracket structure $[.,.] : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ satisfying the requirements of anti-supersymmetry and super Jacobi identity. If $\mathscr{G}$ is finite-dimensional and the dimensions of $\mathscr{G}_B$ and $\mathscr{G}_F$ are $m = \#B$ and $n = \#F$, respectively, then $\mathscr{G}$ is said to have dimension $(m|n)$. We shall identify grading indices by the same indices in the power of (-1), i.e., we use $(-1)^x$ instead of $(-1)^{|x|}$, where $(-1)^x$ equals 1 or -1 if the Lie sub-superalgebra element is even or odd, respectively.[2]

Let us turn our attention to the model setting with Lie supergroups. First of all, it should be noted that the original WZW model based on a Lie supergroup G in Dewitt's notation was first presented in [22]. Accordingly, the action of the YB deformed WZW model on a Lie supergroup G may be expressed as[3]

$$S^{YB}_{WZW}(g) = \frac{1}{2}\int_{\Sigma} d\sigma^+ d\sigma^-\, (-1)^a J^a_+ \Omega_{ab} L^b_- + \frac{\kappa}{12}\int_{B_3} d^3\sigma\, \varepsilon^{\alpha\beta\gamma}\, (-1)^{a+bc} L^a_\alpha \Omega_{ad}\, f^d{}_{bc}\, L^b_\beta L^c_\gamma, \tag{2.1}$$

where $\sigma^\alpha = (\sigma^+, \sigma^-)$ are the standard lightcone variables such that their relationship with the worldsheet coordinates (τ, σ) is given by $\sigma^\pm = (\tau \pm \sigma)/\sqrt{2}$. Here, the left-invariant super one-form $L_\alpha = g^{-1}\, \partial_\alpha g$ is written in terms of an element $g(\tau, \sigma)$ of the Lie supergroup G. L_α is a $\mathscr{G}$-valued function, that is, it can be written as $L_\alpha = (-1)^a L^a_\alpha T_a$, in which $T_a, a = 1, ..., dim\, G$ are the basis of Lie superalgebra $\mathscr{G}$ of G. A key ingredient contained in both terms of the action (2.1) is the most general non-degenerate invariant supersymmetric bilinear form Ω_{ab} on the Lie algebra $\mathscr{G}$ which satisfies the following condition [22]:

$$f^d{}_{ab}\, \Omega_{dc} + (-1)^{bc} f^d{}_{ac}\, \Omega_{db} \;=\; 0. \tag{2.2}$$

Note that the bilinear form Ω_{ab} is defined as inner product $< . \,, \; . >$ for the basis T_a of $\mathscr{G}$, and $f^c{}_{ab}$ are the structure constants which determine the (anti-)commutation relations $[T_a, T_b] = f^c{}_{ab}\; T_c$. The deformed currents $J_\pm = (-1)^a J^a_\pm T_a$ are defined in the following form

[1] The even elements are sometimes called bosonic, and the odd elements fermionic. From now on, we use B and F instead of 0 and 1, respectively.

[2] Note that this notation was first used by Dewitt in [21]. Throughout this paper we work with Dewitt's notation.

[3] The last term in (2.1) is the standard WZ term integrated over a 3-dimensional space B_3 parameterized by (τ, σ, ξ) and whose boundary is the worldsheet Σ, where the extra direction is labeled by ξ. In this term, $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol in three dimensions.

$$J_{\pm} = (1+\omega\eta^2)\frac{1 \pm \tilde{A}R}{1-\eta^2 R^2} L_{\pm}, \tag{2.3}$$

where $\eta, \tilde{A}$ and κ are three independent real parameters such that the deformation is measured by η and $\tilde{A}$. The last parameter κ is regarded as the level. When $\eta = \tilde{A} = 0$ and $\kappa = 1$, the action (2.1) is nothing but that of the original WZW model on the Lie supergroup [22]. The operator R in (2.3) is a linear map from the Lie superalgebra $\mathscr{G}$ to itself, $R : \mathscr{G} \to \mathscr{G}$. It is a skew-supersymmetric solution of the (m)GCYBE on $\mathscr{G}$. That is to say, for any $X, Y \in \mathscr{G}$ it satisfies

$$[R(X), R(Y)] - R\big([R(X), Y] + [X, R(Y)]\big) = \omega[X, Y]. \tag{2.4}$$

Here ω is a constant parameter which can be normalized by rescaling R. Equation (2.4) can be generalized to the mGCYBE if one sets $\omega = \pm 1$, while the case with $\omega = 0$ is the homogeneous GCYBE. Moreover, the skew-supersymmetric condition of the linear R-operator requires that

$$< R(X), Y > + < X, R(Y) >= 0. \tag{2.5}$$

In what follows we will focus on a class of linear R-operators constructed from a classical r-matrix $r \in \mathcal{G} \otimes \mathcal{G}$ by means of the general formula[4]

$$R(X) =< r\ ,\ 1 \otimes X >, \tag{2.6}$$

for any $X \in \mathscr{G}$. Here the r-matrix defined as $r = r^{ab}\ T_a \otimes T_b$ is a solution of the following standard (m)GCYBE [20]

$$[[r, r]] \equiv [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \omega\ \Omega, \tag{2.7}$$

where $r_{12} = r \otimes 1$, $r_{23} = 1 \otimes r$ and $r_{13} = r^{ab}\ T_a \otimes 1 \otimes T_b$; moreover, $\Omega \in \Lambda^3(\mathscr{G})$ is the canonical triple tensor Casimir of G. Notice that the standard form of the (m)GCYBE is equivalent to (2.4). When the r-matrix is a skew-supersymmetric solution of (2.7), i.e., $r^{ab} = -(-1)^{ab}\ r^{ba}$, one can write

$$\begin{aligned} r &= \frac{1}{2} r^{ab}\big(T_a \otimes T_b - (-1)^{ab}\ T_b \otimes T_a\big) \\ &= \frac{1}{2} r^{ab}\ T_a \wedge T_b. \end{aligned} \tag{2.8}$$

We furthermore note that the r-matrix is considered to be even as $r \in \mathscr{G}_B \wedge \mathscr{G}_B \oplus \mathscr{G}_F \wedge \mathscr{G}_F$ so that it has the following matrix representation[5]

$$r^{ab} = \left(\begin{array}{c|c} r_B & 0 \\ \hline 0 & r_F \end{array}\right). \tag{2.10}$$

According to this, $r^{ab} = 0$ if $|a| \neq |b|$. In other words, fermions with bosons can't be mixed (grading is preserved). By using the fact that in r^{ab}, $|a| + |b| = 0$, and by expanding X and R in terms of the bases of $\mathscr{G}$ as $X = (-1)^a\ x^a T_a$ and $R = (-1)^b\ R_a{}^b T_b$, and then by substituting (2.8) in (2.6) we find

$$R_a{}^b = -(-1)^{ac}\ \Omega_{ac}\ r^{cb}. \tag{2.11}$$

Matrices such as Ω_{ab} and $R_a{}^b$ are also considered similar to (2.10), that is, one considers for them $|a| + |b| = 0$. Accordingly, the (m)GCYBE (2.4) can be rewritten into the following form:

$$(-1)^k\ R_a{}^c\ f^k{}_{cd} R_b{}^d - (-1)^b\ R_a{}^c\ f^d{}_{cb} R_d{}^k - (-1)^a\ R_b{}^c\ f^d{}_{ac} R_d{}^k = \omega\ f^k{}_{ab}. \tag{2.12}$$

It is also useful to obtain matrix form of the above equation by using the matrix representations of the structure constants, $f^c{}_{ab} = -(\mathcal{Y}^c)_{ab}$, giving us[6]

$$(-1)^d\ R\ \mathcal{Y}^k R^{st} - (-1)^c\ R(\mathcal{Y}^d R_d{}^k) - (\mathcal{Y}^d R_d{}^k) R^{st} = (-1)^k\ \omega \mathcal{Y}^k, \tag{2.13}$$

where index d in the first term of the left hand side denotes the column of matrix $\mathcal{Y}^k$, while in the second sentence, c corresponds to the row of matrix $\mathcal{Y}^d$. In the next sections, we employ the above formulation in order to obtain the linear R-operators and r-matrices on the $gl(1|1)$ and $(\mathcal{C}^3 + \mathcal{A})$ Lie superalgebras. By using the obtained R-operators we will find all YB deformations of WZW models based on these Lie supergroups.

4 We note that the inner product is evaluated on the second site of the r-matrix.

5 For a Lie superalgebra $\mathscr{G} = \mathscr{G}_B \oplus \mathscr{G}_F$ of dimension $(m|n)$ we define the basis of $\mathscr{G}$ as $\{T_a\}_{a=1}^{m+n} = \{t_i, S_\alpha\}$ where $\{t_i\}_{i=1}^m$ and $\{S_\alpha\}_{\alpha=m+1}^{m+n}$ are the bosonic and fermionic basis, respectively. Accordingly, the r-matrix can be written into the form

$$r = r_B^{ij}\ t_i \otimes t_j + r_F^{\alpha\beta}\ S_\alpha \otimes S_\beta. \tag{2.9}$$

6 Here the superscript "st" in R^{st} stands for supertranspose [21].

3. YB deformations of WZW model on the $GL(1|1)$ Lie supergroup

In this section we first solve the (m)GCYBE (2.13) in order to obtain the R-operators and inequivalent r-matrices for the $gl(1|1)$ Lie superalgebra. Using the resulting R-operators we construct the YB deformed backgrounds of the $GL(1|1)$ WZW model.

3.1. R-operators and r-matrices of the $gl(1|1)$

First of all, let us introduce the $gl(1|1)$ Lie superalgebra. In Backhouse's classification [23], the $gl(1|1)$ has been denoted by $(\mathcal{C}^2_{-1}+\mathcal{A})$; in fact, traditional notation for the $(C^2_{-1}+A)$ Lie superalgebra is the $gl(1|1)$. On the other, in Ref. [24] we classified all four-dimensional Drinfeld superdoubles of the type (2|2) and showed that there are just three classes of non-isomorphic Drinfeld superdoubles of the type (2|2) so that two of them are isomorphic to the Lie superalgebras $gl(1|1)$ and $(\mathcal{C}^3+\mathcal{A})$, another is an Abelian Lie superalgebra. These possess two bosonic generators and two fermionic ones. We shall denote the former by T_1, T_2 and use T_3, T_4 for fermionic generators. From now on we consider T_1, T_2 and T_3, T_4 as bosonic and fermionic generators, respectively. For the $gl(1|1)$, the relations between these elements are, in the non-standard basis, given by [23]

$$[T_1, T_3] = T_3, \quad [T_1, T_4] = -T_4, \quad \{T_3, T_4\} = T_2. \tag{3.1}$$

It should be noted that in Ref. [25] two of us obtained all Lie superbialgebra structures on the $gl(1|1)$ and their corresponding r-matrices in the standard basis. According to DeWitt's notation [21], in the standard basis the structure constants f^B_{FF} are considered to be pure imaginary. As we showed a moment ago in (3.1) here we work in the non-standard basis, so our results on the Lie superbialgebra structures and corresponding r-matrices will be different from those of [25]. The $gl(1|1)$ Lie superalgebra possesses a non-degenerate supersymmetric ad-invariant metric Ω_{ab} which is defined for any pair of bases $T_a, T_b \in gl(1|1)$ such that by using (2.2) and also the structure constants of (3.1) one gets [22]

$$\Omega_{ab} = \begin{pmatrix} \beta & \alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}, \tag{3.2}$$

for some real constants α, β. The metric is needed e.g. to write down the action of WZW model on the $GL(1|1)$ Lie supergroup.

Before proceeding to solve the (m)GCYBE (2.13) for the $gl(1|1)$, let us first assume that the most general skew-supersymmetric r-matrix $r \in \mathcal{G}_{(2|2)} \otimes \mathcal{G}_{(2|2)}$ has the following form:

$$r = m_1 T_1 \wedge T_2 + m_2 T_3 \wedge T_4 + \frac{1}{2} m_3 T_3 \wedge T_3 + \frac{1}{2} m_4 T_4 \wedge T_4. \tag{3.3}$$

Comparing this with (2.8) one can obtain the matrix representation of r^{ab}, giving us

$$r^{ab} = \begin{pmatrix} 0 & m_1 & 0 & 0 \\ -m_1 & 0 & 0 & 0 \\ 0 & 0 & m_3 & m_2 \\ 0 & 0 & m_2 & m_4 \end{pmatrix}, \tag{3.4}$$

where m_i are some real parameters. In addition, the matrix representations of the $gl(1|1)$ are easily obtained to be

$$\mathcal{Y}^1 = 0, \quad \mathcal{Y}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{Y}^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Y}^4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Inserting (3.2) and (3.4) into (2.11) one can obtain the general form of the corresponding R-operator. Thus, by substituting the resulting R-operator and also the representations (3.5) into equation (2.13), the general solution of the (m)GCYBE is split into four families ${R_{I_a}}^b$, ${R_{II_a}}^b$, ${R_{III_a}}^b$ and ${R_{IV_a}}^b$ such that the solutions are, in terms of the constants $\alpha, \beta, \omega, m_i$, given by

$$\begin{aligned} {R_{I_a}}^b &= \begin{pmatrix} \alpha m_1 & -\beta m_1 & 0 & 0 \\ 0 & -\alpha m_1 & 0 & 0 \\ 0 & 0 & \pm\sqrt{-\omega} & 0 \\ 0 & 0 & 0 & \mp\sqrt{-\omega} \end{pmatrix}, \quad {R_{II_a}}^b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\omega}{\alpha m_3} \\ 0 & 0 & -\alpha m_3 & 0 \end{pmatrix}, \\ {R_{III_a}}^b &= \pm\sqrt{-\omega} \begin{pmatrix} 1 & -\frac{\beta}{\alpha} & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & \pm\frac{\alpha m_4}{\sqrt{-\omega}} \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad {R_{IV_a}}^b = \pm\sqrt{-\omega} \begin{pmatrix} -1 & \frac{\beta}{\alpha} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \mp\frac{\alpha m_3}{\sqrt{-\omega}} & -1 \end{pmatrix}. \end{aligned} \tag{3.6}$$

Again by employing (2.11) and (3.2) one can obtain the corresponding r-matrices in the form of (2.9), giving us

$$
\begin{aligned}
r_{I} &= m_1\, T_1 \wedge T_2 \pm \frac{\sqrt{-\omega}}{\alpha}\, T_3 \wedge T_4,\\
r_{II} &= \frac{m_3}{2}\, T_3 \wedge T_3 - \frac{\omega}{2\alpha^2 m_3}\, T_4 \wedge T_4,\\
r_{III} &= \pm\frac{\sqrt{-\omega}}{\alpha}\Big(T_1 \wedge T_2 + T_3 \wedge T_4\Big) + \frac{m_4}{2}\, T_4 \wedge T_4,\\
r_{IV} &= \mp\frac{\sqrt{-\omega}}{\alpha}\Big(T_1 \wedge T_2 - T_3 \wedge T_4\Big) + \frac{m_3}{2}\, T_3 \wedge T_3.
\end{aligned}
\tag{3.7}
$$

The next step is that to determine the inequivalent r-matrices for the $gl(1|1)$. In fact, we need to specify the exact value of the parameters m_i of the solutions (3.7). In Ref. [17] as a Proposition we proved that two r-matrices r and r' of a Lie algebra $\mathscr{G}$ are equivalent if one can be obtained from the other by means of a change of basis which is an automorphism A of Lie algebra $\mathscr{G}$. Here we generalize the Proposition to the super case.

Proposition 3.1. *Two r-matrices r and r' of a Lie superalgebra $\mathscr{G}$ are equivalent if there exists an automorphism A of $\mathscr{G}$ such that*

$$
r^{ab} = (-1)^d\, (A^{st})^a{}_c\, r'^{cd}\, A_d{}^b.
\tag{3.8}
$$

The proof of this Proposition is similar to those of [17].

According to formula (3.8) in order to obtain the inequivalent r-matrices one must use the automorphism group of Lie superalgebra $\mathscr{G}$ which preserves (a) the parity of the generators (they can't mix fermions with bosons), and (b) the structure constants $f^c{}_{ab}$. Therefore it is crucial for our further considerations to identify the supergroup of automorphisms of the $gl(1|1)$ Lie superalgebra. We define the action of the automorphism A on $\mathscr{G}$ by the transformation $T'_a = (-1)^b\, A_a{}^b\, T_b$. The set of automorphisms of $gl(1|1)$ is generated by two transformations:

$$
T'_1 = -T_1 + cT_2, \quad T'_2 = abT_2, \quad T'_3 = -aT_4, \quad T'_4 = -bT_3,
\tag{3.9}
$$

and

$$
T'_1 = T_1 + cT_2, \quad T'_2 = abT_2, \quad T'_3 = -aT_3, \quad T'_4 = -bT_4,
\tag{3.10}
$$

where a, b, c are arbitrary real numbers such that $ab \neq 0$. The bases $\{T'_a\}$ obey the same (anti-)commutation relations as $\{T_a\}$. When taken into account, the above transformations lead to a conclusion that the parameters m_i in (3.7) can be scaled out to take the value of 1 or 0. Now by using the automorphism transformations and by employing formula (3.8), one can determine the inequivalent r-matrices for the $gl(1|1)$. Finally we arrive at eleven families of inequivalent r-matrices whose representatives can be described by means of the following Theorem.

Theorem 3.1. *Any r-matrix of the $gl(1|1)$ Lie superalgebra as a solution of the (m)GCYBE belongs just to one of the following eleven inequivalent classes*[7]

$$
\begin{aligned}
r_{i} &= T_1 \wedge T_2,\\
r_{ii} &= \frac{1}{2} T_3 \wedge T_3,\\
r_{iii} &= -\frac{1}{2} T_3 \wedge T_3,\\
r_{iv} &= T_3 \wedge T_4,\\
r_{v} &= T_1 \wedge T_2 + m_2 T_3 \wedge T_4, \quad m_2 = \frac{\sqrt{-\omega}}{\alpha} > 0, \quad m_2 \neq 1,\\
r_{vi} &= \frac{1}{2}\big(T_3 \wedge T_3 + T_4 \wedge T_4\big),\\
r_{vii} &= \frac{1}{2}\big(T_3 \wedge T_3 - T_4 \wedge T_4\big),\\
r_{viii} &= -\frac{1}{2}\big(T_3 \wedge T_3 + T_4 \wedge T_4\big),\\
r_{ix} &= T_1 \wedge T_2 + T_3 \wedge T_4,\\
r_{x} &= T_1 \wedge T_2 + T_3 \wedge T_4 + \frac{1}{2} T_4 \wedge T_4,\\
r_{xi} &= T_1 \wedge T_2 + T_3 \wedge T_4 - \frac{1}{2} T_4 \wedge T_4.
\end{aligned}
$$

[7] As we mentioned at the beginning of subsection (3.1), all Lie superbialgebra structures on the $gl(1|1)$ and their corresponding r-matrices have been, in the standard basis, obtained in [25]. There, it has been shown that among seventeen families of inequivalent Lie superbialgebra structures, only six of them are of coboundary type, while in the present work we have obtained eleven families of inequivalent r-matrices. The reason behind this is that if one solves super co-Jacobi and mixed super Jacobi identities for the $gl(1|1)$ in the non-standard basis, then he/she sees that the solutions will be different from those of [25].

It should be noted that among eleven inequivalent classes of the r-matrices, only r_i, r_{ii} and r_{iii} satisfy the standard GCYBE with $\omega = 0$, while the rest are solutions of the mGCYBE with $\omega = -\alpha^2$ except for r_v and r_{vii} which is $\omega = \alpha^2$ for r_{vii}. The parameter m_2 is present in r_v as it can't be removed by means of the transformations (3.9) and (3.10). It means that for different values of m_2 we have inequivalent r-matrices. However, as we will see, the m_2 plays a role of the deformation parameter in the YB deformed background of the $GL(1|1)$ WZW model.

Before closing this subsection, let us look at the unimodularity condition on the solutions of the (m)GCYBE for the $gl(1|1)$, Theorem 3.1. As we know the r-matrix is the initial input for construction of the YB deformed backgrounds. When the r-matrix satisfies the unimodularity condition that is given by [26]

$$r^{ab}\,[T_a, T_b] = 0, \tag{3.11}$$

then, the resulting deformed background is a solution to type IIB supergravity. If not, the background does not satisfy the on-shell condition of the supergravity and becomes a solution to a generalized supergravity. Below we determine which of the r-matrices classified in Theorem 3.1 are unimodular and or non-unimodular. Using (3.11) together with (3.1) we find that the r-matrices r_{iv}, r_v, r_{ix}, r_x and r_{xi} are non-unimodular, while the rest denote the unimodular r-matrices. In the following, by calculating the linear R-operators corresponding to the inequivalent r-matrices of the $gl(1|1)$ we will deform the $GL(1|1)$ WZW model.

3.2. YB deformed backgrounds of the $GL(1|1)$ WZW model

Before proceeding to construct out the YB deformed backgrounds of the $GL(1|1)$ WZW model, let us have an overview of undeformed WZW model structure based on the $GL(1|1)$ Lie supergroup. In Ref. [22], it was constructed the $GL(1|1)$ WZW model in order to study super Poisson-Lie symmetry [27] of the model. As mentioned in section 2, by setting $\eta = \tilde{A} = 0$ and $\kappa = 1$ in (2.3) one gets the original WZW model from the action (2.1). Let us introduce a supergroup element represented by

$$g = e^{\chi T_4}\, e^{y T_1}\, e^{x T_2}\, e^{\psi T_3}, \tag{3.12}$$

where $x(\tau,\sigma)$ and $y(\tau,\sigma)$ denote bosonic fields while $\psi(\tau,\sigma)$ and $\chi(\tau,\sigma)$ stand for fermionic fields. Using (3.12), the components of left-invariant super one-form $L^a_\pm$ on the $GL(1|1)$ can be evaluated as [22]

$$\begin{aligned} &L^1_\pm = \partial_\pm y, \qquad\qquad L^2_\pm = \partial_\pm x - \partial_\pm \chi\; \psi e^y, \\ &L^3_\pm = -\partial_\pm \psi - \partial_\pm y\; \psi, \qquad L^4_\pm = -\partial_\pm \chi\; e^y. \end{aligned} \tag{3.13}$$

A key ingredient in writing down the action of a WZW model is the most general supersymmetric ad-invariant form such that for the $gl(1|1)$ has been given by equation (3.2). Finally by using (3.1), (3.2) and (3.13) one can write down the action of WZW model based on the $GL(1|1)$ similar to what was done in Ref. [22]. The corresponding supersymmetric metric and anti-supersymmetric two-form field (B-field) are given by

$$\begin{aligned} ds^2 &= (-1)^{\mu\nu}\, G_{\mu\nu} dx^\mu\, dx^\nu = \beta dy^2 + 2dydx - 2e^y\, d\psi d\chi, \\ B &= \frac{1}{2}(-1)^{\mu\nu}\, B_{\mu\nu} dx^\mu \wedge dx^\nu = -e^y\, d\psi \wedge d\chi. \end{aligned} \tag{3.14}$$

Here we have assumed that the constant α of Ω_{ab} in (3.2) is set to be 1. From now on we consider $\alpha = 1$. Equation (3.14) as a background of the WZW model should be conformally invariant. To check this, one first looks at the one-loop beta function equations [28]

$$\begin{aligned} &\mathcal{R}_{\mu\nu} + \frac{1}{4} H_{\mu\rho\sigma} H^{\sigma\rho}{}_{\nu} + 2 \overrightarrow{\nabla}_\mu \overrightarrow{\nabla}_\nu \Phi \;=\; 0, \\ &(-1)^{\lambda}\, \nabla^{\lambda} \left(e^{-2\Phi} H_{\lambda\mu\nu}\right) \;=\; 0, \\ &4\Lambda - \mathcal{R} - \frac{1}{12} H_{\mu\nu\rho} H^{\rho\nu\mu} + 4 \overrightarrow{\nabla}_\mu \Phi\, \overrightarrow{\nabla}^\mu \Phi - 4 \overrightarrow{\nabla}_\mu \overrightarrow{\nabla}^\mu \Phi = 0, \end{aligned} \tag{3.15}$$

where the covariant derivatives $\overrightarrow{\nabla}_\mu$, scalar curvature $\mathcal{R}$ and Ricci tensor $\mathcal{R}_{\mu\nu}$ are calculated from the metric $G_{\mu\nu}$ that is also used for lowering and raising indices, and $H_{\mu\nu\rho}$ is the field strength corresponding to the B-field which is defined by

$$H_{\mu\nu\rho} = (-1)^{\mu} \frac{\overrightarrow{\partial}}{\partial x^\mu} B_{\nu\rho} + (-1)^{\nu+\mu(\nu+\rho)} \frac{\overrightarrow{\partial}}{\partial x^\nu} B_{\rho\mu} + (-1)^{\rho+\rho(\mu+\nu)} \frac{\overrightarrow{\partial}}{\partial x^\rho} B_{\mu\nu}. \tag{3.16}$$

For the background (3.14) one easily verifies equations (3.15) with a constant dilaton field, $\Phi = \varphi_0$, and vanishing cosmological constant.

Let us turn into the main goal of this subsection which is nothing but calculating the YB deformations of the $GL(1|1)$ WZW model. As we mentioned earlier, having R-operators one can calculate the deformed currents. Now we use formulas (2.11) and (3.2) to obtain all linear R-operators corresponding to the inequivalent r-matrices of Theorem 3.1. In order to calculate the currents $J_\pm$ one may write down relation (2.3) in the following form

$$J^a_\pm - (-1)^{b+c}\, \eta^2 J^b_\pm\, R_b{}^c\, R_c{}^a = (1+\omega\eta^2)\big[L^a_\pm \pm (-1)^b\, \tilde{A} L^b_\pm\, R_b{}^a\big]. \tag{3.17}$$

Finally by using the resulting linear R-operators satisfying the (m)GCYBE, and also by utilizing relation (3.17) together with (3.13) one obtains the YB deformations of the $GL(1|1)$ WZW model. The deformed backgrounds including metric and B-field together with the

Table 1
YB deformed backgrounds of the $GL(1|1)$ WZW model.

Background symbol	Backgrounds including metric and B-field	Comments
$GL(1\|1)_{i}^{(\eta,\tilde{A},\kappa)}$	$ds^2=\frac{1}{1-\eta^2}\Big[\beta dy^2+2dydx+2\eta^2\psi e^y dyd\chi\Big]-2e^y d\psi d\chi$ $B=\frac{\tilde{A}}{1-\eta^2}\ \psi e^y dy\wedge d\chi-\kappa e^y d\psi\wedge d\chi$	$\omega=0$
$GL(1\|1)_{ii}^{(\tilde{A},\kappa)}$	$ds^2=\beta dy^2+2dydx-2e^y d\psi d\chi$ $B=-\frac{1}{2}\tilde{A}e^{2y}\ d\chi\wedge d\chi-\kappa e^y d\psi\wedge d\chi$	$\omega=0$
$GL(1\|1)_{iii}^{(\tilde{A},\kappa)}$	$ds^2=\beta dy^2+2dydx-2e^y d\psi d\chi$ $B=\frac{1}{2}\tilde{A}e^{2y}\ d\chi\wedge d\chi-\kappa e^y d\psi\wedge d\chi$	$\omega=0$
$GL(1\|1)_{iv}^{(\eta,\kappa)}$	$ds^2=(1-\eta^2)\Big(\beta dy^2+2dydx\Big)-2\eta^2\psi e^y dyd\chi-2e^y d\psi d\chi$ $B=-\kappa e^y d\psi\wedge d\chi$	$\omega=-1$
$GL(1\|1)_{v}^{(\eta,\tilde{A},\kappa)}$	$ds^2=\frac{1}{1-\eta^2}\Big[\beta(1-m_2^2\eta^2)dy^2+2dydx+2\eta^2(1-m_2^2)\psi e^y dyd\chi\Big]-2e^y d\psi d\chi$ $B=\frac{\tilde{A}(1-m_2^2\eta^2)}{1-\eta^2}\ \psi e^y dy\wedge d\chi-\kappa e^y d\psi\wedge d\chi$	$m_2=\sqrt{-\omega}$
$GL(1\|1)_{vi}^{(\eta,\tilde{A},\kappa)}$	$ds^2=(1-\eta^2)\Big[\beta dy^2+2dydx+\frac{2\eta^2}{1+\eta^2}\psi e^y dyd\chi-\frac{2}{1+\eta^2}e^y d\psi d\chi\Big]$ $B=\frac{-\tilde{A}(1-\eta^2)}{1+\eta^2}\big[\psi dy\wedge d\psi+\frac{1}{2}e^{2y}d\chi\wedge d\chi\big]-\kappa e^y d\psi\wedge d\chi$	$\omega=-1$
$GL(1\|1)_{vii}^{(\eta,\tilde{A},\kappa)}$	$ds^2=(1+\eta^2)\Big[\beta dy^2+2dydx-\frac{2\eta^2}{1-\eta^2}\psi e^y dyd\chi-\frac{2}{1-\eta^2}e^y d\psi d\chi\Big]$ $B=\frac{\tilde{A}(1+\eta^2)}{1-\eta^2}\ [\psi dy\wedge d\psi-\frac{1}{2}e^{2y}d\chi\wedge d\chi]-\kappa e^y d\psi\wedge d\chi$	$\omega=1$
$GL(1\|1)_{viii}^{(\eta,\tilde{A},\kappa)}$	$ds^2=(1-\eta^2)\Big[\beta dy^2+2dydx+\frac{2\eta^2}{1+\eta^2}\psi e^y dyd\chi-\frac{2}{1+\eta^2}e^y d\psi d\chi\Big]$ $B=\frac{\tilde{A}(1-\eta^2)}{1+\eta^2}\ [\psi dy\wedge d\psi+\frac{1}{2}e^{2y}d\chi\wedge d\chi]-\kappa e^y d\psi\wedge d\chi$	$\omega=-1$
$GL(1\|1)_{ix}^{(\tilde{A},\kappa)}$	$ds^2=\beta dy^2+2dydx-2e^y d\psi d\chi$ $B=\tilde{A}\ \psi e^y dy\wedge d\chi-\kappa e^y d\psi\wedge d\chi$	$\omega=-1$
$GL(1\|1)_{x}^{(\tilde{A},\kappa)}$	$ds^2=\beta dy^2+2dydx-2e^y d\psi d\chi$ $B=-\tilde{A}\ \psi dy\wedge d\psi-(\kappa+\tilde{A})e^y d\psi\wedge d\chi$	$\omega=-1$
$GL(1\|1)_{xi}^{(\tilde{A},\kappa)}$	$ds^2=\beta dy^2+2dydx-2e^y d\psi d\chi$ $B=\tilde{A}\ \psi dy\wedge d\psi-(\kappa+\tilde{A})e^y d\psi\wedge d\chi$	$\omega=-1$

related comments are summarized in Table 1. Notice that the symbol of each background, e.g. $GL(1|1)_{i}^{(\eta,\tilde{A},\kappa)}$, indicates the deformed background derived by r_i; roman numbers i, ii etc. distinguish between several possible deformed backgrounds of the $GL(1|1)$ WZW model, and the $(\kappa,\eta,\tilde{A})$ indicate the deformation parameters of each background.

As it is seen from Table 1, in some of the backgrounds such as $GL(1|1)_{ii}^{(\tilde{A},\kappa)}$, $GL(1|1)_{iii}^{(\tilde{A},\kappa)}$, $GL(1|1)_{ix}^{(\tilde{A},\kappa)}$, $GL(1|1)_{x}^{(\tilde{A},\kappa)}$ and $GL(1|1)_{xi}^{(\tilde{A},\kappa)}$, the metrics are invariant under the deformation, up to two-form B-fields. That is, the effect coming from the deformations is reflected only as the coefficient of B-field. With the exceptions of the $GL(1|1)_{ii}^{(\tilde{A},\kappa)}$, $GL(1|1)_{iii}^{(\tilde{A},\kappa)}$ and $GL(1|1)_{iv}^{(\eta,\kappa)}$, for the rest of the backgrounds we have ignored the total derivative terms that appeared in the B-fields part.

4. YB deformations of WZW model on the (C^3+A) Lie supergroup

Similarly to the performance of calculations for the $gl(1|1)$, in this section we first solve the (m)GCYBE (2.13) to obtain the R-operators and inequivalent r-matrices for the $(\mathcal{C}^3+\mathcal{A})$ Lie superalgebra. We then get YB deformations of the WZW model based on the (C^3+A) Lie supergroup by utilizing the inequivalent r-matrices satisfying the (m)GCYBE. This is the subject of the present section.

4.1. R-operators and r-matrices of the $(\mathcal{C}^3+\mathcal{A})$

The $(\mathcal{C}^3+\mathcal{A})$ Lie superalgebra is spanned by the set of generators $\{T_1,T_2;T_3,T_4\}$ which fulfill the following non-zero (anti-)commutation rules [23]:

$$[T_1,T_4]=T_3,\quad \{T_4,T_4\}=T_2. \tag{4.1}$$

Notice that the Lie superbialgebra structures on the $(\mathcal{C}^3+\mathcal{A})$ along with their corresponding r-matrices, in the standard basis, were obtained in [29]. Here we work in the non-standard basis; accordingly, our results on the r-matrices will be different from those of [29].

Analogously, we consider an element $r\in(\mathcal{C}^3+\mathcal{A})\otimes(\mathcal{C}^3+\mathcal{A})$ as in (3.3), or equivalently, (3.4). On the other hand, using (2.2) one easily checks that the non-degenerate ad-invariant metric on the $(\mathcal{C}^3+\mathcal{A})$ is the same (3.2). The general form of the corresponding R-operator can be found by inserting (3.2) and (3.4) into (2.11). Calculating the matrix representations $(\mathcal{Y}^c)_{ab}$ of the $(\mathcal{C}^3+\mathcal{A})$ and then putting the resulting R-operator into (2.13), the most general solution can be determined like

$$R_a{}^b = \begin{pmatrix} m_1 & -\beta m_1 & 0 & 0 \\ 0 & -m_1 & 0 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & -m_3 & -m_2 \end{pmatrix}. \tag{4.2}$$

Here the condition (2.13) has led to the following constraints:

$$\omega = m_2(m_2 + 2m_1), \quad m_4 = 0. \tag{4.3}$$

Again by employing (2.11), the corresponding r-matrix to the above solution is obtained to be

$$r = m_1 T_1 \wedge T_2 + m_2 T_3 \wedge T_4 + \frac{1}{2} m_3 \, T_3 \wedge T_3. \tag{4.4}$$

In the following, in order to find inequivalent r-matrices we need to specify the exact value of the parameters m_i of the above solution. For this purpose, one must use the formula (3.8). The use of this formula requires that we know the automorphism transformation of the given Lie superalgebra. For the $(\mathcal{C}^3 + \mathcal{A})$ the automorphism transformation preserving the (anti-)commutation rules (4.1) is given by

$$T_1' = aT_1 + cT_2, \quad T_2' = b^2 T_2, \quad T_3' = -abT_3, \quad T_4' = -dT_3 - bT_4, \tag{4.5}$$

for some constants a, b, c, d. After performing the transformation (4.5) on formula (3.8), one concludes that r-matrices of the $(\mathcal{C}^3 + \mathcal{A})$ are split into eight inequivalent classes. For the sake of clarity the results are summarized in Theorem 4.1.

Theorem 4.1. *Any r-matrix of the $(\mathcal{C}^3 + \mathcal{A})$ Lie superalgebra as a solution of the (m)GCYBE belongs just to one of the following eight inequivalent classes*

$$\begin{aligned}
r_i &= \frac{1}{2} T_3 \wedge T_3, \\
r_{ii} &= -\frac{1}{2} T_3 \wedge T_3, \\
r_{iii} &= T_3 \wedge T_4, \\
r_{iv} &= T_1 \wedge T_2, \\
r_v &= T_1 \wedge T_2 + \frac{1}{2} T_3 \wedge T_3, \\
r_{vi} &= T_1 \wedge T_2 - \frac{1}{2} T_3 \wedge T_3, \\
r_{vii} &= T_1 \wedge T_2 + m_2 \, T_3 \wedge T_4, \omega = m_2(m_2 + 2), \quad m_2 \neq 0, -2 \\
r_{viii} &= T_1 \wedge T_2 - 2 \, T_3 \wedge T_4.
\end{aligned}$$

It is noteworthy that only the r-matrices r_{iii} and r_{vii} satisfy the mGCYBE with $\omega = 1$ and $\omega = m_2(m_2 + 2)$, respectively, while the rest are solutions of the GCYBE. At the end of this subsection it should be noted that all inequivalent r-matrices above are unimodular, that is, they satisfy the unimodularity condition (3.11).

4.2. YB deformed backgrounds of the $(C^3 + A)$ WZW model

We start this subsection by introducing the $(C^3 + A)$ WZW model. The $(C^3 + A)$ WZW model based on the $(C^3 + A)$ Lie supergroup was originally created in Ref. [30] in order to study its super Poisson-Lie T-dualizability [27]. In order to write the model explicitly we need to find the super one-form $L^a_\pm$'s. To this purpose we use a general element of $(C^3 + A)$ as in (3.12). Then we find [30]

$$\begin{aligned}
L^1_\pm &= \partial_\pm y, \qquad L^2_\pm = \partial_\pm x - \partial_\pm \chi \, \frac{\chi}{2}, \\
L^3_\pm &= -\partial_\pm \psi + \partial_\pm \chi \, y, \qquad L^4_\pm = -\partial_\pm \chi.
\end{aligned} \tag{4.6}$$

As mentioned before, one must set the parameters $\eta = \tilde{A} = 0$ and $\kappa = 1$ in (2.3) to get the original WZW model from the action (2.1). Using (4.1), (4.6) and the fact that the ad-invariant metric on the $(\mathcal{C}^3 + \mathcal{A})$ is the same (3.2), one computes the action of WZW model on the $(C^3 + A)$ Lie supergroup. From the action one can easily read off the corresponding metric and anti-supersymmetric fields, giving us

$$\begin{aligned}
ds^2 &= \beta dy^2 + 2dydx + \chi dy d\chi - 2d\psi d\chi, \\
B &= \frac{\chi}{2} \, dy \wedge d\chi.
\end{aligned} \tag{4.7}$$

Indeed, this background satisfies the one-loop beta function equations (3.15) with $\Phi = \varphi_0$ and $\Lambda = 0$.

We are looking for our main goal in this section, which is nothing but calculating the YB deformations of the $(C^3 + A)$ WZW model. First, employing formulas (2.11) and (3.2) we obtain all linear R-operators corresponding to the inequivalent r-matrices of Theorem 4.1. Then, making use of the relations (3.17) and (4.6) one obtains the deformed currents $J_\pm$. Finally we have used the action (2.1) to classify all YB deformed backgrounds of the $(C^3 + A)$ WZW model. The results including metric and B-field are summarized in Table 2. As it is seen, only in the backgrounds $(C^3 + A)_i^{(\tilde{A},\kappa)}$ and $(C^3 + A)_{ii}^{(\tilde{A},\kappa)}$, the metrics remained unchanged under transformation, up to the B-fields. In addition, for all backgrounds we have ignored the total derivative terms that appeared in the B-fields part, except for the mentioned backgrounds.

Table 2
YB deformed backgrounds of the (C^3+A) WZW model.

Background symbol	Backgrounds including metric and B-field	Comments
$(C^3+A)_{i}^{(\tilde{A},\kappa)}$	$ds^2=\beta dy^2+2dydx+\chi dyd\chi-2d\psi d\chi$ $B=-\frac{1}{2}\tilde{A}\ d\chi\wedge d\chi+\frac{1}{2}\kappa\chi\ dy\wedge d\chi$	$\omega=0$
$(C^3+A)_{ii}^{(\tilde{A},\kappa)}$	$ds^2=\beta dy^2+2dydx+\chi dyd\chi-2d\psi d\chi$ $B=+\frac{1}{2}\tilde{A}\ d\chi\wedge d\chi+\frac{1}{2}\kappa\chi\ dy\wedge d\chi$	$\omega=0$
$(C^3+A)_{iii}^{(\eta,\tilde{A},\kappa)}$	$ds^2=(1+\eta^2)\Big[\beta dy^2+2dydx+\chi dyd\chi-\frac{2}{(1-\eta^2)}d\psi d\chi\Big]$ $B=-\frac{\tilde{A}(1+\eta^2)}{(1-\eta^2)}y\ d\chi\wedge d\chi+\frac{1}{2}\kappa\chi\ dy\wedge d\chi$	$\omega=1$
$(C^3+A)_{iv}^{(\eta,\tilde{A},\kappa)}$	$ds^2=\frac{1}{(1-\eta^2)}\Big[\beta dy^2+2dydx+\chi dyd\chi\Big]-2d\psi d\chi$ $B=\frac{1}{2}\Big[\kappa+\frac{\tilde{A}}{(1-\eta^2)}\Big]\chi dy\wedge d\chi$	$\omega=0$
$(C^3+A)_{v}^{(\eta,\tilde{A},\kappa)}$	$ds^2=\frac{1}{(1-\eta^2)}\Big[\beta dy^2+2dydx+\chi dyd\chi\Big]-2d\psi d\chi$ $B=\frac{1}{2}\Big[\kappa+\frac{\tilde{A}}{(1-\eta^2)}\Big]\chi dy\wedge d\chi-\frac{1}{2}\tilde{A}\ d\chi\wedge d\chi$	$\omega=0$
$(C^3+A)_{vi}^{(\eta,\tilde{A},\kappa)}$	$ds^2=\frac{1}{(1-\eta^2)}\Big[\beta dy^2+2dydx+\chi dyd\chi\Big]-2d\psi d\chi$ $B=\frac{1}{2}\Big[\kappa+\frac{\tilde{A}}{(1-\eta^2)}\Big]\chi dy\wedge d\chi+\frac{1}{2}\tilde{A}\ d\chi\wedge d\chi$	$\omega=0$
$(C^3+A)_{vii}^{(\eta,\tilde{A},\kappa)}$	$ds^2=\frac{(1+\omega\eta^2)}{(1-\eta^2)}\Big[\beta dy^2+2dydx+\chi\ dyd\chi\Big]-\frac{2(1+\omega\eta^2)}{(1-m_2^2\eta^2)}d\psi d\chi$ $B=\frac{1}{2}\Big[\kappa+\frac{\tilde{A}(1+\omega\eta^2)}{(1-\eta^2)}\Big]\chi dy\wedge d\chi-\frac{\tilde{A}m_2(1+\omega\eta^2)}{(1-m_2^2\eta^2)}yd\chi\wedge d\chi$	$\omega=m_2(m_2+2)$ $m_2\neq 0,-2$
$(C^3+A)_{viii}^{(\eta,\tilde{A},\kappa)}$	$ds^2=\frac{1}{(1-\eta^2)}\Big[\beta dy^2+2dydx+\chi\ dyd\chi\Big]-\frac{2}{1-4\eta^2}d\psi d\chi$ $B=\frac{1}{2}\Big[\kappa+\frac{\tilde{A}}{(1-\eta^2)}\Big]\chi dy\wedge d\chi+\frac{2\tilde{A}}{(1-4\eta^2)}yd\chi\wedge d\chi$	$\omega=0$

Table 3
The dilaton fields making the $GL(1|1)$ deformed backgrounds conformal up to one-loop order.

Background symbol	Dilaton field	Comments
$GL(1\|1)_{i}^{(\eta,\tilde{A},\kappa)}$	$\Phi=\frac{\Gamma}{8(1-\eta^2)^2}y^2+c_1y+c_0$	$\Gamma=\Big[\tilde{A}+\kappa(1-\eta^2)\Big]^2-1$
$GL(1\|1)_{ii}^{(\tilde{A},\kappa)}$	$\Phi=\frac{1}{8}(\kappa^2-1)y^2+c_1y+c_0$	
$GL(1\|1)_{iii}^{(\tilde{A},\kappa)}$	$\Phi=\frac{1}{8}(\kappa^2-1)y^2+c_1y+c_0$	
$GL(1\|1)_{iv}^{(\eta,\kappa)}$	$\Phi=\frac{1}{8}\big[\kappa^2-(1-\eta^2)^2\big]y^2+c_1y+c_0$	
$GL(1\|1)_{v}^{(\eta,\tilde{A},\kappa)}$	$\Phi=\frac{\Gamma}{8(1-\eta^2)^2}y^2+c_1y+c_0$	$\Gamma=\Big[\tilde{A}(1-m_2^2\eta^2)+\kappa(1-\eta^2)\Big]^2-(1-m_2^2\eta^2)^2$
$GL(1\|1)_{vi}^{(\eta,\tilde{A},\kappa)}$	$\Phi=\frac{\Gamma}{2}y^2+c_1y+c_0$	$\Gamma=\tilde{A}^2-\frac{(1+\eta^2)^2}{4(1-\eta^2)^2}\big[(1-\eta^2)^2-\kappa^2\big]$
$GL(1\|1)_{vii}^{(\eta,\tilde{A},\kappa)}$	$\Phi=\frac{\Gamma}{2}y^2+c_1y+c_0$	$\Gamma=-\tilde{A}^2-\frac{(1-\eta^2)^2}{4(1+\eta^2)^2}\big[(1+\eta^2)^2-\kappa^2\big]$
$GL(1\|1)_{viii}^{(\eta,\tilde{A},\kappa)}$	$\Phi=\frac{\Gamma}{2}y^2+c_1y+c_0$	$\Gamma=\tilde{A}^2-\frac{(1+\eta^2)^2}{4(1-\eta^2)^2}\big[(1-\eta^2)^2-\kappa^2\big]$
$GL(1\|1)_{ix}^{(\tilde{A},\kappa)}$	$\Phi=\frac{1}{8}\big[(\tilde{A}+\kappa)^2-1\big]y^2+c_1y+c_0$	
$GL(1\|1)_{x}^{(\tilde{A},\kappa)}$	$\Phi=\frac{1}{8}\big[(\tilde{A}+\kappa)^2-1\big]y^2+c_1y+c_0$	
$GL(1\|1)_{xi}^{(\tilde{A},\kappa)}$	$\Phi=\frac{1}{8}\big[(\tilde{A}+\kappa)^2-1\big]y^2+c_1y+c_0$	

5. Conformal invariance of the YB deformed backgrounds

Our goal in this section is to investigate the conformal invariance conditions of the deformed models. In fact, we shall show that the WZW models based on the $GL(1|1)$ and (C^3+A) Lie supergroups can be considered as conformal theories within the classes of the YB deformations preserving the conformal invariance up to the one-loop order. Accordingly, using the equations (3.15) we check the conformal invariance conditions of the deformed backgrounds (Tables 1 and 2). From solving the equations we find the general form of the dilaton fields that make the deformed backgrounds conformal up to the one-loop order. The results obtained for the deformations of the $GL(1|1)$ WZW model are represented in Table 3. It is noteworthy that in all cases the cosmological constant vanishes. Also, the results obtained from solving equations (3.15) for the deformed backgrounds of the (C^3+A) WZW model are summarized in Table 4. In some cases of the (C^3+A) deformed backgrounds, we have shown that dilaton fields can depend on both bosonic coordinates. Note that c_0 and c_1 in Tables 3 and 4 are some arbitrary constants.

6. Summary and concluding remarks

We have generalized the formulation of YB deformation of WZW model proposed by Delduc, Magro and Vicedo from Lie groups to Lie supergroups. As showed, this generalization enabled us to find the various kinds of the solutions to the (m)GCYBE. As two influential examples, we classified the inequivalent r-matrices as solutions of the (m)GCYBE for the $gl(1|1)$ and $(\mathcal{C}^3+\mathcal{A})$ Lie superalgebras in the non-standard basis. Using these solutions we could construct YB deformations of the WZW models based on the $GL(1|1)$ and (C^3+A) Lie supergroups. We furthermore showed that the deformed backgrounds are conformally invariant up to the one-loop order which is the most important feature of the resulting models. With this interpretation, we have shown that the WZW models on the aforementioned

Table 4
The dilaton fields making the $(C^3 + A)$ deformed backgrounds conformal up to one-loop order.

Background symbol	Dilaton field	Comments
$(C^3+A)_{i}^{(\tilde{A},\kappa)}$	$\Phi = c_1\, y + c_0$;	$\Lambda = 0$,
$(C^3+A)_{ii}^{(\tilde{A},\kappa)}$	$\Phi = c_1\, y + c_0$;	$\Lambda = 0$,
$(C^3+A)_{iii}^{(\eta,\tilde{A},\kappa)}$	$\Phi = c_1\, y + c_0$;	$\Lambda = 0$,
	$\Phi = \left(\frac{\Lambda(1+\eta^2)}{\beta}\right)^{\frac{1}{2}} x + c_0$;	$\tilde{A} = -\frac{\kappa(1-\eta^2)}{2(1+\eta^2)}$
$(C^3+A)_{iv}^{(\eta,\tilde{A},\kappa)}$	$\Phi = c_1\, y + c_0$;	$\Lambda = 0$,
	$\Phi = \left(\frac{\Lambda}{\beta(1-\eta^2)}\right)^{\frac{1}{2}} x + c_0$;	$\tilde{A} = -5(1-\eta^2)$
$(C^3+A)_{v}^{(\eta,\tilde{A},\kappa)}$	$\Phi = c_1\, y + c_0$;	$\Lambda = 0$,
	$\Phi = \left(\frac{\Lambda}{\beta(1-\eta^2)}\right)^{\frac{1}{2}} x + c_0$;	$\tilde{A} = -5(1-\eta^2)$
$(C^3+A)_{vi}^{(\eta,\tilde{A},\kappa)}$	$\Phi = c_1\, y + c_0$;	$\Lambda = 0$,
	$\Phi = \left(\frac{\Lambda}{\beta(1-\eta^2)}\right)^{\frac{1}{2}} x + c_0$;	$\tilde{A} = -5(1-\eta^2)$
$(C^3+A)_{vii}^{(\eta,\tilde{A},\kappa)}$	$\Phi = c_1\, y + c_0$;	$\Lambda = 0$,
	$\Phi = \left(\frac{\Lambda\left[1+m_2(m_2+2)\eta^2\right]}{\beta(1-\eta^2)}\right)^{\frac{1}{2}} x + c_0$;	$\tilde{A} = -\frac{\kappa(1-\eta^2)(1-m_2^2\eta^2)}{\left[1+m_2(m_2+2)\eta^2\right]\left[1+2m_2-m_2(m_2+2)\eta^2\right]}$
$(C^3+A)_{viii}^{(\eta,\tilde{A},\kappa)}$	$\Phi = c_1\, y + c_0$;	$\Lambda = 0$,
	$\Phi = \left(\frac{\Lambda}{\beta(1-\eta^2)}\right)^{\frac{1}{2}} x + c_0$;	$\tilde{A} = \frac{\kappa}{3}(1-\eta^2)(1-4\eta^2)$

supergroups can be considered as conformal theories within the classes of the YB deformations preserving the conformal invariance up to one-loop order.

As mentioned earlier, here we have worked with two of the WZW models based on the $GL(1|1)$ and $(C^3 + A)$ Lie supergroups. The $GL(1|1)$ WZW model is interesting from the point of view of physics, because in some of the articles it has attracted considerable attention: By studying maximally symmetric branes in the $GL(1|1)$ WZW model it was shown that such branes are localized along (twisted) super-conjugacy classes [31] (see also [32]). The correlators of the model through a free field representation were constructed out in [33], then, by investigating some properties of the theory it was shown that some of the model correlators can be contained logarithmic singularities. Generally, WZW models on Lie supergroups present themselves as an ideal playground to extend many of the beautiful results of unitary rational conformal field theory to logarithmic models. Even the simplest models are mathematically rich and physically relevant. In addition, the existence of super Poisson-Lie symmetry is the most important feature of the $GL(1|1)$ WZW model [22].

We hope that in future it will be possible to find other YB deformed WZW models, especially for physically interesting backgrounds. As a future direction, it would be interesting to get the YB deformations of the WZW models on Lie supergroups in higher dimensions such as the $OSP(1|2)$ and $OSP(2|2)$ by following our present analysis and method. However, our results in the present work can still provide insights into (generalized) supergravity solutions. For this purpose, one must generalize the generalized supergravity equations to supermanifolds. Some of these problems are currently under investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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