Published for SISSA by Springer

Received: May 22, 2023
Revised: June 27, 2023
Accepted: June 27, 2023
Published: July 13, 2023

$(0,4)$ Projective superspaces. Part I. Interacting linear sigma models

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Abstract: We describe the projective superspace approach to supersymmetric models with off-shell $(0,4)$ supersymmetry in two dimensions. In addition to the usual superspace coordinates, projective superspace has extra bosonic variables — one doublet for each SU(2) in the R-symmetry SU(2) $\times$ SU(2) which are interpreted as homogeneous coordinates on $\mathbf{CP}^1 \times \mathbf{CP}^1$. The superfields are analytic in the $\mathbf{CP}^1$ coordinates and this analyticity plays an important role in our description. For instance, it leads to stringent constraints on the interactions one can write down for a given superfield content of the model. As an example, we describe in projective superspace Witten's ADHM sigma model — a linear sigma model with non-derivative interactions whose target is $\mathbf{R}^4$ with a Yang-Mills instanton solution. The hyperkähler nature of target space and the twistor description of instantons by Ward, and Atiyah, Hitchin, Drinfeld and Manin are natural outputs of our construction.

Keywords: D-Branes, Extended Supersymmetry, Superspaces

ArXiv ePrint: 2303.14675

https://doi.org/10.1007/JHEP07(2023)117

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1 Introduction

Two dimensional quantum field theories with chiral supersymmetry have appeared in a variety of physical and mathematical contexts. The most familiar example is the construction of heterotic string models which have $(0, 1)$ supersymmetry on the worldsheet [1]. Conformal theories with $(0, 2)$ supersymmetry were explored [2, 3] in the context of compactifications of the type $\mathbf{R}^4 \times K$ where K is a compact Calabi-Yau threefold. $(0, 2)$ Landau-Ginzburg models were also found to furnish a large class of $(0, 2)$ heterotic sigma models [4]. $(0, 4)$ worldsheet conformal theories are also interesting: they describe compactifications to six dimensions [5–7] and are useful in worldsheet descriptions of five-brane instantons [8, 9].

Since the brane revolution, many two dimensional spacetime models with chiral supersymmetry have been constructed — these appear as low-energy effective theories on two dimensional intersections of D-branes or on D1-branes probing manifolds with special holonomy. Depending on the brane setup, the models on the intersection may have $(0, 1)$, $(0, 2)$, $(0, 4)$ or even $(0, 8)$ supersymmetry [10, 11]. Typically, D-branes have gauge fields as part of their low-energy dynamics and the chiral supersymmetric theory is a gauged linear sigma model.

For example, a D1-brane probing a Spin(7) manifold has $(0, 1)$ supersymmetry on its worldvolume whereas it has $(0, 2)$ supersymmetry when probing a Calabi-Yau fourfold. The intersection of two stacks of D5-branes on a two-dimensional plane has $(0, 8)$ supersymmetry on the common intersection [12]; including a probe D1-brane on the common intersection gives $(0, 4)$ susy on the intersection [13–16]; various T-dual configurations have also been explored, for instance, see [17]. Another system of D-branes which has $(0, 4)$ susy is the $\mathrm{D1} \subset \mathrm{D5} \subset \mathrm{D9}$ system which is a D1-brane probe of a gauge theory instanton on $\mathbf{R}^4$ realized by the $\mathrm{D5} \subset \mathrm{D9}$ system [18], or instantons on an ALE space realized by taking the four transverse directions of the D9-brane relative to the D5-brane [19]. Other brane realizations include the worldvolume theory on M5-branes wrapped on a coassociative submanifold of a G_2-manifold which has $(0, 2)$ supersymmetry [20] and M5-branes wrapped on a four dimensional submanifold of a Calabi-Yau threefold which has $(0, 4)$ supersymmetry [21–23].

Superspace has proven to be powerful in understanding supersymmetric theories primarily because it realizes the supersymmetry algebra off-shell. The advantage of an off-shell realization is that, as long as the constraints on superfields do not themselves introduce interactions, we have a clean separation of kinematics and dynamics and the sum of two supersymmetric actions is automatically supersymmetric. This has been useful in uncovering the geometric structures hidden in supersymmetric theories and also understanding dualities between very different-looking models [24, 25]. However, the presence of so-called E-terms (which are deformations of the chirality constraints of fermionic superfields, see [26], and also appendix A.2 of this paper) can mix dynamics with kinematics, and then supersymmetry restricts the structure of the action even in superspace; we shall see that this plays a crucial role in our construction of interacting models.

Superspace descriptions of $(0, 1)$, $(1, 1)$, $(0, 2)$, $(1, 2)$ and $(2, 2)$ theories exist [27–34] and are well-understood. For theories with a higher amount of supersymmetry (for instance $(4, 4)$ in two dimensions or more generally, theories with eight supercharges in other

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dimensions), no description of off-shell charged hypermultiplets in ordinary superspace is known (the naive ordinary superspace constraints for the charged hypermultiplet put it on-shell, see [35, section 4.6]).

There are at least two approaches that address these issues, harmonic superspace [36, 37] and the closely related isotwistor superspace [38–40], and projective superspace [41, 42]. All approaches introduce a new set of bosonic coordinates u which are coordinates on an S^2. In the harmonic approach the u are viewed as harmonic coordinates on $S^2 \simeq \mathrm{SU}(2)/\mathrm{U}(1)$ where SU(2) is the R-symmetry group or a subgroup thereof, and one considers superfields which are harmonic functions on S^2. For a detailed description of harmonic superspace for $(4,4)$ theories, see [43–46]. In the projective approach, the S^2 is viewed as $\mathbf{CP}^1 \simeq \{\mathbf{C}^2 \smallsetminus 0\}/\mathbf{C}^\star$ and the u are homogeneous coordinates on the $\mathbf{CP}^1$ and the superfields are analytic functions on $\mathbf{CP}^1$. These two approaches are in fact related [47, 48].

Projective superspace has been successful in describing many supersymmetric models with eight supercharges [42, 49–56]. In projective superspace, one can write down new kinds of superfields and superspace constraints which depend on the coordinates u. More precisely, they are fibred over the coset space $\mathbf{CP}^1$. Superfields over projective superspace typically contain an infinite number of ordinary superfields (the coefficients in a Taylor expansion in u) and these turn out to be crucial in realizing the off-shell version of the hypermultiplet. Dynamically, most of these superfields turn out to be auxiliary and thus do not change the on-shell content of the hypermultiplet.

In section 2, we construct $(0,4)$ projective superspace and the realization of the $(0,4)$ supersymmetry algebra on it. We also describe the various superfields that are relevant to us, a general $(0,4)$ supersymmetric action in projective superspace, and the R-symmetry properties of superfields and actions. Projective superspace for $(0,4)$ supersymmetric theories was introduced in [57, 58] and was used to give off-shell formulations of $(0,4)$ supersymmetric nonlinear sigma models involving hypermultiplets. In this paper, we describe linear sigma models with manifest off-shell $(0,4)$ supersymmetry.

The R-symmetry of the $(0,4)$ supersymmetry algebra is $\mathrm{SU}(2) \times \mathrm{SU}(2)'$ and thus one has two projective superspaces with the $\mathbf{CP}^1$s corresponding to the two SU(2) subgroups. The hypermultiplets are also of two kinds, transforming as a doublet under either SU(2) or $\mathrm{SU}(2)'$. We call them standard hypermultiplets and, following [59], twisted hypermultiplets respectively. We describe these in detail in section 3. We shall see that a hyper can be realized either as a linear polynomial in the homogeneous coordinates (the $\mathcal{O}(1)$ superfield) or as a power series in a local coordinate on the $\mathbf{CP}^1$ (the 'polar' superfield). The $\mathcal{O}(1)$ superfield is treated in some detail in [57, 58]. A model with $(0,p)$ supersymmetry can admit fermionic multiplets with chirality opposite to that of the supercharges. These are the fermi multiplets; we realize them in projective superspace in section 4.

In $(0,2)$ models, we have interactions of the nonlinear sigma model type or the non-derivative type. Non-derivative interactions between chiral multiplets, gauge multiplets and fermi multiplets are described by modifying their superspace constraints with the so-called E-terms, or by including superpotential-like J-terms in the Lagrangian (see appendix A.2 of this paper). In section 5, we describe the E-term type non-derivative interactions for $(0,4)$ models containing standard hypers, twisted hypers and fermis (see equation (2.14) for a general description); it turns out that $(0,4)$ superpotential-like J-terms are not possible.

In a companion paper [60], we describe gauge multiplets and their interactions with hypers and fermis in projective superspace. Further extensions include coupling the various different types of hypermultiplets to $(0,4)$ supergravity in projective superspace (see [61] for related work in ordinary $(0,4)$ superspace).

In section 6, we describe in projective superspace a prominent $(0,4)$ supersymmetric model: a linear sigma model which flows down to a sigma model with target being an instanton solution in four dimensions. The couplings of the linear sigma model and the constraints they satisfy as a consequence of $(0,4)$ supersymmetry encode the data that enters the ADHM construction of instantons [62]. This was demonstrated in $(0,1)$ superspace by Witten [59], and it was given a D-brane interpretation by Douglas [18]. A manifest $(0,4)$ construction was given in harmonic superspace in [63, 64] (see [65] for some partial results in ordinary $(0,4)$ superspace). At the end of this Introduction, we give a short summary of our description of this model in $(0,4)$ projective superspace.

The appendix includes a quick review of $(0,1)$ and $(0,2)$ superspaces (appendix A), a realization of the $(4,4)$ hypermultiplet in $(4,4)$ projective superspace and its reduction to $(0,4)$ projective superspace (appendix B), and finally a detailed derivation of the ordinary space component actions for the general $(0,4)$ supersymmetric interacting linear sigma model (appendix C).

The ADHM sigma model in $(0,4)$ projective superspace: a precis. The target space $\mathbf{R}^4$ of the sigma model is realized in the $(0,4)$ projective superspace construction by a pair of twisted hypermultiplets $\boldsymbol{H}_{Y'}$, $Y'=1',2'$, with a symplectic reality condition $\overline{\boldsymbol{H}}^{Y'}=\varepsilon^{Y'Z'}\boldsymbol{H}_{Z'}$ where $\varepsilon^{Y'Z'}$ is the Levi-Civita symbol. Projective superspace has an auxiliary doublet coordinate $v^{a'}$, $a'=1',2'$, corresponding to the R-symmetry SU(2) under which each twisted hyper is a doublet (see section 2). Explicitly, the twisted hypers can be written as $\boldsymbol{H}_{Y'}=\sum_{a'}H_{a'Y'}v^{a'}$ where the a' index on $H_{a'Y'}$ indicates that we have an R-symmetry SU(2) doublet for each Y'. The 4-tuple $(v^{1'},v^{2'},\boldsymbol{H}_{1'},\boldsymbol{H}_{2'})$ can be interpreted homogeneous coordinates for the twistor space $\mathbf{CP}^3$ in which case the definitions $\boldsymbol{H}_{Y'}=\sum_{a'}H_{a'Y'}v^{a'}$ are simply the twistor space incidence relations. These naturally give twistor space $\mathbf{CP}^3$ the structure of a $\mathbf{CP}^1$ fibration over $\mathbf{S}^4$, where the $v^{a'}$ describe the $\mathbf{CP}^1$ and the $H_{a'Y'}$ describe $\mathbf{R}^4$ (whose compactification is the $\mathbf{S}^4$). The $\mathbf{CP}^1$ parametrizes the hyperkähler structure of $\mathbf{R}^4$ and thus the hyperkähler nature of the target space is manifest.

The SU(n) instanton bundle (of winding number k) is realized by k arctic standard hypermultiplets and $2k+n$ arctic fermi superfields. Our description of arctic standard hypers naturally associates two k dimensional vector spaces V_S and $\widehat{V}_S$ to them, and similarly, a $2k+n$ dimensional vector space V_F to the arctic fermis. The superspace constraints on the standard hypers and fermis involve maps $\widehat{\boldsymbol{C}}:V_S\to V_F$ and $\boldsymbol{C}:V_F\to\widehat{V}_S$ which are linear in the $v^{a'}$ and the twisted hypers $\boldsymbol{H}_{Y'}$, e.g., $\boldsymbol{C}=K_{a'}v^{a'}+L^{Y'}\boldsymbol{H}_{Y'}$ with $K_{a'}$ and $L^{Y'}$ constants. Supersymmetry invariance then demands that $\boldsymbol{C}\widehat{\boldsymbol{C}}=0$ and $\overline{\boldsymbol{C}}=\widehat{\boldsymbol{C}}$. The first equation (which is a condensed form of the ADHM equations) implies that

$$V_S \xrightarrow{\widehat{C}} V_F \xrightarrow{C} \widehat{V}_S\,, \tag{1.1}$$

is a monad, and the second equation gives the bundle described by the above monad a symplectic structure. With an additional non-degeneracy condition on $\boldsymbol{C}$, we precisely get the holomorphic bundles on twistor space which correspond to instantons on $\mathbf{R}^4$ [62, 67, 68]. In the rest of section 6, we review the analysis of the classical sigma model given in [59] which explicitly gives the expression for the instanton gauge field in terms of the couplings $\boldsymbol{C}$. We also adopt bases for the vector spaces V_S, $\widehat{V}_S$ and V_F and write the ADHM equations $\boldsymbol{C}\overline{\boldsymbol{C}} = 0$ in a more traditional form, and also describe the symmetries of the instanton moduli space. Our construction can be trivially extended to describe self-dual solutions on $\mathbf{R}^{4k'}$ with $k' > 1$ [66], as suggested in [59].

2 (0, 4) projective superspace

2.1 Introduction

The $(0,4)$ supersymmetric algebra consists of four real supercharges $\mathcal{Q}_{\mu+}$, $\mu = 1, \ldots, 4$, of right-handed chirality. It is useful to write these real supercharges in terms of a 2×2 matrix $\mathcal{Q}_{aa'+}$ that satisfies the reality conditions

$$\mathcal{Q}_{aa'+} = \overline{\mathcal{Q}}_{+}^{bb'}\,\varepsilon_{ba}\varepsilon_{b'a'}\,, \tag{2.1}$$

where $\overline{\mathcal{Q}}_{+}^{bb'} = \overline{(\mathcal{Q}_{bb'+})}$. Here, $a = 1, 2$ and $a' = 1', 2'$ are SU(2)-doublet indices. The R-symmetry group is then $\mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbf{Z}_2$. We will be interested in the representations of the supersymmetry algebra which are charged under just one of the SU(2)s and hence it is useful to consider the double cover $\mathrm{Spin}(4) \approx \mathrm{SU}(2) \times \mathrm{SU}(2) := F \times F'$. The a and a' indices are lowered using the invariant tensors ε_{ab} and $\varepsilon_{a'b'}$ which satisfy $\varepsilon^{ab}\varepsilon_{bc} = -\delta^a{}_c$, $\varepsilon^{a'b'}\varepsilon_{b'c'} = -\delta^{a'}{}_{c'}$ and $\varepsilon_{12} = \varepsilon_{1'2'} = +1$.

The supersymmetry algebra is

$$\{\mathcal{Q}_{aa'+}\,, \mathcal{Q}_{bb'+}\} = -2\mathrm{i}\varepsilon_{ab}\varepsilon_{a'b'}\partial_{++}\;. \tag{2.2}$$

Ordinary $(0,4)$ superspace $\mathbf{R}^{1,1|0,4}$ is described by the supercoordinates $\underline{z} = (x^{\pm\pm}, \theta^{aa'+})$ where $x^{\pm\pm} = \frac{1}{2}(x^0 \pm x^1)$. The corresponding supercovariant derivatives are $\partial_{\pm\pm} = \partial_0 \pm \partial_1$ and $\mathrm{D}_{aa'+}$ with the algebra

$$\{\mathrm{D}_{aa'+}\,, \mathrm{D}_{bb'+}\} = 2\mathrm{i}\varepsilon_{ab}\varepsilon_{a'b'}\partial_{++}\,, \quad [\mathrm{D}_{aa'+}\,, \partial_{\pm\pm}] = 0\;. \tag{2.3}$$

The derivatives $\mathrm{D}_{aa'+}$ also satisfy the same reality condition as for the supersymmetry generators (2.1). We loosely refer to (2.3) as the supersymmetry algebra though it differs from (2.2) by a sign. The supersymmetry generators $\mathcal{Q}_{aa'+}$ and the derivatives $\mathrm{D}_{aa'+}$ mutually anticommute: $\{\mathcal{Q}_{aa'+}\,, \mathrm{D}_{bb'+}\} = 0$.

In this paper, we work exclusively with the derivatives $\mathrm{D}_{aa'+}$ rather than the supersymmetry generators $\mathcal{Q}_{aa'+}$. Supersymmetry transformations of some component of a superfield Φ can be expressed in terms of $\mathrm{D}_{aa'+}$ because of the following fact which can be easily verified by using the explicit superspace expressions for $\mathcal{Q}_{aa'+}$ and $\mathrm{D}_{aa'+}$:

$$\delta\Phi_{|} = \left(\epsilon^{aa'+}\mathcal{Q}_{aa'+}\Phi\right)_{|} = \left(\epsilon^{aa'+}\mathrm{D}_{aa'+}\Phi\right)_{|}\,, \tag{2.4}$$

where $\epsilon^{aa'+}$ are constant Grassmann parameters, and $(X)_|$ stands for the operation of setting the Grassmann coordinates $\theta^{aa'+}$ to zero in the expression X. The $\mathrm{d}\theta^{aa'+}$ that appear in the superspace measure can also replaced by the corresponding $\mathrm{D}_{aa'+}$ up to total derivatives.[1]

It is convenient to define

$$\mathrm{D}_+ := \mathrm{D}_{11'+}\,, \quad \overline{\mathrm{D}}_+ := \mathrm{D}_{22'+}\,, \quad \mathrm{Q}_+ := \mathrm{D}_{21'+}\,, \quad \overline{\mathrm{Q}}_+ := -\mathrm{D}_{12'+}\,. \tag{2.5}$$

These derivatives span two (anti)commuting $(0,2)$ subalgebras:

$$\{\mathrm{D}_+, \overline{\mathrm{D}}_+\} = 2\mathrm{i}\partial_{++}\,, \quad \{\mathrm{Q}_+, \overline{\mathrm{Q}}_+\} = 2\mathrm{i}\partial_{++}\,, \text{ with other anticommutators equal to zero.} \tag{2.6}$$

2.2 Algebras, superfields and actions

Consider two sets of commuting coordinates u^a and $v^{a'}$ which are doublets under the R-symmetry SU(2) subgroups F and F' respectively. These are most usefully interpreted in our context as homogeneous coordinates on $\mathbf{CP}^1 \times \mathbf{CP}^{1\prime}$ (we label the second $\mathbf{CP}^1$ as $\mathbf{CP}^{1\prime}$ to indicate its relation to F'). The superspace with the coordinates $(x^{\pm\pm}, \theta^{aa'+}, u^a, v^{a'})$ is $\mathbf{R}^{1,1|0,4} \times \mathbf{CP}^1 \times \mathbf{CP}^{1\prime}$ which we refer to as *projective superspace*. The subspaces $\mathbf{R}^{1,1|0,4} \times \mathbf{CP}^1$ and $\mathbf{R}^{1,1|0,4} \times \mathbf{CP}^{1\prime}$ are important for us.

We also introduce *conjugate* doublets $\widetilde{u}^a$ and $\widetilde{v}^{a'}$ which satisfy

$$\varepsilon_{ab}\widetilde{u}^a u^b = 1\,, \quad \varepsilon_{a'b'}\widetilde{v}^{a'} v^{b'} = 1\,. \tag{2.7}$$

A shift symmetry. Note that there is more than one solution to the equation $\varepsilon_{ab}\widetilde{u}^a u^b = 1$. If $\widetilde{u}^b_0$ is one solution, then so is $\widetilde{u}^b_0 + \omega u^b$ for any $\omega \in \mathbf{C}$. Thus there is a shift symmetry on the $\widetilde{u}^a$:

$$\widetilde{u}^a \to \widetilde{u}^a + \omega u^a\,, \quad \text{for} \quad \omega \in \mathbf{C}\,. \tag{2.8}$$

There is a similar shift symmetry for the conjugate doublet $\widetilde{v}^{a'}$.

Derivatives on projective superspace. Consider the derivatives

$$\mathbf{D}_{a'+} := u^a \mathrm{D}_{aa'+}\,, \quad \widetilde{\mathbf{D}}_{a'+} := \widetilde{u}^a \mathrm{D}_{aa'+}\,, \quad \mathbf{D}_{a+} := v^{a'} \mathrm{D}_{aa'+}\,, \quad \widetilde{\mathbf{D}}_{a+} := \widetilde{v}^{a'} \mathrm{D}_{aa'+}\,, \tag{2.9}$$

where $\widetilde{u}^a$ and $\widetilde{v}^{a'}$ are any solutions to the equations (2.7). The algebra of the derivatives (2.9) is obtained from (2.3):

$$\begin{aligned} &\{\mathbf{D}_{a'+}, \mathbf{D}_{b'+}\} = 0\,, \quad \{\widetilde{\mathbf{D}}_{a'+}, \widetilde{\mathbf{D}}_{b'+}\} = 0\,, \quad \{\mathbf{D}_{a'+}, \widetilde{\mathbf{D}}_{b'+}\} = -2\mathrm{i}\varepsilon_{a'b'}\partial_{++}\,, \\ &\{\mathbf{D}_{a+}, \mathbf{D}_{b+}\} = 0\,, \quad \{\widetilde{\mathbf{D}}_{a+}, \widetilde{\mathbf{D}}_{b+}\} = 0\,, \quad \{\mathbf{D}_{a+}, \widetilde{\mathbf{D}}_{b+}\} = -2\mathrm{i}\varepsilon_{ab}\partial_{++}\,. \end{aligned} \tag{2.10}$$

Note that the shift symmetry (2.8) shifts the derivatives $\widetilde{\mathbf{D}}_{a'+}$ by $\omega \mathbf{D}_{a'+}$ but it leaves the algebra (2.10) unchanged. We shall see below that the action is also invariant under the shift symmetry up to total derivative terms.

We also introduce the fully contracted derivative

$$\mathbf{D}_+ = u^a v^{a'} \mathrm{D}_{aa'+} = u^a \mathbf{D}_{a+} = v^{a'} \mathbf{D}_{a'+} \quad \text{which satisfies} \quad \mathbf{D}_+^2 = 0\,, \tag{2.11}$$

due to the anticommutation relations (2.3). We can recover the algebra in (2.10) by writing $\mathbf{D}_+^2 = u^a u^b \{\mathbf{D}_{a+}, \mathbf{D}_{b+}\}$ or $\mathbf{D}_+^2 = v^{a'} v^{b'} \{\mathbf{D}_{a'+}, \mathbf{D}_{b'+}\}$.

[1] This is standard procedure, see e.g. [35].

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Projective superfields. An F-*projective* superfield $\mathbf{\Phi}(\underline{z}, u)$ is a function of the superspace coordinates $\underline{z} = (x_{\pm\pm}, \theta^{aa'+})$ and the $\mathbf{CP}^1$ coordinates u^a which satisfy the following:

(1) $\mathbf{\Phi}$ is holomorphic in a domain in $\mathbf{CP}^1$,

(2) $\mathbf{\Phi}$ satisfies the *projective* constraints $\mathbf{D}_{a'+}\mathbf{\Phi}(\underline{z}, u) = 0$,

(3) $\mathbf{\Phi}$ may be in non-trivial representations of the R-symmetry group $\mathrm{SU}(2) \times \mathrm{SU}(2)'$ and the Lorentz group $\mathrm{SO}(1,1)$.

An F'-projective superfield is analogously a function of the superspace coordinates $\underline{z}$ and the $\mathbf{CP}^{1\prime}$ coordinate $v^{a'}$ and is annihilated by $\mathbf{D}_{a+}$. We discuss the different types of projective superfields in section 2.4.

The F-projective constraints $\mathbf{D}_{a'+}\mathbf{\Phi}(\underline{z}, u) = 0$ can be encoded more economically in terms of the fully contracted derivative (2.11) $\mathbf{D}_+ = v^{a'}\mathbf{D}_{a'+}$:

$$\mathbf{D}_+\mathbf{\Phi} = 0 \; . \tag{2.12}$$

Since $\mathbf{\Phi}$ depends only on u and not on v, $\mathbf{D}_+\mathbf{\Phi} = v^{a'}\mathbf{D}_{a'+}\mathbf{\Phi}$ implies $\mathbf{D}_{a'+}\mathbf{\Phi} = 0$. The advantage of (2.12) is that it takes the same form for F'-projective superfields $\mathbf{\Phi}(\underline{z}, v)$ as well, since we can now recover $\mathbf{D}_{a+}\mathbf{\Phi} = 0$ using $\mathbf{D}_+ = u^a\mathbf{D}_{a+}$. We frequently use the derivative $\mathbf{D}_+$ in the paper.

Actions. The constraints $\mathbf{D}_{a'+}\mathbf{\Phi} = 0$ on a projective superfield $\mathbf{\Phi}$ imply that $\mathbf{\Phi}$ depends on only half of the Grassmann coordinates. The appropriate superspace measure which ensures $(0,4)$ invariance of an action composed of projective superfields is then quadratic in the derivatives $\widetilde{\mathbf{D}}_{a'+}$, i.e., $\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}$. The $(0,4)$ supersymmetric action is then given by

$$\mathcal{S}[\mathbf{\Phi}] = \int \mathrm{d}^2x \left(\frac{1}{2\pi\mathrm{i}} \oint_\gamma \varepsilon_{ab} u^a \mathrm{d}u^b \; \widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+} \; \boldsymbol{K}_{--}(\mathbf{\Phi}) \right)\Big| \; , \tag{2.13}$$

where

1. $|$ sets all the Grassmann coordinates to zero (we frequently omit the $|$ from our expressions).

2. $\boldsymbol{K}_{--}$ is the superspace Lagrangian which satisfies $\mathbf{D}_{a'+}\boldsymbol{K}_{--} = 0$. It must carry the $--$ Lorentz representation (left-moving part of a vector) in order to compensate the $++$ in the projective superspace measure.

3. The contour $\gamma \in \mathbf{CP}^1$ is chosen to avoid possible singularities in $\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\boldsymbol{K}_{--}$.

The action is invariant (up to total spacetime derivatives) under the shift symmetry (2.8) $\widetilde{\mathbf{D}}_{a'+} \to \widetilde{\mathbf{D}}_{a'+} + \omega\mathbf{D}_{a'+}$ since the Lagrangian $\boldsymbol{K}_{--}$ satisfies $\mathbf{D}_{a'+}\boldsymbol{K}_{--} = 0$. Since the superspace measure $\varepsilon_{ab}u^a\mathrm{d}u^b\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}$ is invariant under F and F', the action (2.13) is manifestly invariant under F and F' if the superspace Lagrangian is invariant.

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Non-derivative interactions. Suppose a projective superfield Φ_s is in the spin s representation of the Lorentz group SO(1,1). The requirement that $\mathbf{D}_{a'+}\boldsymbol{\Phi}_s = 0$ can be relaxed to have a non-zero right hand side:

$$\mathbf{D}_{a'+}\boldsymbol{\Phi}_s = \boldsymbol{S}_{a',s+1}\,, \tag{2.14}$$

where $\boldsymbol{S}_{a',s+1}$ is a function of other superfields in the model and is in the spin $s+1$ representation of SO(1,1). This allows us to introduce interactions (the so-called E-terms) as we will see later in section 5:

The modified constraints (2.14) are consistent with the algebra $\{\mathbf{D}_{a'+}, \mathbf{D}_{b'+}\} = 0$ only if the function satisfies

$$\mathbf{D}_{a'+}\boldsymbol{S}_{b',s+1} + \mathbf{D}_{b'+}\boldsymbol{S}_{a',s+1} = 0\,. \tag{2.15}$$

To ensure (0,4) invariance of the action, we require that the superspace Lagrangian $\boldsymbol{K}_{--}(\boldsymbol{\Phi})$ satisfies $\mathbf{D}_{a'+}\boldsymbol{K}_{--} = 0$ even if $\mathbf{D}_{a'+}\boldsymbol{\Phi}$ is not zero. This further constrains the $\boldsymbol{S}_{a',s+1}$.

Thus, any (0,4) supersymmetric model must satisfy the following constraints:

1. The (0,4) algebra $\mathbf{D}_+^2 = 0$ must be satisfied on every superfield in the model,
2. The superspace Lagrangian $\boldsymbol{K}_{--}$ must satisfy $\mathbf{D}_{a'+}\boldsymbol{K}_{--} = 0$ to ensure (0,4) supersymmetry of the action.

These criteria place stringent constraints on the superfield content and the interactions in a model.

2.3 Projective superspace in inhomogeneous coordinates

A primer on $\mathbf{CP}^1$. The projective space $\mathbf{CP}^1$ is constructed as the quotient space $\{\mathbf{C}^2 \smallsetminus 0\}/\sim$, where $\sim$ is the following equivalence relation on the coordinates of $\mathbf{C}^2$: $(u^1, u^2) \sim (\lambda u^1, \lambda u^2)$, $\lambda \in \mathbf{C}^\star$. We describe $\mathbf{CP}^1$ in terms of two charts U_1 and U_2:

$$\mathsf{U}_a := \{(u^1, u^2) \in \mathbf{C}^2 \mid u^a \neq 0\}\,. \tag{2.16}$$

The map $\mathcal{S} \in \mathrm{SU}(2)$ which acts on the homogeneous coordinates as

$$\mathcal{S}:\ \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} -u^2 \\ u^1 \end{pmatrix}, \tag{2.17}$$

interchanges the two charts. Using the equivalence $(u^1, u^2) \sim (\lambda u^1, \lambda u^2)$, $\lambda \in \mathbf{C}^\star$, we can scale out the non-zero coordinate in each of the charts and obtain a description in terms of *inhomogeneous* coordinates:

$$\mathsf{U}_1 = \{(1, -\rho) \mid \rho \in \mathbf{C}\}\,, \quad \mathsf{U}_2 = \{(\zeta, 1) \mid \zeta \in \mathbf{C}\}\,, \tag{2.18}$$

with $\rho = -u^2/u^1$ and $\zeta = u^1/u^2$. On the intersection $\mathsf{U}_{12} := \mathsf{U}_1 \cap \mathsf{U}_2$, the local coordinates ζ and ρ are related by the $\mathcal{S}$ map (2.17)

$$\mathcal{S}:\ \zeta \longmapsto -1/\zeta = \rho\,. \tag{2.19}$$

We can thus express all our results exclusively in terms of one of the inhomogeneous coordinates, say ζ, by appending the point $\zeta = \infty$ to the chart U_2. We frequently adopt this usage to avoid cluttering of notation.

The derivatives $\mathbf{D}_{a'+}$, $\widetilde{\mathbf{D}}_{a'+}$. We next express the derivatives $\mathbf{D}_{a'+}$ and $\widetilde{\mathbf{D}}_{a'+}$ in terms of the local coordinates ζ and ρ in the charts U_2 and U_1 respectively. In the chart U_2, we have $u^2 \neq 0$ and $u^a = (u^2)(\zeta, 1)$. Thus, we can choose $\widetilde{u}^a = (u^2)^{-1}(1,0)$ which indeed satisfies $\widetilde{u}^a u^b \varepsilon_{ab} = 1$. Using the scale invariance $u^a \to \lambda u^a$, we can set $u^2 = 1$ as discussed above (2.18). The derivatives $\mathbf{D}_{a'+} = u^a \mathrm{D}_{aa'+}$ and $\widetilde{\mathbf{D}}_{a'+} = \widetilde{u}^a \mathrm{D}_{aa'+}$ are then given by

$$\text{In } \mathsf{U}_2: \quad \mathbf{D}_{1'+} = \zeta \mathrm{D}_+ + \mathrm{Q}_+\,, \quad \mathbf{D}_{2'+} = -\zeta \overline{\mathrm{Q}}_+ + \overline{\mathrm{D}}_+\,, \quad \widetilde{\mathbf{D}}_{1'+} = \mathrm{D}_+\,, \quad \widetilde{\mathbf{D}}_{2'+} = -\overline{\mathrm{Q}}_+\,, \tag{2.20}$$

where we have used the expressions (2.5) for $\mathrm{D}_{aa'+}$. A similar description can be obtained in the chart U_1 in which $u^1 \neq 0$. Writing $u^a = u^1(1, -\rho)$, choosing $\widetilde{u}^a = (u^1)^{-1}(0,-1)$ and setting $u^1 = 1$ by scale invariance, we have

$$\text{In } \mathsf{U}_1: \quad \mathbf{D}_{1'+} = \mathrm{D}_+ - \rho \mathrm{Q}_+\,, \quad \mathbf{D}_{2'+} = -\overline{\mathrm{Q}}_+ - \rho \overline{\mathrm{D}}_+\,, \quad \widetilde{\mathbf{D}}_{1'+} = -\mathrm{Q}_+\,, \quad \widetilde{\mathbf{D}}_{2'+} = -\overline{\mathrm{D}}_+\,. \tag{2.21}$$

Observe that, in the intersection U_{12}, the derivatives $\mathbf{D}_{a'+}(\rho)$ defined in U_1 are related to the $\mathbf{D}_{a'+}(\zeta)$ defined in U_2 as

$$\mathbf{D}_{a'+}(\rho) = (-\rho)\mathbf{D}_{a'+}(\zeta(\rho))\,, \tag{2.22}$$

which is the gluing rule for a global section of the line bundle $\mathcal{O}(1) \to \mathbf{CP}^1$ (we have used that $\zeta(\rho) = -\rho^{-1}$ on the overlap).

Similarly, the derivatives $\widetilde{\mathbf{D}}_{a'+}(\rho)$ in U_1 and $\widetilde{\mathbf{D}}_{a'+}(\zeta)$ in U_2 are related on the overlap as

$$\widetilde{\mathbf{D}}_{a'+}(\rho) = (-\rho)^{-1}\widetilde{\mathbf{D}}_{a'+}(\zeta(\rho)) + \rho^{-1}\mathbf{D}_{a'+}(\rho) = (-\rho)^{-1}\widetilde{\mathbf{D}}_{a'+}(\zeta(\rho)) - \mathbf{D}_{a'+}(\zeta(\rho))\,, \tag{2.23}$$

where we have used (2.22) in going to the last expression. The transformation (2.23) can be viewed as the usual transformation of a section of $\mathcal{O}(-1)$ plus a shift term proportional to $\mathbf{D}_{a'+}$ generated by the shift symmetry (2.8). This allows us to define $\widetilde{\mathbf{D}}_{a'+}$ globally on $\mathbf{CP}^1$, not as a section of $\mathcal{O}(-1)$ but as a section of the affine bundle modelled on $\mathcal{O}(-1)$.

Note: there is an alternate way of writing the (0,4) algebra using the derivatives $\mathbf{D}_{a'+}$ and $\frac{\partial}{\partial\zeta}$:

$$\{\mathbf{D}_{a'+}\,, [\partial_\zeta\,, \mathbf{D}_{b'+}]\} = -2\mathrm{i}\varepsilon_{a'b'}\partial_{++}\,. \tag{2.24}$$

Thus, one may use the derivatives $\mathbf{D}_{a'+}$ and $\partial/\partial\zeta$ instead of $\mathbf{D}_{a'+}$ and $\widetilde{\mathbf{D}}_{a'+}$ in describing projective superspace. Observe that $\partial_\zeta \mathbf{D}_{b'+}$ coincides with $\widetilde{\mathbf{D}}_{b'+}$ in U_1 and $\partial_\rho \mathbf{D}_{b'+}$ coincides with $\widetilde{\mathbf{D}}_{b'+}$ in U_2. Further, $\partial_\zeta \mathbf{D}_{a'+}$ also satisfies the rule (2.23). However, this is expected since the derivative of a section of $\mathcal{O}(1)$ transforms as a section of the affine bundle modelled on $\mathcal{O}(-1)$.

A (0,2) action which is (0,4) supersymmetric. Plugging in the derivatives (2.20) in the action (2.13), we get

$$\mathcal{S}[\mathbf{\Phi}] = \int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \mathrm{D}_+\overline{\mathrm{Q}}_+ \boldsymbol{K}_{--}(\mathbf{\Phi})\,. \tag{2.25}$$

We can also rewrite the above action in (0,2) superspace. Using $-\overline{\mathrm{D}}_+ = -\zeta^{-1}\overline{\mathrm{D}}_+ + \zeta^{-1}\mathbf{D}_{2'+}$ and $\mathbf{D}_{a'+}\boldsymbol{K}_{--} = 0$, we get

$$\mathcal{S}[\mathbf{\Phi}] = \int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \mathrm{D}_+\overline{\mathrm{D}}_+ \boldsymbol{K}_{--}(\mathbf{\Phi}) = \int \mathrm{d}^2x\; \mathrm{D}_+\overline{\mathrm{D}}_+ \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \boldsymbol{K}_{--}(\mathbf{\Phi})\,. \tag{2.26}$$

F'-projective superspace. For completeness, we explicitly describe some analogous aspects of F'-projective superspace. We have the inhomogeneous coordinate ζ' for the $\mathbf{CP}^1$ corresponding to the F' doublet $v^{a'}$. We then choose $v^{a'} = (\zeta', 1)$ and $\widetilde{v}^{a'} = (1, 0)$ using the scale invariance $v^{a'} \to \lambda' v^{a'}$, $\lambda' \in \mathbf{C}^\star$. The F'-projective derivatives $\mathbf{D}_{a+}$ and $\widetilde{\mathbf{D}}_{a+}$ are then

$$\mathbf{D}_{1+} = \zeta' \mathrm{D}_+ - \overline{\mathrm{Q}}_+ \,, \quad \mathbf{D}_{2+} = \zeta' \mathrm{Q}_+ + \overline{\mathrm{D}}_+ \,, \quad \widetilde{\mathbf{D}}_{1+} = \mathrm{D}_+ \,, \quad \widetilde{\mathbf{D}}_{2+} = \mathrm{Q}_+ \,. \tag{2.27}$$

A $(0,4)$ supersymmetric action in $(0,2)$ superspace for F'-projective superfields $\mathbf{\Phi}'$ is given by

$$\mathcal{S}[\mathbf{\Phi}'] = -\int \mathrm{d}^2 x \ \mathrm{D}_+ \overline{\mathrm{D}}_+ \oint_{\gamma'} \frac{\mathrm{d}\zeta'}{2\pi \mathrm{i} \zeta'} \boldsymbol{K}'_{--}(\mathbf{\Phi}') \,. \tag{2.28}$$

The actions we consider in this paper will only have a single contour integral over either ζ or ζ'.

The fully contracted derivative $\mathbf{D}_+ = u^a v^{a'} \mathrm{D}_{aa'+}$ (2.11) in terms of ζ and ζ' is

$$\mathbf{D}_+ = \zeta\zeta' \mathrm{D}_{11'+} + \zeta \mathrm{D}_{12'+} + \zeta' \mathrm{D}_{21'+} + \mathrm{D}_{22'+} = \zeta\zeta' \mathrm{D}_+ - \zeta \overline{\mathrm{Q}}_+ + \zeta' \mathrm{Q}_+ + \overline{\mathrm{D}}_+ \,. \tag{2.29}$$

2.4 Analytic structure of projective superfields

Recall that F-projective superfields are holomorphic in a connected open subset of $\mathbf{CP}^1$ and that they are annihilated by the derivatives $\mathbf{D}_{a'+}$. We now describe the different types of projective superfields which differ in their analytic structure on the $\mathbf{CP}^1$. F'-projective superfields are defined analogously.

$\mathcal{O}(p)$ superfields. The superfield is a homogeneous polynomial in the u^a of degree $p > 0$:

$$\boldsymbol{\eta}(\underline{z}, u) = \eta_{a_1 \cdots a_p}(\underline{z}) u^{a_1} \cdots u^{a_p} = \sum_{i=0}^{p} \eta_i(\underline{z}) (u^1)^i (u^2)^{p-i} \,. \tag{2.30}$$

The components $\eta_{a_1 \cdots a_p}(\underline{z})$ are ordinary $(0,4)$ superfields, i.e., functions on $\mathbf{R}^{1,1|0,4}$. Note that $\boldsymbol{\eta}$ is a global section of the line bundle $\mathcal{O}(p) \to \mathbf{CP}^1$. We thus call such superfields **$\mathcal{O}(p)$ superfields**. In the chart U_2 where $u^2 \neq 0$ we can write $\boldsymbol{\eta}$ as

$$\boldsymbol{\eta}(\underline{z}, u) = (u^2)^p \boldsymbol{\eta}(\underline{z}, \zeta) = (u^2)^p \sum_{j=0}^{p} \eta_j(\underline{z}) \, \zeta^j \,, \tag{2.31}$$

which becomes a polynomial in the inhomogeneous coordinate $\zeta = u^1/u^2$ when we set $u^2 = 1$.

Meromorphic $\mathcal{O}(n)$ superfields. The $\mathcal{O}(n)$ superfields discussed above are global holomorphic sections of $\mathcal{O}(n) \to \mathbf{CP}^1$. We can consider more general superfields which are only local sections of $\mathcal{O}(n)$ and cannot be extended to all of $\mathbf{CP}^1$. A familiar class of examples are the meromorphic sections of $\mathcal{O}(n)$ which are rational functions of u^a:

$$\boldsymbol{\eta}(\underline{z}, u) = \frac{\boldsymbol{P}(\underline{z}, u)}{\boldsymbol{Q}(\underline{z}, u)} = \frac{P_{i_1 \cdots i_p}(\underline{z}) u^{i_1} \cdots u^{i_p}}{Q_{i_1 \cdots i_q}(\underline{z}) u^{i_1} \cdots u^{i_q}} \,, \tag{2.32}$$

where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are homogeneous polynomials of degree p and q respectively. The domain of definition $\mathcal{D}_{\boldsymbol{\eta}}$ of $\boldsymbol{\eta}$ on $\mathbf{CP}^1$ is restricted to the open set where $\boldsymbol{Q}(\underline{z}, u) \neq 0$. The degree of homogeneity of $\boldsymbol{\eta}$ is then $n = p - q$ and thus $\boldsymbol{\eta}$ is a local section of $\mathcal{O}(p-q) \to \mathbf{CP}^1$ defined on $\mathcal{D}_{\boldsymbol{\eta}}$. In terms of the inhomogeneous coordinate ζ, we have

$$\boldsymbol{\eta}(\underline{z}, \zeta) = \frac{a_0(\underline{z}) + a_1(\underline{z})\zeta + \cdots + a_p(\underline{z})\zeta^p}{b_0(\underline{z}) + b_1(\underline{z})\zeta + \cdots + b_q(\underline{z})\zeta^q}\,, \tag{2.33}$$

where the $a_i(\underline{z})$ are appropriate combinations of $P_{i_1\cdots i_p}(\underline{z})$ and similarly, $b_i(\underline{z})$ are combinations of the $Q_{i_1\cdots i_q}$.

Local superfields. Consider superfields which are formal power series in ζ or ζ^{-1} or both. These appear as series expansions of local holomorphic sections in the neighbourhoods of $\zeta = 0$, $\zeta = \infty$ or in the annulus $\mathbf{CP}^1 \smallsetminus \{0, \infty\}$. Consider a power series in ζ:

$$\boldsymbol{\Upsilon}(\underline{z}, \zeta) = \sum_{j=0}^{\infty} \Upsilon_j(\underline{z})\zeta^j\ . \tag{2.34}$$

Such superfields shall be termed **arctic** since they are well-defined at the north pole $\zeta = 0$ of $\mathbf{CP}^1$ (and possibly in a neighbourhood of $\zeta = 0$ as well). Similarly, a superfield which is a power series in ζ^{-1} is designated **antarctic**.

Finally, a superfield which is defined in the annulus and is real under the extended complex conjugation given below in section 2.6 is called **equatorial**.

2.5 R-symmetry in projective superspace

We consider the R-symmetry transformation of the various objects in projective superspace for the subgroup $F = \mathrm{SU}(2)$ in this subsection [42] (the discussion for $F' = \mathrm{SU}(2)'$ proceeds analogously). The homogeneous coordinates $u^a = (u^1, u^2)$ on $\mathbf{CP}^1$ transforms as a doublet under F:

$$u^c \to (g\cdot u)^c = g^c{}_d u^d\,, \quad g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix} \quad \text{with} \quad a\bar{a} + b\bar{b} = 1\ . \tag{2.35}$$

Accordingly, the inhomogeneous coordinate $\zeta = u^1/u^2$ transforms fractional-linearly:

$$\zeta \to g\cdot\zeta = \frac{a\zeta + b}{-\bar{b}\zeta + \bar{a}}\ . \tag{2.36}$$

Also, a doublet $u_a = \varepsilon_{ab}u^b$ with a lower index a transforms as

$$u_a \to (g\cdot u)_a := u_b(g^{-1})^b{}_a\ . \tag{2.37}$$

Factor of automorphy. We define a *factor of automorphy* $j : F \times \mathbf{CP}^1 \to \mathbf{C}$ for the action of F on $\mathbf{CP}^1$ as follows. Let $g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix} \in F$ and $\zeta \in \mathbf{CP}^1$. Then we have

$$j(g, \zeta) := (\bar{a} - \bar{b}\zeta)\ . \tag{2.38}$$

It is easy to check that $j(g,\zeta)$ satisfies $j(g_1g_2,\zeta) = j(g_1, g_2\cdot\zeta)\, j(g_2,\zeta)$. Suppose we have an object $\mathbf{\Phi}(\zeta)$ that depends holomorphically on ζ. The transformation of $\mathbf{\Phi}$ by a F-transformation g is denoted by $g\cdot\mathbf{\Phi}$. An object $\mathbf{\Phi}(\zeta)$ is said to have F-weight n if it satisfies

$$\mathbf{\Phi}(\zeta) = j(g,\zeta)^n \times [g\cdot\mathbf{\Phi}](g\cdot\zeta)\,, \quad g\in F\,, \tag{2.39}$$

That is, $\mathbf{\Phi}$ is a local section of the line bundle $\mathcal{O}(n)\to\mathbf{CP}^1$. Note that weight 0 objects are simply local functions on $\mathbf{CP}^1$.

Next, we describe the R-symmetry of $\mathcal{O}(n)$ superfields and arctic superfields.

$\mathcal{O}(n)$ superfields. Consider an $\mathcal{O}(n)$ superfield $\boldsymbol{\eta}$ given by $\boldsymbol{\eta}(u) = \eta_{a_1\dots a_n}u^{a_1}\cdots u^{a_n}$. Since all F-doublet indices are contracted in $\boldsymbol{\eta}(u)$, it is invariant under F. That is,

$$\boldsymbol{\eta}(u) = [g\cdot\boldsymbol{\eta}](g\cdot u)\,, \quad g\in F\,, \tag{2.40}$$

where $[g\cdot\boldsymbol{\eta}](g\cdot u)$ on the right hand side is a new $\mathcal{O}(n)$ superfield $[g\cdot\boldsymbol{\eta}]$ obtained by transforming the components $\eta_{a_1\cdots a_n}$, and evaluated at the transformed coordinates $g\cdot u$. In terms of the inhomogeneous coordinate ζ, we have

$$\boldsymbol{\eta}(u) := (u^2)^n\boldsymbol{\eta}(\zeta)\,, \quad \text{with} \quad \boldsymbol{\eta}(\zeta) := \sum_{j=0}^{n}\eta_j\zeta^j\,, \tag{2.41}$$

where η_j are appropriate combinations of the $\eta_{a_1\cdots a_n}$. Similarly,

$$[g\cdot\boldsymbol{\eta}](g\cdot u) = (\bar{a}u^2 - \bar{b}u^1)^n \times {}^g\boldsymbol{\eta}(g\cdot\zeta) = (u^2)^n j(g,\zeta)^n \times [g\cdot\boldsymbol{\eta}](g\cdot\zeta)\ . \tag{2.42}$$

This leads to

$$\boldsymbol{\eta}(\zeta) = j(g,\zeta)^n[g\cdot\boldsymbol{\eta}](g\cdot\zeta)\ . \tag{2.43}$$

We define the transformation of a $\mathcal{O}(n)$ superfield $\boldsymbol{\eta}(\zeta)$ by an element $g\in F$ as

$$\boldsymbol{\eta}(\zeta) \to\ [g^{-1}\cdot\boldsymbol{\eta}](\zeta) := j(g,\zeta)^n\,\boldsymbol{\eta}(g\cdot\zeta)\,, \tag{2.44}$$

where the right hand side must be expanded about $\zeta=0$ so that it is a function of ζ rather than $g\cdot\zeta$. Thus, an $\mathcal{O}(n)$ superfield has weight n (note that this is also the degree of the line bundle $\mathcal{O}(n)\to\mathbf{CP}^1$). Meromorphic sections of $\mathcal{O}(n)$ also transform similarly under R-symmetry.

An example. We are primarily interested in describing hypermultiplets which correspond to $n=1$. In this case the components η_a of $\boldsymbol{\eta}(\underline{z},u) = \eta_a(\underline{z})u^a$ transform as an F-doublet. We check that $\boldsymbol{\eta}$ satisfies (2.43) for $n=1$:

$$j(g,\zeta)\times[g\cdot\boldsymbol{\eta}](g\cdot\zeta) = (\bar{a}\eta_1+\bar{b}\eta_2)(a\zeta+b)+(-b\eta_1+a\eta_2)(\bar{a}-\bar{b}\zeta) = \eta_1\zeta+\eta_2 = \boldsymbol{\eta}(\zeta)\,, \tag{2.45}$$

where we have used the SU(2) transformation of a doublet η_a with a lower index as described in eq. (2.37). It can be easily checked that the conjugate $\overline{\boldsymbol{\eta}} = \overline{\eta}^1 - \zeta\overline{\eta}^2$ (cf. (2.64)) also transforms as an $\mathcal{O}(1)$ multiplet.

Arctic superfields. Arctic superfields are typically defined only in a neighbourhood of $\zeta = 0$ and not globally on $\mathbf{CP}^1$. As a result, we may only consider infinitesimal R-symmetry transformations of arctic superfields since they retain ζ in a neighbourhood of $\zeta = 0$. These we obtain by setting $a = 1 + \mathrm{i}\alpha$ and $b = \beta$, with α and β infinitesimal, in the formula for the F-transformation g in (2.35). The determinant condition $a\overline{a} + b\overline{b} = 1$ then gives $\mathrm{i}(\alpha - \overline{\alpha}) = 0$ to first order in the infinitesimals, i.e., α is real. The infinitesimal F-transformation of ζ is then (cf. [53])

$$\delta\zeta = \beta + 2\mathrm{i}\alpha\zeta + \overline{\beta}\zeta^2 \ . \tag{2.46}$$

The F-transformation of an arctic superfield $\boldsymbol{\Upsilon}(\zeta) = \sum_0^\infty \Upsilon_j\,\zeta^j$ of *weight* k is then given by the infinitesimal version of $[g^{-1} \cdot \boldsymbol{\Upsilon}](\zeta) = j(g,\zeta)^k \times \boldsymbol{\Upsilon}(g \cdot \zeta)$:

$$\delta\boldsymbol{\Upsilon}(\zeta) = -k(\mathrm{i}\alpha + \overline{\beta}\zeta)\boldsymbol{\Upsilon}(\zeta) + \frac{\partial\boldsymbol{\Upsilon}}{\partial\zeta}\,\delta\zeta\,, \quad k \in \mathbf{Z}\ . \tag{2.47}$$

It is important to note that arctic superfields can be assigned any integral weight k *a priori* since arctics go to arctics under infinitesimal transformations for any k in eq. (2.47).[2] Further, it is easy to check that $\zeta^k\overline{\boldsymbol{\Upsilon}}(-\zeta^{-1})$ also transforms as an object of weight k but is no longer an antarctic superfield.

The components Υ_j transform under (2.47) as

$$\delta\Upsilon_j = (j+1)\beta\Upsilon_{j+1} + (2j-k)\alpha\Upsilon_j + (j-1-k)\overline{\beta}\Upsilon_{j-1}\ . \tag{2.48}$$

Let us look at $k = 1$ which will be required in our study of hypermultiplets. We shall show below that, with our choice of action for the arctic superfield, the components $\Upsilon_{j\geq 2}$ will turn out to be auxiliary and will be set to zero by their equations of motion. The arctic superfield then truncates to an $\mathcal{O}(1)$ superfield after substituting $\Upsilon_{j\geq 2} = 0$. It is then clear that the components Υ_0 and Υ_1 decouple from the $\Upsilon_{j\geq 2}$ components in (2.48) and Υ_0 and Υ_1 transform as

$$\delta\Upsilon_0 = -\alpha\Upsilon_0 + \beta\Upsilon_1\,, \quad \delta\Upsilon_1 = \alpha\Upsilon_1 - \overline{\beta}\Upsilon_0\ . \tag{2.49}$$

These are the transformation rules for an F-doublet (Υ_1, Υ_0) and this is the standard transformation of a hypermultiplet under SU(2) R-symmetry.

The derivatives $\mathbf{D}_{a'+}$,$\widetilde{\mathbf{D}}_{a'+}$. Since $\mathbf{D}_{a'+} = u^a\mathrm{D}_{aa'+}$, the same manipulations we did for $\mathcal{O}(n)$ superfields works here and it follows from (2.43) that $\mathbf{D}_{a'+}$ has F-weight $+1$. Let us next discuss the F-weight of $\widetilde{\mathbf{D}}_{a'+}$. Recall from the discussion above equation (2.20) that our chosen solution for the equation $\varepsilon_{ab}\widetilde{u}^a u^b = 1$ is

$$\widetilde{u}^a = (u^2)^{-1}\begin{pmatrix}1\\0\end{pmatrix}\,, \quad \text{given} \quad u^a = \begin{pmatrix}u^1\\u^2\end{pmatrix} = u^2\begin{pmatrix}\zeta\\1\end{pmatrix}\ . \tag{2.50}$$

[2] Explicitly, we have

$$\delta\boldsymbol{\Upsilon} = [g^{-1}\cdot\boldsymbol{\Upsilon}](\zeta) - \boldsymbol{\Upsilon}(\zeta) = (1 - k\mathrm{i}\alpha - k\overline{\beta}\zeta)\left(\boldsymbol{\Upsilon}(\zeta) + \frac{\partial\boldsymbol{\Upsilon}}{\partial\zeta}\delta\zeta\right) - \boldsymbol{\Upsilon}(\zeta) = -k(\mathrm{i}\alpha + \overline{\beta}\zeta)\boldsymbol{\Upsilon}(\zeta) + \frac{\partial\boldsymbol{\Upsilon}}{\partial\zeta}\delta\zeta\ .$$

Clearly, the right hand side is also an arctic superfield.

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Under F-transformations, since u^2 transforms as $u^2 \to j(g,\zeta)u^2$, u^a and $\widetilde{u}^a$ transform as

$$\widetilde{u}^a \to j(g,\zeta)^{-1}(u^2)^{-1}\begin{pmatrix}1\\0\end{pmatrix}, \quad u^a \to j(g,\zeta)u^2\begin{pmatrix}\frac{a\zeta+b}{\bar{a}-\bar{b}\zeta}\\1\end{pmatrix}. \tag{2.51}$$

From this, it is clear that $\widetilde{\mathbf{D}}_{a'+}$ has F-weight -1. This is consistent with the algebra $\{\mathbf{D}_{a'+}, \widetilde{\mathbf{D}}_{b'+}\} = -2\mathrm{i}\varepsilon_{a'b'}\partial_{++}$ since the right hand side is independent of ζ and hence, has weight 0.

However, the transformation (2.51) of $\widetilde{u}^a$ does not look like that of an F-doublet. The latter looks like

$$\widetilde{u}^a \to \begin{pmatrix}a(u^2)^{-1}\\-\bar{b}(u^2)^{-1}\end{pmatrix}. \tag{2.52}$$

How do we reconcile (2.51) and (2.52)? Recall that we had a shift symmetry (2.8) $\delta\widetilde{u}^a = \omega u^a$ in the space of $\widetilde{u}^a$ that satisfy $\varepsilon_{ab}\widetilde{u}^a u^b = 1$. We could add a shift in one of the transformations, say (2.51) and see if that can be matched with (2.52) for a particular value of the shift parameter. Indeed, writing

$$j(g,\zeta)^{-1}(u^2)^{-1}\begin{pmatrix}1\\0\end{pmatrix} + \omega j(g,\zeta)u_2\begin{pmatrix}\frac{a\zeta+b}{\bar{a}-\bar{b}\zeta}\\1\end{pmatrix} = (u^2)^{-1}\begin{pmatrix}a\\-\bar{b}\end{pmatrix}, \tag{2.53}$$

we get a solution for ω

$$\omega = -\bar{b}(u^2)^{-2}j(g,\zeta)^{-1}. \tag{2.54}$$

In analogy with (2.44), we define the transformations of the $\mathbf{D}_{a'+}$ and $\widetilde{\mathbf{D}}_{a'+}$ expressed in inhomogeneous coordinates as

$$\mathbf{D}_{a'+}(\zeta) \to j(g,\zeta)\mathbf{D}_{a'+}(g\cdot\zeta), \quad \widetilde{\mathbf{D}}_{a'+}(\zeta) \to j(g,\zeta)^{-1}\widetilde{\mathbf{D}}_{a'+}(g\cdot\zeta) - \bar{b}\,\mathbf{D}_{a'+}(g\cdot\zeta). \tag{2.55}$$

The projective superspace measure. Recall that the $(0,4)$ projective superspace action (2.13) is

$$\mathcal{S}[\mathbf{\Phi}] = \int \mathrm{d}^2x \frac{1}{2\pi\mathrm{i}}\oint_\gamma \varepsilon_{ab}u^a\mathrm{d}u^b\,\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\boldsymbol{K}_{--}(\mathbf{\Phi}). \tag{2.56}$$

As discussed after (2.13), the action is manifestly F and F' invariant provided the superspace Lagrangian $\boldsymbol{K}_{--}$ is invariant. In terms of F-weight, it has weight 0 since the measure $u^a\mathrm{d}u^b$ has two factors of u^a and $\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}$ has two factors of $\widetilde{u}^a$. Let us elaborate in terms of inhomogeneous coordinates. The action takes the form

$$\mathcal{S}[\mathbf{\Phi}] = \int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\boldsymbol{K}_{--}(\mathbf{\Phi}). \tag{2.57}$$

The measure $\mathrm{d}\zeta$ transforms with F-weight 2 under an F-transformation $\zeta \to g\cdot\zeta$:

$$\mathrm{d}\zeta = j(g,\zeta)^2\,\mathrm{d}(g\cdot\zeta). \tag{2.58}$$

The superderivatives $\widetilde{\mathbf{D}}_{a'+}$ effectively transform with F-weight -1 (cf. the first term in the transformation of $\widetilde{\mathbf{D}}_{a'+}$ in (2.55); the second term in (2.55) is proportional to $\mathbf{D}_{a'+}$ which annihilates $\boldsymbol{K}_{--}$). As a result, the combination $\mathrm{d}\zeta\,\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}$ has weight 0, i.e., the superspace measure is invariant (up to total derivatives). Since integrating a weight 0 object with the invariant measure yields an F-invariant answer, the action is R-symmetric if the superspace Lagrangian $\boldsymbol{K}_{--}$ has weight 0.

2.6 Extended complex conjugation

Recall the $\mathcal{S}$ map (2.17) which takes $\zeta \to -1/\zeta$. The *antipodal* map $\mathcal{I}$ that takes a point on $\mathbf{CP}^1$ to its antipode is the composition of the $\mathcal{S}$ map and complex conjugation:

$$\mathcal{I}: \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \longmapsto \begin{pmatrix} -\overline{(u^2)} \\ \overline{(u^1)} \end{pmatrix}, \quad \text{that is} \quad \mathcal{I}: \zeta \longmapsto -\frac{1}{\overline{\zeta}} . \tag{2.59}$$

The antipodal map can be used to define a new real structure [51] on the (sheaf of) sections of a line bundle as the action of the antipodal map on a section followed by ordinary complex conjugation of the resulting section.

For instance, the antipodal map acts on an arctic superfield $\boldsymbol{\Upsilon}(\zeta) = \sum_{j\geq 0} \Upsilon_j \zeta^j$ (which is a local section of some line bundle on $\mathbf{CP}^1$) as

$$\sum_{j\geq 0} \Upsilon_j \zeta^j \to \sum_{j\geq 0} \Upsilon_j (-\overline{\zeta})^{-j} . \tag{2.60}$$

Ordinary complex conjugation of the resulting local section is

$$\sum_{j\geq 0} \Upsilon_j (-\overline{\zeta})^{-j} \to \sum_{j\geq 0} \overline{\Upsilon}_j (-\zeta)^{-j} . \tag{2.61}$$

Thus, the extended complex conjugate of an arctic superfield $\boldsymbol{\Upsilon}(\zeta)$ is

$$\overline{\boldsymbol{\Upsilon}}(-1/\zeta) := \sum_{j\geq 0} \overline{\Upsilon}_j (-1/\zeta)^j . \tag{2.62}$$

Let us compute the extended complex conjugate of an $\mathcal{O}(p)$ superfield $\boldsymbol{\eta}$. Since $\boldsymbol{\eta}$ is globally defined on $\mathbf{CP}^1$, and the antipodal map contains the $\mathcal{S}$ map, we can use the F-transformation rule (2.44) for $\mathcal{O}(p)$ superfields to obtain the extended complex conjugate:

$$\boldsymbol{\eta}(\underline{z}, \zeta) = \sum_{j=0}^{p} \eta_j(\underline{z}) \zeta^j \quad \overset{\mathcal{I}}{\longmapsto} \quad (-\overline{\zeta})^p \sum_{j=0}^{p} \eta_j(\underline{z}) (-\overline{\zeta})^{-j} \quad \overset{\text{c.c.}}{\longmapsto} \quad \sum_{j=0}^{p} \overline{\eta}_j(\underline{z}) (-\zeta)^{p-j} . \tag{2.63}$$

The difference between the above and (2.62) is that there is an additional factor of $(-\overline{\zeta})^p$ in the antipodal map step. This factor makes the new section also a global section of $\mathcal{O}(p)$. Thus, the extended complex conjugate of an $\mathcal{O}(p)$ superfield $\boldsymbol{\eta}$ is

$$\overline{\boldsymbol{\eta}}(\underline{z}, \zeta) := \sum_{j=0}^{p} \overline{\eta}_j(\underline{z}) (-\zeta)^{p-j} . \tag{2.64}$$

A reality condition. As is obvious from (2.64), the extended conjugate of an $\mathcal{O}(p)$ superfield is also an $\mathcal{O}(p)$ superfield. Notice that applying the extended complex conjugate twice on $\boldsymbol{\eta}$ gives

$$\overline{\overline{\boldsymbol{\eta}}} = (-1)^p \boldsymbol{\eta} . \tag{2.65}$$

Thus, we can impose a *reality* condition on an $\mathcal{O}(p)$ superfield only when p is even:

$$\boldsymbol{\eta}(\zeta) = \overline{\boldsymbol{\eta}}(\zeta) , \quad \text{that is,} \quad \sum_{j=0}^{p} \eta_j(\underline{z}) \zeta^j = \sum_{j=0}^{p} \overline{\eta}_j(\underline{z}) (-\zeta)^{p-j} . \tag{2.66}$$

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Extended complex conjugates of $\mathbf{D}_{a'+}$ and $\widetilde{\mathbf{D}}_{a'+}$. Next, consider the derivatives $\mathbf{D}_{a'+}$ and $\widetilde{\mathbf{D}}_{a'+}$. Since they are globally defined (see the equations (2.22), (2.23) and the discussion around them), we use the global F-transformation rules in (2.55) to get the conjugates. The factor of automorphy for the $\mathcal{S}$-map is $j(\mathcal{S},\zeta) = -\zeta$. The complex conjugate of $\mathbf{D}_{a'+}$ is then

$$\begin{aligned} \mathbf{D}_{a'+} \ &\overset{\mathcal{I}}{\longmapsto} \ -\bar{\zeta}(-\bar{\zeta}^{-1}\mathrm{D}_{1a'+} + \mathrm{D}_{2a'+}) \ \overset{\text{c.c.}}{\longmapsto} \ -\zeta(-\zeta^{-1}\overline{\mathrm{D}}_{1a'+} + \overline{\mathrm{D}}_{2a'+}) \\ &= \varepsilon^{a'b'}(\mathrm{D}_{2b'+} + \zeta \mathrm{D}_{1b'+}) = \varepsilon^{a'b'}\mathbf{D}_{b'+} \ . \end{aligned} \tag{2.67}$$

The complex conjugate of $\widetilde{\mathbf{D}}_{a'+}$ is obtained as follows. First, we apply the antipodal map:

$$\begin{aligned} \widetilde{\mathbf{D}}_{a'+} \ &\overset{\mathcal{I}}{\longmapsto} \ (-\bar{\zeta})^{-1}\widetilde{\mathbf{D}}_{a'+}(-\bar{\zeta}^{-1}) - \mathbf{D}_{a'+}(-\bar{\zeta}^{-1}) \\ &= (-\bar{\zeta})^{-1}\mathrm{D}_{1a'+} - (-\bar{\zeta}^{-1}\mathrm{D}_{1a'+} + \mathrm{D}_{2a'+}) = -\mathrm{D}_{2a'+} \ . \end{aligned} \tag{2.68}$$

where we have used the fact that since the $\widetilde{\mathbf{D}}_{a'+}$ are independent of ζ, the expressions for $\widetilde{\mathbf{D}}_{a'+}(-\bar{\zeta}^{-1})$ are the same as in (2.20), i.e., $\widetilde{\mathbf{D}}_{a'+}(-\bar{\zeta}^{-1}) = \mathrm{D}_{1a'+}$. Next, doing ordinary complex conjugation, we get

$$-\mathrm{D}_{2a'+} \ \overset{\text{c.c.}}{\longmapsto} \ \varepsilon^{a'b'}\mathrm{D}_{1b'+} = \varepsilon^{a'b'}\widetilde{\mathbf{D}}_{b'+} \ . \tag{2.69}$$

Thus, we have

$$\overline{\mathbf{D}}{}^{a'}_{+} = \varepsilon^{a'b'}\mathbf{D}_{b'+} \, , \quad \overline{\widetilde{\mathbf{D}}}{}^{a'}_{+} = \varepsilon^{a'b'}\widetilde{\mathbf{D}}_{b'+} \ . \tag{2.70}$$

We may need to consider a slightly different version of the complex conjugates of the derivatives when they act on arctic superfields for the following reason. Under extended complex conjugation, an arctic superfield goes to an antarctic superfield (see (2.62)). We would like this to be true for the derivative of an arctic as well. However, applying (2.70) on $\mathbf{D}_{a'+}\boldsymbol{\Upsilon}$ gives $\varepsilon^{a'b'}\mathbf{D}_{b'+}\overline{\boldsymbol{\Upsilon}}$ which is not antarctic due to a term proportional to ζ in $\mathbf{D}_{b'+}$. On the other hand, treating $\mathbf{D}_{a'+}\boldsymbol{\Upsilon}$ as a new arctic superfield with components

$$\mathbf{D}_{a'+}\boldsymbol{\Upsilon}(\zeta) = \sum_{j\geq 0}(\zeta \mathrm{D}_{1a'+} + \mathrm{D}_{2a'+})\Upsilon_j\zeta^j = \sum_{j\geq 0}\zeta^j(\mathrm{D}_{2a'+}\Upsilon_j + \mathrm{D}_{1a'+}\Upsilon_{j-1}) \, , \tag{2.71}$$

we can apply the conjugation rule (2.62) to the above and obtain

$$\sum_{j\geq 0}(-1/\zeta)^j\varepsilon^{a'b'}(-\mathrm{D}_{1b'+}\overline{\Upsilon}_j + \mathrm{D}_{2a'+}\overline{\Upsilon}_{j-1}) \, , \tag{2.72}$$

as the conjugate antarctic superfield corresponding to $\mathbf{D}_{a'+}\boldsymbol{\Upsilon}$. Clearly, (2.72) can be written as

$$\varepsilon^{a'b'}(-\mathrm{D}_{1b'+} - \zeta^{-1}\mathrm{D}_{2b'+})\sum_{j\geq 0}(-1/\zeta)^j\overline{\Upsilon}_j = -\zeta^{-1}\varepsilon^{a'b'}\mathbf{D}_{b'+}\overline{\boldsymbol{\Upsilon}}(-1/\zeta) \, , \tag{2.73}$$

which suggests that we modify the conjugate of the derivative $\mathbf{D}_{a'+}$ when acting on arctic superfields to

$$\mathbf{D}_{a'+} \to \breve{\mathbf{D}}^{a'+} = -\zeta^{-1}\varepsilon^{a'b'}\mathbf{D}_{b'+} \ . \tag{2.74}$$

Similarly, we have

$$\widetilde{\mathbf{D}}_{a'+}\boldsymbol{\Upsilon}(\zeta) = \sum_{j\geq 0}\zeta^j\mathrm{D}_{1a'+}\Upsilon_j \ . \tag{2.75}$$

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Applying (2.62) to the above, we get

$$\sum_{j\geq 0}(-1/\zeta)^j \varepsilon^{a'b'}\mathrm{D}_{2b'+}\overline{\varUpsilon}_j \ . \tag{2.76}$$

Note the identity

$$\mathrm{D}_{2b'+} = -\zeta \mathrm{D}_{1b'+} + \mathbf{D}_{b'+} = -\zeta\widetilde{\mathbf{D}}_{b'+} + \mathbf{D}_{b'+} \ . \tag{2.77}$$

This allows us to write (2.76) as

$$\sum_{j\geq 0}(-1/\zeta)^j \varepsilon^{a'b'}\mathrm{D}_{2b'+}\overline{\varUpsilon}_j = \varepsilon^{a'b'}(-\zeta\widetilde{\mathbf{D}}_{b'+} + \mathbf{D}_{b'+})\overline{\mathbf{\Upsilon}}(-1/\zeta)\,, \tag{2.78}$$

which suggests the modification

$$\widetilde{\mathbf{D}}_{a'+} \to \breve{\widetilde{\mathbf{D}}}{}^{a'+} = \varepsilon^{a'b'}(-\zeta\widetilde{\mathbf{D}}_{b'+} + \mathbf{D}_{b'+}) \ . \tag{2.79}$$

Thus, on arctic superfields, we can postulate the following modified extended complex conjugates of the derivatives:

$$\breve{\mathbf{D}}^{a'}_{+} = -\zeta^{-1}\varepsilon^{a'b'}\mathbf{D}_{b'+}\,, \quad \breve{\widetilde{\mathbf{D}}}{}^{a'}_{+} = \varepsilon^{a'b'}(-\zeta\widetilde{\mathbf{D}}_{b'+} + \mathbf{D}_{b'+}) \ . \tag{2.80}$$

The notation $\breve{}$ for the above notion of the extended complex conjugate of a derivative has been used earlier in [47] and has been called 'smile conjugation'; we continue to use the same notation in this paper. Note that the smile conjugation simply treats $\mathbf{D}_{a'+}(\zeta)$ and $\widetilde{\mathbf{D}}_{a'+}(\zeta)$ as local sections and applies the conjugation rule (2.62).

3 Hypermultiplets

The dynamical degrees of freedom of a $(0,4)$ hypermultiplet consists of two $(0,2)$ chiral superfields ϕ and χ such that $(\phi,\overline{\chi})$ form an SU(2) doublet. The SU(2) in question can be either F or F' and the corresponding hypers are called standard and twisted hypermultiplets respectively. A standard hypermultiplet[3] can be described in $(0,4)$ projective superspace either by an $\mathcal{O}(1)$ superfield [57] or by a pair of F-arctic superfields $(\mathbf{\Upsilon},\mathbf{\Upsilon}_{--})$. The analogous notation for the twisted hypers is $\mathcal{O}(1)'$ and F'-arctic respectively. We describe free hypermultiplets in this section and study interactions in section 5.

3.1 Standard hypermultiplets

3.1.1 $\mathcal{O}(1)$ superfield

We start with a complex $\mathcal{O}(1)$ superfield $\boldsymbol{\eta} = \eta_a u^a$. In terms of the inhomogeneous coordinate ζ, we have $u^a = (\zeta, 1)$ and

$$\boldsymbol{\eta}(\zeta) = \eta_2 + \zeta\eta_1\,, \quad \overline{\boldsymbol{\eta}}(\zeta) = \overline{\eta}^1 - \zeta\overline{\eta}^2 \ . \tag{3.1}$$

[3]See also [65] for a discussion in ordinary superspace.

The projective constraints $\mathbf{D}_{a'+}\boldsymbol{\eta} = 0$ give the following constraints on η_1 and η_2:

$$\overline{\mathrm{Q}}_+\eta_1 = 0\,, \quad \mathrm{D}_+\eta_1 = 0\,, \quad \overline{\mathrm{D}}_+\eta_2 = 0\,, \quad \mathrm{Q}_+\eta_2 = 0\,, \quad \overline{\mathrm{Q}}_+\eta_2 = \overline{\mathrm{D}}_+\eta_1\,, \quad \mathrm{Q}_+\eta_1 = -\mathrm{D}_+\eta_2\;. \tag{3.2}$$

We see that $\overline{\eta}^1$ and η_2 are $(0,2)$ chiral superfields since $\overline{\mathrm{D}}_+$ annihilates them. See appendix A.2 for a review of $(0,2)$ superspace.

The superpartner fermions are defined as[4]

$$\sqrt{2}\xi_{a'+} := \widetilde{\mathbf{D}}_{a'+}\boldsymbol{\eta}\,, \quad \sqrt{2}\overline{\xi}^{a'}_+ := -\varepsilon^{a'b'}\widetilde{\mathbf{D}}_{b'+}\overline{\boldsymbol{\eta}}\;. \tag{3.3}$$

The superpartners $\xi_{a'+}$ are in the doublet of F'; they are also independent of ζ since the above combinations are globally defined weight 0 superfields, i.e., global holomorphic functions on $\mathbf{CP}^1$ which are indeed constants in ζ. Using the expressions $\widetilde{\mathbf{D}}_{a'+} = \mathrm{D}_{1a'+}$ and that $\mathbf{D}_{a'+}\boldsymbol{\eta} = 0$, we can arrive at the following $(0,2)$ superspace definitions for the $\xi_{a'+}$:

$$\sqrt{2}\xi_{1'+} = \mathrm{D}_+\eta_2\,, \quad \sqrt{2}\overline{\xi}^{1'}_+ = -\overline{\mathrm{D}}_+\overline{\eta}^2\,, \quad -\sqrt{2}\xi_{2'+} = \overline{\mathrm{D}}_+\eta_1\,, \quad \sqrt{2}\overline{\xi}^{2'}_+ = \mathrm{D}_+\overline{\eta}^1\;. \tag{3.4}$$

The next superfield in the multiplet would be $\widetilde{\mathbf{D}}_{a'+}\widetilde{\mathbf{D}}_{b'+}\boldsymbol{\eta}$ which (1) is globally defined on $\mathbf{CP}^1$, (2) has F-weight -1, (3) is antisymmetric in $a'b'$, and (4) is a Lorentz vector. The only superfield which satisfies all these properties is $\varepsilon_{a'b'}\partial_{++}\widetilde{\boldsymbol{\eta}}$, where $\widetilde{\boldsymbol{\eta}} = \widetilde{u}^a\eta_a$. Thus, we have

$$\widetilde{\mathbf{D}}_{a'+}\widetilde{\mathbf{D}}_{b'+}\boldsymbol{\eta} = -2\mathrm{i}\varepsilon_{a'b'}\partial_{++}\widetilde{\boldsymbol{\eta}}\,, \quad \widetilde{\mathbf{D}}_{a'+}\widetilde{\mathbf{D}}_{b'+}\overline{\boldsymbol{\eta}} = -2\mathrm{i}\varepsilon_{a'b'}\partial_{++}\overline{\widetilde{\boldsymbol{\eta}}}\;. \tag{3.5}$$

The above equations (3.5) can be explicitly checked by using the expressions for $\widetilde{\mathbf{D}}_{a'+}$ in (2.20), the complex conjugate derivatives in (2.70), and the projective constraints (3.2).

The $(0,4)$ supersymmetric action that describes the (free) hypermultiplet is

$$\mathcal{S} = -\frac{\mathrm{i}}{2}\int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\left(\zeta^{-1}\overline{\boldsymbol{\eta}}\partial_{--}\boldsymbol{\eta}\right)\;. \tag{3.6}$$

Using the fact that the superspace Lagrangian is annihilated by $\mathbf{D}_{a'+}$, we can write it as an action in $(0,2)$ superspace as in (2.26). We get

$$\mathcal{S} = \frac{\mathrm{i}}{2}\int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\mathrm{D}_+\overline{\mathrm{D}}_+\left(\zeta^{-1}\overline{\boldsymbol{\eta}}\partial_{--}\boldsymbol{\eta}\right)\;. \tag{3.7}$$

Next, we can obtain the component action by first performing the ζ-integral, pushing in the derivatives and using the definitions (3.4) and that $\overline{\eta}^1$ and η_2 are $(0,2)$ chiral superfields:

$$\begin{aligned}\mathcal{S} &= \frac{\mathrm{i}}{2}\mathrm{D}_+\overline{\mathrm{D}}_+\int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\left((\zeta^{-1}\overline{\eta}^1 - \overline{\eta}^2)\partial_{--}(\zeta\eta_1 + \eta_2)\right)\,,\\ &= \frac{\mathrm{i}}{2}\int \mathrm{d}^2x\, \mathrm{D}_+\overline{\mathrm{D}}_+\left(\overline{\eta}^1\partial_{--}\eta_1 - \overline{\eta}^2\partial_{--}\eta_2\right)\,,\\ &= \int \mathrm{d}^2x\left(-\partial_\mu\overline{\eta}^a\partial^\mu\eta_a - \mathrm{i}\overline{\xi}^{a'}_+\partial_{--}\xi_{a'+}\right)\;.\end{aligned} \tag{3.8}$$

[4]The conjugate fermions are obtained as follows. $\widetilde{\mathbf{D}}_{a'+}\boldsymbol{\eta}$ is best thought of as $[\widetilde{\mathbf{D}}_{a'+}, \boldsymbol{\eta}]$ which, under conjugation, goes to $[\overline{\boldsymbol{\eta}}, \overline{(\widetilde{\mathbf{D}}_{a'+})}] = -\varepsilon^{a'c'}\widetilde{\mathbf{D}}_{c'+}\overline{\boldsymbol{\eta}}$.

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(See appendix A.2 for a derivation of the component action from the $(0,2)$ action in the second line in (3.8).) We can also obtain the same component action as above by pushing in the derivatives $\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}$ in (3.6), use the definitions (3.3) and (3.5), and finally perform the ζ integral (see appendix C.2).

The $\mathcal{O}(1)$ superfield can be described in ordinary $(0,4)$ superspace as well. Writing $\mathbf{D}_{a'+} = u^a \mathrm{D}_{aa'+}$ and $\boldsymbol{\eta} = u^a \eta_a$, the projective constraints $\mathbf{D}_{a'+}\boldsymbol{\eta} = 0$ are equivalent to

$$\mathrm{D}_{aa'+}\eta_b + \mathrm{D}_{ba'+}\eta_a = 0 \; . \tag{3.9}$$

As noted in [57], in contrast to an $\mathcal{O}(1)$ superfield in $(4,4)$ projective superspace, the above $(0,4)$ constraints do not put the $\mathcal{O}(1)$ superfield on-shell. Only the antisymmetric part in ab of $D_{aa'+}\eta_b$ is non-zero and it gives the superpartner fermions defined in (3.3) (or equivalently (3.4)):

$$\mathrm{D}_{aa'+}\eta_b =: \sqrt{2}\varepsilon_{ab}\xi_{a'+} \, , \quad \sqrt{2}\varepsilon^{ab}\overline{\xi}^{a'}_{+} = -\varepsilon^{ac}\varepsilon^{a'c'}\mathrm{D}_{cc'+}\overline{\eta}^{b} \; . \tag{3.10}$$

Note that the scalars η_a are in an F-doublet whereas the fermions $\xi_{a'+}$ are in an F'-doublet.

Recall from (3.2) that $\overline{\eta}^1$ and η_2 are annihilated by $\overline{\mathrm{D}}_+$ and Q_+. Thus, we can write down a manifestly $(0,4)$ supersymmetric action with the measure $\mathrm{D}_+\overline{\mathrm{Q}}_+$:

$$\mathcal{S} = \frac{\mathrm{i}}{2}\int \mathrm{d}^2x \, \mathrm{D}_+\overline{\mathrm{Q}}_+ \left(\overline{\eta}^1 \partial_{--}\eta_2\right) \; . \tag{3.11}$$

This is the projective superspace action (3.6) after plugging in $\widetilde{\mathbf{D}}_{1'+} = \mathrm{D}_+$, $\widetilde{\mathbf{D}}_{2'+} = -\overline{\mathrm{Q}}_+$ and performing the ζ integral; therefore, it also coincides with the $(0,2)$ action (3.7). The above action is not manifestly R-symmetric, but a manifestly R-symmetric action also exists which agrees with any of the above actions (up to total spacetime derivatives):

$$\mathcal{S} = \frac{\mathrm{i}}{2}\int \mathrm{d}^2x \, \varepsilon^{a'b'}\mathrm{D}_{aa'+}\mathrm{D}_{bb'+}\left(\overline{\eta}^a \partial_{--}\varepsilon^{bc}\eta_c\right) \; . \tag{3.12}$$

However, the above action is not manifestly supersymmetric since the measure does not involve all four superspace derivatives.

The F-projective superspace action (3.6) does not seem to be invariant under R-symmetry since the Lagrangian does not seem to transform with F-weight 0. To write a manifestly R-symmetric action in projective superspace, we use the arctic realization of the hypermultiplet, one that arises naturally from $(4,4)$ projective superspace (see appendix B.2).

3.1.2 Arctic superfield

Consider two arctic multiplets $\boldsymbol{\Upsilon}$ and $\boldsymbol{\Upsilon}_{--}$ with ζ-expansions

$$\boldsymbol{\Upsilon}(\zeta) = \sum_{j=0}^{\infty}\Upsilon_j\zeta^j \, , \quad \boldsymbol{\Upsilon}_{--}(\zeta) = \sum_{j=0}^{\infty}\Upsilon_{j--}\zeta^j \; . \tag{3.13}$$

The projective constraints $\mathbf{D}_{a'+}\boldsymbol{\Upsilon} = 0$ give

$$\mathrm{Q}_+\Upsilon_0 = 0 \, , \quad \overline{\mathrm{D}}_+\Upsilon_0 = 0 \, , \quad \mathrm{Q}_+\Upsilon_{j+1} = -\mathrm{D}_+\Upsilon_j \, , \quad \overline{\mathrm{Q}}_+\Upsilon_j = \overline{\mathrm{D}}_+\Upsilon_{j+1} \quad \text{for} \quad j \geq 0 \, , \tag{3.14}$$

and similarly for $\boldsymbol{\Upsilon}_{--}$. The zeroth components $\varUpsilon_0$ and $\varUpsilon_{0--}$ are $(0,2)$ chiral superfields since $\overline{\mathrm{D}}_+\varUpsilon_0 = \overline{\mathrm{D}}_+\varUpsilon_{0--} = 0$ whereas the $\varUpsilon_j$, $\varUpsilon_{j--}$, $j \geq 1$, are unconstrained as $(0,2)$ superfields.

The $(0,4)$ supersymmetric action that describes the (free) standard hypermultiplet is

$$\mathcal{S} = \int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+} \left(\frac{\mathrm{i}}{2}\overline{\boldsymbol{\Upsilon}}\partial_{--}\boldsymbol{\Upsilon} - \zeta\overline{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}_{--} + \zeta^{-1}\overline{\boldsymbol{\Upsilon}}_{--}\boldsymbol{\Upsilon}\right). \tag{3.15}$$

In fact, the above action is equivalent to that of an $\mathcal{O}(1)$ superfield when we go partially on-shell by performing the ζ-integral in the last two terms and integrating out the fields $\varUpsilon_{j--}$ for $j \geq 1$:

$$\begin{aligned} &-\mathrm{D}_+\overline{\mathrm{D}}_+ \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\left(-\zeta\overline{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}_{--} + \zeta^{-1}\overline{\boldsymbol{\Upsilon}}_{--}\boldsymbol{\Upsilon}\right), \\ &= -\mathrm{D}_+\overline{\mathrm{D}}_+\left(\overline{\varUpsilon}_1\varUpsilon_{0--} + \overline{\varUpsilon}_{0--}\varUpsilon_1 + \sum_{j=1}^{\infty}(-1)^{j+1}\left(-\overline{\varUpsilon}_{j+1}\varUpsilon_{j--} - \overline{\varUpsilon}_{j--}\varUpsilon_{j+1}\right)\right). \end{aligned} \tag{3.16}$$

Since the $\varUpsilon_{j--}$, $j \geq 1$, are unconstrained as $(0,2)$ superfields, we can integrate them out in the above superspace action. This imposes $\overline{\varUpsilon}_{j+1} = \varUpsilon_{j+1} = 0$ for $j \geq 1$ and retains only the ζ^0 and ζ^1 terms in $\boldsymbol{\Upsilon}$. Integrating out $\varUpsilon_{j+1}$, we get $\varUpsilon_{j--} = \frac{\mathrm{i}}{2}\partial_{--}\varUpsilon_{j+1}$ for $j \geq 1$. We cannot integrate out $\varUpsilon_{0--}$ in the same way and set $\varUpsilon_1 = 0$ since $\varUpsilon_{0--}$ is constrained as a $(0,2)$ superfield, $\overline{\mathrm{D}}_+\varUpsilon_{0--} = 0$. Instead, integrating out the constrained superfield $\varUpsilon_{0--}$ constrains $\varUpsilon_1$ to satisfy $\overline{\mathrm{D}}_+\overline{\varUpsilon}_1 = 0$.[5]

Thus, we have two $(0,2)$ chiral superfields $\varUpsilon_0$ and $\overline{\varUpsilon}_1$ which we relabel as η_2 and $\overline{\eta}^1$ respectively to make contact with the $\mathcal{O}(1)$ superfield terminology (3.1). Thus, $\boldsymbol{\Upsilon}$ becomes an $\mathcal{O}(1)$ superfield when we go partially on-shell by integrating out the auxiliary superfield $\boldsymbol{\Upsilon}_{--}$:

$$\boldsymbol{\Upsilon} = \varUpsilon_0 + \zeta\varUpsilon_1 = \zeta\eta_1 + \eta_2\,, \quad \overline{\boldsymbol{\Upsilon}} = \overline{\eta}^2 - \zeta^{-1}\overline{\eta}^1 = -\zeta^{-1}(\overline{\eta}^1 - \zeta\overline{\eta}^2)\,, \tag{3.17}$$

and the action (3.15) becomes the $\mathcal{O}(1)$ action (3.8):

$$\mathcal{S} = \frac{\mathrm{i}}{2}\int \mathrm{d}^2x\, \mathrm{D}_+\overline{\mathrm{D}}_+\left(\overline{\eta}^1\partial_{--}\eta_1 - \overline{\eta}^2\partial_{--}\eta_2\right) = \int \mathrm{d}^2x\left(-\partial_\mu\overline{\eta}^a\partial^\mu\eta_a - \mathrm{i}\overline{\xi}^{a'}_+\partial_{--}\xi_{a'+}\right). \tag{3.18}$$

Since integrating out $\boldsymbol{\Upsilon}_{--}$ gives an $\mathcal{O}(1)$ superfield, it is consistent to give an F-weight of $+1$ to $\boldsymbol{\Upsilon}$. However, the action does not seem to have F-weight 0 and hence does not appear R-symmetric. But the action in ordinary space (3.18) is certainly R-symmetric! Let us see how to understand the R-symmetry of (3.15).

The terms depending on $\boldsymbol{\Upsilon}_{--}$ can be made to have weight 0 by declaring that $\boldsymbol{\Upsilon}_{--}$ is a weight -1 superfield. However, the kinetic term is still a problem. Since $\boldsymbol{\Upsilon}_{--}$ is an auxiliary superfield, we can give it a non-standard R-symmetry transformation so that it

[5]Here is the procedure to integrate out a constrained superfield: we first relax the constraint on $\varUpsilon_{0--}$ and introduce a Lagrange multiplier superfield Λ_-: $-\mathrm{D}_+\overline{\mathrm{D}}_+\left(\overline{\varUpsilon}_1\varUpsilon_{0--} + \Lambda_-(\overline{\mathrm{D}}_+\varUpsilon_{0--})\right)$. Integrating out Λ_- re-imposes the constraint $\overline{\mathrm{D}}_+\varUpsilon_{0--} = 0$ whereas integrating out $\varUpsilon_{0--}$ gives $\overline{\varUpsilon}_1 = -\overline{\mathrm{D}}_+\Lambda_-$, which indeed satisfies $\overline{\mathrm{D}}_+\overline{\varUpsilon}_1 = 0$. We can conclude the same by going down to components, or at an intermediate stage by pushing in $\overline{\mathrm{D}}_+$ in the first term in the Lagrangian (3.16) to get $-\mathrm{D}_+\left((\overline{\mathrm{D}}_+\overline{\varUpsilon}_1)\varUpsilon_{0--}\right)$. Since the remaining measure D_+ does not kill $\varUpsilon_{0--}$, we can integrate it out to conclude that $\overline{\mathrm{D}}_+\overline{\varUpsilon}_1 = 0$.

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cancels that of the kinetic term (this is motivated from the $(4,4) \to (0,4)$ reduction in appendix B.2):

$$\boldsymbol{\Upsilon}_{--}(\zeta) \to j(g,\zeta)^{-1}\boldsymbol{\Upsilon}_{--}(g\cdot\zeta) - \frac{\mathrm{i}}{2}\bar{b}\partial_{--}\boldsymbol{\Upsilon}(g\cdot\zeta)\,, \quad \text{where} \quad g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in F\,. \tag{3.19}$$

Recall that we must only perform infinitesimal F-transformations on arctic superfields (see the discussion above eq. (2.46)). It is easy to check that the Lagrangian (3.15) transforms with weight zero when we transform $\boldsymbol{\Upsilon}_{--}$ according to the above rule (see appendix B.2 for an explicit demonstration).

We could write down the $(0,4)$ descendants directly by acting on $\boldsymbol{\Upsilon}$ and $\boldsymbol{\Upsilon}_{--}$ with the derivatives $\widetilde{\mathbf{D}}_{a'+}$. We could then compute the component action (3.18) by pushing the derivatives in the measure $\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}$ into the Lagrangian in the $(0,4)$ action (3.15) and using the definition of the $(0,4)$ descendants. This procedure results in the same conclusions, namely that $\boldsymbol{\Upsilon}$ is truncated to an $\mathcal{O}(1)$ superfield and $\boldsymbol{\Upsilon}_{--}$ is auxiliary, and hence we do not describe it here. However, see appendix C.2 for an illustration of this method for an arctic fermi superfield.

3.2 Twisted hypermultiplets

A twisted hypermultiplet is described by a complex $\mathcal{O}(1)'$ superfield $\boldsymbol{H}(\zeta')$ that is specified as

$$\boldsymbol{H}(\zeta') = \zeta' H_{1'} + H_{2'}\,, \quad \overline{\boldsymbol{H}}(\zeta') = -\zeta'\overline{H}^{2'} + \overline{H}^{1'}\,. \tag{3.20}$$

The F'-projective constraints $\mathbf{D}_{a+}\boldsymbol{H} = 0$ are given by

$$\begin{aligned} &\overline{\mathrm{Q}}_+ H_{1'} = \mathrm{D}_+ H_{2'}\,, && \mathrm{Q}_+ H_{2'} = -\overline{\mathrm{D}}_+ H_{1'}\,, \\ &\overline{\mathrm{Q}}_+ H_{2'} = 0\,, && \overline{\mathrm{D}}_+ H_{2'} = 0\,, \qquad \mathrm{Q}_+ H_{1'} = 0\,, \qquad \mathrm{D}_+ H_{1'} = 0\,. \end{aligned} \tag{3.21}$$

$\overline{H}^{1'}$ and $H_{2'}$ are $(0,2)$ chiral superfields since $\overline{\mathrm{D}}_+$ annihilates them. As for the standard hyper, the superpartner fermions are defined by

$$\mathrm{D}_{aa'+}H_{b'} = \sqrt{2}\varepsilon_{a'b'}\xi_{a+}\,. \tag{3.22}$$

The above definition makes it clear that the superpartner fermions ξ_{a+} of $H_{a'}$ are in the doublet of F. Explicitly, we have

$$\sqrt{2}\xi_{1+} = \mathrm{D}_+ H_{2'}\,, \quad \sqrt{2}\bar{\xi}^1_+ = -\overline{\mathrm{D}}_+\overline{H}^{2'}\,, \quad \sqrt{2}\xi_{2+} = -\overline{\mathrm{D}}_+ H_{1'}\,, \quad \sqrt{2}\bar{\xi}^2_+ = \mathrm{D}_+\overline{H}^{1'}\,. \tag{3.23}$$

The $(0,4)$ supersymmetric action that describes the twisted hypermultiplet is

$$\mathcal{S} = -\int \mathrm{d}^2x \oint_{\gamma'} \frac{\mathrm{d}\zeta'}{2\pi\mathrm{i}} \widetilde{\mathbf{D}}_{1+}\widetilde{\mathbf{D}}_{2+}\left(\frac{\mathrm{i}}{2}\zeta'^{-1}\overline{\boldsymbol{H}}\partial_{--}\boldsymbol{H}\right) = \int \mathrm{d}^2x \oint_{\gamma'} \frac{\mathrm{d}\zeta'}{2\pi\mathrm{i}\zeta'}\mathrm{D}_+\overline{\mathrm{D}}_+\left(\frac{\mathrm{i}}{2}\zeta'^{-1}\overline{\boldsymbol{H}}\partial_{--}\boldsymbol{H}\right). \tag{3.24}$$

Performing the ζ' integral, we get

$$\mathcal{S} = \frac{\mathrm{i}}{2}\int \mathrm{d}^2x\, \mathrm{D}_+\overline{\mathrm{D}}_+\left(\overline{H}^{1'}\partial_{--}H_{1'} - \overline{H}^{2'}\partial_{--}H_{2'}\right) = \int \mathrm{d}^2x\,(-\partial_\mu\overline{H}^{a'}\partial^\mu H_{a'} - \mathrm{i}\bar{\xi}^a_+\partial_{--}\xi_{a+})\,, \tag{3.25}$$

which is the action for two $(0,2)$ chiral multiplets $\overline{H}^{1'}$ and $H_{2'}$. The $H_{a'}$ form an F'-doublet and hence, the above multiplet describes a twisted hyper. The description in terms of F'-arctic superfields is analogous to that of the standard hyper.

3.3 Other $(0,4)$ scalar multiplets

The work [61, 65] describes four different scalar multiplet representations of the $(0,4)$ supersymmetry algebra. Their $(0,4)$ algebra is in terms of the superderivatives D_{a+} and $\overline{\mathrm{D}}^{a}_{+}$ where a is an SU(2) doublet index (in [61, 65], the SU(2) doublet index is written as i instead of a). This SU(2) is one of the SU(2) subgroups of the R-symmetry $F \times F'$ and we identify it with F. The F' subgroup is not manifest but can be restored by defining the derivatives $\mathrm{D}_{aa'+}$ such that $\mathrm{D}_{a1'+} = \mathrm{D}_{a+}$ and $\mathrm{D}_{a2'+} = -\varepsilon_{ab}\overline{\mathrm{D}}^{b}_{+}$. The scalar multiplets (SM) are described as follows.

1. SM-I: a pair of complex scalar fields $\mathcal{A}$, $\mathcal{B}$, and an SU(2) doublet of fermions ψ^{-a}.
2. SM-II: a real scalar field ϕ, a real SU(2) triplet of scalar fields $\phi_a{}^b$, a doublet of complex fermions λ^-_a.
3. SM-III: an SU(2) doublet of scalar fields $\mathcal{A}_a$, and a pair of complex fermions ρ^-, π^-.
4. SM-IV: an SU(2) doublet of scalar fields $\mathcal{B}_a$, a real fermion ψ^-, a real SU(2) triplet of fermions ψ_a^{-b}.

The pair of scalar fields $(\mathcal{A}, \mathcal{B})$ in SM-I and the pair of complex fermions (ρ^-, π^-) can be interpreted as doublets under a different SU(2) which we identify with the F' subgroup of the R-symmetry group. The Lorentz spinor superscript index on the fermions can be lowered using the Levi-Civita symbol $\varepsilon_{+-} = -\varepsilon_{-+} = 1$. Thus, the fermions occurring above are right-handed fermions that occur as superpartners in the various hypermultiplets described in this section.

Given the above identifications, it is clear that SM-I describes the content of a twisted hyper, i.e., an $\mathcal{O}(1)'$ superfield, and SM-III describes a standard hyper, i.e., an $\mathcal{O}(1)$ superfield. It can be easily checked that the supersymmetry transformations of the various components given in [61, 65] agree with those of the $\mathcal{O}(1)$ and $\mathcal{O}(1)'$ superfields given in this section. The multiplet SM-II is described by a real $\mathcal{O}(2)$ superfield and SM-IV is described by a real $\mathcal{O}(2)'$ superfield. We give a short analysis of the $\mathcal{O}(2)$ superfield below.

3.4 $\mathcal{O}(2)$ superfields

A real $\mathcal{O}(2)$ superfield $\boldsymbol{X}$ can be described in terms of a rank two tensor X_{ab} as

$$\boldsymbol{X} = u^a u^b X_{ab} = \zeta^2 X_{11} + \zeta(X_{12} + X_{21}) + X_{22} \ . \tag{3.26}$$

It satisfies the projective constraints $\mathbf{D}_{a'+}\boldsymbol{X} = 0$ and the reality constraint $\overline{\boldsymbol{X}} = \boldsymbol{X}$ (where the extended complex conjugate for an $\mathcal{O}(2)$ superfield is defined in equation (2.64) of section 2.6). In terms of X_{ab}, they become

$$\mathrm{D}_{(a|a'+}X_{bc)} = 0 \,, \quad \overline{X}^{ab} = \varepsilon^{ac}\varepsilon^{bd}X_{cd} \ . \tag{3.27}$$

The first equation says that the totally symmetric part of the rank three tensor $\mathrm{D}_{aa'+}X_{bc} = 0$. The totally antisymmetric part is also trivially zero. The mixed symmetric part then defines the superpartner fermions $\xi_{aa'+}$:

$$\mathrm{D}_{aa'+}X_{bc} = \sqrt{2}(\varepsilon_{ab}\xi_{ca'+} + \varepsilon_{ac}\xi_{ba'+} + \varepsilon_{bc}\xi_{aa'+}) \ . \tag{3.28}$$

These fermions satisfy the reality constraint $\bar{\xi}_{+}^{aa'} = \varepsilon^{ab}\varepsilon^{a'b'}\xi_{bb'+}$. Thus, the independent fermion content is in the doublet $\xi_{a1'+}$. This way of describing the fermion content breaks the F' R-symmetry.

The multiplet X_{bc} decomposes into the symmetric part $S_{bc} = \frac{1}{2}(X_{bc} + X_{cb})$ and the antisymmetric part $X = \frac{1}{2}\varepsilon^{bc}X_{bc}$ which are as irreducible representations of SU(2). These satisfy

$$\mathrm{D}_{aa'+}S_{bc} = \sqrt{2}(\varepsilon_{ab}\xi_{ca'+} + \varepsilon_{ac}\xi_{ba'+})\,, \quad \mathrm{D}_{aa'+}X = \sqrt{2}\xi_{aa'+}\ . \tag{3.29}$$

Using the Schoutens' 'identity' $\varepsilon_{ab}\xi_{ca'+} + \varepsilon_{ca}\xi_{ba'+} + \varepsilon_{bc}\xi_{aa'+} = 0$ (which is merely the statement that a totally antisymmetric rank three tensor where the indices run over two values vanishes identically), we equivalently have

$$\mathrm{D}_{aa'+}S_{bc} = \sqrt{2}(2\varepsilon_{ab}\xi_{ca'+} + \varepsilon_{bc}\xi_{aa'+})\,, \quad \mathrm{D}_{aa'+}X = \sqrt{2}\xi_{aa'+}\ . \tag{3.30}$$

Note that the $\mathcal{O}(2)$ superfield contains only the symmetric part S_{bc} of X_{bc} and does not contain the antisymmetric part X. Moreover, the superpartners of X are the same fermions as those of S_{bc}.

Finally, defining $S_b{}^c = S_{bd}\varepsilon^{dc}$, we see that $S_b{}^b = 0$. Thus, the scalars fields $S_b{}^c$ and X form the scalar content of SM-II and the fermions $\xi_{a1'+}$ form the fermion content of SM-II. The multiplet SM-IV is similarly described by an $\mathcal{O}(2)'$ superfield.

4 Fermi multiplets

In this section, we describe matter fermi multiplets. We focus on F-projective fermi superfields below; the F'-case follows analogously. Like hypermultiplets, fermi multiplets can be realized either as $\mathcal{O}(n)$ superfields or F-arctic superfields. We only describe arctic superfields here since all our constructions use only those and not the $\mathcal{O}(n)$ superfields.

4.1 Arctic fermi superfields

Start with a weight 0 F-arctic superfield $\boldsymbol{\Upsilon}_- = \sum_0^\infty \Upsilon_{j-}\zeta^j$ satisfying

$$\mathbf{D}_{a'+}\boldsymbol{\Upsilon}_- = 0\ . \tag{4.1}$$

The constraints in terms of Υ_{j-} are

$$\mathrm{Q}_+\Upsilon_{0-} = 0\,, \quad \mathrm{Q}_+\Upsilon_{j+1,-} + \mathrm{D}_+\Upsilon_{j-} = 0\,, \quad \overline{\mathrm{D}}_+\Upsilon_{0-} = 0\,, \quad \overline{\mathrm{D}}_+\Upsilon_{j+1,-} - \overline{\mathrm{Q}}_+\Upsilon_{j-} = 0\ . \tag{4.2}$$

The Υ_{j-} for $j \geq 1$ are unconstrained $(0,2)$ superfields while Υ_{0-} satisfies the chirality constraint $\overline{\mathrm{D}}_+\Upsilon_{0-} = 0$. We relabel Υ_{0-} as ψ_-. The action is

$$\begin{aligned}\mathcal{S} &= -\frac{1}{2}\int \mathrm{d}^2x \oint \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\left(\breve{\boldsymbol{\Upsilon}}_-\boldsymbol{\Upsilon}_-\right) = \frac{1}{2}\int \mathrm{d}^2x \oint \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\mathrm{D}_+\overline{\mathrm{D}}_+\left(\breve{\boldsymbol{\Upsilon}}_-\boldsymbol{\Upsilon}_-\right), \\ &= \frac{1}{2}\int \mathrm{d}^2x\,\mathrm{D}_+\overline{\mathrm{D}}_+(\overline{\psi}_-\psi_-) + \frac{1}{2}\sum_{j=1}^{\infty}(-1)^j\int \mathrm{d}^2x\,\mathrm{D}_+\overline{\mathrm{D}}_+\left(\overline{\Upsilon}_{j-}\Upsilon_{j-}\right), \\ &= \frac{1}{2}\int \mathrm{d}^2x\,\mathrm{D}_+\overline{\mathrm{D}}_+\left(\overline{\psi}_-\psi_-\right)\ . \end{aligned} \tag{4.3}$$

In the last step, we have integrated out the (0,2) unconstrained superfields $\varUpsilon_{j-}$ with $j \geq 1$. Note that this is consistent with the F-transformations discussed in section 2.5 only for weight $k = 0$. In more detail, the F transformation rules for the fields $\varUpsilon_{j-}$ in (2.48) preserve the auxiliary field equations $\varUpsilon_{j-} = 0$, $j \geq 1$, only for weight 0.

To get the component action, we push the measure derivatives $\mathrm{D}_+\overline{\mathrm{D}}_+$ into the Lagrangian:

$$\mathcal{S} = \int \mathrm{d}^2x\, (\overline{G}G + \mathrm{i}\partial_{++}\overline{\psi}_-\psi_-)\,, \tag{4.4}$$

where the auxiliary field G is defined as $-\sqrt{2}G = \mathrm{D}_+\psi_-$. In appendix C.2, we define the ordinary space components of $\boldsymbol{\Upsilon}_-$ directly without going to $(0,2)$ superspace by acting on $\boldsymbol{\Upsilon}_-$ with $\widetilde{\mathbf{D}}_{a'+}$ successively. We also compute the above component action by directly pushing in the $(0,4)$ measure $\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}$ in (4.3) and using the definitions of the components that were just alluded to, and finally perform the ζ-integral.

4.2 Other fermi superfields

We can also write down fermi superfields that are globally defined on $\mathbf{CP}^1$, i.e., spinorial $\mathcal{O}(n)$ superfields. Of particular interest are the complex $\mathcal{O}(1)$ fermi superfields $\boldsymbol{\psi}_-$ and real $\mathcal{O}(2)$ fermi superfields $\boldsymbol{X}_-$. The $\mathcal{O}(1)$ superfield contains an F-doublet of fermis ψ_{a-} and an F'-doublet of auxiliary scalars $F_{a'}$ defined by

$$\mathrm{D}_{aa'+}\psi_{b-} = \sqrt{2}\varepsilon_{ab}F_{a'}\ . \tag{4.5}$$

Similarly, a real $\mathcal{O}(2)$ superfield is described by a rank two tensor X_{ab-} (compare with (3.26)) with superpartner auxiliary scalars $F_{ab'}$:

$$\mathrm{D}_{aa'+}X_{bc-} = \sqrt{2}(\varepsilon_{ab}F_{ca'} + \varepsilon_{ac}F_{ba'} + \varepsilon_{bc}F_{aa'})\ . \tag{4.6}$$

In [65], four types of $(0,4)$ fermi multiplets are described, namely, MSM-I, -II, -III and -IV (MSM is short for Minus Spinor Multiplet). It is easy to repeat the steps of section 3.3 to arrive at the fact that MSM-I is an $\mathcal{O}(1)$ superfield, MSM-II is an $\mathcal{O}(2)'$ superfield, MSM-III is an $\mathcal{O}(1)'$ superfield and MSM-IV is an $\mathcal{O}(2)$ superfield.

5 Interactions

The criteria for $(0,4)$ supersymmetry are closure of the algebra $\mathbf{D}_+^2 = 0$ on all the superfields and the invariance of the action (see the comments at the end of section 2.2). In this section, we use these criteria to discover possible $(0,4)$ supersymmetric interactions between twisted hypers, standard hypers and fermis.

As indicated in the Introduction (section 1), interactions could be E-terms, gauge interactions, or of the nonlinear sigma model type. Nonlinear sigma models have been discussed for $\mathcal{O}(1)$ standard hypers in $(0,4)$ projective superspace [58] and arctic standard hypers in $(4,4)$ projective superspace [51]. We have not explored all the possibilities for E-term interactions. In this paper, we consider the combination of F-arctic standard hypers, F-arctic fermis and $\mathcal{O}(1)'$ twisted hypers with the R-charge assignments given previously (of course, everything we say can be used for the mirror combination where we swap the two R-symmetry groups).

Consider F-arctic fermi multiplets $\boldsymbol{\Upsilon}_-$, arctic standard hypermultiplets $(\boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}_{--})$ and $\mathcal{O}(1)'$ twisted hypermultiplets $\boldsymbol{H}$ with the following projective constraints:

$$\begin{aligned}
&\mathbf{D}_+\boldsymbol{\Upsilon} = 0\,, \qquad \mathbf{D}_+\boldsymbol{\Upsilon}_- = -\sqrt{2}\widehat{\boldsymbol{C}}\boldsymbol{\Upsilon}\,, \qquad \mathbf{D}_+\boldsymbol{\Upsilon}_{--} = \frac{1}{\sqrt{2}}\boldsymbol{C}\boldsymbol{\Upsilon}_-\,, \qquad \mathbf{D}_+\boldsymbol{H} = 0\,, \\
&\mathbf{D}_+\overline{\boldsymbol{\Upsilon}} = 0\,, \qquad \mathbf{D}_+\overline{\boldsymbol{\Upsilon}}_- = \sqrt{2}\zeta\overline{\boldsymbol{\Upsilon}}\overline{\widehat{\boldsymbol{C}}}\,, \qquad \mathbf{D}_+\overline{\boldsymbol{\Upsilon}}_{--} = \frac{1}{\sqrt{2}}\zeta\overline{\boldsymbol{\Upsilon}}_-\overline{\boldsymbol{C}}\,, \qquad \mathbf{D}_+\overline{\boldsymbol{H}} = 0\,.
\end{aligned} \tag{5.1}$$

where $\mathbf{D}_+ = u^a v^{a'} \mathrm{D}_{aa'+}$ is the fully contracted derivative (see (2.29) in section 2), $\boldsymbol{C} = v^{a'} C_{a'}$ and $\widehat{\boldsymbol{C}} = v^{a'} \widehat{C}_{a'}$ are $\mathcal{O}(1)'$ superfields which are functions of the various superfields in the model. The second line in (5.1) is obtained by applying extended complex conjugation on the first line and using the appropriate definitions of extended complex conjugates from section 2.6.

The closure of the supersymmetry algebra $\mathbf{D}_+^2 = 0$ on $\boldsymbol{\Upsilon}_-$ and $\boldsymbol{\Upsilon}_{--}$ give

$$\mathbf{D}_+\boldsymbol{C} = 0\,, \quad \mathbf{D}_+\widehat{\boldsymbol{C}} = 0\,, \quad \boldsymbol{C}\widehat{\boldsymbol{C}} = 0\,. \tag{5.2}$$

The action $\mathcal{S}$ for the above superfields splits into an action $\mathcal{S}_F$ in F-projective superspace for the standard hypers and the fermis, and an action $\mathcal{S}_{F'}$ in F'-projective superspace for the twisted hypers, i.e., $\mathcal{S} = \mathcal{S}_F + \mathcal{S}_{F'}$ with

$$\begin{aligned}
\mathcal{S}_F &= \int \mathrm{d}^2x \oint \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+} \left(\frac{\mathrm{i}}{2}\overline{\boldsymbol{\Upsilon}}\partial_{--}\boldsymbol{\Upsilon} - \zeta\overline{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}_{--} + \zeta^{-1}\overline{\boldsymbol{\Upsilon}}_{--}\boldsymbol{\Upsilon} - \frac{1}{2}\overline{\boldsymbol{\Upsilon}}_-\boldsymbol{\Upsilon}_-\right)\,, \\
\mathcal{S}_{F'} &= \int \mathrm{d}^2x \oint \frac{\mathrm{d}\zeta'}{2\pi\mathrm{i}} \widetilde{\mathbf{D}}_{1+}\widetilde{\mathbf{D}}_{2+} \left(-\frac{\mathrm{i}}{2}\zeta'^{-1}\overline{\boldsymbol{H}}\partial_{--}\boldsymbol{H}\right)\,.
\end{aligned} \tag{5.3}$$

The action $\mathcal{S}_F$ and $\mathcal{S}_{F'}$ in (5.3) are $(0,4)$ invariant if the Lagrangians are annihilated by $\mathbf{D}_+$. This is obvious for $\mathcal{S}_{F'}$. The action of $\mathbf{D}_+$ on the Lagrangian in $\mathcal{S}_F$ is

$$\begin{aligned}
&\mathbf{D}_+\left(\frac{\mathrm{i}}{2}\overline{\boldsymbol{\Upsilon}}\partial_{--}\boldsymbol{\Upsilon} - \zeta\overline{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}_{--} + \zeta^{-1}\overline{\boldsymbol{\Upsilon}}_{--}\boldsymbol{\Upsilon} - \frac{1}{2}\overline{\boldsymbol{\Upsilon}}_-\boldsymbol{\Upsilon}_-\right) \\
&\qquad = -\frac{1}{\sqrt{2}}\zeta\overline{\boldsymbol{\Upsilon}}(\boldsymbol{C} + \overline{\widehat{\boldsymbol{C}}})\boldsymbol{\Upsilon}_- + \frac{1}{\sqrt{2}}\overline{\boldsymbol{\Upsilon}}_-(\overline{\boldsymbol{C}} - \widehat{\boldsymbol{C}})\boldsymbol{\Upsilon}\,.
\end{aligned} \tag{5.4}$$

For the right hand side to be zero, the following conditions then have to be satisfied:

$$\overline{\boldsymbol{C}} = \widehat{\boldsymbol{C}}\,, \quad \boldsymbol{C} = -\overline{\widehat{\boldsymbol{C}}}\,, \quad \text{i.e.,} \quad \overline{C}^{a'} = \varepsilon^{a'b'}\widehat{C}_{b'}\,. \tag{5.5}$$

(the two conditions are consistent with each other since we have $\overline{\overline{\boldsymbol{\Phi}}} = -\boldsymbol{\Phi}$ for an $\mathcal{O}(1)'$ superfield $\boldsymbol{\Phi}$.)

Upon using (5.5), the constraints $\boldsymbol{C}\widehat{\boldsymbol{C}} = 0$ in (5.2) become

$$\boldsymbol{C}\overline{\boldsymbol{C}} = 0 \quad \Leftrightarrow \quad C_{a'}\overline{C}^{b'} = \frac{1}{2}C_{c'}\overline{C}^{c'}\,\delta_{a'}^{b'}\,. \tag{5.6}$$

$\boldsymbol{C}$ and $\widehat{\boldsymbol{C}}$ are *a priori* functions of both standard and twisted hypers. We restrict ourselves to the case where $\boldsymbol{C}$ and $\widehat{\boldsymbol{C}}$ are polynomials in the standard and twisted hypers. Recall that the F-weights of $\mathbf{D}_+$, $\boldsymbol{\Upsilon}$, $\boldsymbol{\Upsilon}_-$ and $\boldsymbol{\Upsilon}_{--}$ are $+1$, $+1$, 0 and -1 respectively. Since the F-weight has to be preserved in the constraint equations (5.1) above, $\boldsymbol{C}$ and $\widehat{\boldsymbol{C}}$ should have F-weight 0. Further, since we restrict $\boldsymbol{C}$ and $\overline{\boldsymbol{C}}$ to be polynomials in the superfields, they must simply be independent of the standard hypers $\boldsymbol{\Upsilon}$.

The reality constraints (5.5) are also consistent with $\boldsymbol{C}$ and $\widehat{\boldsymbol{C}}$ being independent of standard hypers. However, note that $\boldsymbol{C}$ and $\widehat{\boldsymbol{C}}$ can be chosen to be more general F-weight 0 functions of the standard hypers (e.g. rational functions) and these may have good Taylor expansions around both $\zeta = 0$ and $\zeta = \infty$. Then it is possible to satisfy the reality constraint (5.5) even when $\boldsymbol{C}$ and $\widehat{\boldsymbol{C}}$ depend on arctic standard hypers non-trivially.

Since $\boldsymbol{C}$ is an $\mathcal{O}(1)'$ superfield which is assumed to be a polynomial in the twisted hypers and is annihilated by $\mathbf{D}_{a+}$, it must be linear in the $\mathcal{O}(1)'$ twisted hypers $\boldsymbol{H}$. Thus, $\boldsymbol{C}$ must take the form

$$\boldsymbol{C} = \boldsymbol{K} + L\boldsymbol{H}\,, \tag{5.7}$$

where $\boldsymbol{K}$ is $\mathcal{O}(1)'$ and constant, and L is constant.

Recall from sections 3.1, 3.2 and 4 that the dynamical components of the arctic standard hyper are $(\eta_a, \xi_{a'+})$, those of the twisted hyper are $(H_{a'}, \xi_{a+})$ and that of the fermi is (ψ_-). The full component action for these fields that follows from the projective superspace action (5.3) is worked out in appendix C. We give the result here:

$$\begin{aligned}\mathcal{S} = &\int \mathrm{d}^2x\,\left(-\overline{\partial_\mu H}^{a'}\partial_\mu H_{a'} - \mathrm{i}\bar{\xi}^a_+\partial_{--}\xi_{a+} - \overline{\partial_\mu\eta}^a\partial_\mu\eta_a - \mathrm{i}\bar{\xi}^{a'}_+\partial_{--}\xi_{a'+} - \mathrm{i}\overline{\psi}_-\partial_{++}\psi_-\right)\\ &+\int \mathrm{d}^2x\,\left(-\frac{1}{2}\overline{\eta}^a C_{a'}\overline{C}^{a'}\eta_a + \left(-\bar{\xi}^{a'}_+ C_{a'}\psi_- - \overline{\eta}^a L\xi_{a+}\psi_- + \text{c.c.}\right)\right)\,.\end{aligned} \tag{5.8}$$

6 Example: ADHM sigma model

In this section we consider an interacting model with standard hypers, fermis and twisted hypers. This is a particular $(0,4)$ linear sigma model which flows to a nonlinear sigma model with target space a k-instanton solution in Yang-Mills theory in four dimensions. This model was written in $(0,1)$ superspace in [18, 59] and in harmonic superspace in [63, 64].

This linear sigma model for $\mathrm{U}(n)$ instantons is realised by the following nested D-brane configuration in Type IIB theory [18]: 1 D1-brane $\subset$ k D5-branes $\subset$ n D9-branes. The k D5-branes appear as k-instanton configurations in the D9-brane $\mathrm{U}(n)$ gauge theory and the D1-brane probes this configuration. The $1+1$ dimensional linear sigma model is the theory on the D1-brane worldsheet.

The D1-brane worldsheet theory includes a $\mathrm{U}(1)$ gauge multiplet arising from the D1-D1 open string spectrum. However, the $\mathrm{U}(1)$ multiplet does not have an effect on the computation of the instanton connection on target space in the classical theory on the D1-brane [18]. We describe the classical $\mathrm{U}(n)$ instanton model without the $\mathrm{U}(1)$ gauge multiplet in section 6.1 and show that it reproduces the calculation in [18], and redo the analysis more carefully in the companion paper [60] with the gauge multiplet included. The novelty of the projective superspace approach is that twistor space and the relevant holomorphic bundles on twistor space required for describing instantons [62, 67, 68] appear explicitly in the description of the model which we describe below.

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For $\mathrm{SO}(n)$ instantons, we add an $\mathrm{O9}^{-}$-plane to the above D-brane configuration. The orientifold projection requires an even number of D5-branes which we take to be $2k$, and after the projection pairs of D5-branes are stuck and cannot be separated. The projection reduces the D9-brane gauge group to $\mathrm{SO}(n)$, that of the D5-branes to $\mathrm{Sp}(k)$ and projects out the vector multiplet on the D1-brane. For $\mathrm{Sp}(n)$ instantons, we start with $2n$ D9-branes, k D5-branes and 2 D1-branes and add an $\mathrm{O9}^{+}$-plane which results in an $\mathrm{Sp}(n)$ gauge group on the D9-branes, an $\mathrm{SO}(k)$ gauge group on the D5-branes and an $\mathrm{Sp}(1)$ gauge group on the D1-branes (again, the two D1-branes cannot be separated). These facts may be found in, e.g., [69]. Since the $\mathrm{Sp}(n)$ instanton sigma model requires a gauge multiplet, and both $\mathrm{SO}(n)$ and $\mathrm{Sp}(n)$ models require orientifolds, we describe both sigma models together in the companion paper [60].

6.1 $\mathrm{U}(n)$ instantons

The superfield content consists of

1. $2k'$ twisted hypers $\boldsymbol{H}_{Y'}$, $Y' = 1', \ldots, 2k'$ (we consider $2k' = 2$ for most of the discussion),
2. k standard hypers $(\boldsymbol{\Upsilon}_Y, \boldsymbol{\Upsilon}_{Y--})$, $Y = 1, \ldots, k$,
3. $2k + n$ fermis $\boldsymbol{\Upsilon}_{A-}$, $A = 1, \ldots, 2k + n$.

The above superfields (for $2k' = 2$) are a subset of the low-energy spectrum of the various Dp-Dq open strings in the D-brane configuration described above. Since we are interested in the low-energy theory on the D1-brane, we retain only those fields that appear from the D1-Dp open string sectors for $p = 1, 5, 9$. The two twisted hypers $\boldsymbol{H}_{Y'}$ arise from the D1-D1 strings in the directions transverse to the D1-brane and D5-branes. The k standard hypers $\boldsymbol{\Upsilon}_Y$ arise from D1-D5 strings and the $2k + n$ fermis $\boldsymbol{\Upsilon}_{A-}$ arise from the D1-D5 strings ($2k$ fermis) and the D1-D9 strings (n fermis). Part of the couplings $\boldsymbol{C}$ described below arise from the D5-D9 open string degrees of freedom which are frozen from the point of view of the D1-brane, and they contain the instanton moduli.

We suppress the flavour indices Y', Y and A on the twisted hypers, standard hypers and fermis respectively unless we wish to explicitly exhibit the flavour properties of the superfields. We work with a given symplectic structure $\omega^{Y'Z'}$ on the space of twisted hypers. This allows for a reality condition:

$$\overline{\boldsymbol{H}}^{Y'} = \omega^{Y'Z'} \boldsymbol{H}_{Z'}\,, \quad \text{i.e.,} \quad \overline{H}^{a'Y'} = \varepsilon^{a'b'} \omega^{Y'Z'} H_{b'Z'}\,. \tag{6.1}$$

(note that according to the above condition $\overline{\overline{\boldsymbol{H}}}_{Y'} = -\boldsymbol{H}_{Y'}$ since $\omega^{Y'Z'}\omega_{Z'X'} = -\delta^{Y'}_{X'}$. This is consistent with the result $\overline{\overline{\boldsymbol{H}}} = -\boldsymbol{H}$ for an $\mathcal{O}(1)'$ multiplet). The most general $(0,4)$ constraints are those given in (5.1):

$$\mathbf{D}_+ \boldsymbol{\Upsilon} = 0\,, \quad \mathbf{D}_+ \boldsymbol{\Upsilon}_- = -\sqrt{2}\widehat{\boldsymbol{C}}\boldsymbol{\Upsilon}\,, \quad \mathbf{D}_+ \boldsymbol{\Upsilon}_{--} = \frac{1}{\sqrt{2}} \boldsymbol{C}\,\boldsymbol{\Upsilon}_-\,, \quad \mathbf{D}_+ \boldsymbol{H} = 0\,, \tag{6.2}$$

where recall from section 5 that $\boldsymbol{C} = v^{a'} C_{a'}$, $\widehat{\boldsymbol{C}} = v^{a'} \widehat{C}_{a'}$ are $\mathcal{O}(1)'$ superfields. As discussed in section 5, $\boldsymbol{C}$ and $\widehat{\boldsymbol{C}}$ are independent of the standard hypers $\boldsymbol{\Upsilon}$ and are linear in the $\mathcal{O}(1)'$

twisted hypers $\boldsymbol{H}$. The constraints on the couplings $\boldsymbol{C}$ and $\widehat{\boldsymbol{C}}$ that follow from the closure of the $(0,4)$ superalgebra are (5.2) which we reproduce here for convenience:

$$\mathbf{D}_+\boldsymbol{C} = 0\,, \quad \mathbf{D}_+\widehat{\boldsymbol{C}} = 0\,, \quad \boldsymbol{C}\widehat{\boldsymbol{C}} = 0\,. \tag{6.3}$$

$\boldsymbol{C}$ and $\widehat{\boldsymbol{C}}$ and are $k \times (2k+n)$ and $(2k+n) \times k$ matrices respectively; with the flavour indices explicitly displayed, the matrices are resp. written as $\boldsymbol{C}_Y^A$ and $\widehat{\boldsymbol{C}}_A^Y$. Recall from the discussion around (5.7) that $\boldsymbol{C}$ has to be of the form

$$\boldsymbol{C} = \boldsymbol{K} + L^{Y'}\boldsymbol{H}_{Y'}\,, \tag{6.4}$$

where $\boldsymbol{H}_{Y'}$ are the twisted hypermultiplets. The coupling $\boldsymbol{K}$ is a constant $k \times (2k+n)$ matrix $\mathcal{O}(1)'$ superfield and the $L^{Y'}$ are constant $k \times (2k+n)$ matrices (one matrix for each $Y' \in \{1,\ldots,2k'\}$).

Twistor space. Let us consider two twisted hypers, i.e., $2k' = 2$ (everything we say for two twisted hypers can be extended to general k'). The twisted hyper superfields $H_{a'Y'}$ are coordinates on the target space $\mathbf{R}^4$. The $\mathrm{SU}(2)'$ doublet $v^{a'}$ together with the projective superfields $\boldsymbol{H}_{Y'}$ can be interpreted as homogeneous coordinates $\boldsymbol{Z} = (v^{1'}, v^{2'}, \boldsymbol{H}_{1'}, \boldsymbol{H}_{2'})$ for a $\mathbf{CP}^3$ which is in fact the twistor space of $\mathbf{S}^4$ (the one-point compactification of the target $\mathbf{R}^4$). The symplectic structure $\omega^{Y'Z'}$ on the space of twisted hypers and the symplectic structure $\varepsilon^{a'b'}$ on the space of F'-doublets together give an antiholomorphic involution $v^{a'} \to \varepsilon^{a'b'}\overline{v}_{b'}$, $\boldsymbol{H}_{Y'} \to \omega_{Y'Z'}\overline{\boldsymbol{H}}^{Z'}$, on the $\mathbf{CP}^3$ which squares to -1. The $(v^{a'}, H_{Y'b'})$ serve as coordinates on the correspondence space and the *incidence relations* $\boldsymbol{H}_{Y'} = H_{Y'a'}v^{a'}$ are simply the definition of the $\boldsymbol{H}_{Y'}$ as projective superfields.

Monads on twistor space. Next, we show that the couplings $\boldsymbol{C}$ and $\widehat{\boldsymbol{C}}$ encode the data of a *monad* on $\mathbf{CP}^3$. Let V_S and $\widehat{V}_S$ be the vector spaces of $\boldsymbol{\Upsilon}$ and $\boldsymbol{\Upsilon}_{--}$ respectively with $\dim V_S = \dim \widehat{V}_S = k$ and V_F be the vector space of fermis with $\dim V_F = 2k+n$. Then, the couplings $\boldsymbol{C}$ and $\widehat{\boldsymbol{C}}$ can be interpreted as elements of $\mathrm{Hom}(V_F, \widehat{V}_S)$ and $\mathrm{Hom}(V_S, V_F)$ respectively, as is clear from the constraints (6.2). Recall that these maps are linear in the homogeneous coordinates $\boldsymbol{Z} = \{v^{a'}, \boldsymbol{H}_{Y'}\}$ since $\boldsymbol{C} = K_{a'}v^{a'} + L^{Y'}\boldsymbol{H}_{Y'}$. We thus have

$$V_S \xrightarrow{\widehat{\boldsymbol{C}}} V_F \xrightarrow{\boldsymbol{C}} \widehat{V}_S\,. \tag{6.5}$$

The constraint $\boldsymbol{C}\widehat{\boldsymbol{C}} = 0$ that follows from the closure of the algebra (6.3) makes (6.5) a complex. We further require that $\widehat{\boldsymbol{C}}$ is injective and $\boldsymbol{C}$ is surjective: this imposes non-degeneracy conditions on the couplings $K_{a'}$ and $L^{Y'}$. Then the above complex is precisely a monad and the cohomology at V_F, i.e., $\ker \boldsymbol{C}/\operatorname{im}\widehat{\boldsymbol{C}}$ is a holomorphic rank n vector bundle $\mathcal{E}$ on $\mathbf{CP}^3$ which is trivial when restricted to lines in $\mathbf{CP}^3$, and has $c_2(\mathcal{E}) = k$. Thus, the data that goes into choosing the off-shell superfield content of our linear sigma model is precisely the same data that goes into defining a holomorphic bundle on twistor space $\mathbf{CP}^3$ that is trivial on lines.

We get a symplectic structure on the bundle $\mathcal{E}$ also from the requirement that the action is $(0,4)$ supersymmetric. Some reality conditions (which were implicit in the previous sections) are necessary on the vector spaces V_F, V_S and $\widehat{V}_S$ to write down an action for the projective superfields $\boldsymbol{\Upsilon}_-$, $\boldsymbol{\Upsilon}$ and $\boldsymbol{\Upsilon}_{--}$. They are (1) a hermitian structure on V_F that identifies $V_F^* \simeq V_F^\vee$, and (2) the identification $\widehat{V}_S^* \simeq V_S^\vee$, where V^* and $V^\vee$ stand for the complex conjugate and dual of a vector space V respectively. With these at hand, the $(0,4)$ invariance of the action gives the following constraint (5.5) on the couplings $\boldsymbol{C}$ and $\widehat{\boldsymbol{C}}$:

$$\overline{\boldsymbol{C}} = \widehat{\boldsymbol{C}}\,, \tag{6.6}$$

where the bar on $\overline{\boldsymbol{C}}$ acts the hermitian conjugate on the matrix components and extended conjugate on the $\mathcal{O}(1)'$ superfield. This imposes a symplectic structure on the bundle $\mathcal{E}$ obtained from the monad (6.5). By the Penrose-Ward-Atiyah correspondence [67, 68], the bundle $\mathcal{E}$ on twistor space with the symplectic structure described above corresponds to a self-dual $\mathrm{SU}(n)$ connection on $\mathbf{R}^4$ (more precisely, on the one-point compactification S^4 of $\mathbf{R}^4$). The ADHM construction [62] gives an explicit expression for the instanton gauge field in terms of the data described above. The constraints $\boldsymbol{C}\overline{\boldsymbol{C}} = 0$ are precisely the ADHM equations that describe the instanton moduli space [62].

Next, we show that the model flows to an $\mathrm{SU}(n)$ instanton solution in the infrared by explicitly obtaining the expression for the instanton gauge field given by the ADHM construction [62]. The material in the rest of this section is not new and follows the calculations in [18, 59]. In section 6.2 below, we choose particular bases for the vector spaces of superfields to give the usual standard characterization of the ADHM instanton moduli space in terms of finite dimensional matrices. Again, most of the material is standard except for a formula of the virtual dimension of the instanton moduli space on $\mathbf{R}^{4k'}$ for $k' \geq 2$.

Instantons on $\mathbf{R}^4$. The potential energy density of the model described above can be read off from the general expression in (5.8) and is positive-definite:

$$V = \frac{1}{2}\overline{\eta}^a C_{a'}\overline{C}^{a'}\eta_a = \frac{1}{2}\overline{\eta}^{aY} C^A_{a'Y}\overline{C}^{a'Z}_A\eta_{aZ}\ . \tag{6.7}$$

Recall that $C_{a'} = K_{a'} + L^{Y'}H_{a'Y'}$ and the η_{aY} are components of the arctic standard hyper $\boldsymbol{\Upsilon}_Y = \zeta\eta_{1Y} + \eta_{2Y}$ once we eliminate the auxiliary superfields accompanying higher powers of ζ (see (3.17) and the discussion around it). Suppose the constant matrices $K_{a'}$ and $L^{Y'}$ are sufficiently generic so that $\frac{1}{2}C_{a'}\overline{C}^{a'} \equiv f^{-1}$ is an invertible $k\times k$ matrix, i.e., all its eigenvalues are non-zero, for any value of $H_{a'Y'}$. Then, the vacuum corresponds to setting the $\eta_{aY} = 0$ for every flavour $Y = 1,\ldots,k$.

About this vacuum, the potential V vanishes and in particular does not give a mass for the twisted hyper scalars: there is a classical moduli space of vacua $\mathbf{R}^4$ parametrized by the four twisted hyper scalars with the reality condition (6.1). Under the genericity assumption on $K_{a'}$ and $L^{Y'}$, the eigenvalues of the standard hyper mass matrix f^{-1} are all (1) positive since f^{-1} is a positive-definite matrix, and (2) strictly positive since f^{-1} is invertible. We list them as $(m_1^2, m_2^2, \ldots, m_k^2)$. Then, the mass of the standard hyper scalars η_{aY} for a given Y is m_Y. The Yukawa couplings can also be read off from (5.8):

$$-\,\overline{\xi}^{a'}_+ C_{a'}\psi_- - \overline{\eta}^a L\xi_{a+}\psi_- - \overline{\psi}_-\overline{C}^{a'}\xi_{a'+} - \overline{\psi}_-\overline{\xi}^a_+\overline{L}\eta_a\ . \tag{6.8}$$

On the classical vacuum moduli space characterised by $\eta_{aY} = \overline{\eta}^{aY} = 0$ and arbitrary $H_{Y'a'}$, the twisted hyper fermions $\xi_{Y'a+}$ again have no mass terms. Let us look at the mass terms for the standard hyper fermions $\xi_{Ya'+}$:

$$- \overline{\xi}_+^{a'} C_{a'} \psi_- - \overline{\psi}_- \overline{C}^{a'} \xi_{a'+} = -\overline{\xi}_+^{a'Y} C^A_{a'Y} \psi_{A-} - \overline{\psi}_-^A \overline{C}_A^{a'Y} \xi_{a'Y+} \,, \tag{6.9}$$

where we have displayed the flavour indices explicitly. Recall that we have diagonalized $f^{-1} = C_{1'}\overline{C}^{1'} = C_{2'}\overline{C}^{2'}$. By using an appropriate $\mathrm{U}(2k+n)$ transformation, we can further cast the $2k \times (2k+n)$ matrix $\begin{pmatrix} C^A_{1'Y} \\ C^A_{2'Y} \end{pmatrix}$ into a block form with a non-trivial $2k \times 2k$ block and a zero $2k \times n$ block:

$$\begin{pmatrix} C^A_{1'Y} \\ C^A_{2'Y} \end{pmatrix} = \begin{pmatrix} \bigstar_{2k\times 2k} & \mathbf{0}_{2k\times n} \end{pmatrix} \,, \tag{6.10}$$

where the non-trivial $2k \times 2k$ block is $\mathrm{diag}(m_1, \ldots, m_k, m_1, \ldots, m_k)$. For a fixed flavour Y of the standard hyper, the two fermions $\xi_{1'Y+}$, $\xi_{2'Y+}$ and the two fermis $\psi_{Y,-}$, $\psi_{k+Y,-}$ interact through the 2×2 mass matrix $\begin{pmatrix} m_Y & 0 \\ 0 & m_Y \end{pmatrix}$ and become massive with mass m_Y. Recall that the standard hyper scalars η_{aY} also have the same mass m_Y. The zero block of size $2k \times n$ implies that the n fermis ψ_{A-}, $A = 2k+1, \ldots, 2k+n$ are massless. Thus, for generic values of the couplings $K_{a'}$ and $L^{Y'}$, we have k massive standard hypers, $2k$ massive fermis and n massless fermis about any point of the classical vacuum moduli space that is parametrized by the massless twisted hypers.

The n massless fermis can be characterised more generally as the solutions of the equation

$$\sum_{A=1}^{2k+n} \boldsymbol{C}^A_Y \psi_{A-} = 0 \,. \tag{6.11}$$

Let the n massless solutions be arranged into the $(2k+n) \times n$ matrix ${\mathcal{V}_A}^i$ with the normalisation ${(\mathcal{V}^\dagger)_j}^A {\mathcal{V}_A}^i = \delta^i_j$. The most general massless solution is then

$$\psi_{A-} = \sum_{i=1}^{n} {\mathcal{V}_A}^i \lambda_{i-} \,. \tag{6.12}$$

Plugging in the above expression for ψ_{A-} in its kinetic term, we get the kinetic term for the massless modes λ_{i-}:

$$\overline{\psi}^A_- \partial_{++} \psi_{A-} = \overline{\lambda}^i_- {(\mathcal{V}^\dagger)_i}^A \partial_{++} ({\mathcal{V}_A}^j \lambda_{j-}) = \overline{\lambda}^i_- \left[{\delta_i}^j \partial_{++} + \partial_{++} \overline{H}^{Y'a'} {(\mathcal{V}^\dagger)_i}^A \frac{\partial {\mathcal{V}_A}^j}{\partial \overline{H}^{Y'a'}} \right] \lambda_{j-} \,. \tag{6.13}$$

We see that the massless fermis have now acquired an additional connection which is the pullback of a connection $\mathcal{A}$ on target space $\mathbf{R}^4$:

$$(\mathcal{A}_{Y'a'})_i{}^j := \mathrm{i} {(\mathcal{V}^\dagger)_i}^A \frac{\partial {\mathcal{V}_A}^j}{\partial \overline{H}^{Y'a'}} \,. \tag{6.14}$$

This is the connection for a k-instanton solution with $\mathrm{U}(n)$ gauge group, a fact that follows from standard results in the ADHM construction. Since we have assumed the instanton to be non-degenerate, the U(1) part of the connection is trivial and $\mathcal{A}_{Y'a'}$ is in fact an SU(n) instanton connection. We study the degenerate cases carefully in [60] where we shall find that the U(1) gauge multiplet on the D1-brane worldsheet plays an important role.

6.2 The instanton moduli space and symmetries

The constraints $\boldsymbol{C}\overline{\boldsymbol{C}} = 0$ and the fermi zero modes (6.11) (and in turn, the formula for the instanton gauge field) are unaffected by GL(k, $\mathbf{C}$) transformations on the space of standard hypermultiplets and U($2k+n$) transformations of the space of fermis:

$$\boldsymbol{C} \to S \cdot \boldsymbol{C} \cdot U^\dagger\,, \quad S \in \mathrm{GL}(k,\mathbf{C})\,, \quad U \in \mathrm{U}(2k+n)\;. \tag{6.15}$$

Thus, two different solutions of $\boldsymbol{C}\overline{\boldsymbol{C}} = 0$ that are related by a GL(k, $\mathbf{C}$) $\times$ U($2k+n$) transformation as in (6.15) correspond to the same instanton solution. This redundancy allows us to choose a simple form for the coupling $\boldsymbol{C}$ and the equations $\boldsymbol{C}\overline{\boldsymbol{C}} = 0$.

Plugging in the explicit form $\boldsymbol{C} = \boldsymbol{K} + L^{Y'}\boldsymbol{H}_{Y'}$, we get

$$\begin{aligned} 0 = \boldsymbol{C}\overline{\boldsymbol{C}} &= \boldsymbol{K}\overline{\boldsymbol{K}} + \boldsymbol{K}\overline{L}_{Z'}\,\overline{\boldsymbol{H}}^{Z'} + \boldsymbol{H}_{Y'}L^{Y'}\overline{\boldsymbol{K}} + L^{Y'}\overline{L}_{Z'}\boldsymbol{H}_{Y'}\overline{\boldsymbol{H}}^{Z'}\,,\\ &= \boldsymbol{K}\overline{\boldsymbol{K}} + \boldsymbol{K}\overline{L}_{Z'}\,\omega^{Z'X'}\boldsymbol{H}_{X'} + \boldsymbol{H}_{Y'}L^{Y'}\overline{\boldsymbol{K}} + L^{Y'}\overline{L}_{Z'}\boldsymbol{H}_{Y'}\omega^{Z'X'}\boldsymbol{H}_{X'}\;. \end{aligned} \tag{6.16}$$

We have used the reality condition (6.1) on the twisted hypers in going to the second line above. Terms with different numbers of twisted hypers must vanish separately. Let us study each of them in turn:

1. The constant part $\boldsymbol{K}\overline{\boldsymbol{K}}$ of (6.16) satisfies $\boldsymbol{K}\overline{\boldsymbol{K}} = 0$. Displaying the SU(2)$'$ indices explicitly, we have

   $$K_{b'}\overline{K}^{c'}\varepsilon_{c'a'} + K_{a'}\overline{K}^{c'}\varepsilon_{c'b'} = 0\,, \quad \text{i.e.,} \quad K_{b'}\overline{K}^{c'} = \mu\,\delta_{b'}{}^{c'}\,, \tag{6.17}$$

   where μ is a positive-definite $k \times k$ matrix.

2. The vanishing of the terms linear in $\boldsymbol{H}_{Y'}$ in (6.16) requires

   $$\boldsymbol{K}\overline{L}_{Z'}\omega^{Z'Y'} + L^{Y'}\overline{\boldsymbol{K}} = 0\,, \quad \text{or, with SU(2)}' \text{ indices,} \quad K_{b'}\overline{L}_{Z'}\omega^{Z'Y'} = -L^{Y'}\overline{K}^{c'}\varepsilon_{c'b'}\;. \tag{6.18}$$

3. The term quadratic in the twisted hypers $L^{Y'}\overline{L}_{Z'}\omega^{Z'X'}\boldsymbol{H}_{Y'}\boldsymbol{H}_{X'}$ vanishes when

   $$L^{Y'}\overline{L}_{Z'}\omega^{Z'X'} + L^{X'}\overline{L}_{Z'}\omega^{Z'Y'} = 0\,, \quad \text{that is} \quad L^{Y'}\overline{L}_{Z'}\omega^{Z'X'} = \nu^{Y'X'}\,, \tag{6.19}$$

   where $\nu^{Y'X'}$ is antisymmetric in $Y'X'$ and is an arbitrary hermitian $k \times k$ matrix for each $X', Y' \in \{1, \ldots, 2k'\}$. For the special case $k' = 1$, i.e., when there are two twisted hypers, the antisymmetric matrix $\nu^{Y'X'}$ is proportional to the symplectic form $\omega^{Y'X'}$:

   $$L^{Y'}\overline{L}_{Z'}\omega^{Z'X'} = \nu\,\omega^{Y'X'}\,, \quad \text{that is} \quad L^{Y'}\overline{L}_{Z'} = \nu\,\delta^{Y'}{}_{Z'}\,, \tag{6.20}$$

   where ν is now a single positive-definite $k \times k$ matrix.

The couplings $\boldsymbol{K}$ and $L^{Y'}$ transform under the $\mathrm{GL}(k,\mathbf{C})\times\mathrm{U}(2k+n)$ (6.15) as

$$\boldsymbol{K}\to S\cdot\boldsymbol{K}\cdot U^\dagger\,,\quad L^{Y'}\to S\cdot L^{Y'}\cdot U^\dagger\,, \tag{6.21}$$

with the same $\mathrm{GL}(k,\mathbf{C})$ matrix S and $\mathrm{U}(2k+n)$ matrix U for all Y'. This freedom can be used to choose a convenient form for $L^{Y'}$ and $\boldsymbol{K}$ as follows.

First, the $L^{Y'}$ satisfy the constraints (6.19) $L^{Y'}\overline{L}_{Z'}\omega^{Z'X'}+L^{X'}\overline{L}_{Z'}\omega^{Z'Y'}=0$. Suppose we choose the symplectic form canonically to be

$$\omega_{Y'}{}^{Z'}=\mathrm{diag}_{k'/2}\{\omega_2,\omega_2,\dots,\omega_2\}\,,\quad \text{with}\quad \omega_2=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\,, \tag{6.22}$$

where diag_ℓ indicates that length of the diagonal matrix is ℓ. Let us look at the pair of matrices $L^{1'}$, $L^{2'}$. They satisfy

$$L^{1'}\overline{L}_{1'}=L^{2'}\overline{L}_{2'}=\nu^{1'2'}\,,\quad L^{1'}\overline{L}_{2'}=0\,. \tag{6.23}$$

By an appropriate $\mathrm{GL}(k,\mathbf{C})$ transformation S (6.21), we can transform $\nu^{1'2'}$ into the $k\times k$ identity matrix. Then, the $2k\times(2k+n)$ matrix $\begin{pmatrix}L^{1'}\\L^{2'}\end{pmatrix}$ satisfies

$$\begin{pmatrix}L^{1'}\\L^{2'}\end{pmatrix}\begin{pmatrix}\overline{L}_{1'}&\overline{L}_{2'}\end{pmatrix}=\begin{pmatrix}\mathbb{1}_k&0_k\\0_k&\mathbb{1}_k\end{pmatrix}\,, \tag{6.24}$$

where $\mathbb{1}_k$ and 0_k are the $k\times k$ identity and zero matrices respectively. Using an appropriate $\mathrm{U}(2k+n)$ transformation U (6.21), we can cast the above $2k\times(2k+n)$ matrix into the form

$$\begin{pmatrix}L^{1'}\\L^{2'}\end{pmatrix}=\begin{pmatrix}\mathbb{1}_k&0_k&0_{k\times n}\\0_k&\mathbb{1}_k&0_{k\times n}\end{pmatrix}\,. \tag{6.25}$$

There is a residual $\mathrm{U}(k)\times\mathrm{U}(n)$ subgroup of $\mathrm{GL}(k,\mathbf{C})\times\mathrm{U}(2k+n)$ which preserves the above configuration (6.25) which corresponds to

$$S=\mathcal{U}\,,\quad U=\begin{pmatrix}\mathcal{U}&0_k&0_{k\times n}\\0_k&\mathcal{U}&0_{k\times n}\\0_{n\times k}&0_{n\times k}&\widetilde{\mathcal{U}}\end{pmatrix}\,,\quad\text{where}\quad \mathcal{U}\in\mathrm{U}(k)\,,\quad \widetilde{\mathcal{U}}\in\mathrm{U}(n)\,. \tag{6.26}$$

The reality constraint (6.18) for $Y'=1',2'$, i.e.,

$$-\boldsymbol{K}\overline{L}_{2'}+L^{1'}\overline{\boldsymbol{K}}=0\,,\quad \boldsymbol{K}\overline{L}_{1'}+L^{2'}\overline{\boldsymbol{K}}=0\,, \tag{6.27}$$

is solved by the following expression for $\boldsymbol{K}$:

$$\boldsymbol{K}=\begin{pmatrix}\zeta'B_1^{(1')}+B_2^{(1')\dagger}&-\zeta'B_2^{(1')}+B_1^{(1')\dagger}&\zeta'I^{(1')}+J^{(1')\dagger}\end{pmatrix}\,. \tag{6.28}$$

where $I^{(1')}$, $J^{(1')\dagger}$ are $k\times n$ matrices and $B_1^{(1')}$, $B_2^{(1')}$ are $k\times k$ matrices. The remaining matrices $L^{Y'}$, $Y'=3',4',\dots,2k'$ can also be simplified to a form similar to (6.28) using the constraints

$$L^{1'}\overline{L}_{2y'-1}-L^{2y'}\overline{L}_{2'}=0\,,\quad L^{2'}\overline{L}_{2y'-1}+L^{2y'}\overline{L}_{1'}=0\,,\quad y'=2',\dots,k'\,, \tag{6.29}$$

where we have introduced the index $y' = 2', \ldots, k'$, such that the pairs $\{2y'-1, 2y'\}$ cover the index $Y' \in \{3', 4', \ldots, 2k'\}$ (later, we will append the value $y' = 1'$ as well). We then get the simplified form

$$\begin{pmatrix} L^{2y'-1} \\ L^{2y'} \end{pmatrix} = \begin{pmatrix} B_1^{(y')} & -B_2^{(y')} & I^{(y')} \\ B_2^{(y')\dagger} & B_1^{(y')\dagger} & J^{(y')\dagger} \end{pmatrix} . \tag{6.30}$$

Thus, the degrees of freedom that remain after fixing the $\mathrm{GL}(k, \mathbf{C}) \times \mathrm{U}(2k+n)$ symmetries are

$$\{B_1^{(y')}\,,\ B_2^{(y')}\,,\ I^{(y')}\,,\ J^{(y')}\}\,, \quad \text{for} \quad y' = 1', \ldots, k' \,. \tag{6.31}$$

There are $k'(2k^2+2k^2+2kn+2kn) = k'(4k^2+4kn)$ real degrees of freedom. The remaining constraints on the $K_{a'}$ and $L^{Y'}$, $Y' = 3', 4', \ldots$, are

$$\begin{aligned} &K_{a'}\overline{K}^{c'}\varepsilon_{c'b'} + K_{a'}\overline{K}^{c'}\varepsilon_{c'b'} = 0\,, \\ &K_{a'}\overline{L}_{Z'}\omega^{Z'Y'} + L^{Y'}\overline{K}^{c'}\varepsilon_{c'a'} = 0\,, \qquad L^{Y'}\overline{L}_{Z'}\omega^{Z'X'} + L^{X'}\overline{L}_{Z'}\omega^{Z'Y'} = 0\,. \end{aligned} \tag{6.32}$$

In terms of the matrices $B_1^{(y')}$, $B_2^{(y')}$, $I^{(y')}$ and $J^{(y')}$, $y' = 1', \ldots, k'$, we have the equations

$$\begin{aligned} &[B_1^{(y')}, B_2^{(z')}] + [B_1^{(z')}, B_2^{(y')}] + I^{(y')}J^{(z')} + I^{(z')}J^{(y')} = 0\,, \\ &[B_1^{(y')}, B_1^{(z')\dagger}] + [B_2^{(y')}, B_2^{(z')\dagger}] + I^{(y')}I^{(z')\dagger} - J^{(z')\dagger}J^{(y')} = 0\,, \quad \text{for all} \quad y', z' = 1', \ldots, k' \,. \end{aligned} \tag{6.33}$$

Let us get a count of the number of such equations. The above equations are symmetric in y', z'. For $y' = z'$, the last equation in (6.33) is manifestly real whereas the first equation is complex. Thus, for $y' = z'$, we have $k' \times 3k^2$ real equations. For $y' \neq z'$, it is sufficient to restrict $y' < z'$, and both equations in (6.33) are complex. This gives a count of $\frac{1}{2}k'(k'-1) \times 4k^2$. In total, the number of equations is $k^2k'(2k'+1)$. For $k' = 1$, the target space is $\mathbf{R}^4$ and the above equations are precisely the ADHM equations.

We must also remember that the instanton connection (6.14) is invariant under the residual $\mathrm{U}(k)$ transformations (6.26). We treat the residual $\mathrm{U}(n)$ in (6.26) as a symmetry of framings at ∞ of the instanton solution. The $B_1^{(y')}$, $B_2^{(y')}$ are inert under framing whereas the $I^{(y')}$ and $J^{(y')}$ transform as

$$I^{(y')} \to I^{(y')}\widetilde{\mathcal{U}}\,, \quad J^{(y')} \to \widetilde{\mathcal{U}}^\dagger J^{(y')} \,. \tag{6.34}$$

Thus, the moduli space of framed instantons is described by

$$\left\{\textsc{Fields} \,\middle|\, \textsc{Equations}\right\} \big/ \textsc{Symmetries}\,, \tag{6.35}$$

with

1. Fields: $B_1^{(y')}, B_2^{(y')}, I^{(y')}, J^{(y')}$, $y' = 1', \ldots, k'$,
2. Equations: the equations (6.33), and
3. Symmetries: the residual $\mathrm{U}(k)$ symmetry in (6.26) which acts on the various fields as

$$B_1^{(y')} \to \mathcal{U}B_1^{(y')}\mathcal{U}^\dagger\,, \quad B_2^{(y')} \to \mathcal{U}B_2^{(y')}\mathcal{U}^\dagger\,, \quad I^{(y')} \to \mathcal{U}I^{(y')}\,, \quad J^{(y')} \to J^{(y')}\mathcal{U}^\dagger \,. \tag{6.36}$$

The virtual dimension of the moduli space of framed instantons is then

$$\begin{aligned}\dim_{\mathbf{R}}\{\textsc{Fields}\} - \dim_{\mathbf{R}}\{\textsc{Equations}\} - \dim_{\mathbf{R}}\{\textsc{Symmetries}\}\\ = k'(4k^2+4kn) - k^2k'(2k'+1) - k^2 = 4k'kn - k^2(2k'-1)(k'-1)\ .\end{aligned} \tag{6.37}$$

When $k' = 1$, this becomes $4kn$ which is the virtual dimension (in fact, the dimension itself) of the $\mathrm{SU}(n)$ k-instanton moduli space on $\mathbf{R}^4$.

Acknowledgments

The authors thank N. Nekrasov, S. Shatashvili and J. P. Ang for useful discussions during early stages of the work. The authors are grateful to the referee for useful comments and suggestions. N. P. would like to acknowledge the support of the Jawaharlal Nehru Postdoctoral Fellowship for part of the duration of the work. The work of M. R. was supported in part by NSF grants PHY-19-15093 and PHY-22-15093.

A (0, 1) and (0, 2) superspace

A.1 (0, 1) superspace

$(0,1)$ superspace has coordinates $(x^{\pm\pm}, \theta^+)$ where θ^+ is a real Grassmann variable. The corresponding supercovariant derivatives are $(\partial_{\pm\pm}, \mathcal{D}_+)$ which satisfy the algebra

$$\mathcal{D}_+^2 = \mathrm{i}\partial_{++}\ . \tag{A.1}$$

with all other commutators being zero.

Multiplets of the $(0,1)$ supersymmetry algebra are not constrained. The most common ones are the scalar multiplet (spin 0), the fermi multiplet (spin $\frac{1}{2}$, left-handed) and the gauge multiplet (spin 1). The multiplets are irreducible representations of the algebra when they are real (or hermitian).

A real scalar superfield ϕ has components

$$\phi_|\ , \quad \mathrm{i}\xi_+ = (\mathcal{D}_+\phi)_|\ , \tag{A.2}$$

where $\phi_|$ is a real scalar field and ξ_+ is a real right-handed fermion. We follow the usual convention of denoting the lowest component of a superfield by the same symbol and drop the 'slash' $|$ from here on. A supersymmetric action with the lowest number of derivatives is

$$\mathcal{S}_{\text{scalar}} = \frac{\mathrm{i}}{2}\int \mathrm{d}^2x\, \mathcal{D}_+\left(-(\mathcal{D}_+\phi)\,\partial_{--}\phi\right) = \frac{1}{2}\int \mathrm{d}^2x \left(-\partial^\mu\phi\,\partial_\mu\phi - \mathrm{i}\xi_+\partial_{--}\xi_+\right)\ . \tag{A.3}$$

A real fermi superfield ψ_- has the components

$$\psi_-\ , \quad F = \mathcal{D}_+\psi_-\ , \tag{A.4}$$

where ψ_- is a real left-handed fermion and F is a real auxiliary field, with the action

$$\mathcal{S}_{\text{fermi}} = \frac{1}{2}\int \mathrm{d}^2x\, \mathcal{D}_+\left(\psi_-\mathcal{D}_+\psi_-\right) = \frac{1}{2}\int \mathrm{d}^2x \left(-\mathrm{i}\psi_-\partial_{++}\psi_- + F^2\right)\ . \tag{A.5}$$

One can add a potential term in the action via a term that is linear in the fermi superfields $\psi_{\alpha-}$ in the theory:

$$\mathcal{S}_{\text{potential}} = \int \mathrm{d}^2 x\, \mathcal{D}_+ \left(\psi_{\alpha-} M^\alpha\right) = \int \mathrm{d}^2 x \left(F_\alpha M^\alpha - \psi_{\alpha-} \frac{\partial M^\alpha}{\partial \phi_i} \xi_{i+}\right) , \tag{A.6}$$

where $M^\alpha := M^\alpha(\phi)$ are functions of the scalar superfields in the theory.

A.2 (0, 2) superspace

$(0,2)$ superspace has coordinates $(x^{\pm\pm}, \theta^+, \overline{\theta}^+)$ where θ^+ and $\overline{\theta}^+$ are left-handed spinors. We denote the corresponding supercovariant derivatives by $(\partial_{++}, \mathrm{D}_+, \overline{\mathrm{D}}_+)$. They satisfy the algebra

$$\mathrm{D}_+^2 = \overline{\mathrm{D}}_+^2 = 0\,, \quad \{\mathrm{D}_+\,, \overline{\mathrm{D}}_+\} = 2\mathrm{i}\partial_{++}\ . \tag{A.7}$$

We review various constrained superfields that are required to write down supersymmetric actions in superspace.

Chiral. A scalar chiral superfield (or, simply a chiral superfield) ϕ is a Lorentz scalar and satisfies $\overline{\mathrm{D}}_+\phi = 0$ and has components

$$\phi\,, \quad \overline{\phi}\,, \quad \sqrt{2}\,\xi_+ := \mathrm{D}_+\phi\,, \quad -\sqrt{2}\,\overline{\xi}_+ := \overline{\mathrm{D}}_+\overline{\phi}\,, \tag{A.8}$$

and consequently, $\overline{\mathrm{D}}_+\mathrm{D}_+\phi = 2\mathrm{i}\partial_{++}\phi$. The action for a free chiral superfield is

$$\mathcal{S}_{\text{chiral}} = -\frac{\mathrm{i}}{2}\int \mathrm{d}^2 x\, \mathrm{D}_+\overline{\mathrm{D}}_+ \left(\overline{\phi}\,\partial_{--}\phi\right) = \int \mathrm{d}^2 x \left(-\overline{\partial^\mu\phi}\,\partial_\mu\phi - \mathrm{i}\overline{\xi}_+\partial_{--}\xi_+\right)\ . \tag{A.9}$$

Fermi. A Fermi superfield ψ_- is a left-handed spinor and satisfies the constraint $\overline{\mathrm{D}}_+\psi_- = 0$. It has components

$$\psi_-\,, \quad \overline{\psi}_-\,, \quad -\sqrt{2}\,G := \mathrm{D}_+\psi_-\,, \quad -\sqrt{2}\,\overline{G} := \overline{\mathrm{D}}_+\overline{\psi}_-\ . \tag{A.10}$$

The action for a free Fermi multiplet ψ_- is

$$\mathcal{S}_{\text{Fermi}} = \frac{1}{2}\int \mathrm{d}^2 x\, \mathrm{D}_+\overline{\mathrm{D}}_+ \left(\overline{\psi}_-\psi_-\right) = \int \mathrm{d}^2 x \left(-\mathrm{i}\overline{\psi}_-\partial_{++}\psi_- + \overline{G}G\right)\ . \tag{A.11}$$

We see that the left-handed fermion ψ_- satisfies the equation of motion $\partial_{++}\psi_- = 0$ and hence is right-moving on-shell. The field G is auxiliary with equation of motion $G = 0$.

Potential terms. Let ϕ_i collectively denote all the $(0,2)$ chiral superfields in the theory and $\psi_{\alpha-}$ the $(0,2)$ Fermi superfields. We can modify the constraint $\overline{\mathrm{D}}_+\psi_{\alpha-} = 0$ to

$$\overline{\mathrm{D}}_+\psi_{\alpha-} = \sqrt{2}E_\alpha(\phi)\,, \tag{A.12}$$

where the $E_\alpha(\phi)$ are holomorphic functions of the chiral multiplets ϕ_i. This modification results in additional interaction terms in the action for the fermi superfields:

$$\begin{aligned}\mathcal{S}_{\text{Fermi}} &= \frac{1}{2}\int \mathrm{d}^2 x\, \mathrm{D}_+\overline{\mathrm{D}}_+ \left(\overline{\psi}^\alpha_-\psi_{\alpha-}\right), \\ &= \int \mathrm{d}^2 x \left(-\mathrm{i}\overline{\psi}^\alpha_-\partial_{++}\psi_{\alpha-} + \overline{G}^\alpha G_\alpha - \overline{E}^\alpha(\overline{\phi})E_\alpha(\phi) + \overline{\psi}^\alpha_- \frac{\partial E_\alpha}{\partial\phi_i}\,\xi_{i+} + \frac{\partial\overline{E}^\alpha}{\partial\overline{\phi}^i}\,\overline{\xi}^i_+\,\psi_{\alpha-}\right)\ .\end{aligned} \tag{A.13}$$

We can also write a superpotential term, known as a "J-term" in $(0,2)$ literature:

$$\begin{aligned}\mathcal{S}_J &= -\frac{1}{\sqrt{2}}\int \mathrm{d}^2x\, \mathrm{D}_+\left(J^\alpha(\phi)\psi_{\alpha-}\right) + \text{h.c.}\,,\\ &= \int \mathrm{d}^2x \left(J^\alpha(\phi)G_\alpha + \overline{G}^\alpha \overline{J}_\alpha(\overline{\phi}) - \frac{\partial J^\alpha}{\partial \phi_j}\xi_{j+}\psi_{\alpha-} - \overline{\psi}^\alpha_- \frac{\partial \overline{J}_\alpha}{\partial \overline{\phi}^j}\overline{\xi}^j_+\right)\,. \end{aligned} \tag{A.14}$$

Since the superspace measure in the J-term involves only half the supercovariant derivatives, its invariance under $(0,2)$ supersymmetry requires the integrand to be chiral, i.e., $\overline{\mathrm{D}}_+(\psi_{\alpha-}J^\alpha) = 0$. This implies

$$E\cdot J := \sum_\alpha E_\alpha J^\alpha = 0\,. \tag{A.15}$$

If the above constraint is not satisfied, supersymmetry is softly broken down from $(0,2)$ to $(0,1)$, even though the J-term is written in $(0,2)$ superspace.

Reduction to $\mathcal{N} = (0,1)$ superspace. Define the derivatives

$$\mathcal{D}_+ = \frac{\mathrm{D}_+ + \overline{\mathrm{D}}_+}{\sqrt{2}}\,,\ \mathcal{Q}_+ = \frac{\mathrm{D}_+ - \overline{\mathrm{D}}_+}{\sqrt{2}} \quad \text{with} \quad \mathcal{D}_+^2 = \mathrm{i}\partial_{++}\,,\ \mathcal{Q}_+^2 = -\mathrm{i}\partial_{++}\,,\ \{\mathcal{D}_+, \mathcal{Q}_+\} = 0\,. \tag{A.16}$$

$\mathcal{D}_+$ is the real $(0,1)$ super derivative and $\mathcal{Q}_+$ is the generator of the extra (non-manifest) supersymmetry.

The $(0,2)$ chiral and fermi multiplets (and their antichiral counterparts) become complex $(0,1)$ scalar and fermi multiplets with components

$$\begin{aligned}\text{Chiral}: &\quad \phi\,,\quad \mathcal{D}_+\phi = \xi_+\,,\quad \mathcal{D}_+\overline{\phi} = -\overline{\xi}_+\,,\\ \text{Fermi}: &\quad \psi_-\,,\quad \mathcal{D}_+\psi_- = G + E =: F\,,\quad \mathcal{D}_+\overline{\psi}_- = \overline{G} + \overline{E} =: \overline{F}\,.\end{aligned} \tag{A.17}$$

We have $\mathrm{D}_+\overline{\mathrm{D}}_+ = -\mathrm{i}\mathcal{D}_+\mathcal{Q}_+ + \mathrm{i}\partial_{++}$. We can discard the second term since it gives rise to a total derivative term in the action. Using that $\mathcal{Q}_+$ acts as $-\mathrm{i}\mathcal{D}_+$ on superfields satisfying $\overline{\mathrm{D}}_+(\cdot) = 0$, we can write the $(0,2)$ actions in $(0,1)$ superspace:

$$\begin{aligned}\mathcal{S}_{\text{chiral}} &= \frac{\mathrm{i}}{2}\int \mathrm{d}^2x\, \mathcal{D}_+\left(-\mathcal{D}_+\overline{\phi}^i\,\partial_{--}\phi_i - \partial_{--}\overline{\phi}^i\mathcal{D}_+\phi_i\right)\,,\\ \mathcal{S}_{\text{fermi}} &= \int \mathrm{d}^2x\,\mathcal{D}_+\left(\overline{\psi}^\alpha_-\left(\frac{1}{2}\mathcal{D}_+\psi_{\alpha-} - \mu_\alpha\right) + \left(\frac{1}{2}\mathcal{D}_+\overline{\psi}^\alpha - \overline{\mu}^\alpha\right)\psi_{\alpha-}\right)\,,\end{aligned} \tag{A.18}$$

where $\mu_\alpha = E_\alpha + \overline{J}_\alpha$.

B (4,4) projective superspace and (4,4) → (0,4)

B.1 Definitions

We start with the $(4,4)$ real supercharges $(\mathcal{Q}_{m+}, \mathcal{Q}_{\tilde{m}-})$ with $m,\ \tilde{m} = 1,2,3,4$. The R-symmetry group is

$$\mathrm{Spin}(4)_L \times \mathrm{Spin}(4)_R \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)'_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)''_R\,.$$

We restrict our attention to the subgroup $F \times F' \times F''$ where $F = \mathrm{SU}(2)_\Delta$, the diagonal subgroup of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, $F' = \mathrm{SU}(2)'_L$ and $F'' = \mathrm{SU}(2)''_R$. The supercharges can then be written as $(\mathcal{Q}_{aa'+}, \mathcal{Q}_{aa''-})$ where a, a' and a'' are doublet indices of F, F' and F'' respectively. This restriction of the R-symmetry group to a subgroup seems to be required to obtain the vector multiplet via gauged supercovariant derivatives and the relevant superspace constraints [70].

The algebra of $(4,4)$ supercovariant derivatives $\mathrm{D}_{aa'+}$ and $\mathrm{D}_{aa''-}$ is

$$\{\mathrm{D}_{aa'+}\,,\mathrm{D}_{bb'+}\} = 2\mathrm{i}\varepsilon_{ab}\varepsilon_{a'b'}\partial_{++}\,,\ \{\mathrm{D}_{aa''-}\,,\mathrm{D}_{bb''-}\} = 2\mathrm{i}\varepsilon_{ab}\varepsilon_{a''b''}\partial_{--}\,,\ \{\mathrm{D}_{aa'+}\,,\mathrm{D}_{bb''-}\} = 0\ . \tag{B.1}$$

The reality conditions on the derivatives are

$$\mathrm{D}_{aa'\pm} = \overline{\mathrm{D}}{}^{bb'}_{\pm}\,\varepsilon_{ba}\varepsilon_{b'a'}\ . \tag{B.2}$$

It will be useful to define the $(2,2)$ subalgebra spanned by the derivatives

$$\mathrm{D}_+ := \mathrm{D}_{11'+}\,,\quad \mathrm{D}_- := \mathrm{D}_{11''-}\,,\quad \overline{\mathrm{D}}_+ := \mathrm{D}_{22'+}\,,\quad \overline{\mathrm{D}}_- := \mathrm{D}_{22''-}\,, \tag{B.3}$$

which satisfy

$$\{\mathrm{D}_\pm, \overline{\mathrm{D}}_\pm\} = 2\mathrm{i}\partial_{\pm\pm}\ . \tag{B.4}$$

The non-manifest $(4,4)$ supersymmetry generators are then $\mathrm{Q}_+ := \mathrm{D}_{21'+}$, $\overline{\mathrm{Q}}_+ := -\mathrm{D}_{12'+}$ and $\mathrm{Q}_- := \mathrm{D}_{21''-}$, $\overline{\mathrm{Q}}_- := -\mathrm{D}_{12''-}$.

The general projective superspace corresponding to $F \times F' \times F''$ is described by introducing a doublet for each of the SU(2)s in the R-symmetry group: $u^a = (\zeta, 1)$, $v^{a'} = (\zeta', 1)$ and $w^{a''} = (\zeta'', 1)$ for the subgroups $F = \mathrm{SU}(2)_\Delta$, $F' = \mathrm{SU}(2)'_L$ and $F'' = \mathrm{SU}(2)'_R$ respectively.

We then define the following projective supercovariant derivatives:

$$\begin{aligned}
\mathbf{D}_{a'+} &:= u^a\mathrm{D}_{aa'+}\,, &&\text{i.e.,}\quad \mathbf{D}_{1'+} = \zeta\mathrm{D}_+ + \mathrm{Q}_+\,, && \mathbf{D}_{2'+} = -\zeta\overline{\mathrm{Q}}_+ + \overline{\mathrm{D}}_+\,,\\
\mathbf{D}_{a+} &:= v^{a'}\mathrm{D}_{aa'+}\,, &&\text{i.e.,}\quad \mathbf{D}_{1+} = \zeta'\mathrm{D}_+ - \overline{\mathrm{Q}}_+\,, && \mathbf{D}_{2+} = \zeta'\mathrm{Q}_+ + \overline{\mathrm{D}}_+\,,\\
\mathbf{D}_{a''-} &:= u^a\mathrm{D}_{aa''-}\,, &&\text{i.e.,}\quad \mathbf{D}_{1''-} = \zeta\mathrm{D}_- + \mathrm{Q}_-\,, && \mathbf{D}_{2''-} = -\zeta\overline{\mathrm{Q}}_- + \overline{\mathrm{D}}_-\,,\\
\mathbf{D}_{a-} &:= w^{a''}\mathrm{D}_{aa''-}\,, &&\text{i.e.,}\quad \mathbf{D}_{1-} = \zeta''\mathrm{D}_- - \overline{\mathrm{Q}}_-\,, && \mathbf{D}_{2-} = \zeta''\mathrm{Q}_- + \overline{\mathrm{D}}_-\ .
\end{aligned} \tag{B.5}$$

We also introduce the doublets $\widetilde{u}^a$, $\widetilde{v}^{a'}$ and $\widetilde{w}^{a''}$ as was done in the main text above eq. (2.9). We again choose $\widetilde{u}^a = (1,0)$, $\widetilde{v}^{a'} = (1,0)$ and $\widetilde{w}^{a''} = (1,0)$ and define the linearly independent derivatives:

$$\begin{aligned}
\widetilde{\mathbf{D}}_{a'+} &:= \widetilde{u}^a\mathrm{D}_{aa'+}\,, &&\text{i.e.,}\quad \widetilde{\mathbf{D}}_{1'+} = \mathrm{D}_+\,, && \widetilde{\mathbf{D}}_{2'+} = -\overline{\mathrm{Q}}_+\,,\\
\widetilde{\mathbf{D}}_{a+} &:= \widetilde{v}^{a'}\mathrm{D}_{aa'+}\,, &&\text{i.e.,}\quad \widetilde{\mathbf{D}}_{1+} = \mathrm{D}_+\,, && \widetilde{\mathbf{D}}_{2+} = \mathrm{Q}_+\,,\\
\widetilde{\mathbf{D}}_{a''-} &:= \widetilde{u}^a\mathrm{D}_{aa''-}\,, &&\text{i.e.,}\quad \widetilde{\mathbf{D}}_{1''-} = \mathrm{D}_-\,, && \widetilde{\mathbf{D}}_{2''-} = -\overline{\mathrm{Q}}_-\,,\\
\widetilde{\mathbf{D}}_{a-} &:= \widetilde{w}^{a''}\mathrm{D}_{aa''-}\,, &&\text{i.e.,}\quad \widetilde{\mathbf{D}}_{1-} = \mathrm{D}_-\,, && \widetilde{\mathbf{D}}_{2-} = \mathrm{Q}_-\ .
\end{aligned} \tag{B.6}$$

We consider projective superfields which are functions of one projective coordinate from the left moving sector (ζ or ζ') and one projective coordinate from the right-moving sector (ζ

or ζ'') and are annihilated by the corresponding set of projective derivatives. For example, an (F, F'') projective superfield Φ is a function of ζ and ζ'' and is annihilated by $\mathbf{D}_{a'+}(\zeta)$ and $\mathbf{D}_{a-}(\zeta'')$. The $(4,4)$ supersymmetric action is

$$\mathcal{S}[\mathbf{\Phi}] = \int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \oint_{\gamma''} \frac{\mathrm{d}\zeta''}{2\pi\mathrm{i}} \widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\widetilde{\mathbf{D}}_{1-}\widetilde{\mathbf{D}}_{2-} \boldsymbol{K}(\mathbf{\Phi}) \ . \tag{B.7}$$

Using that $\boldsymbol{K}(\mathbf{\Phi})$ is annihilated by $\mathbf{D}_{a'+}$ and $\mathbf{D}_{a-}$ and $\widetilde{\mathbf{D}}_{2'+} = \zeta^{-1}\mathbf{D}_{2'+} - \zeta^{-1}\overline{\mathrm{D}}_+$, $\widetilde{\mathbf{D}}_{2-} = \zeta''^{-1}\mathbf{D}_{2-} - \zeta''^{-1}\overline{\mathrm{D}}_-$, we can replace the measure by the $(2,2)$ measure and do the ζ, ζ'' integrals to get an action in $(2,2)$ superspace:

$$\mathcal{S}[\mathbf{\Phi}] = \int \mathrm{d}^2x\, \mathrm{D}_+\overline{\mathrm{D}}_+\mathrm{D}_-\overline{\mathrm{D}}_- \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \oint_{\gamma''} \frac{\mathrm{d}\zeta''}{2\pi\mathrm{i}\zeta''} \boldsymbol{K}(\mathbf{\Phi}) \ . \tag{B.8}$$

There are many choices for projective superfields: they can be a polynomial or a power series in each of the projective coordinates that they depend on. A polynomial $\mathcal{O}(n)$ superfield with respect to F, F' and F'' will be respectively denoted as $\mathcal{O}(n)$, $\mathcal{O}(n')$ and $\mathcal{O}(n'')$. Power series superfields are typically denoted as F-arctic, F-antarctic, F'-arctic and so on. Below, we discuss the (F, F) arctic superfield, i.e., an arctic superfield which is a power series only in ζ and is annihilated by $\mathbf{D}_{a'+}(\zeta)$ and $\mathbf{D}_{a''-}(\zeta)$.

B.2 $(4,4)$ standard hypermultiplet

Consider an (F, F) arctic superfield $\mathbf{\Upsilon}(\zeta) = \sum_{i=0}^{\infty} \Upsilon_i \zeta^i$ with alternate notation Φ and Σ for Υ_0 and Υ_1 respectively. The constraints $\mathbf{D}_{a'+}\mathbf{\Upsilon} = \mathbf{D}_{a''-}\mathbf{\Upsilon} = 0$ give the $(2,2)$ constraints

$$\begin{gathered}\mathrm{Q}_\pm\Phi = 0 \,, \quad \overline{\mathrm{D}}_\pm\Phi = 0 \,, \quad \overline{\mathrm{Q}}_\pm\Phi = \overline{\mathrm{D}}_\pm\Sigma \Rightarrow \overline{\mathrm{D}}_+\overline{\mathrm{D}}_-\Sigma = 0 \,, \\ \mathrm{Q}_\pm\Upsilon_{j+1} = -\mathrm{D}_\pm\Upsilon_j \text{ for } j \geq 0 \ , \quad \text{and} \quad \overline{\mathrm{Q}}_\pm\Upsilon_j = \overline{\mathrm{D}}_\pm\Upsilon_{j+1} \text{ for } j \geq 1 \ .\end{gathered} \tag{B.9}$$

Φ is chiral as an $(2,2)$ superfield since $\overline{\mathrm{D}}_\pm\Phi = 0$, Σ is complex linear since $\overline{\mathrm{D}}_+\overline{\mathrm{D}}_-\Sigma = 0$, whereas the $\Upsilon_{j\geq 2}$ are unconstrained as $(2,2)$ superfields.

The action for the arctic superfield is

$$\mathcal{S}[\mathbf{\Upsilon}] = \frac{1}{4} \int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\widetilde{\mathbf{D}}_{1''-}\widetilde{\mathbf{D}}_{2''-} (\zeta\overline{\mathbf{\Upsilon}}\mathbf{\Upsilon}) \ . \tag{B.10}$$

This action is R-symmetric since the measure has F-weight -2 ($+2$ from $\mathrm{d}\zeta$, -2 from $\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}$ and -2 from $\widetilde{\mathbf{D}}_{1''-}\widetilde{\mathbf{D}}_{2''-}$) and the Lagrangian has F-weight $+2$ ($+1$ each from $\mathbf{\Upsilon}$ and $\zeta\overline{\mathbf{\Upsilon}}$, see the paragraph after equation (2.47) in section 2.5).

Next, we obtain the $(0,4)$ content by applying $\widetilde{\mathbf{D}}_{a''-}$ to $\mathbf{\Upsilon}$:

$$\mathbf{\Upsilon}_{a''-} \equiv \frac{1}{\sqrt{2}}\widetilde{\mathbf{D}}_{a''-}\mathbf{\Upsilon} \,, \quad \mathbf{\Upsilon}_{--} \equiv -\frac{1}{4}\widetilde{\mathbf{D}}_{1''-}\widetilde{\mathbf{D}}_{2''-}\mathbf{\Upsilon} \ . \tag{B.11}$$

Recall from (2.80) that the conjugate of $\widetilde{\mathbf{D}}_{a''-}$ when acting on arctic superfields is

$$\breve{\widetilde{\mathbf{D}}}{}^{a''}_{-} = \varepsilon^{a''b''}(-\zeta\widetilde{\mathbf{D}}_{b''-} + \mathbf{D}_{b''-}) \ . \tag{B.12}$$

Using this, we get

$$\overline{\mathbf{\Upsilon}}^{a''}_{-} = \frac{1}{\sqrt{2}}\varepsilon^{a''b''}(\zeta\widetilde{\mathbf{D}}_{b''-} - \mathbf{D}_{b''-})\overline{\mathbf{\Upsilon}}\,, \quad \overline{\mathbf{\Upsilon}}_{--} = \frac{1}{4}(-\zeta\widetilde{\mathbf{D}}_{2''-} + \mathbf{D}_{2''-})(\zeta\widetilde{\mathbf{D}}_{1''-} - \mathbf{D}_{1''-})\overline{\mathbf{\Upsilon}}\,. \tag{B.13}$$

Using $\mathbf{D}_{a''-}\overline{\mathbf{\Upsilon}} = 0$ and $\{\mathbf{D}_{2''-}\,, \widetilde{\mathbf{D}}_{1''-}\} = 2\mathrm{i}\partial_{--}$, we get

$$\overline{\mathbf{\Upsilon}}^{a''}_{-} = \frac{1}{\sqrt{2}}\zeta\varepsilon^{a''b''}\widetilde{\mathbf{D}}_{b''-}\overline{\mathbf{\Upsilon}}\,, \quad \overline{\mathbf{\Upsilon}}_{--} = \frac{1}{4}\zeta^2\widetilde{\mathbf{D}}_{1''-}\widetilde{\mathbf{D}}_{2''-}\overline{\mathbf{\Upsilon}} + \frac{\mathrm{i}}{2}\zeta\partial_{--}\overline{\mathbf{\Upsilon}}\,. \tag{B.14}$$

The $(0,4)$ supersymmetric action is obtained by pushing in the $\widetilde{\mathbf{D}}_{a''-}$ derivatives in the measure:

$$\begin{aligned} \mathcal{S}[\mathbf{\Upsilon}] &= \frac{1}{4}\int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\widetilde{\mathbf{D}}_{1''-}\widetilde{\mathbf{D}}_{2''-}\,(\zeta\overline{\mathbf{\Upsilon}}\mathbf{\Upsilon})\,, \\ &= \int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\Big(\zeta^{-1}\overline{\mathbf{\Upsilon}}_{--}\mathbf{\Upsilon} - \frac{\mathrm{i}}{2}(\partial_{--}\overline{\mathbf{\Upsilon}})\mathbf{\Upsilon} - \zeta\overline{\mathbf{\Upsilon}}\mathbf{\Upsilon}_{--} \\ &\qquad - \frac{1}{2}\overline{\mathbf{\Upsilon}}^{1''}_{-}\mathbf{\Upsilon}_{1''-} - \frac{1}{2}\overline{\mathbf{\Upsilon}}^{2''}_{-}\mathbf{\Upsilon}_{2''-}\Big)\,. \end{aligned} \tag{B.15}$$

Let us study the R-symmetry invariance of the above action in more detail. Recall that $\mathbf{\Upsilon}$ and $\widetilde{\mathbf{D}}_{a''-}$ (see (2.80)) transform under F as

$$\mathbf{\Upsilon}(\zeta) \to \mathbf{\Upsilon}'(\zeta) = j(g,\zeta)\mathbf{\Upsilon}(g\cdot\zeta)\,, \quad \widetilde{\mathbf{D}}_{a''-} \to j(g,\zeta)^{-1}\widetilde{\mathbf{D}}_{a''-}(g\cdot\zeta) - \bar{b}\mathbf{D}_{a''-}(g\cdot\zeta)\,, \tag{B.16}$$

where $\mathbf{\Upsilon}'$ is a new superfield which is evaluated at ζ whose expression is given by expanding the right hand side $j(g,\zeta)\mathbf{\Upsilon}(g\cdot\zeta)$ around $\zeta = 0$. The transformations of all the other superfields can be obtained by using the above. We first summarize the results and then detailed calculations. The hypers $\mathbf{\Upsilon}$, $\zeta\overline{\mathbf{\Upsilon}}$ transform as weight 1 objects:

$$\mathbf{\Upsilon}(\zeta) \to j(g,\zeta)\mathbf{\Upsilon}(g\cdot\zeta)\,, \quad \zeta\overline{\mathbf{\Upsilon}}(-\zeta^{-1}) \to j(g,\zeta)\ (g\cdot\zeta)\ \overline{\mathbf{\Upsilon}}(-(g\cdot\zeta)^{-1})\,, \tag{B.17}$$

the fermis $\mathbf{\Upsilon}_{a''-}$, $\overline{\mathbf{\Upsilon}}^{a''}_{-}$ transform as weight 0 objects:

$$\mathbf{\Upsilon}_{a''-}(\zeta) \to \mathbf{\Upsilon}_{a''-}(g\cdot\zeta)\,, \quad \overline{\mathbf{\Upsilon}}^{a''}_{-}(-\zeta^{-1}) \to \overline{\mathbf{\Upsilon}}^{a''}_{-}(-(g\cdot\zeta)^{-1})\,, \tag{B.18}$$

and $\mathbf{\Upsilon}_{--}$, $\zeta^{-1}\overline{\mathbf{\Upsilon}}_{--}$ transform as weight -1 objects, along with an additional shift:

$$\begin{aligned} \mathbf{\Upsilon}_{--}(\zeta) &\to j(g,\zeta)^{-1}\mathbf{\Upsilon}_{--}(g\cdot\zeta) - \frac{\mathrm{i}}{2}\bar{b}\partial_{--}\mathbf{\Upsilon}(g\cdot\zeta)\,, \\ \zeta^{-1}\overline{\mathbf{\Upsilon}}'_{--}(-\zeta^{-1}) &\to j(g,\zeta)^{-1}(g\cdot\zeta)^{-1}\overline{\mathbf{\Upsilon}}_{--}(-(g\cdot\zeta)^{-1}) + \zeta^{-1}b\frac{\mathrm{i}}{2}\partial_{--}\overline{\mathbf{\Upsilon}}(-(g\cdot\zeta)^{-1})\,. \end{aligned} \tag{B.19}$$

Using these, the $(0,4)$ supersymmetric action (B.15) can be checked to be R-symmetric, a fact which was already demonstrated for the $(4,4)$ action (B.10).

The derivation of R-symmetry transformations.

Note: in the following calculations, a $'$ on superfields denotes the transformed superfield and must not be confused with the $'$ on the R-symmetry indices.

Given the transformation of $\boldsymbol{\Upsilon}$ in (B.16), the transformation of $\overline{\boldsymbol{\Upsilon}}$ is

$$\begin{aligned}\overline{\boldsymbol{\Upsilon}}(-\zeta^{-1}) \to \overline{\boldsymbol{\Upsilon}}'(-\zeta^{-1}) &= (a+b\zeta^{-1})\overline{\boldsymbol{\Upsilon}}\left(\frac{-\bar{a}\zeta^{-1}+\bar{b}}{a+b\zeta^{-1}}\right)\\ &= \frac{1}{\zeta}\times(\bar{a}-\bar{b}\zeta)\times\frac{a\zeta+b}{\bar{a}-\bar{b}\zeta}\times\overline{\boldsymbol{\Upsilon}}\left(\frac{-\bar{a}\zeta^{-1}+\bar{b}}{a+b\zeta^{-1}}\right),\end{aligned} \tag{B.20}$$

which implies that

$$\zeta\overline{\boldsymbol{\Upsilon}}(-\zeta^{-1}) \to \zeta\overline{\boldsymbol{\Upsilon}}'(-\zeta^{-1}) = j(g,\zeta)\ (g\cdot\zeta)\ \overline{\boldsymbol{\Upsilon}}(-(g\cdot\zeta)^{-1})\,. \tag{B.21}$$

This tells us that $\zeta\overline{\boldsymbol{\Upsilon}}$ transforms as a weight 1 field as well. $\boldsymbol{\Upsilon}_{--}$ transforms as

$$\begin{aligned}&\boldsymbol{\Upsilon}_{--}(\zeta)\\ &\to -\frac{1}{4}\Big(j(g,\zeta)^{-1}\widetilde{\mathbf{D}}_{1''-}(g\cdot\zeta)-\bar{b}\mathbf{D}_{1''-}(g\cdot\zeta)\Big)\Big(j(g,\zeta)^{-1}\widetilde{\mathbf{D}}_{2''-}(g\cdot\zeta)\Big)\Big(j(g,\zeta)\boldsymbol{\Upsilon}(g\cdot\zeta)\Big),\\ &= j(g,\zeta)^{-1}\boldsymbol{\Upsilon}_{--}(g\cdot\zeta)-\bar{b}\frac{\mathrm{i}}{2}\partial_{--}\boldsymbol{\Upsilon}(g\cdot\zeta),\end{aligned} \tag{B.22}$$

that is,

$$\boldsymbol{\Upsilon}_{--}(\zeta) \to \boldsymbol{\Upsilon}'_{--}(\zeta) = j(g,\zeta)^{-1}\boldsymbol{\Upsilon}_{--}(g\cdot\zeta)-\bar{b}\frac{\mathrm{i}}{2}\partial_{--}\boldsymbol{\Upsilon}(g\cdot\zeta)\,. \tag{B.23}$$

Thus, $\boldsymbol{\Upsilon}_{--}$ transforms as a weight -1 superfield but with an additional shift term proportional to $\partial_{--}\boldsymbol{\Upsilon}$. Finally, we need the transformation of $\overline{\boldsymbol{\Upsilon}}_{--}$. Analogous to (B.20), we have

$$\begin{aligned}\overline{\boldsymbol{\Upsilon}}'_{--}(-\zeta^{-1}) &= \frac{1}{a+b\zeta^{-1}}\overline{\boldsymbol{\Upsilon}}_{--}\left(\frac{-\bar{a}\zeta^{-1}+\bar{b}}{a+b\zeta^{-1}}\right)+b\frac{\mathrm{i}}{2}\partial_{--}\overline{\boldsymbol{\Upsilon}}\left(\frac{-\bar{a}\zeta^{-1}+\bar{b}}{a+b\zeta^{-1}}\right),\\ &= \zeta\frac{\bar{a}-\bar{b}\zeta}{a\zeta+b}\frac{1}{\bar{a}-\bar{b}\zeta}\overline{\boldsymbol{\Upsilon}}_{--}\left(\frac{-\bar{a}\zeta^{-1}+\bar{b}}{a+b\zeta^{-1}}\right)+b\frac{\mathrm{i}}{2}\partial_{--}\overline{\boldsymbol{\Upsilon}}\left(\frac{-\bar{a}\zeta^{-1}+\bar{b}}{a+b\zeta^{-1}}\right),\end{aligned} \tag{B.24}$$

which gives

$$\zeta^{-1}\overline{\boldsymbol{\Upsilon}}'_{--}(-\zeta^{-1}) = j(g,\zeta)^{-1}(g\cdot\zeta)^{-1}\overline{\boldsymbol{\Upsilon}}_{--}(-(g\cdot\zeta)^{-1})+\zeta^{-1}b\frac{\mathrm{i}}{2}\partial_{--}\overline{\boldsymbol{\Upsilon}}(-(g\cdot\zeta)^{-1})\,. \tag{B.25}$$

Again, we see that $\zeta^{-1}\overline{\boldsymbol{\Upsilon}}_{--}$ transforms as a weight -1 superfield, along with an additional shift term proportional to $\partial_{--}\overline{\boldsymbol{\Upsilon}}$. We can also start with the definition of $\overline{\boldsymbol{\Upsilon}}_{--}$ in (B.14) and arrive at the above result. In detail, we have

$$\begin{aligned}&\zeta\widetilde{\mathbf{D}}_{1''-}\widetilde{\mathbf{D}}_{2''-}\overline{\boldsymbol{\Upsilon}}(-\zeta^{-1})\\ &\quad\to \zeta\widetilde{\mathbf{D}}_{1''-}\widetilde{\mathbf{D}}_{2''-}\overline{\boldsymbol{\Upsilon}}'(-\zeta^{-1})\\ &\quad= \zeta\Big(j(g,\zeta)^{-1}\widetilde{\mathbf{D}}_{1''-}(g\cdot\zeta)-\bar{b}\mathbf{D}_{1''-}(g\cdot\zeta)\Big)\Big(j(g,\zeta)^{-1}\widetilde{\mathbf{D}}_{2''-}(g\cdot\zeta)\Big)\frac{a\zeta+b}{\zeta}\overline{\boldsymbol{\Upsilon}}(-(g\cdot\zeta)^{-1})\\ &\quad= j(g,\zeta)^{-1}\ g\cdot\zeta\ \widetilde{\mathbf{D}}_{1''-}\widetilde{\mathbf{D}}_{2''-}\overline{\boldsymbol{\Upsilon}}(-(g\cdot\zeta)^{-1})+2\mathrm{i}\bar{b}\ g\cdot\zeta\ \partial_{--}\overline{\boldsymbol{\Upsilon}}(-(g\cdot\zeta)^{-1})\,.\end{aligned} \tag{B.26}$$

Also using the transformation of $\overline{\boldsymbol{\Upsilon}}$ from (B.21), we get

$$\begin{aligned}
&\zeta\widetilde{\mathbf{D}}_{1''-}\widetilde{\mathbf{D}}_{2''-}\overline{\boldsymbol{\Upsilon}}'(-\zeta^{-1}) + 2\mathrm{i}\partial_{--}\overline{\boldsymbol{\Upsilon}}'(-\zeta^{-1}) \\
&= j(g,\zeta)^{-1}\ g\cdot\zeta\ \widetilde{\mathbf{D}}_{1''-}\widetilde{\mathbf{D}}_{2''-}\overline{\boldsymbol{\Upsilon}}(-(g\cdot\zeta)^{-1}) + 2\mathrm{i}\left(\bar{b}\ g\cdot\zeta + \frac{a\zeta+b}{\zeta}\right)\partial_{--}\overline{\boldsymbol{\Upsilon}}(-(g\cdot\zeta)^{-1}) \\
&= j(g,\zeta)^{-1}\left(g\cdot\zeta\ \widetilde{\mathbf{D}}_{1''-}\widetilde{\mathbf{D}}_{2''-}\overline{\boldsymbol{\Upsilon}}(-(g\cdot\zeta)^{-1}) + 2\mathrm{i}\partial_{--}\overline{\boldsymbol{\Upsilon}}_{--}(-g\cdot\zeta)^{-1})\right) \\
&\quad + 2\mathrm{i}\left(\bar{b}\ g\cdot\zeta + \frac{a\zeta+b}{\zeta} - j(g,\zeta)^{-1}\right)\partial_{--}\overline{\boldsymbol{\Upsilon}}(-(g\cdot\zeta)^{-1})\ .
\end{aligned} \tag{B.27}$$

The quantity in the parentheses in the last line simplifies to give b/ζ. Plugging this into (B.27) and dividing by 4, we get (B.25).

The fermis $\boldsymbol{\Upsilon}_{a''-}$ transform as weight 0 objects:

$$\begin{aligned}
\boldsymbol{\Upsilon}_{a''-}(\zeta) &\to \left(j(g,\zeta)^{-1}\widetilde{\mathbf{D}}_{a''-}(g\cdot\zeta) - \bar{b}\mathbf{D}_{a''-}(g\cdot\zeta)\right)\left(j(g,\zeta)\boldsymbol{\Upsilon}(g\cdot\zeta)\right) \\
&= \widetilde{\mathbf{D}}_{a''-}(g\cdot\zeta)\boldsymbol{\Upsilon}(g\cdot\zeta) = \boldsymbol{\Upsilon}_{a''-}(g\cdot\zeta)\ .
\end{aligned} \tag{B.28}$$

The conjugates $\overline{\boldsymbol{\Upsilon}}^{a''}_{-}$ also transform with weight 0, a fact which can be seen either by complex conjugating the expressions in (B.28) or by direct calculation using the expression for $\overline{\boldsymbol{\Upsilon}}^{a''}_{-}$ in (B.14):

$$\begin{aligned}
\zeta\widetilde{\mathbf{D}}_{a''-}\overline{\boldsymbol{\Upsilon}}(-\zeta^{-1}) \to \zeta\widetilde{\mathbf{D}}_{a''-}\overline{\boldsymbol{\Upsilon}}'(-\zeta^{-1}) & \\
&= \zeta\left(j(g,\zeta)^{-1}\widetilde{\mathbf{D}}_{a''-}(g\cdot\zeta) - \bar{b}\mathbf{D}_{a''-}(g\cdot\zeta)\right)\frac{a\zeta+b}{\zeta}\overline{\boldsymbol{\Upsilon}}(-(g\cdot\zeta)^{-1}) \\
&= g\cdot\zeta\widetilde{\mathbf{D}}_{a''-}(g\cdot\zeta)\overline{\boldsymbol{\Upsilon}}(-(g\cdot\zeta)^{-1})\,,
\end{aligned} \tag{B.29}$$

which gives

$$\overline{\boldsymbol{\Upsilon}}^{a''}_{-}(-\zeta^{-1}) \to \overline{\boldsymbol{\Upsilon}}^{a''}_{-}(-(g\cdot\zeta)^{-1})\ . \tag{B.30}$$

C Component actions

In this appendix, we derive the action for the ordinary space components of the various superfields in two ways: (1) by reducing to $(0,2)$ superspace and using standard results from appendix A.2, and (2) by reducing directly to ordinary space by pushing in the $\widetilde{\mathbf{D}}_{a'+}$ in the superspace measure.

C.1 $(0,4) \to (0,2) \to (0,0)$

Recall from sections 3 and 4 the ζ and ζ' expansions of the various superfields:

$$\boldsymbol{\Upsilon} = \sum_{j=0}^{\infty}\Upsilon_j\,,\quad \boldsymbol{\Upsilon}_{--} = \sum_{j=0}^{\infty}\Upsilon_{j--}\,,\quad \boldsymbol{\Upsilon}_{-} = \sum_{j=0}^{\infty}\Upsilon_{j-}\,,\quad \boldsymbol{H} = \zeta' H_{1'} + H_{2'}\ . \tag{C.1}$$

Also recall that we relabelled some low-lying components of the above superfields since they were constrained as $(0,2)$ superfields:

$$\Upsilon_0 \to \eta_2\,,\quad \Upsilon_1 \to \eta_1\,,\quad \text{and}\quad \Upsilon_{0-} \to \psi_-\ . \tag{C.2}$$

We reproduce here the projective superspace constraints on the various superfields given in (5.1):

$$\mathbf{D}_+\boldsymbol{\Upsilon}=0\,,\quad \mathbf{D}_+\boldsymbol{\Upsilon}_-=-\sqrt{2}\widehat{\boldsymbol{C}}\boldsymbol{\Upsilon}\,,\quad \mathbf{D}_+\boldsymbol{\Upsilon}_{--}=\frac{1}{\sqrt{2}}\boldsymbol{C}\boldsymbol{\Upsilon}_-\,,\quad \mathbf{D}_+\boldsymbol{H}=0\,,$$
$$\mathbf{D}_+\overline{\boldsymbol{\Upsilon}}=0\,,\quad \mathbf{D}_+\overline{\boldsymbol{\Upsilon}}_-=\sqrt{2}\zeta\overline{\boldsymbol{\Upsilon}}\overline{\widehat{\boldsymbol{C}}}\,,\quad \mathbf{D}_+\overline{\boldsymbol{\Upsilon}}_{--}=\frac{1}{\sqrt{2}}\zeta\overline{\boldsymbol{\Upsilon}}_-\overline{\boldsymbol{C}}\,,\quad \mathbf{D}_+\overline{\boldsymbol{H}}=0\,,\tag{C.3}$$

The actions are given by

$$\mathcal{S}_F=\int \mathrm{d}^2x\oint\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\left(\frac{\mathrm{i}}{2}\overline{\boldsymbol{\Upsilon}}\partial_{--}\boldsymbol{\Upsilon}-\zeta\overline{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}_{--}+\zeta^{-1}\overline{\boldsymbol{\Upsilon}}_{--}\boldsymbol{\Upsilon}-\frac{1}{2}\overline{\boldsymbol{\Upsilon}}_-\boldsymbol{\Upsilon}_-\right)\,,$$
$$\mathcal{S}_{F'}=\int \mathrm{d}^2x\oint\frac{\mathrm{d}\zeta'}{2\pi\mathrm{i}}\widetilde{\mathbf{D}}_{1+}\widetilde{\mathbf{D}}_{2+}\left(-\frac{\mathrm{i}}{2}\zeta'^{-1}\overline{\boldsymbol{H}}\partial_{--}\boldsymbol{H}\right)\,,\tag{C.4}$$

The closure of the projective superspace algebra $\mathbf{D}_+^2=0$ on $\boldsymbol{\Upsilon}_{--}$ gives the constraints

$$\mathbf{D}_+\boldsymbol{C}=\mathbf{D}_+\widehat{\boldsymbol{C}}=0\,,\quad \boldsymbol{C}\widehat{\boldsymbol{C}}=0\,,\quad \text{i.e.,}\quad \mathbf{D}_{a+}\boldsymbol{C}=\mathbf{D}_{a+}\widehat{\boldsymbol{C}}=0\,,\quad C_{(a'}\widehat{C}_{b')}=0\,,\tag{C.5}$$

and the $(0,4)$ invariance of the above actions gives

$$\overline{\boldsymbol{C}}=\widehat{\boldsymbol{C}}\,,\quad \text{that is}\,,\quad \widehat{C}_{a'}=\overline{C}^{b'}\varepsilon_{b'a'}\ .\tag{C.6}$$

The assumption that the $\boldsymbol{C}$ are polynomials in the various superfields constrains $\boldsymbol{C}$ to take the form $\boldsymbol{C}=\boldsymbol{K}+L\boldsymbol{H}$. The constraints (C.3) lead to the following E-terms for the $(0,2)$ superfield $\psi_-=\Upsilon_{0-}$ and Υ_{0--}:

$$\overline{\mathrm{D}}_+\psi_-=-\sqrt{2}E=-\sqrt{2}\widehat{C}_{2'}\eta_2,\,,\quad \overline{\mathrm{D}}_+\Upsilon_{0--}=\frac{1}{\sqrt{2}}C_{2'}\psi_-\ .\tag{C.7}$$

Integrating out the auxiliary superfield $\boldsymbol{\Upsilon}_{--}$ proceeds in the same way as in the free case, with one important difference due to the E-term for Υ_{0--} above. Unconstraining Υ_{0--} in the standard way (see Footnote 5), we get

$$-\mathrm{D}_+\overline{\mathrm{D}}_+\left(\overline{\Upsilon}_1\Upsilon_{0--}+\Lambda_-\left(\overline{\mathrm{D}}_+\Upsilon_{0--}-\frac{1}{\sqrt{2}}C_{2'}\psi_-\right)\right)\ .\tag{C.8}$$

Integrating out Υ_{0--} gives $\overline{\Upsilon}_1=-\overline{\mathrm{D}}_+\Lambda_-$ which implies that $\overline{\Upsilon}_1$ is a $(0,2)$ chiral superfield which we labelled as $\overline{\eta}^1$. In addition, there is now a $(0,2)$ J-term:

$$-\frac{1}{\sqrt{2}}\mathrm{D}_+\overline{\mathrm{D}}_+(-\Lambda_-C_{2'}\psi_-)=-\frac{1}{\sqrt{2}}\mathrm{D}_+(\overline{\eta}^1C_{2'}\psi_--\sqrt{2}\Lambda_-C_{2'}\widehat{C}_{2'}\eta_2)=-\frac{1}{\sqrt{2}}\mathrm{D}_+(\overline{\eta}^1C_{2'}\psi_-)\,,\tag{C.9}$$

where, in the last equality, we have used the constraint $C_{2'}\widehat{C}_{2'}=0$ that follows from (C.5).

Rewriting the projective superspace measure in (C.4) as $-\zeta^{-1}\mathrm{D}_+\overline{\mathrm{D}}_+$ and performing the ζ- and ζ'-integrals, we get the following $(0,2)$ superspace actions:

$$\mathcal{S}_{F'}=\int\mathrm{d}^2x\mathrm{D}_+\overline{\mathrm{D}}_+\left(\frac{\mathrm{i}}{2}\overline{H}^{1'}\partial_{--}H_{1'}-\frac{\mathrm{i}}{2}\overline{H}^{2'}\partial_{--}H_{2'}\right)\,,$$
$$\mathcal{S}_F=\int\mathrm{d}^2x\mathrm{D}_+\overline{\mathrm{D}}_+\left(\frac{\mathrm{i}}{2}\overline{\eta}^1\partial_{--}\eta_1-\frac{\mathrm{i}}{2}\overline{\eta}^2\partial_{--}\eta_2+\frac{1}{2}\overline{\psi}_-\psi_-\right)+\int\mathrm{d}^2x\mathrm{D}_+\left(-\frac{1}{\sqrt{2}}\overline{\eta}^1C_{2'}\psi_-\right)+\text{c.c.},\tag{C.10}$$

with $\overline{\mathrm{D}}_+\psi_-=\sqrt{2}E=-\sqrt{2}\widehat{C}_{2'}\eta_2$.

Now we further push in the derivatives in the $(0,2)$ actions above and compute the component actions according to appendix A.2. Recall that the superspace components of $H_{a'}$ are

$$\sqrt{2}\xi_{1+} = \mathrm{D}_+ H_{2'}\,, \quad \sqrt{2}\overline{\xi}^{1}_{+} = -\overline{\mathrm{D}}_+ \overline{H}^{2'}\,, \quad \sqrt{2}\xi_{2+} = -\overline{\mathrm{D}}_+ H_{1'}\,, \quad \sqrt{2}\overline{\xi}^{\,2}_{+} = \mathrm{D}_+ \overline{H}^{1'}\,, \tag{C.11}$$

and the superspace components of η_a are

$$\sqrt{2}\xi_{1'+} = \mathrm{D}_+ \eta_2\,, \quad \sqrt{2}\overline{\xi}^{1'}_{+} = -\overline{\mathrm{D}}_+ \overline{\eta}^2\,, \quad -\sqrt{2}\xi_{2'+} = \overline{\mathrm{D}}_+ \eta_1\,, \quad \sqrt{2}\overline{\xi}^{2'}_{+} = \mathrm{D}_+ \overline{\eta}^1\,. \tag{C.12}$$

The components of the fermi ψ_- are

$$\mathrm{D}_+ \psi_- = -\sqrt{2} G\,, \quad \overline{\mathrm{D}}_+ \overline{\psi}_- = -\sqrt{2}\overline{G}\,. \tag{C.13}$$

Let us work out the twisted hyper part of $\mathcal{S}_{F'}$ first. We have

$$\begin{aligned} \mathcal{S}_{F'}[H_{a'}] &= \int \mathrm{d}^2 x\, \mathrm{D}_+ \left(-\frac{\mathrm{i}}{\sqrt{2}} \overline{H}^{1'} \partial_{--} \xi_{2+} + \frac{\mathrm{i}}{\sqrt{2}} \overline{\xi}^1_+ \nabla_{--} H_{2'} \right)\,, \\ &= \int \mathrm{d}^2 x\, (-\overline{\partial_\mu H}^{a'} \partial_\mu H_{a'} - \mathrm{i}\overline{\xi}^a_+ \partial_{--} \xi_{a+})\,. \end{aligned} \tag{C.14}$$

The standard hyper part of $\mathcal{S}_F$ is given by

$$\begin{aligned} \mathcal{S}_F[\eta_a] &= \int \mathrm{d}^2 x\, \mathrm{D}_+ \left(-\frac{\mathrm{i}}{\sqrt{2}} \overline{\eta}^1 \partial_{--} \xi_{2'+} + \frac{\mathrm{i}}{\sqrt{2}} \overline{\xi}^{1'}_+ \partial_{--} \eta_2 \right)\,, \\ &= \int \mathrm{d}^2 x\, (-\overline{\partial_\mu \eta}^a \partial_\mu \eta_a - \mathrm{i}\overline{\xi}^{a'}_+ \partial_{--} \xi_{a'+})\,, \end{aligned} \tag{C.15}$$

whereas the fermi part of $\mathcal{S}_F$ is given by

$$\begin{aligned} \mathcal{S}_F[\psi_-] &= \frac{1}{\sqrt{2}} \int \mathrm{d}^2 x\, \mathrm{D}_+ (-\overline{G}\psi_- - \overline{\psi}_- E) + \int \mathrm{d}^2 x \left(\mathrm{D}_+ \left(-\frac{1}{\sqrt{2}} \overline{\eta}^1 C_{2'} \psi_- \right) + \text{c.c.} \right)\,, \\ &= \int \mathrm{d}^2 x \Bigg(\mathrm{i}(\partial_{++}\overline{\psi}_-)\psi_- + \overline{G}G - \overline{\eta}^2 \overline{\widehat{C}}^{2'} \widehat{C}_{2'} \eta_2 \\ &\qquad + \frac{1}{\sqrt{2}} \overline{\eta}^2 (\overline{\mathrm{D}}_+ \overline{\widehat{C}}^{2'}) \psi_- - \frac{1}{\sqrt{2}} \overline{\psi}_- (\mathrm{D}_+ \widehat{C}_{2'}) \eta_2 - \overline{\xi}^{1'}_+ \overline{\widehat{C}}^{2'} \psi_- - \overline{\psi}_- \widehat{C}_{2'} \xi_{1'+} \Bigg) \\ &\quad + \int \mathrm{d}^2 x \left(-\overline{\xi}^{2'}_+ C_{2'} \psi_- - \frac{1}{\sqrt{2}} \overline{\eta}^1 (\mathrm{D}_+ C_{2'}) \psi_- + \overline{\eta}^1 C_{2'} G + \text{c.c.} \right)\,, \end{aligned} \tag{C.16}$$

Integrating out the auxiliary fields G, $\overline{G}$, we get

$$\overline{G} = -\overline{\eta}^1 C_{2'}\,, \quad G = -\overline{C}^{2'} \eta_1\,, \tag{C.17}$$

and the fermi action becomes

$$\begin{aligned} \mathcal{S}_F[\psi_-] &= \int \mathrm{d}^2 x \Bigg(\mathrm{i}(\mathrm{D}_{++}\overline{\psi}_-)\psi_- - \overline{\eta}^1 C_{2'} \overline{C}^{2'} \eta_1 - \overline{\eta}^2 \overline{\widehat{C}}^{2'} \widehat{C}_{2'} \eta_2 \\ &\qquad + \frac{1}{\sqrt{2}} \overline{\eta}^2 (\overline{\mathrm{D}}_+ \overline{\widehat{C}}^{2'}) \psi_- - \frac{1}{\sqrt{2}} \overline{\psi}_- (\mathrm{D}_+ \widehat{C}_{2'}) \eta_2 - \overline{\xi}^{1'}_+ \overline{\widehat{C}}^{2'} \psi_- - \overline{\psi}_- \widehat{C}_{2'} \xi_{1'+} \Bigg) \\ &\quad + \int \mathrm{d}^2 x \left(-\overline{\xi}^{2'}_+ C_{2'} \psi_- - \frac{1}{\sqrt{2}} \overline{\eta}^1 (\mathrm{D}_+ C_{2'}) \psi_- + \text{c.c.} \right)\,. \end{aligned} \tag{C.18}$$

Let us look at the potential terms:

$$-V = -\overline{\eta}^1 C_{2'}\overline{C}^{2'}\eta_1 - \overline{\eta}^2\overline{\widehat{C}}^{2'}\widehat{C}_{2'}\eta_2 = -\overline{\eta}^1 C_{2'}\overline{C}^{2'}\eta_1 - \overline{\eta}^2 C_{1'}\overline{C}^{1'}\eta_2\,, \tag{C.19}$$

where, in the second step, we have used the reality constraint (C.6). The above form does not seem invariant under R-symmetry. However, it follows from $\boldsymbol{C}\overline{\boldsymbol{C}} = 0$ that $C_{1'}\overline{C}^{1'} = C_{2'}\overline{C}^{2'}$ which allows us to write the potential in manifest R-symmetry form:

$$-V = -\frac{1}{2}\overline{\eta}^a C_{a'}\overline{C}^{a'}\eta_a\ . \tag{C.20}$$

Next, let us collect all the Yukawa couplings from the fermi action (C.18):

$$\left(\frac{1}{\sqrt{2}}\overline{\eta}^2(\overline{\mathrm{D}}_+C_{1'})\psi_- - \overline{\xi}^{1'}_+C_{1'}\psi_- - \overline{\xi}^{2'}C_{2'}\psi_- - \frac{1}{\sqrt{2}}\overline{\eta}^1\mathrm{D}_+C_{2'}\psi_-\right) + \text{c.c.} \tag{C.21}$$

We have $C_{a'} = K_{a'} + LH_{a'}$. Then, we get

$$\left(-\overline{\eta}^2L\xi_{2+}\psi_- - \overline{\xi}^{1'}_+(K_{1'}+LH_{1'})\psi_- - \overline{\xi}^{2'}(K_{2'}+LH_{2'})\psi_- - \overline{\eta}^1L\xi_{1+}\psi_-\right) + \text{c.c.} \tag{C.22}$$

We thus get the manifest R-symmetric form of the Yukawa couplings

$$\left(-\overline{\xi}^{a'}_+K_{a'}\psi_- - \overline{\eta}^aL\xi_{a+}\psi_- - \overline{\xi}^{a'}_+LH_{a'}\psi_-\right) + \text{c.c.}\,, \tag{C.23}$$

where the first term and its complex conjugate together are mass terms which contain the fermis and the superpartners of the standard hypers. The other terms are Yukawa couplings which involve the standard hypers, the twisted hypers and the fermis.

C.2 $(0,4) \to (0,0)$

In this subsection, we directly go from $(0,4)$ superspace to ordinary space. We give two illustrative examples, an $\mathcal{O}(1)$ standard hyper and an arctic fermi.

$\mathcal{O}(1)$ standard hyper. Recall from (3.3) and (3.5) that the $(0,4)$ descendants of $\boldsymbol{\eta}$ are, at the first level,

$$\sqrt{2}\xi_{a'+} := \widetilde{\mathbf{D}}_{a'+}\boldsymbol{\eta}\,,\quad \sqrt{2}\overline{\xi}^{a'}_+ := -\varepsilon^{a'b'}\widetilde{\mathbf{D}}_{b'+}\overline{\boldsymbol{\eta}}\,, \tag{C.24}$$

and at the second level,

$$\widetilde{\mathbf{D}}_{a'+}\widetilde{\mathbf{D}}_{b'+}\boldsymbol{\eta} = -2\mathrm{i}\varepsilon_{a'b'}\partial_{++}\widetilde{\boldsymbol{\eta}}\,,\quad \widetilde{\mathbf{D}}_{a'+}\widetilde{\mathbf{D}}_{b'+}\overline{\boldsymbol{\eta}} = -2\mathrm{i}\varepsilon_{a'b'}\partial_{++}\widetilde{\overline{\boldsymbol{\eta}}}\ . \tag{C.25}$$

The action is

$$\mathcal{S} = -\frac{\mathrm{i}}{2}\int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\left(\zeta^{-1}\overline{\boldsymbol{\eta}}\partial_{--}\boldsymbol{\eta}\right)\ . \tag{C.26}$$

Pushing in the derivatives in the measure and using (C.24) and (C.25), we get

$$\begin{aligned}
\mathcal{S} &= -\mathrm{i}\int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\left(-\mathrm{i}\partial_{++}\widetilde{\overline{\boldsymbol{\eta}}}\partial_{--}\boldsymbol{\eta} + \overline{\xi}^{1'}_+\partial_{--}\xi_{1'+} + \overline{\xi}^{2'}_+\partial_{--}\xi_{2'+} - \mathrm{i}\overline{\boldsymbol{\eta}}\partial_{--}\partial_{++}\widetilde{\boldsymbol{\eta}}\right),\\
&= \int \mathrm{d}^2x \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\left(\partial_{++}\overline{\eta}^2\partial_{--}\eta_2 - \mathrm{i}\overline{\xi}^{a'}_+\partial_{--}\xi_{a'+} - \overline{\eta}^1\partial_{--}\partial_{++}\eta_1\right),\\
&= \int \mathrm{d}^2x \left(-\partial_\mu\overline{\eta}^a\partial^\mu\eta_a - \mathrm{i}\overline{\xi}_+^{a'}\partial_{--}\xi_{a'+}\right)\ .
\end{aligned} \tag{C.27}$$

where, in going to the second line, we have used the explicit expressions $\widetilde{\boldsymbol{\eta}} = \eta_1$ and $\widetilde{\overline{\boldsymbol{\eta}}} = -\overline{\eta}^2$ (to compute $\widetilde{\overline{\boldsymbol{\eta}}}$, we follow the same steps as for $\widetilde{\mathbf{D}}_{a'+}$ in section 2.6).

The arctic fermi superfield. We look at the weight 0 arctic fermi superfield. The descendants are

$$\begin{aligned} \sqrt{2}\boldsymbol{F}_{a'} &= \widetilde{\mathbf{D}}_{a'+}\boldsymbol{\Upsilon}_- \,, & \sqrt{2}\overline{\boldsymbol{F}}^{a'} &= -\varepsilon^{a'b'}\zeta\widetilde{\mathbf{D}}_{b'+}\overline{\boldsymbol{\Upsilon}}_- \,, \\ \boldsymbol{X}_+ &= \widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\boldsymbol{\Upsilon}_- \,, & \overline{\boldsymbol{X}}_+ &= -\zeta^2\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\overline{\boldsymbol{\Upsilon}}_- - 2\mathrm{i}\zeta\partial_{++}\overline{\boldsymbol{\Upsilon}}_- \,. \end{aligned} \tag{C.28}$$

The action is

$$\mathcal{S} = -\frac{1}{2}\int \mathrm{d}^2x \oint \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\widetilde{\mathbf{D}}_{1'+}\widetilde{\mathbf{D}}_{2'+}\left(\overline{\boldsymbol{\Upsilon}}_-\boldsymbol{\Upsilon}_-\right) \,. \tag{C.29}$$

Pushing in the derivatives in the measure, we get

$$\mathcal{S} = -\frac{1}{2}\int \mathrm{d}^2x \oint \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\left(-2\mathrm{i}\zeta^{-1}\partial_{++}\overline{\boldsymbol{\Upsilon}}_-\boldsymbol{\Upsilon}_- - \zeta^{-2}\overline{\boldsymbol{X}}_+\boldsymbol{\Upsilon}_- + \overline{\boldsymbol{\Upsilon}}_-\boldsymbol{X}_+ - 2\zeta^{-1}\overline{\boldsymbol{F}}^{a'}\boldsymbol{F}_{a'}\right) \,. \tag{C.30}$$

The superfields $\boldsymbol{X}_+$ and $\boldsymbol{F}_{a'}$ are auxiliary and can be integrated out. The terms involving $\boldsymbol{X}_+$ are

$$\oint \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}(\overline{\boldsymbol{\Upsilon}}_-\boldsymbol{X}_+ - \zeta^{-2}\overline{\boldsymbol{X}}_+\boldsymbol{\Upsilon}_-) = -\sum_{j\geq 0}(-1)^j\overline{\Upsilon}_{j+1,-}X_{j+} + \text{c.c.} \,. \tag{C.31}$$

Integrating out X_{j+} gives

$$\Upsilon_{j,-} = 0 \quad \text{for} \quad j \geq 1 \,. \tag{C.32}$$

Thus, the weight 0 superfield $\boldsymbol{\Upsilon}_-$ which was locally defined on $\mathbf{CP}^1$ becomes a constant on $\mathbf{CP}^1$, which is nothing but a globally defined weight 0 superfield. Integrating out $\boldsymbol{F}_{a'}$ just sets them to zero. Relabelling $\Upsilon_{0-} \to \psi_-$, the action becomes

$$\mathcal{S} = \int \mathrm{d}^2x\,(-\mathrm{i})\,\overline{\psi}_-\partial_{++}\psi_- \,. \tag{C.33}$$

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