Phys. Lett. B 852 (2024) 138610

Contents lists available at ScienceDirect

Physics Letters B

journal homepage: www.elsevier.com/locate/physletb

Letter

Low-energy limit of N-photon amplitudes in a constant field

Naser Ahmadiniaz [a,*], Misha A. Lopez-Lopez [a,b], Christian Schubert [c]

[a] Helmholtz-Zentrum Dresden-Rossendorf, Bautzner Landstraße 400, 01328 Dresden, Germany
[b] Institut für Theoretische Physik, Technische Universität Dresden, 01062 Dresden, Germany
[c] Facultad de Ciencias Físico-Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Avenida Francisco J. Mújica, 58060 Morelia, Michoacán, Mexico

ARTICLE INFO

Editor: A. Ringwald

Keywords:
Worldline formalism
Photon amplitudes
Low energy
Constant field

ABSTRACT

While the QED photon amplitudes at full momentum so far have been calculated only up to the six-photon level, in the low-energy limit there are explicit formulas for all helicity components even at the N-photon level, obtained by Martin et al. in 2002. Here we use the worldline formalism to extend that result to the N-photon amplitudes in a generic constant field. For both scalar and spinor QED, we obtain compact representations for the low-energy limits of these amplitudes involving only simple algebra and a single global proper-time integral with trigonometric integrand.

1. Introduction

Although the QED photon amplitudes are prototypical for all gauge-boson correlators and have been studied for almost a century [1–9] (for reviews, see [10–12]), for arbitrary momenta results exist only up to the six-point level [13]. Only in the low-energy limit, where all photon energies are small compared to the electron mass, has it been feasible to derive explicit expressions for all helicity components even at the N-photon level [14]. In the present letter, we generalize these formulas to the inclusion of a generic constant external field. While in [14] these amplitudes were derived from the Euler-Heisenberg Lagrangian [2] and its scalar QED analogue, the Weisskopf Lagrangian [3], for the purpose of the constant-field generalization we find it more convenient to proceed along the lines of a later rederivation [15] based on a direct amplitude calculation using the worldline formalism [16–23]. We will thus start with a short review of the calculation of general photon amplitudes in that formalism, first in vacuum and then in the constant-field background. We then proceed to the calculation of their low-energy limits, first for scalar and then for spinor QED.

2. Worldline representation of the QED N - photon amplitudes

We start with a short summary of the computation of photon amplitudes in the worldline formalism (for details, see [22,24]). In scalar QED, the formalism leads to the following path-integral representation of the one-loop N - photon amplitude [20],

$$\Gamma_{\rm scal}(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N)=(-ie)^N\int_0^\infty\frac{dT}{T}\,{\rm e}^{-m^2T}\int_{x(0)=x(T)}Dx\,{\rm e}^{-\int_0^T d\tau\frac{\dot x^2}{4}}\times V^\gamma_{\rm scal}[k_1,\varepsilon_1]\cdots V^\gamma_{\rm scal}[k_N,\varepsilon_N]\,. \tag{1}$$

Here m is the mass of the loop scalar, the path integral runs over all closed loops of periodicity T in (euclidean) spacetime, and each of the photons is represented by a vertex operator

$$V^\gamma_{\rm scal}[k,\varepsilon]=\int_0^T d\tau\,\varepsilon\cdot\dot x\,{\rm e}^{ik\cdot x}=\int_0^T d\tau\,{\rm e}^{ik\cdot x+\varepsilon\cdot\dot x}\Big|_\varepsilon\,, \tag{2}$$

where the polarisation vector ε^μ need not necessarily obey on-shell conditions. A formal gaussian integration of the path integral in (1) leads to the following master formula for this amplitude [18–20]:

$$\Gamma_{\rm scal}(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N)=(-ie)^N\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\,{\rm e}^{-m^2T}\times\prod_{i=1}^N\int_0^T d\tau_i\exp\Big\{\sum_{i,j=1}^N\Big[\frac12 G_{ij}k_i\cdot k_j-i\dot G_{ij}\varepsilon_i\cdot k_j+\frac12\ddot G_{ij}\varepsilon_i\cdot\varepsilon_j\Big]\Big\}\Big|_{\varepsilon_1\varepsilon_2\dots\varepsilon_N}\,. \tag{3}$$

Here we have abbreviated $G_{ij}\equiv G(\tau_i,\tau_j)$ etc., where G is the “bosonic worldline Green’s function” defined by

* Corresponding author.
E-mail address: n.ahmadiniaz@hzdr.de (N. Ahmadiniaz).

https://doi.org/10.1016/j.physletb.2024.138610
Received 15 December 2023; Received in revised form 22 March 2024; Accepted 25 March 2024
Available online 28 March 2024

$$G(\tau,\tau') \equiv \mid \tau-\tau' \mid - \frac{(\tau-\tau')^2}{T}\,, \tag{4}$$

and a 'dot' denotes a derivative acting on the first variable,

$$\dot G(\tau,\tau') = \mathrm{sgn}(\tau-\tau') - 2\frac{(\tau-\tau')}{T}, \quad \ddot G(\tau,\tau') = 2\delta(\tau-\tau') - \frac{2}{T}\,. \tag{5}$$

The exponential must still be expanded and only the terms be retained that contain each polarisation vector ε_i linearly:

$$\exp\{\cdot\}\Big|_{\varepsilon_1\varepsilon_2\dots\varepsilon_N} \equiv (-i)^N P_N(\dot G_{ij},\ddot G_{ij}) \exp\left[\frac{1}{2}\sum_{i,j=1}^N G_{ij} k_i\cdot k_j\right], \tag{6}$$

with certain polynomials P_N.

The generalization to the spinor-loop case in the modern approach is done through the addition of a Grassmann path integral, to be evaluated with the "fermionic worldline Green's function" $G_F(\tau,\tau') \equiv \mathrm{sgn}(\tau-\tau')$ [25,18–20]. Moreover, these spin-induced terms can be conveniently generated by the following "Bern-Kosower loop-replacement" rule: after removing all second derivatives $\ddot G_{ij}$ from the polynomial P_N by suitable partial integrations, the integrand for the spinor loop case can be obtained by simultaneously replacing every "τ-cycle" $\dot G_{i_1 i_2}\dot G_{i_2 i_3}\cdots \dot G_{i_n i_1}$ by the corresponding "super τ-cycle" $\dot G_{i_1 i_2}\dot G_{i_2 i_3}\cdots \dot G_{i_n i_1} - G_{F i_1 i_2}G_{F i_2 i_3}\cdots G_{F i_n i_1}$ (and multiplying the whole amplitude by a factor of -2 for the difference in statistics and number of degrees of freedom).

Subsequently, it emerged that this whole procedure can be generalized to the inclusion of a constant external field in a very economical way, namely by a modification of the path-integral determinant [21] and the introduction of generalized worldline Green's functions $\mathcal{G}_B, \mathcal{G}_F$ that take the external field into account [29,30]

$$\mathcal{G}_B(\tau,\tau') \equiv \frac{T}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}}\,\mathrm{e}^{-i\mathcal{Z}\dot G(\tau,\tau')} + i\mathcal{Z}\dot G(\tau,\tau') - 1\right), \tag{7}$$

$$\mathcal{G}_F(\tau,\tau') \equiv G_F(\tau,\tau')\frac{\mathrm{e}^{-i\mathcal{Z}\dot G(\tau,\tau')}}{\cos\mathcal{Z}}\,, \tag{8}$$

where $\mathcal{Z}_{\mu\nu} \equiv eF_{\mu\nu}T$, $F_{\mu\nu} = \mathrm{const}$ is the external field, and the subscripts B and F stand for "bosonic" and "fermionic" respectively. These expressions for the constant-field Green's functions should be understood as power series in the Lorentz matrix $\mathcal{Z}$. The master formula for the photon amplitudes in vacuum (3) can then be generalized to the constant field case as follows [29,30],

$$\Gamma_{\mathrm{scal}}(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N;F) = (-ie)^N \int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\,\mathrm{e}^{-m^2T}\det{}^{\frac12}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right]$$
$$\times\prod_{i=1}^N\int_0^T d\tau_i \exp\left\{\sum_{i,j=1}^N\left[\frac12 k_i\cdot\mathcal{G}_{Bij}\cdot k_j - i\varepsilon_i\cdot\dot{\mathcal{G}}_{Bij}\cdot k_j + \frac12\varepsilon_i\cdot\ddot{\mathcal{G}}_{Bij}\cdot\varepsilon_j\right]\right\}\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}. \tag{9}$$

This representation has already been tested on the calculations of the tadpole [31,32] and vacuum polarization amplitudes [33,34], as well as on magnetic photon splitting [35,36], and found to be vastly more efficient than the traditional methods based on second quantization.

Similar master formulas have been derived for the N - photon amplitudes in plane-wave backgrounds [37] and in combined constant field – plane-wave backgrounds [38].

3. Low-energy limit of the N - photon amplitudes in vacuum

In this section, the external field $F_{\mu\nu}$ is temporarily turned off to revisit earlier results, particularly examining the multiphoton amplitude in vacuum. This is in preparation for the next section, which will explore the presence of a constant external field. The low-energy limit of the photon amplitudes is defined by the condition that all photon energies be small compared to the mass of the loop scalar or fermion,

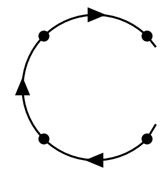

Fig. 1. Worldline Feynman diagram representing an integrated bicycle factor.

$$\omega_i \ll m, \quad i = 1,\dots,N\,. \tag{10}$$

This condition then justifies truncating all the vertex operators to their terms linear in the momentum. Noting that the leading, momentum-independent term in this expansion integrates to zero for a closed loop, and adding a suitable total-derivative term, we can write the vertex operator of a low-energy photon as

$$V^{\gamma\,(\mathrm{LE})}_{\mathrm{scal}}[f] = \frac{i}{2}\int_0^T d\tau\, x(\tau)\cdot f\cdot\dot x(\tau) = \frac{i}{2}\int_0^T d\tau\,\mathrm{e}^{x(\tau)\cdot f\cdot\dot x(\tau)}\Big|_f\,, \tag{11}$$

where $f_{\mu\nu} = k_\mu\varepsilon_\nu - \varepsilon_\mu k_\nu$ is the photon field-strength tensor. The Wick contraction of a product of such objects produces products of "Lorentz cycles"

$$Z_n(i_1 i_2\dots i_n) \equiv \left(\frac12\right)^{\delta_{n2}}\mathrm{tr}\left(\prod_{j=1}^n f_{i_j}\right), \tag{12}$$

with coefficients that, by suitable partial integrations, can be written as integrals of the "τ - cycles" $\dot G_{i_1 i_2}\dot G_{i_2 i_3}\cdots\dot G_{i_n i_1}$ introduced above. The result can be further simplified by observing that, in the one-dimensional worldline theory, each of the resulting "bicycle" factors

$$\int_0^T d\tau_{i_1}\cdots\int_0^T d\tau_{i_n}\,\dot G_{i_1 i_2}\dot G_{i_2 i_3}\cdots\dot G_{i_n i_1}\,\mathrm{tr}\,(f_{i_1}f_{i_2}\cdots f_{i_n}), \tag{13}$$

can be identified with the one-loop n - point Feynman diagram depicted in Fig. 1.

Products of such factors thus in the worldline theory correspond to disconnected diagrams, and by standard combinatorics can be reduced to the exponential of the sum of all connected diagrams. In this way, and with a rescaling $\tau_i = Tu_i$, we arrive at

$$\left\langle V^{\gamma\,(\mathrm{LE})}_{\mathrm{scal}}[f_1]\cdots V^{\gamma\,(\mathrm{LE})}_{\mathrm{scal}}[f_N]\right\rangle = (iT)^N\exp\left\{\sum_{n=1}^\infty b_{2n}\sum_{\{i_1\dots i_{2n}\}} Z^{\mathrm{dist}}_{2n}(\{i_1 i_2\dots i_{2n}\})\right\}\Big|_{f_1\dots f_N}, \tag{14}$$

where $Z^{\mathrm{dist}}_k(\{i_1 i_2\dots i_k\})$ denotes the sum over all distinct Lorentz cycles which can be formed with a given subset of indices, e.g. $Z^{\mathrm{dist}}_4(\{ijkl\}) = Z_4(ijkl) + Z_4(ijlk) + Z_4(ikjl)$, and b_n denotes the basic "cycle integral"

$$b_n \equiv \int_0^1 du_1 du_2\dots du_n\,\dot G_{12}\dot G_{23}\cdots\dot G_{n1}\,. \tag{15}$$

This integral can be expressed in terms of the Bernoulli numbers $\mathcal{B}_n$ [30]:

$$b_n = \begin{cases} -2^n\frac{\mathcal{B}_n}{n!} & n \ \text{even}\,,\\ 0 & n \ \text{odd}\,.\end{cases} \tag{16}$$

Eq. (14) can be further simplified using the combinatorial fact that

$$\mathrm{tr}\left[(f_1+\dots+f_N)^n\right]\Big|_{\text{all different}} = 2n\sum_{\{i_1\dots i_n\}} Z^{\mathrm{dist}}_n(\{i_1 i_2\dots i_n\})\,, \tag{17}$$

($n\neq 0$). Introducing $f_{\mathrm{tot}} \equiv \sum_{i=1}^N f_i$, using all this in (1) and eliminating the T-integral, we arrive at the following formula for the low-energy limit of the one-loop N-photon ($N \geq 4$) amplitude in scalar QED [15]:

$$\Gamma^{(\mathrm{LE})}_{\mathrm{scal}}(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N) = \frac{e^N\Gamma(N-2)}{(4\pi)^2 m^{2N-4}}\exp\Big\{\sum_{n=1}^{\infty}\frac{b_{2n}}{4n}\mathrm{tr}\,(f_{\mathrm{tot}}^{2n})\Big\}\Big|_{f_1\dots f_N}, \tag{18}$$

(here we have exempted the trivial case $N=2$ to be able to set $D=4$).

Further, the above-mentioned Bern-Kosower loop replacement rule allows us to generalise this result to spinor QED simply by replacing the cycle integral (16) by the "super - cycle integral"

$$\int_0^1 du_1 du_2\dots du_n\Big(\dot G_{12}\dot G_{23}\cdots\dot G_{n1} - G_{F12}G_{F23}\cdots G_{Fn1}\Big) = (2-2^n)\,b_n\,. \tag{19}$$

The only other change is a global factor of (-2) for statistics and degrees of freedom. Therefore [15]

$$\Gamma^{(\mathrm{LE})}_{\mathrm{spin}}(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N) = (-2)\frac{e^N\Gamma(N-2)}{(4\pi)^2 m^{2N-4}} \times\exp\Big\{\sum_{n=1}^{\infty}(1-2^{2n-1})\frac{b_{2n}}{2n}\mathrm{tr}\,(f_{\mathrm{tot}}^{2n})\Big\}\Big|_{f_1\dots f_N} \tag{20}$$

Formulas equivalent to (18) and (20) were derived in [14] from the Euler-Heisenberg Lagrangian and its scalar QED equivalent, the Weisskopf Lagrangian, however as already mentioned we have found the above procedure more suitable for our present purpose of generalization to the constant-field background.

Note that in the above derivation on-shell conditions have not yet been used. In [14] it was further shown how to obtain explicit expressions for all the helicity components of the on-shell amplitudes, using the spinor helicity formalism. And for the special case of the "all plus" (or "all minus") amplitudes this calculation was carried further to the two-loop level [15], taking advantage of the explicit expressions obtained in [39] for the two-loop corrections to the Euler-Heisenberg and Weisskopf Lagrangians in the special case of a self-dual field.

Two interesting aspects of the on-shell N-photon amplitudes in the low-energy limit (at any loop order) have emerged in that work. First, in the helicity basis the low-energy amplitudes obey a "double Furry theorem", that is, they are non-vanishing only if the number of positive and negative helicity photons are separately even [14]. Second, in the spinor-helicity formalism the full dependence of the low-energy amplitudes on the momenta and polarizations can, for any number of photons and any given helicity component, be absorbed into a single invariant, effectively reducing the amplitude to a single number. This holds at any loop order, and made it feasible to use these amplitudes for a study of the asymptotic properties of the photon S-matrix [26–28].

4. Low-energy limit of the N - photon amplitudes in a constant field: scalar QED

Let us now return to the scalar QED case, and show how to modify the procedure of the previous section to take an additional constant external field $F_{\mu\nu}$ into account. Apart from the global determinant factor exhibited in (9), the only change is that the Wick contraction of the N low-energy vertex operators (11) now must be performed using the generalized Green's function (7). One still arrives at the same representation in terms of bicycle factors (13), only that now, since the function replacing $\dot G(\tau,\tau')$,

$$\dot{\mathcal{G}}_B(\tau,\tau') = \frac{i}{\mathcal{Z}}\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}}\,\mathrm{e}^{-i\mathcal{Z}\dot G(\tau,\tau')}-1\right), \tag{21}$$

is non-trivial in Lorentz space, the Lorentz and τ - cycle factors will not any more factorize, but rather combine to form a single Lorentz trace $\dot{\mathcal{G}}_B(i_1 i_2\dots i_n)$, defined as

$$\dot{\mathcal{G}}_B(i_1 i_2\dots i_n) \equiv \Big(\frac12\Big)^{\delta_{n1}+\delta_{n2}}\mathrm{tr}(f_{i_1}\cdot\dot{\mathcal{G}}_{Bi_1i_2}\cdot f_{i_2}\cdot\dot{\mathcal{G}}_{Bi_2i_3}\cdots f_{i_n}\cdot\dot{\mathcal{G}}_{Bi_ni_1}). \tag{22}$$

Thus in the presence of the constant field (14) generalizes to

$$\Big\langle V^{\gamma(\mathrm{LE})}_{\mathrm{scal}}[f_1]\cdots V^{\gamma(\mathrm{LE})}_{\mathrm{scal}}[f_N]\Big\rangle_F = (iT)^N\exp\Big\{\sum_{n=1}^{\infty}\sum_{\{i_1\dots i_n\}}\prod_{k=1}^{n}\int_0^1 du_{i_k}\,\dot{\mathcal{G}}_B^{\mathrm{dist}}(\{i_1 i_2\dots i_n\})\Big\}\Big|_{f_1\dots f_N} \tag{23}$$

(note that n is now running over all integers, not only over the even ones) and the challenge is to compute the generalization of the cycle integral (15),

$$I^{\mathrm{cyc}}_{\mathrm{scal}}(f_1,f_2,\dots,f_n;F) \equiv \int_0^1 du_1\cdots\int_0^1 du_n\,\dot{\mathcal{G}}_B(12\dots n)\,. \tag{24}$$

This can be done in the following way. Restricting ourselves now to the case of a generic constant field (both Maxwell invariants nonzero), and choosing a Lorentz frame where both the magnetic and the electric field point along the z axis, the (euclidean) field strength tensor takes the form[1]

$$F = \begin{pmatrix} 0 & B & 0 & 0\\ -B & 0 & 0 & 0\\ 0 & 0 & 0 & iE\\ 0 & 0 & -iE & 0\end{pmatrix}. \tag{25}$$

Defining $z_+\equiv eBT$ and $z_-\equiv ieET$ and the matrices g_+, g_-, and r_+, r_- by

$$g_+ \equiv \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad g_- \equiv \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},$$
$$r_+ \equiv \begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad r_- \equiv \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}, \tag{26}$$

one can then show the factorization

$$\det{}^{\frac12}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right] = \frac{z_+z_-}{\sinh z_+\sinh z_-}, \tag{27}$$

and the decomposition

$$\dot{\mathcal{G}}_B(\tau_i,\tau_j) = \sum_{\alpha=\pm}S_{Bij}(z_\alpha)\,g_\alpha^{\mu\nu} - i\sum_{\alpha=\pm}A_{Bij}(z_\alpha)\,r_\alpha^{\mu\nu}, \tag{28}$$

with coefficient functions

$$S_{Bij}(z) = \frac{\sinh(z\dot G_{ij})}{\sinh z}, \tag{29}$$

$$A_{Bij}(z) = \frac{\cosh(z\dot G_{ij})}{\sinh z} - \frac1z\,. \tag{30}$$

For our present purpose, it is essential to note that those can be written in terms of the single function

$$H^B_{ij}(z) \equiv \frac{e^{z\dot G_{ij}}}{\sinh z} - \frac1z \tag{31}$$

as

$$S_{Bij}(z) = \frac12\Big[H^B_{ij}(z) + H^B_{ij}(-z)\Big], \tag{32}$$

$$A_{Bij}(z) = \frac12\Big[H^B_{ij}(z) - H^B_{ij}(-z)\Big]. \tag{33}$$

Thus, defining

$$\mathfrak{g}^\alpha_\beta \equiv \frac12(g_\beta - \alpha i r_\beta), \tag{34}$$

[1] Our conventions follow [22].

with $\alpha, \beta = \pm$, we can rewrite (28) as

$$\dot{\mathcal{G}}_B(\tau_i, \tau_j) = \sum_{\alpha,\beta=\pm} H^B_{ij}(\alpha z_\beta)\mathfrak{g}^\alpha_\beta\,. \tag{35}$$

The function H^B has the following remarkable property of reproducing itself under folding,

$$H^{B(2)}_{ik}(z,z') \equiv \int_0^T d\tau_j H^B_{ij}(z)H^B_{jk}(z') = \frac{H^B_{ik}(z)}{z'-z} + \frac{H^B_{ik}(z')}{z-z'}\,, \tag{36}$$

$$\begin{aligned} H^{B(3)}_{il}(z,z',z'') &\equiv \int_0^T d\tau_j \int_0^T d\tau_k H^B_{ij}(z)H^B_{jk}(z')H^B_{kl}(z'') \\ &= \frac{H^B_{il}(z)}{(z'-z)(z''-z)} + \frac{H^B_{il}(z')}{(z-z')(z''-z')} \\ &\quad + \frac{H^B_{il}(z'')}{(z-z'')(z'-z'')}\,, \\ &\vdots \end{aligned}$$

$$H^{B(n)}_{i_1 i_{n+1}}(z_1,\dots,z_n) = \sum_{k=1}^n \frac{H^B_{i_1 i_{n+1}}(z_k)}{\prod_{l\neq k}(z_l - z_k)}\,. \tag{37}$$

Using this in (22), (24) we obtain

$$I^{\rm cyc}_{\rm scal}(f_1,\dots,f_n;F) = \Big(\frac12\Big)^{\delta_{n1}+\delta_{n2}} \sum_{\alpha_1,\beta_1=\pm}\cdots\sum_{\alpha_n,\beta_n=\pm} {\rm tr}\,(f_1\mathfrak{g}^{\alpha_1}_{\beta_1} f_2\cdots \mathfrak{g}^{\alpha_n}_{\beta_n}) \times \sum_{k=1}^n \frac{H^B_{11}(\alpha_k z_{\beta_k})}{\prod_{l\neq k}(\alpha_l z_{\beta_l} - \alpha_k z_{\beta_k})}\,, \tag{38}$$

where now only the coincidence limit of $H^B_{ij}(z)$ appears,

$$H^B_{ii}(z) = \coth z - \frac{1}{z}\,. \tag{39}$$

Note that the product in the denominator can contain zero factors, but those are spurious and cancelled by zeroes of the numerator (for an example, see the limit $z' \to z$ of the right-hand side of (36)).

Putting the pieces together, we arrive at the following generalization of the vacuum formula (18),

$$\Gamma^{\rm (LE)}_{\rm scal}(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N;F) = \frac{e^N}{(4\pi)^2}\int_0^\infty \frac{dT}{T}T^{N-2}\,{\rm e}^{-m^2T}\frac{z_+z_-}{\sinh z_+ \sinh z_-} \times \exp\Big\{\sum_{n=1}^\infty \frac{1}{2n} I^{\rm cyc}_{\rm scal}(f_{\rm tot},\dots,f_{\rm tot};F)\Big\}\Big|_{f_1\dots f_N}\,. \tag{40}$$

5. Low-energy limit of the N - photon amplitudes in a constant field: spinor QED

The transition to the spinor QED case can be done by the following adaption of the Bern-Kosower replacement rule to the constant-field case [30],

$$\dot{\mathcal{G}}_B(i_1 i_2\dots i_n) \to \dot{\mathcal{G}}_B(i_1 i_2\dots i_n) - \mathcal{G}_F(i_1 i_2\dots i_n)\,, \tag{41}$$

with

$$\mathcal{G}_F(i_1 i_2\dots i_n) \equiv \Big(\frac12\Big)^{\delta_{n1}+\delta_{n2}} {\rm tr}(f_{i_1}\cdot \mathcal{G}_{Fi_1i_2}\cdot f_{i_2}\cdot \mathcal{G}_{Fi_2i_3}\cdots f_{i_n}\cdot \mathcal{G}_{Fi_ni_1})\,, \tag{42}$$

where $\mathcal{G}_F(\tau,\tau')$ was given in (8). It permits a matrix decomposition analogous to (35),

$$\mathcal{G}_F(\tau_i,\tau_j) = \sum_{\alpha,\beta=\pm} H^F_{ij}(\alpha z_\beta)\mathfrak{g}^\alpha_\beta\,, \tag{43}$$

now in terms of the function

$$H^F_{ij}(z) \equiv G_{Fij}\frac{e^{z\dot{G}_{ij}}}{\cosh z}\,. \tag{44}$$

Remarkably, this function turns out to obey exactly the same multiple-folding formula as H^B, eq. (37). Thus without further ado we can generalize (38) to

$$\begin{aligned} I^{\rm cyc}_{\rm spin}&(f_1,\dots,f_n;F) \\ &\equiv \int_0^1 du_1\cdots\int_0^1 du_n\,\big(\dot{\mathcal{G}}_B(12\dots n) - \mathcal{G}_F(12\dots n)\big) \\ &= \Big(\frac12\Big)^{\delta_{n1}+\delta_{n2}}\sum_{\alpha_1,\beta_1=\pm}\cdots\sum_{\alpha_n,\beta_n=\pm}{\rm tr}\,(f_1\mathfrak{g}^{\alpha_1}_{\beta_1}f_2\cdots\mathfrak{g}^{\alpha_n}_{\beta_n}) \\ &\quad\times\sum_{k=1}^n\frac{H^B_{11}(\alpha_k z_{\beta_k}) - H^F_{11}(\alpha_k z_{\beta_k})}{\prod_{l\neq k}(\alpha_l z_{\beta_l} - \alpha_k z_{\beta_k})}\,, \end{aligned} \tag{45}$$

which now involves only the function

$$H^B_{ii}(z) - H^F_{ii}(z) = \coth z - \tanh z - \frac{1}{z}\,. \tag{46}$$

This brings us to our main result, the spinor QED generalization of the master formula (40) for the low-energy limit of the N - photon amplitudes in the constant field:

$$\Gamma^{\rm (LE)}_{\rm spin}(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N;F) = -2\frac{e^N}{(4\pi)^2}\int_0^\infty\frac{dT}{T}T^{N-2}\,{\rm e}^{-m^2T}\frac{z_+z_-}{\tanh z_+\tanh z_-} \times\exp\Big\{\sum_{n=1}^\infty\frac{1}{2n}I^{\rm cyc}_{\rm spin}(f_{\rm tot},\dots,f_{\rm tot};F)\Big\}\Big|_{f_1\dots f_N}\,. \tag{47}$$

6. Summary and outlook

To summarize, we have applied the worldline formalism to the calculation of the low-energy limits of the N - photon amplitudes in a generic constant field, for both scalar and spinor QED. Our final results for these amplitudes, (40) and (47), reduce their explicit calculation to simple algebra and a single integral of Euler-Heisenberg type, which could be easily automatized. They are still valid off-shell.

In a separate publication [40] we apply these formulas to the study of the on-shell four-photon amplitudes in a constant magnetic field. Further generalization from the closed-loop to the open-line case should be feasible along the lines of [41–43].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

[1] H. Euler, B. Kockel, Naturwissenschaften 23 (15) (1935) 246.
[2] W. Heisenberg, H. Euler, Z. Phys. 98 (1936) 714.
[3] V. Weisskopf, Über die Elektrodynamik des Vakuums auf Grund der Quantentheorie des Elektrons, Kong. Dans. Vid. Selsk. Math-fys. Medd. XIV, vol. 6, 1936, reprinted, in: J. Schwinger (Ed.), Quantum Electrodynamics, Dover, New York, 1958.
[4] A. Akhieser, L. Landau, I. Pomeranchouk, Nature 138 (1936) 206.
[5] R. Karplus, M. Neumann, Phys. Rev. 80 (1950) 380.
[6] R. Karplus, M. Neumann, Phys. Rev. 83 (1951) 776.
[7] B. De Tollis, Nuovo Cimento 32 (3) (1964) 757.
[8] B. De Tollis, Nuovo Cimento 35 (4) (1965) 1182.

[9] V. Costantini, B. de Tollis, G. Pistoni, Nuovo Cimento A 2 (1971) 733.
[10] Y. Liang, A. Czarnecki, Can. J. Phys. 90 (2012) 11, arXiv:1111.6126 [hep-ph].
[11] K. Scharnhorst, arXiv:1711.05194 [physics.hist-ph].
[12] N. Ahmadiniaz, C. Lopez-Arcos, M.A. Lopez-Lopez, C. Schubert, Nucl. Phys. B 991 (2023) 116216, arXiv:2012.11791 [hep-th].
[13] C. Bernicot, J.P. Guillet, J. High Energy Phys. 0801 (2008) 59.
[14] L.C. Martin, C. Schubert, V.M. Villanueva Sandoval, Nucl. Phys. B 668 (2003) 335, arXiv:hep-th/0301022.
[15] G.V. Dunne, C. Schubert, J. High Energy Phys. 0208 (2002) 053, arXiv:hep-th/0205004.
[16] R.P. Feynman, Phys. Rev. 80 (1950) 440.
[17] R.P. Feynman, Phys. Rev. 84 (1951) 108.
[18] A.M. Polyakov, Gauge Fields and Strings, Harwood, 1987.
[19] Z. Bern, D.A. Kosower, Nucl. Phys. B 379 (1992) 451.
[20] M.J. Strassler, Nucl. Phys. B 385 (1992) 145, arXiv:hep-ph/9205205.
[21] M.G. Schmidt, C. Schubert, Phys. Lett. B 318 (1993) 438, arXiv:hep-th/9309055.
[22] C. Schubert, Phys. Rep. 355 (2001) 73, arXiv:hep-th/0101036.
[23] J.P. Edwards, C. Schubert, arXiv:1912.10004 [hep-th].
[24] O. Corradini, C. Schubert, J.P. Edwards, N. Ahmadiniaz, arXiv:1512.08694 [hep-th].
[25] E.S. Fradkin, Nucl. Phys. 76 (1966) 588.
[26] I. Huet, D.G.C. McKeon, C. Schubert, J. High Energy Phys. 1012 (2010) 036, arXiv:1010.5315 [hep-th].
[27] I. Huet, M. Rausch de Traubenberg, C. Schubert, Adv. High Energy Phys. 2017 (2017) 6214341, arXiv:1707.07655 [hep-th].
[28] I. Huet, M. Rausch de Traubenberg, C. Schubert, J. High Energy Phys. 1903 (2019) 167, arXiv:1812.08380 [hep-th].
[29] R.Zh. Shaisultanov, Phys. Lett. B 378 (1996) 354, arXiv:hep-th/9512142.
[30] M. Reuter, M.G. Schmidt, C. Schubert, Ann. Phys. (N.Y.) 259 (1997) 313, arXiv:hep-th/9610191.
[31] J.P. Edwards, C. Schubert, Nucl. Phys. B 923 (2017) 339, arXiv:1704.00482 [hep-th].
[32] N. Ahmadiniaz, F. Bastianelli, O. Corradini, J.P. Edwards, C. Schubert, Nucl. Phys. B 924 (2017) 377, arXiv:1704.05040 [hep-th].
[33] W. Dittrich, R. Shaisultanov, Phys. Rev. D 62 (2000) 045024, arXiv:hep-th/0001171.
[34] C. Schubert, Nucl. Phys. B 585 (2000) 407, arXiv:hep-ph/0001288.
[35] S.L. Adler, C. Schubert, Phys. Rev. Lett. 77 (1996) 1695, arXiv:hep-th/9605035.
[36] S.L. Adler, Adventures in Theoretical Physics: Selected Papers with Commentaries, World Scientific, 2006, arXiv:hep-ph/0505177.
[37] James P. Edwards, C. Schubert, Phys. Lett. B 22 (2021) 136696, e-Print: arXiv:2105.08173 [hep-th].
[38] C. Schubert, R. Shaisultanov, Phys. Lett. B 843 (2023) 137969, arXiv:2303.08907 [hep-th].
[39] G.V. Dunne, C. Schubert, Phys. Lett. B 526 (2002) 55, arXiv:hep-th/0111134.
[40] M. Ahmadiniaz, M.A. Lopez-Lopez, C. Schubert, R. Schützhold, M.A. Trejo, in preparation.
[41] A. Ahmad, N. Ahmadiniaz, O. Corradini, S.P. Kim, C. Schubert, Nucl. Phys. B 919 (2017) 9, arXiv:1612.02944 [hep-ph].
[42] N. Ahmadiniaz, V.M. Banda Guzmán, F. Bastianelli, O. Corradini, J.P. Edwards, C. Schubert, J. High Energy Phys. 08 (2020) 049, arXiv:2004.01391 [hep-th].
[43] N. Ahmadiniaz, V.M. Banda Guzmán, F. Bastianelli, O. Corradini, J.P. Edwards, C. Schubert, J. High Energy Phys. 01 (2022) 050, arXiv:2107.00199 [hep-th].