Kinetic mixing effect in noncommutative $B − L$ gauge theory

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ABSTRACT: It is well established that the $\mathrm{SU}(P)_L$ gauge symmetry for $P \geq 3$ can address the question of fermion generation number due to the anomaly cancellation, but it neither commutes nor closes algebraically with electric and baryon-minus-lepton charges. Hence, two U(1) factors that determine such charges are required, yielding a complete gauge symmetry, $\mathrm{SU}(P)_L \otimes \mathrm{U}(1)_X \otimes \mathrm{U}(1)_N$, apart from the color group. The resulting theory manifestly provides neutrino mass, dark matter, inflation, and baryon asymmetry of the universe. Furthermore, this gauge structure may present kinetic mixing effects associated to the U(1) gauge fields, which affect the electroweak precision test such as the ρ parameter and Z couplings as well as the new physics processes. We will construct the model, examine the interplay between the kinetic mixing and those due to the symmetry breaking, and obtain the physical results in detail.

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Contents

1 Introduction

The standard model of fundamental particles and interactions has been very successful in describing the observed phenomena, but it is incomplete. First of all, the experimental evidences of neutrino oscillations caused by nonzero small neutrino masses and flavor mixing require new physics beyond the standard model [1, 2]. Additionally, the cosmological challenges of particle physics such as inflation, dark matter, and baryon asymmetry also acquire the standard model extension [3]. Hence, it is worthwhile to look for a theory that addresses all these puzzles.

The standard model actually contains a hidden/accident symmetry U(1)$_{B-L}$. If one includes, e.g., three right-handed neutrinos it behaves as a gauge symmetry free from all

the anomalies. The resulting theory based on $\mathrm{SU}(3)_C \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y \otimes \mathrm{U}(1)_{B-L}$ can provide consistent neutrino masses via induced seesaw mechanism [4–12]. This theory also generates suitable baryon asymmetry converted from the leptogenesis resulting from the seesaw mechanism [13]. However, within this framework, it is not naturally to understand dark matter. Indeed, a matter parity can be induced as residual gauge symmetry due to the $\mathrm{U}(1)_{B-L}$ breaking. However, the theory does not contain any odd field responsible for dark matter candidate. Let us note that the Majoron associated with $B - L$ breaking is actually eaten by the new neutral gauge boson, which should rapidly decay into quarks and leptons. The $B - L$ Higgs field and right-handed neutrinos are also unstable, since they decay to ordinary particles.

In this work, we discuss a class of models based upon gauge symmetry, $\mathrm{SU}(3)_C \otimes \mathrm{SU}(P)_L \otimes \mathrm{U}(1)_X \otimes \mathrm{U}(1)_N$, called 3-$P$-1-1, for $P = 3, 4$. Here, $\mathrm{SU}(2)_L$ is extended to $\mathrm{SU}(P)_L$ which offers a natural solution for the question of generation number [14–19]. It is easily verified that the electric charge Q and baryon-minus-lepton charge $B-L$ neither commute nor close algebraically with $\mathrm{SU}(P)_L$ [20–25]. Hence, the two Abelian factors $\mathrm{U}(1)_{X,N}$ are resulted from algebraic closure condition, in which the new charges X and N are related to Q and $B - L$ via the Cartan generators of $\mathrm{SU}(P)_L$, respectively. Besides the answer of generation number, the model manifestly accommodates dark matter which is unified with normal matter to form $\mathrm{SU}(P)_L$ multiplets. This is a consequence of the noncommutative $B - L$ symmetry and matter partiy as a residual gauge symmetry. Such dark fields have "wrong" $B-L$ charge in comparison to the standard model definition, that is old under the matter parity, providing dark matter candidates. They may be a fermion, scalar or gauge boson. The abundance of dark matter observed today can either be thermally produced as a WIMP or results from a standard leptogenesis similarly to the baryon asymmetry [26]. Therefore, in the second case both the dark and normal matter asymmetries are produced due to the CP-violating decay of the lightest right-handed neutrino. In a scenario, the $\mathrm{U}(1)_N$ breaking field successfully inflates the early universe, and its decay reheats the universe producing such right-handed neutrinos, as desirable [26].

The 3-3-1-1 model has been extensively investigated in the literature [20–26], but the 3-4-1-1 model has not considered yet. In this work, we construct the 3-4-1-1 model with general fermion and scalar contents, obtain the matter parity, and interpret dark matter candidates. Since the theory contains two $\mathrm{U}(1)$ factors, the kinetic mixing between the corresponding gauge bosons is not avoidable [27–32]. Therefore, we diagonalize the gauge boson sector when including the kinetic mixing term. The effect of the kinetic mixing is present in the ρ parameter and the coupling of Z with fermions, which can alter the electroweak precision test. It significantly modifies the neutral meson mixings and rare meson decays. The last aim of this work is to probe the new physics of the model at the LHC. This work also revisits the kinetic mixing effect in the 3-3-1-1 model, which was previously studied [33].

The rest of this work is organized as follows. In section 2, we introduce the model and show dark matter. In section 3, we diagonalize the gauge sector. In section 4, we examine the ρ parameter, mixing parameters, and the Z couplings. In section 5, we investigate the FCNCs. The search for the new physics is presented in section 6. In section 7, the kinetic mixing effect in a previous study is revisited. Finally, we conclude this work in section 8.

2 The model

In this section we propose the 3-4-1-1 model, while the 3-3-1-1 model [25, 33] was well established and skipped.

2.1 Gauge symmetry

As stated, the gauge symmetry is given by

$$\mathrm{SU}(3)_C \otimes \mathrm{SU}(4)_L \otimes \mathrm{U}(1)_X \otimes \mathrm{U}(1)_N. \tag{2.1}$$

The electric and baryon-minus-lepton charges are embedded as

$$Q = T_3 + \beta T_8 + \gamma T_{15} + X, \tag{2.2}$$
$$B - L = b T_8 + c T_{15} + N, \tag{2.3}$$

where T_i ($i = 1, 2, 3, \ldots, 15$), X, and N are $\mathrm{SU}(4)_L$, $\mathrm{U}(1)_X$, and $\mathrm{U}(1)_N$ charges, respectively.

Nontrivial commutation relations are obtained by

$$[Q, T_1 \pm iT_2] = \pm(T_1 \pm iT_2), \tag{2.4}$$
$$[Q, T_4 \pm iT_5] = \mp q(T_4 \pm iT_5), \tag{2.5}$$
$$[Q, T_6 \pm iT_7] = \mp(1 + q)(T_6 \pm iT_7), \tag{2.6}$$
$$[Q, T_9 \pm iT_{10}] = \mp p(T_9 \pm iT_{10}), \tag{2.7}$$
$$[Q, T_{11} \pm iT_{12}] = \mp(1 + p)(T_{11} \pm iT_{12}), \tag{2.8}$$
$$[Q, T_{13} \pm iT_{14}] = \mp(p - q)(T_{13} \pm iT_{14}), \tag{2.9}$$
$$[B - L, T_4 \pm iT_5] = \mp(1 + n)(T_4 \pm iT_5), \tag{2.10}$$
$$[B - L, T_6 \pm iT_7] = \mp(1 + n)(T_6 \pm iT_7), \tag{2.11}$$
$$[B - L, T_9 \pm iT_{10}] = \mp(1 + m)(T_9 \pm iT_{10}), \tag{2.12}$$
$$[B - L, T_{11} \pm iT_{12}] = \mp(1 + m)(T_{11} \pm iT_{12}), \tag{2.13}$$
$$[B - L, T_{13} \pm iT_{14}] = \mp(m - n)(T_{13} \pm iT_{14}), \tag{2.14}$$

where we define the basic electric charges as $q = -(1 + \sqrt{3}\beta)/2$ and $p = -(1 + \sqrt{6}\gamma - q)/3$ and the basic baryon-minus-lepton charges as $n = -(2 + \sqrt{3}b)/2$ and $m = -(2 + \sqrt{6}c - n)/3$. Hence, (q, p) and (n, m) will determine the Q and $B - L$ charges of new particles, respectively.

2.2 Particle presentation

The fermions transform under the 3-4-1-1 gauge symmetry as

$$\psi_{aL} \equiv \begin{pmatrix} \nu \\ e \\ E^{q,n} \\ F^{p,m} \end{pmatrix}_{aL} \sim \left(1, 4, \frac{p + q - 1}{4}, \frac{m + n - 2}{4}\right), \tag{2.15}$$

$$Q_{\alpha L} \equiv \begin{pmatrix} d \\ -u \\ J^{-q-1/3,-n-2/3} \\ K^{-p-1/3,-m-2/3} \end{pmatrix}_{\alpha L} \sim \left(3, 4^*, -\frac{p+q+1/3}{4}, -\frac{m+n+2/3}{4} \right), \quad (2.16)$$

$$Q_{3L} \equiv \begin{pmatrix} u \\ d \\ J^{q+2/3,n+4/3} \\ K^{p+2/3,m+4/3} \end{pmatrix}_{3L} \sim \left(3, 4, \frac{p+q+5/3}{4}, \frac{m+n+10/3}{4} \right), \quad (2.17)$$

$$\nu_{aR} \sim (1,1,0,-1), \qquad\qquad e_{aR} \sim (1,1,-1,-1), \quad (2.18)$$

$$E_{aR} \sim (1,1,q,n), \qquad\qquad F_{aR} \sim (1,1,p,m), \quad (2.19)$$

$$u_{aR} \sim (3,1,2/3,1/3), \qquad\qquad d_{aR} \sim (3,1,-1/3,1/3), \quad (2.20)$$

$$J_{\alpha R} \sim (3,1,-q-1/3,-n-2/3), \quad K_{\alpha R} \sim (3,1,-p-1/3,-m-2/3), \quad (2.21)$$

$$J_{3R} \sim (3,1,q+2/3,n+4/3), \qquad K_{3R} \sim (3,1,p+2/3,m+4/3), \quad (2.22)$$

where $a = 1,2,3$ and $\alpha = 1,2$ denote generation indices. Additionally, ν_R, E, F, J, and K are new fields, included to complete the representations. This fermion content is independent of all the anomalies (cf. appendix A).

In order for gauge symmetry breaking and mass generation, we introduce the scalar content,

$$\eta = \begin{pmatrix} \eta_1^{0,0} \\ \eta_2^{-1,0} \\ \eta_3^{q,n+1} \\ \eta_4^{p,m+1} \end{pmatrix} \sim \left(1, 4, \frac{p+q-1}{4}, \frac{m+n+2}{4} \right), \quad (2.23)$$

$$\rho = \begin{pmatrix} \rho_1^{1,0} \\ \rho_2^{0,0} \\ \rho_3^{q+1,n+1} \\ \rho_4^{p+1,m+1} \end{pmatrix} \sim \left(1, 4, \frac{p+q+3}{4}, \frac{m+n+2}{4} \right), \quad (2.24)$$

$$\chi = \begin{pmatrix} \chi_1^{-q,-n-1} \\ \chi_2^{-q-1,-n-1} \\ \chi_3^{0,0} \\ \chi_4^{p-q,m-n} \end{pmatrix} \sim \left(1, 4, \frac{p-3q-1}{4}, \frac{m-3n-2}{4} \right), \quad (2.25)$$

$$\Xi = \begin{pmatrix} \Xi_1^{-p,-m-1} \\ \Xi_2^{-p-1,-m-1} \\ \Xi_3^{-p+q,-m+n} \\ \Xi_4^{0,0} \end{pmatrix} \sim \left(1, 4, \frac{-3p+q-1}{4}, \frac{-3m+n-2}{4} \right), \quad (2.26)$$

$$\phi \sim (1,1,0,2), \quad (2.27)$$

where the superscipts stand for $(Q, B-L)$ respectively, while the subscripts indicate $\mathrm{SU}(4)_L$ components. The scalars obtain such quantum numbers, provided that they couple left-handed fermions to corresponding right-handed counterparts, except that ϕ couples to $\nu_R \nu_R$ (see below).

2.3 Total Lagrangian

The total Lagrangian has the form,

$$\mathcal{L} = \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{Yukawa}} - V, \tag{2.28}$$

where the first part combines kinetic terms and gauge interactions, given by

$$\mathcal{L}_{\text{kinetic}} = \sum_F \bar{F} i \gamma^\mu D_\mu F + \sum_S (D^\mu S)^\dagger (D_\mu S)$$
$$-\frac{1}{4} G_{r\mu\nu} G_r^{\mu\nu} - \frac{1}{4} A_{i\mu\nu} A_i^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} C_{\mu\nu} C^{\mu\nu} - \frac{\delta}{2} B_{\mu\nu} C^{\mu\nu}. \tag{2.29}$$

The covariant derivative is

$$D_\mu = \partial_\mu + i g_s T_r G_{r\mu} + i g T_i A_{i\mu} + i g_X X B_\mu + i g_N N C_\mu, \tag{2.30}$$

and we denote the coupling constants (g_s, g, g_X, g_N), generators (T_r, T_i, X, N), and gauge bosons (G_r, A_i, B, C) corresponding to the 3-4-1-1 subgroups, respectively. Above, F and S run over the fermion and scalar multiplets, while the parameter δ is dimensionless, called kinetic mixing.[1]

The second and last parts are the Yukawa interactions and scalar potential, given respectively by

$$\mathcal{L}_{\text{Yukawa}} = h_{ab}^\nu \bar{\psi}_{aL} \eta \nu_{bR} + h_{ab}^e \bar{\psi}_{aL} \rho e_{bR} + h_{ab}^E \bar{\psi}_{aL} \chi E_{bR} + h_{ab}^F \bar{\psi}_{aL} \Xi F_{bR} + h_{ab}^{\prime\nu} \bar{\nu}_{aR}^c \nu_{bR} \phi$$
$$+ h_{33}^J \bar{Q}_{3L} \chi J_{3R} + h_{33}^K \bar{Q}_{3L} \Xi K_{3R} + h_{\alpha\beta}^J \bar{Q}_{\alpha L} \chi^* J_{\beta R} + h_{\alpha\beta}^K \bar{Q}_{\alpha L} \Xi^* K_{\beta R} \tag{2.31}$$
$$+ h_{3a}^u \bar{Q}_{3L} \eta u_{aR} + h_{\alpha a}^u \bar{Q}_{\alpha L} \rho^* u_{aR} + h_{3a}^d \bar{Q}_{3L} \rho d_{aR} + h_{\alpha a}^d \bar{Q}_{\alpha L} \eta^* d_{aR} + H.c.,$$

$$V = \mu_1^2 \eta^\dagger \eta + \mu_2^2 \rho^\dagger \rho + \mu_3^2 \chi^\dagger \chi + \mu_4^2 \Xi^\dagger \Xi + \mu_5^2 \phi^\dagger \phi + \lambda_1 (\eta^\dagger \eta)^2 + \lambda_2 (\rho^\dagger \rho)^2$$
$$+ \lambda_3 (\chi^\dagger \chi)^2 + \lambda_4 (\Xi^\dagger \Xi)^2 + \lambda_5 (\phi^\dagger \phi)^2 + \lambda_6 (\eta^\dagger \eta)(\rho^\dagger \rho) + \lambda_7 (\eta^\dagger \eta)(\chi^\dagger \chi)$$
$$+ \lambda_8 (\eta^\dagger \eta)(\Xi^\dagger \Xi) + \lambda_9 (\eta^\dagger \eta)(\phi^\dagger \phi) + \lambda_{10} (\rho^\dagger \rho)(\chi^\dagger \chi) + \lambda_{11} (\rho^\dagger \rho)(\Xi^\dagger \Xi)$$
$$+ \lambda_{12} (\rho^\dagger \rho)(\phi^\dagger \phi) + \lambda_{13} (\chi^\dagger \chi)(\Xi^\dagger \Xi) + \lambda_{14} (\chi^\dagger \chi)(\phi^\dagger \phi) + \lambda_{15} (\Xi^\dagger \Xi)(\phi^\dagger \phi)$$
$$+ \lambda_{16} (\eta^\dagger \rho)(\rho^\dagger \eta) + \lambda_{17} (\eta^\dagger \chi)(\chi^\dagger \eta) + \lambda_{18} (\eta^\dagger \Xi)(\Xi^\dagger \eta) + \lambda_{19} (\rho^\dagger \chi)(\chi^\dagger \rho)$$
$$+ \lambda_{20} (\rho^\dagger \Xi)(\Xi^\dagger \rho) + \lambda_{21} (\chi^\dagger \Xi)(\Xi^\dagger \chi) + (\lambda \eta \rho \chi \Xi + H.c.), \tag{2.32}$$

where the Yukawa (h's) and scalar (λ's) couplings are dimensionless, while the μ's parameters have the mass dimension.

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[1]This kinetic mixing term is always presented due to the gauge invariance and cannot be removed by rescaling the corresponding fields. Even if its tree-level value vanishes, it can be radiatively induced [27–32].

2.4 Matter parity

Since Q is conserved, only the neutral components $\eta_1, \rho_2, \chi_3, \Xi_4$, and ϕ develop vacuum expectation values (VEVs),

$$\langle\eta\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} u \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \langle\rho\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \\ 0 \\ 0 \end{pmatrix}, \tag{2.33}$$

$$\langle\chi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ w \\ 0 \end{pmatrix}, \qquad \langle\Xi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 0 \\ V \end{pmatrix}, \qquad \langle\phi\rangle = \frac{1}{\sqrt{2}}\Lambda. \tag{2.34}$$

The VEVs V, w, u, v break the 3-4-1-1 symmetry to $\mathrm{SU}(3)_C \otimes \mathrm{U}(1)_Q \otimes \mathrm{U}(1)_{B-L}$, while the VEV Λ breaks $B - L$ to the matter parity, $\mathrm{U}(1)_{B-L} \to P$, where

$$P = (-1)^{3(B-L)+2s} = (-1)^{3(bT_8+cT_{15}+N)+2s} \tag{2.35}$$

is multiplied by the spin parity $(-1)^{2s}$ as conserved by the Lorentz symmetry, similar to the 3-3-1-1 model [20–25].[2]

Because w, V, Λ provide the masses of new particles, whereas u, v do so for the ordinary particles, we assume $u, v \ll w, V, \Lambda$, to keep a consistency with the standard model.

The matter parity P divides particles into two types:

1. Normal particles according to $P = 1$ (even): ν, e, u, d, $\eta_{1,2}$, $\rho_{1,2}$, χ_3, Ξ_4, ϕ, γ, W, $Z_{1,2,3,4}$, which include the standard model particles.

2. Wrong particles according to $P = P_n^\pm \equiv (-1)^{\pm(3n+1)}$ or $P = P_m^\pm \equiv (-1)^{\pm(3m+1)}$: E, F, J, K, $\eta_{3,4}$, $\rho_{3,4}$, $\chi_{1,2}$, $\Xi_{1,2}$, $W_{13}^{\mp q, \mp(n+1)}$, $W_{14}^{\mp p, \mp(m+1)}$, $W_{23}^{\mp(q+1), \mp(n+1)}$, $W_{24}^{\mp(p+1), \mp(m+1)}$, where the W's fields are non-Hermitian gauge bosons which couple to the mentioned weight-raising/lowering operators. The remainders χ_4, Ξ_3, and $W_{34}^{\pm(q-p), \pm(n-m)}$ have $P = P_n^+ P_m^-$ or conjugated.

Generally, the wrong fields transform nontrivially under the matter parity for $n, m \neq (2k-1)/3$ and $n - m \neq 2k/3$ for every k integer. However, an alternative case is that both $P_{n,m} = -1$ are odd, i.e. $n, m = 2k/3$. In this case all the wrong fields are odd, except that χ_4, Ξ_3, and W_{34} are even which belong to the first type of normal particles.

2.5 Dark matter

It is easy to prove that the wrong particles are always coupled in pairs or self-interacted due to the matter parity conservation, which is analogous to superparticles in supersymmetry [25] (see also Dong and Huong in [34–39]). Hence, the lightest wrong particle (LWP) is stabilized, responsible for dark matter.

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[2]This kind of the matter parity is also recognized in the class of the left-right extensions [34–39].

Indeed, there is no single wrong-field in interactions since P is conserved. If an interaction possesses r_x of P_x^+-fields and s_y of P_y^--fields for $x = n, m$ and $y = n, m$, where r_x and s_y are integer, the P conservation leads to $\sum_x r_x(3x + 1) - \sum_y s_y(3y + 1) = 2k$ for k integer. This happens for any x, y parameter only if $x = y$ and $r_x = s_y$, i.e. P_x^+ and P_x^- always appear in pairs. If an interaction contains t_{xy} of $P_x^+ P_y^-$-fields of this kind, we have $\sum_{xy} t_{xy}(3x - 3y) = 2k$, leading to $t_{xy} = t_{yx}$, i.e. P^+P^- and its conjugate always appear in pairs. Last, a $P_x^+ P_y^-$-field self-interacts with two P_x^- and P_y^+ fields.

In other words, the matter parity can be factorized into two single parities, $P = P_n \otimes P_m$ for $P_n = \{1, P_n^\pm\}$ and $P_m = \{1, P_m^\pm\}$, such that each of them is separately conserved. Here, the normal particles have $P_n = P_m = 1$, the singly wrong fields possess parities $(P_n^\pm, 1)$ or $(1, P_m^\pm)$, and those of doubly wrong fields are (P_n^+, P_m^-) or (P_n^-, P_m^+).

Since the candidate must be color and electrically neutral, we have several dark matter models: (i) $q = 0$ including E^0, η_3^0, χ_1^0, W_{13}^0; (ii) $p = 0$ including F^0, η_4^0, Ξ_1^0, W_{14}^0; (iii) $q = -1$ consisting of ρ_3^0, χ_2^0, W_{23}^0; (iv) $p = -1$ consisting of ρ_4^0, χ_2^0, W_{24}^0. In each case, the remaining basic electric charge is left arbitrary.

The specific dark matter models that combine above cases are

1. $q = p = -1$: the candidate is a scalar combination of $\rho_{3,4}$, χ_2, and Ξ_2, or a gauge boson combination of W_{23} and W_{24}.

2. $q = -1, p = 0$: the candidate is a fermion combination of $F_{1,2,3}$, a scalar combination of η_4, ρ_3, χ_2, and Ξ_1, or a gauge boson combination of W_{14} and W_{23}.

3. $q = 0, p = -1$: the candidate is a fermion combination of $E_{1,2,3}$, a scalar combination of η_3, ρ_4, χ_1, and Ξ_2, or a gauge boson combination of W_{13} and W_{24}.

4. $q = p = 0$: the candidate is a fermion combination of $E_{1,2,3}$ and $F_{1,2,3}$, a scalar combination of $\eta_{3,4}$, χ_1, and Ξ_1, or a gauge boson combination of W_{13} and W_{14}.

The last model is for $p = q \neq 0, -1$. The candidate includes a scalar combination of χ_4^0 and Ξ_3^0, or a vector W_{34}^0.

The key-difference between the 3-4-1-1 model and the previous 3-3-1-1 [20–26] and 3-3-1 [40–51] models is that the present proposal can contain multicomponent dark matter. This is because the matter parity actually contains two single matter parities, characterized by (n, m) parameters, which do not exit in the previous studies.

Take, for instance, the new leptons E, F which are only coupled via W_{34}, χ_4 and Ξ_3 fields. A combination of these scalars is the Goldstone boson of W_{34}, while the orthogonal field is a new Higgs boson. If this new Higgs and W_{34} bosons are heavier than the net mass of E, F, these lightest lepton fields act as two component dark matter. We expect that the Z' resonance sets the E candidate relic, while the Z'' resonance governs the F one. Here Z', Z'' denote two lightest new neutral gauge bosons among the three $Z_{2,3,4}$ as identified below.

2.6 Fermion mass

When the scalars develop VEVs, the fermions gain masses and we write Dirac masses as $-\bar{f}_L m_f f_R + H.c.$ and Majorana masses as $-\frac{1}{2}\bar{f}_{L,R}^c m_f^{L,R} f_{L,R} + H.c.$

The mass matrices of new fermions E_a, F_a, J_a, and K_a are given by

$$[m_E]_{ab} = -h_{ab}^E \frac{w}{\sqrt{2}}, \qquad\qquad [m_F]_{ab} = -h_{ab}^F \frac{V}{\sqrt{2}}, \qquad (2.36)$$

$$[m_J]_{33} = -h_{33}^J \frac{w}{\sqrt{2}}, \qquad\qquad [m_J]_{\alpha\beta} = -h_{\alpha\beta}^J \frac{w}{\sqrt{2}}, \qquad (2.37)$$

$$[m_K]_{33} = -h_{33}^K \frac{V}{\sqrt{2}}, \qquad\qquad [m_K]_{\alpha\beta} = -h_{\alpha\beta}^K \frac{V}{\sqrt{2}}, \qquad (2.38)$$

which all have masses at w, V scale.

The mass matrices of charged-leptons and quarks e_a, u_a and d_a are obtained as

$$[m_e]_{ab} = -h_{ab}^e \frac{v}{\sqrt{2}}, \qquad\qquad\qquad\qquad\qquad\qquad (2.39)$$

$$[m_u]_{3a} = -h_{3a}^u \frac{u}{\sqrt{2}}, \qquad\qquad [m_u]_{\alpha a} = h_{\alpha a}^u \frac{v}{\sqrt{2}}, \qquad (2.40)$$

$$[m_d]_{3a} = -h_{3a}^d \frac{v}{\sqrt{2}}, \qquad\qquad [m_d]_{\alpha a} = -h_{\alpha a}^d \frac{u}{\sqrt{2}}, \qquad (2.41)$$

which provide appropriate masses at u, v scale.

For the neutrinos, $\nu_{aL,R}$, the Dirac and Majorana masses are $[m_\nu]_{ab} = -h_{ab}^\nu \frac{u}{\sqrt{2}}$ and $[m_\nu^R]_{ab} = -\sqrt{2} h_{ab}^{\prime\nu} \Lambda$, respectively. Since $u \ll \Lambda$, the observed neutrinos ($\sim \nu_{aL}$) achieve masses via the type I seesaw mechanism,

$$m_\nu^L \simeq -m_\nu (m_\nu^R)^{-1} (m_\nu)^T \sim u^2/\Lambda, \qquad (2.42)$$

which is small, as expected. The sterile neutrinos ($\sim \nu_{aR}$) obtain large masses, such as m_ν^R.

3 Kinetic mixing

3.1 Canonical basis

Let us write down the kinetic terms of the two U(1) gauge fields as

$$\mathcal{L}_{\text{kinetic}} \supset -\frac{1}{4} B_{\mu\nu}^2 - \frac{1}{4} C_{\mu\nu}^2 - \frac{\delta}{2} B_{\mu\nu} C^{\mu\nu} = -\frac{1}{4}(B_{\mu\nu} + \delta C_{\mu\nu})^2 - \frac{1}{4}(1-\delta^2) C_{\mu\nu}^2, \qquad (3.1)$$

where $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and $C_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$ are the corresponding field strength tensors.

Because of the kinetic mixing term (δ), the two gauge bosons B_μ and C_μ are generally not orthonormalized. We change to the canonical basis by a nonunitary transformation $(B_\mu, C_\mu) \to (B'_\mu, C'_\mu)$, where

$$B' = B + \delta C, \quad C' = \sqrt{1-\delta^2} C. \qquad (3.2)$$

We substitute B, C in terms of B', C' into the covariant derivative. It becomes

$$D_\mu \supset ig_X X B_\mu + ig_N N C_\mu = ig_X X B'_\mu + \frac{i}{\sqrt{1-\delta^2}}(g_N N - g_X X \delta) C'_\mu, \qquad (3.3)$$

which is given in terms of the orthonormalized (canonical) fields (B'_μ, C'_μ).

3.2 Gauge boson mass

The 3-4-1-1 symmetry breaking leads to mixings among A_3, A_8, A_{15}, B', and C'. Their mass Lagrangian arises from $\sum_S (D_\mu \langle S \rangle)^\dagger (D^\mu \langle S \rangle)$, such that

$$\mathcal{L}_{\text{mass}}^{\text{neutral}} = \frac{1}{2} \begin{pmatrix} A_3 & A_8 & A_{15} & B' & C' \end{pmatrix} M^2 \begin{pmatrix} A_3 & A_8 & A_{15} & B' & C' \end{pmatrix}^T, \tag{3.4}$$

where the mass matrix $M^2 = \{m_{ij}^2\}$ is symmetric, possessing the elements,

$$m_{11}^2 = \frac{g^2}{4}(u^2 + v^2), \quad m_{12}^2 = \frac{g^2}{4\sqrt{3}}(u^2 - v^2), \quad m_{13}^2 = \frac{g^2}{4\sqrt{6}}(u^2 - v^2),$$

$$m_{14}^2 = -\frac{g^2 t_X}{4\sqrt{6}}[\beta_1 u^2 + (2\sqrt{6} - \beta_1)v^2],$$

$$m_{15}^2 = \frac{g^2}{4\sqrt{6(1-\delta^2)}}\{[\delta\beta_1 t_X - (\sqrt{2}b + c)t_N]u^2 + [\delta(2\sqrt{6} - \beta_1)t_X + (\sqrt{2}b + c)t_N]v^2\},$$

$$m_{22}^2 = \frac{g^2}{12}(u^2 + v^2 + 4w^2), \quad m_{23}^2 = \frac{g^2}{12\sqrt{2}}(u^2 + v^2 - 2w^2),$$

$$m_{24}^2 = -\frac{g^2 t_X}{12\sqrt{2}}[\beta_1 u^2 - (2\sqrt{6} - \beta_1)v^2 + 2(2\sqrt{2}\beta - \gamma)w^2],$$

$$m_{25}^2 = \frac{g^2}{12\sqrt{2(1-\delta^2)}}\{[\delta\beta_1 t_X - (\sqrt{2}b + c)t_N]u^2 - [\delta(2\sqrt{6} - \beta_1)t_X + (\sqrt{2}b + c)t_N]v^2$$
$$+ 2[\delta(2\sqrt{2}\beta - \gamma)t_X - (2\sqrt{2}b - c)t_N]w^2\},$$

$$m_{33}^2 = \frac{g^2}{24}(u^2 + v^2 + w^2 + 9V^2), \quad m_{34}^2 = -\frac{g^2 t_X}{24}[\beta_1 u^2 - (2\sqrt{6} - \beta_1)v^2 - (2\sqrt{2}\beta - \gamma)w^2 + 9\gamma V^2],$$

$$m_{35}^2 = \frac{g^2}{24\sqrt{1-\delta^2}}\{[\delta\beta_1 t_X - (\sqrt{2}b + c)t_N]u^2 - [\delta(2\sqrt{6} - \beta_1)t_X + (\sqrt{2}b + c)t_N]v^2$$
$$- [\delta(2\sqrt{2}\beta - \gamma)t_X - (2\sqrt{2}b - c)t_N]w^2 + 9(\delta\gamma t_X - ct_N)V^2\},$$

$$m_{44}^2 = \frac{g^2 t_X^2}{24}[\beta_1^2 u^2 + (2\sqrt{6} - \beta_1)^2 v^2 + (2\sqrt{2}\beta - \gamma)^2 w^2 + 9\gamma^2 V^2],$$

$$m_{45}^2 = -\frac{g^2 t_X}{24\sqrt{1-\delta^2}}\{[\delta\beta_1 t_X - (\sqrt{2}b + c)t_N]\beta_1 u^2 + [\delta(2\sqrt{6} - \beta_1)t_X + (\sqrt{2}b + c)t_N](2\sqrt{6} - \beta_1)v^2$$
$$+ [\delta(2\sqrt{2}\beta - \gamma)t_X - (2\sqrt{2}b - c)t_N](2\sqrt{2}\beta - \gamma)w^2 + 9\gamma(\delta\gamma t_X - ct_N)V^2\},$$

$$m_{55}^2 = \frac{g^2}{24(1-\delta^2)}\{[\delta\beta_1 t_X - (\sqrt{2}b + c)t_N]^2 u^2 + [\delta(2\sqrt{6} - \beta_1)t_X + (\sqrt{2}b + c)t_N]^2 v^2$$
$$+ [\delta(2\sqrt{2}\beta - \gamma)t_X - (2\sqrt{2}b - c)t_N]^2 w^2 + 9(\delta\gamma t_X - ct_N)^2 V^2 + 96t_N^2 \Lambda^2\},$$

where we have defined $t_X = g_X/g$, $t_N = g_N/g$, and $\beta_1 = \sqrt{6} + \sqrt{2}\beta + \gamma$.

The mass matrix always provides a zero eigenvalue with corresponding eigenstate (photon field),

$$A = s_W A_3 + c_W \left(\beta t_W A_8 + \gamma t_W A_{15} + \frac{t_W}{t_X} B' \right), \tag{3.5}$$

where $s_W = e/g = t_X / \sqrt{1 + (1 + \beta^2 + \gamma^2)t_X^2}$ is the sine of the Weinberg's angle [52]. Since the field in parentheses of (3.5) is properly the hypercharge field coupled to $Y = Q - T_3$, we define the standard model Z as

$$Z = c_W A_3 - s_W \left(\beta t_W A_8 + \gamma t_W A_{15} + \frac{t_W}{t_X} B' \right). \tag{3.6}$$

The new neutral gauge bosons, called Z'_2, Z'_3, orthogonal to the hypercharge field take the forms,

$$Z'_2 = \frac{1}{\sqrt{1-\beta^2 t_W^2}}\left[(1-\beta^2 t_W^2)A_8 - \beta\gamma t_W^2 A_{15} - \frac{\beta t_W^2}{t_X}B'\right], \tag{3.7}$$

$$Z'_3 = \frac{1}{\sqrt{1+\gamma^2 t_X^2}}\left(A_{15} - \gamma t_X B'\right). \tag{3.8}$$

At this stage, C' is always orthogonal to A, Z, Z'_2, Z'_3.

Let us change to the new basis A, Z, Z'_2, Z'_3, and C', such that $(A_3\, A_8\, A_{15}\, B'\, C')^T = U_1(A\, Z\, Z'_2\, Z'_3\, C')^T$, where

$$U_1 = \begin{pmatrix} s_W & c_W & 0 & 0 & 0 \\ \beta s_W & -\beta s_W t_W & \sqrt{1-\beta^2 t_W^2} & 0 & 0 \\ \gamma s_W & -\gamma s_W t_W & -\frac{\beta\gamma t_W^2}{\sqrt{1-\beta^2 t_W^2}} & \frac{1}{\sqrt{1+\gamma^2 t_X^2}} & 0 \\ \frac{s_W}{t_X} & -\frac{s_W t_W}{t_X} & -\frac{\beta t_W^2}{t_X\sqrt{1-\beta^2 t_W^2}} & -\frac{\gamma t_X}{\sqrt{1+\gamma^2 t_X^2}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.9}$$

The mass matrix M^2 is correspondingly changed to

$$M'^2 = U_1^T M^2 U_1 = \begin{pmatrix} 0 & 0 \\ 0 & M_s'^2 \end{pmatrix}, \qquad M_s'^2 \equiv \begin{pmatrix} m_Z^2 & m_{ZZ'_2}^2 & m_{ZZ'_3}^2 & m_{ZC'}^2 \\ m_{ZZ'_2}^2 & m_{Z'_2}^2 & m_{Z'_2 Z'_3}^2 & m_{Z'_2 C'}^2 \\ m_{ZZ'_3}^2 & m_{Z'_2 Z'_3}^2 & m_{Z'_3}^2 & m_{Z'_3 C'}^2 \\ m_{ZC'}^2 & m_{Z'_2 C'}^2 & m_{Z'_3 C'}^2 & m_{C'}^2 \end{pmatrix}. \tag{3.10}$$

where

$$m_Z^2 = \frac{g^2}{4c_W^2}(u^2+v^2), \qquad m_{ZZ'_2}^2 = \frac{g^2\sqrt{1-(\beta^2+\gamma^2)t_W^2}}{4\sqrt{3}c_W\sqrt{1+\gamma^2 t_X^2}}\{\beta_2 u^2 + (2\sqrt{3}\beta t_X^2 - \beta_2)v^2\},$$

$$m_{ZZ'_3}^2 = \frac{g^2}{4\sqrt{6}c_W\sqrt{1+\gamma^2 t_X^2}}\{(1+\gamma\beta_1 t_X^2)u^2 + [\gamma(2\sqrt{6}-\beta_1)t_X^2-1]v^2\},$$

$$m_{ZC'}^2 = \frac{g^2}{4\sqrt{6}c_W\sqrt{1-\delta^2}}\{[\delta\beta_1 t_X - (\sqrt{2}b+c)t_N]u^2 + [\delta(2\sqrt{6}-\beta_1)t_X + (\sqrt{2}b+c)t_N]v^2\},$$

$$m_{Z'_2}^2 = \frac{g^2[1-(\beta^2+\gamma^2)t_W^2]}{12(1+\gamma^2 t_X^2)}\{\beta_2^2 u^2 + (2\sqrt{3}\beta t_X^2-\beta_2)^2 v^2 + 4[1+(\beta^2+\gamma^2)t_X^2]^2 w^2\},$$

$$m_{Z'_2 Z'_3}^2 = \frac{g^2\sqrt{1-(\beta^2+\gamma^2)t_W^2}}{12\sqrt{2}(1+\gamma^2 t_X^2)}\{(1+\gamma\beta_1 t_X^2)\beta_2 u^2 + [\gamma(2\sqrt{6}-\beta_1)t_X^2-1](2\sqrt{3}\beta t_X^2-\beta_2)v^2$$
$$+2[\gamma(2\sqrt{2}\beta-\gamma)t_X^2-1][1+(\beta^2+\gamma^2)t_X^2]w^2\},$$

$$m_{Z'_2 C'}^2 = \frac{g^2\sqrt{1-(\beta^2+\gamma^2)t_W^2}}{12\sqrt{2}\sqrt{(1-\delta^2)(1+\gamma^2 t_X^2)}}\{[\delta\beta_1 t_X - (\sqrt{2}b+c)t_N]\beta_2 u^2 + [\delta(2\sqrt{6}-\beta_1)t_X + (\sqrt{2}b+c)t_N]$$
$$\times(2\sqrt{3}\beta t_X^2-\beta_2)v^2 + 2[\delta(2\sqrt{2}\beta-\gamma)t_X - (2\sqrt{2}b-c)t_N][1+(\beta^2+\gamma^2)t_X^2]w^2\},$$

$$m_{Z'_3}^2 = \frac{g^2}{24(1+\gamma^2 t_X^2)}\{(1+\gamma\beta_1 t_X^2)^2 u^2 + [\gamma(2\sqrt{6}-\beta_1)t_X^2-1]^2 v^2 + [\gamma(2\sqrt{2}\beta-\gamma)t_X^2-1]^2 w^2$$
$$+9(1+\gamma^2 t_X^2)^2 V^2\},$$

$$m_{Z_3'C'}^2 = \frac{g^2}{24\sqrt{(1-\delta^2)(1+\gamma^2 t_X^2)}}\{[\delta\beta_1 t_X - (\sqrt{2}b+c)t_N](1+\gamma\beta_1 t_X^2)u^2 + [\delta(2\sqrt{6}-\beta_1)t_X + (\sqrt{2}b+c)t_N]$$

$$\times[\gamma(2\sqrt{6}-\beta_1)t_X^2 - 1]v^2 + [\delta(2\sqrt{2}\beta-\gamma)t_X - (2\sqrt{2}b-c)t_N][\gamma(2\sqrt{2}\beta-\gamma)t_X^2 - 1]w^2$$

$$+9(\delta\gamma t_X - ct_N)(1+\gamma^2 t_X^2)V^2\},$$

$$m_{C'}^2 = m_{55}^2,$$

where we have defined $\beta_2 = 1 + (\sqrt{3}\beta + \beta^2 + \gamma^2)t_X^2$.

Since $u, v \ll w, V, \Lambda$, the first row and first column of $M_s'^2$ consist of the elements much smaller than those of the remaining entries. We diagonalize $M_s'^2$ using the seesaw formula [4–12] that separates Z from the heavy fields, given by

$$\left(Z\,Z_2'\,Z_3'\,C'\right)^T = U_2 \left(Z_1\,\mathcal{Z}_2'\,\mathcal{Z}_3'\,\mathcal{C}'\right)^T, \quad M''^2 = U_2^T M_s'^2 U_2 = \begin{pmatrix} m_{Z_1}^2 & 0 \\ 0 & M_s''^2 \end{pmatrix}, \quad (3.11)$$

where Z_1 is physical as decoupled, while $\mathcal{Z}_2'$, $\mathcal{Z}_3'$ and $\mathcal{C}'$ mix via $M_s''^2$, such that

$$U_2 \simeq \begin{pmatrix} 1 & \epsilon_1 & \epsilon_2 & \epsilon_3 \\ -\epsilon_1 & 1 & 0 & 0 \\ -\epsilon_2 & 0 & 1 & 0 \\ -\epsilon_3 & 0 & 0 & 1 \end{pmatrix}, \quad M_s''^2 \simeq \begin{pmatrix} m_{Z_2'}^2 & m_{Z_2'Z_3'}^2 & m_{Z_2'C'}^2 \\ m_{Z_2'Z_3'}^2 & m_{Z_3'}^2 & m_{Z_3'C'}^2 \\ m_{Z_2'C'}^2 & m_{Z_3'C'}^2 & m_{C'}^2 \end{pmatrix}, \quad (3.12)$$

$$m_{Z_1}^2 \simeq m_Z^2 - \epsilon_1 m_{ZZ_2'}^2 - \epsilon_2 m_{ZZ_3'}^2 - \epsilon_3 m_{ZC'}^2. \quad (3.13)$$

We further separate $\epsilon_{1,2,3} \equiv \epsilon_{1,2,3}^0 + \epsilon_{1,2,3}^\delta$, where $\epsilon_{1,2,3}^0$ are the mixing of Z with Z_2', Z_3', and C' due to the symmetry breaking, whereas $\epsilon_{1,2,3}^\delta$ determine those mixings due to the kinetic mixing,

$$\epsilon_1^0 = \frac{1}{4c_W \sqrt{1+\gamma^2 t_X^2}[1+(\beta^2+\gamma^2)t_X^2]^{3/2}}\left\{\frac{\sqrt{3}(1+\gamma^2 t_X^2)[\beta_2 u^2 + (2\sqrt{3}\beta t_X^2 - \beta_2)v^2]}{w^2}\right.$$

$$+\frac{(\beta+2\sqrt{2}\gamma)[1+\gamma(\gamma-2\sqrt{2}\beta)t_X^2]t_X^2(u^2+v^2) + \sqrt{3}[1-\gamma(2\sqrt{2}\beta-\gamma)t_X^2][1+(\beta^2+\gamma^2)t_X^2](u^2-v^2)}{3V^2}$$

$$\left.+\frac{(b\beta+c\gamma)[b-\gamma(c\beta-b\gamma)t_X^2]t_X^2(u^2+v^2)}{4\Lambda^2}\right\}, \quad (3.14)$$

$$\epsilon_2^0 = \frac{1}{c_W \sqrt{1+\gamma^2 t_X^2}[1+(\beta^2+\gamma^2)t_X^2]}\left\{\frac{(\beta+2\sqrt{2}\gamma)t_X^2(u^2+v^2) + \sqrt{3}[1+(\beta^2+\gamma^2)t_X^2](u^2-v^2)}{3\sqrt{2}V^2}\right.$$

$$\left.+\frac{c(b\beta+c\gamma)t_X^2(u^2+v^2)}{16\Lambda^2}\right\}, \quad (3.15)$$

$$\epsilon_3^0 = \frac{(b\beta+c\gamma)t_X^2(u^2+v^2)}{16c_W[1+(\beta^2+\gamma^2)t_X^2]t_N\Lambda^2}, \quad (3.16)$$

$$\epsilon_1^\delta = \frac{\delta\{[b(1+\gamma^2 t_X^2) - \beta(b\beta+2c\gamma)t_X^2]t_N - \delta\beta t_X\}t_X(u^2+v^2)}{16c_W \sqrt{1+\gamma^2 t_X^2}[1+(\beta^2+\gamma^2)t_X^2]^{3/2}t_N^2\Lambda^2}, \quad (3.17)$$

$$\epsilon_2^\delta = \frac{\delta[(ct_N - \delta\gamma t_X) - \gamma(b\beta+c\gamma)t_X^2 t_N]t_X(u^2+v^2)}{16c_W \sqrt{1+\gamma^2 t_X^2}[1+(\beta^2+\gamma^2)t_X^2]t_N^2\Lambda^2}, \quad (3.18)$$

$$\epsilon_3^\delta = \frac{\delta t_X(u^2+v^2)}{16c_W[1+(\beta^2+\gamma^2)t_X^2]t_N\Lambda^2}\left\{\frac{\sqrt{1-\delta^2}}{t_N} - \frac{\delta(b\beta+c\gamma)t_X}{1+\sqrt{1-\delta^2}}\right\}. \quad (3.19)$$

Because $\epsilon_{1,2,3} \sim (u^2,v^2)/(w^2,V^2,\Lambda^2)$, the mixings are very small.

Next, the symmetry breaking is done through three possible ways, corresponding to the assumptions: $w, V \ll \Lambda$, $w \ll V, \Lambda$, or $w, \Lambda \ll V$. Let us consider the first case,

$w, V \ll \Lambda$. We have the element $m^2_{C'}$ much larger than the remainders. The mass matrix M''^2_s can be diagonalized by using the seesaw formula, which yields

$$(\mathcal{Z}'_2 \, \mathcal{Z}'_3 \, \mathcal{C}')^T = U_3 (\mathcal{Z}_2 \, \mathcal{Z}_3 \, Z_4)^T, \quad M'''^2 = U_3^T M''^2_s U_3 = \begin{pmatrix} M^2_{2\times2} & 0 \\ 0 & m^2_{Z_4} \end{pmatrix}, \quad (3.20)$$

where Z_4 is physical as decoupled, while $\mathcal{Z}_2, \mathcal{Z}_3$ mix via $M^2_{2\times2}$. We obtain

$$U_3 \simeq \begin{pmatrix} 1 & 0 & \zeta_1 \\ 0 & 1 & \zeta_2 \\ -\zeta_1 & -\zeta_2 & 1 \end{pmatrix}, \quad M^2_{2\times2} = \begin{pmatrix} m^2_{11} & m^2_{12} \\ m^2_{12} & m^2_{22} \end{pmatrix}, \quad m^2_{Z_4} \simeq m^2_{C'}, \quad (3.21)$$

where $\zeta_{1,2} \equiv \zeta^0_{1,2} + \zeta^\delta_{1,2}$,

$$\zeta^0_1 = -\frac{(2\sqrt{2}b-c)w^2}{24\sqrt{2}\sqrt{1-\beta^2 t^2_W} t_N \Lambda^2}, \quad (3.22)$$

$$\zeta^0_2 = -\frac{9V^2(1+\gamma^2 t^2_X)c - w^2[1-\gamma(2\sqrt{2}\beta-\gamma)t^2_X](2\sqrt{2}b-c)}{96\sqrt{1+\gamma^2 t^2_X} t_N \Lambda^2}, \quad (3.23)$$

$$\zeta^\delta_1 = \frac{\delta w^2[(1-\delta^2+\sqrt{1-\delta^2})(2\sqrt{2}\beta-\gamma)t_X + \delta(2\sqrt{2}b-c)t_N]}{24\sqrt{2}\sqrt{1-\beta^2 t^2_W}(1+\sqrt{1-\delta^2})t^2_N \Lambda^2}, \quad (3.24)$$

$$\zeta^\delta_2 = \frac{\delta}{96\sqrt{1+\gamma^2 t^2_X}(1+\sqrt{1-\delta^2})t^2_N \Lambda^2} \Big\{ 9V^2(1+\gamma^2 t^2_X)[\gamma(1-\delta^2+\sqrt{1-\delta^2})t_X + \delta c t_N] \quad (3.25)$$

$$-w^2[1-\gamma(2\sqrt{2}\beta-\gamma)t^2_X][(1-\delta^2+\sqrt{1-\delta^2})(2\sqrt{2}\beta-\gamma)t_X + \delta(2\sqrt{2}b-c)t_N] \Big\}, \quad (3.26)$$

which are very small, and

$$m^2_{11} \simeq m^2_{Z'_2} - \zeta_1 m^2_{Z'_2 C'} \simeq m^2_{Z'_2}, \quad (3.27)$$

$$m^2_{12} \simeq m^2_{Z'_2 Z'_3} - \zeta_1 m^2_{Z'_3 C'} \simeq m^2_{Z'_2 Z'_3}, \quad (3.28)$$

$$m^2_{22} \simeq m^2_{Z'_3} - \zeta_2 m^2_{Z'_3 C'} \simeq m^2_{Z'_3}. \quad (3.29)$$

Last, we diagonalize $M^2_{2\times2}$ to yield two remaining physical gauge bosons,

$$Z_2 = c_\varphi \mathcal{Z}_2 - s_\varphi \mathcal{Z}_3, \quad Z_3 = s_\varphi \mathcal{Z}_2 + c_\varphi \mathcal{Z}_3. \quad (3.30)$$

The $\mathcal{Z}_2 - \mathcal{Z}_3$ mixing angle and Z_2, Z_3 masses are given by

$$t_{2\varphi} \simeq \frac{4\sqrt{2}w^2[1-\gamma(2\sqrt{2}\beta-\gamma)t^2_X]\sqrt{1+(\beta^2+\gamma^2)t^2_X}}{w^2[7-\gamma^2(2\sqrt{2}\beta-\gamma)^2 t^4_X + (8\beta^2+4\sqrt{2}\beta\gamma+6\gamma^2)t^2_X] - 9V^2(1+\gamma^2 t^2_X)^2}, \quad (3.31)$$

$$m^2_{Z_2,Z_3} = \frac{1}{2}[m^2_{11}+m^2_{22} \mp \sqrt{(m^2_{11}-m^2_{22})^2+4m^4_{12}}]. \quad (3.32)$$

Now we consider two other cases, $w \ll V, \Lambda$ and $w, \Lambda \ll V$. Because $m^2_{Z'_2}, m^2_{Z'_2 Z'_3}, m^2_{Z'_2 C'} \ll m^2_{Z'_3}, m^2_{Z'_3 C'}, m^2_{C'}$, the mass matrix M''^2_s can be diagonalized, obeying

$$(\mathcal{Z}'_2 \, \mathcal{Z}'_3 \, \mathcal{C}')^T = U'_3 (Z_2 \, \mathcal{Z}_3 \, \mathcal{C})^T, \quad M'''^2 = U'^T_3 M''^2_s U'_3 = \begin{pmatrix} m^2_{Z_2} & 0 \\ 0 & M'^2_{2\times2} \end{pmatrix}, \quad (3.33)$$

where Z_2 is physical as decoupled, while $\mathcal{Z}_3$ and $\mathcal{C}$ mix via $M_{2\times2}'^2$, and

$$U_3' \simeq \begin{pmatrix} 1 & \mathcal{E}_1 & \mathcal{E}_2 \\ -\mathcal{E}_1 & 1 & 0 \\ -\mathcal{E}_2 & 0 & 1 \end{pmatrix}, \quad M_{2\times2}'^2 \simeq \begin{pmatrix} m_{Z_3'}^2 & m_{Z_3'C'}^2 \\ m_{Z_3'C'}^2 & m_{C'}^2 \end{pmatrix}, \quad (3.34)$$

$$m_{Z_2}^2 \simeq m_{Z_2'}^2 - \mathcal{E}_1 m_{Z_2'Z_3'}^2 - \mathcal{E}_2 m_{Z_2'C'}^2. \quad (3.35)$$

Further for the case $w \ll V, \Lambda$, we achieve $\mathcal{E}_{1,2} \equiv \mathcal{E}_{1,2}^0 + \mathcal{E}_{1,2}^\delta$, where

$$\mathcal{E}_1^0 = \frac{\sqrt{1+(\beta^2+\gamma^2)t_X^2}\,w^2}{3(1+\gamma^2t_X^2)^2}\left\{\frac{4[\gamma(2\sqrt{2}\beta-\gamma)t_X^2-1]}{3\sqrt{2}V^2} - \frac{[b(1+\gamma^2t_X^2)-c\beta\gamma t_X^2]c}{4\Lambda^2}\right\}, \quad (3.36)$$

$$\mathcal{E}_2^0 = \frac{\sqrt{1+(\beta^2+\gamma^2)t_X^2}[\gamma(c\beta-b\gamma)t_X^2-b]w^2}{12(1+\gamma^2t_X^2)^{3/2}t_N\Lambda^2}, \quad (3.37)$$

$$\mathcal{E}_1^\delta = \frac{\delta\sqrt{1+(\beta^2+\gamma^2)t_X^2}\,t_X[b\gamma(1+\gamma^2t_X^2)t_N+c\beta(1-\gamma^2t_X^2)t_N-\delta\beta\gamma t_X]w^2}{12(1+\gamma^2t_X^2)^2t_N^2\Lambda^2}, \quad (3.38)$$

$$\mathcal{E}_2^\delta = \frac{\delta\sqrt{1+(\beta^2+\gamma^2)t_X^2}\,w^2}{12(1+\gamma^2t_X^2)^{3/2}t_N\Lambda^2}\left\{\frac{\delta[b+\gamma(b\gamma-c\beta)t_X^2]}{1+\sqrt{1-\delta^2}} + \frac{\sqrt{1-\delta^2}\beta t_X}{t_N}\right\}, \quad (3.39)$$

which are very small. Otherwise, for the case $w, \Lambda \ll V$, we have

$$\mathcal{E}_1 = -\frac{\sqrt{1+(\beta^2+\gamma^2)t_X^2}(\delta\gamma t_X - ct_N)[\beta(\delta+c\gamma t_X t_N)t_X - b(1+\gamma^2t_X^2)t_N]w^2}{[\beta(\delta+c\gamma t_X t_N)t_X - b(1+\gamma^2t_X^2)t_N]^2w^2 + 12(1+\gamma^2t_X^2)^2t_N^2\Lambda^2}, \quad (3.40)$$

$$\mathcal{E}_2 = \frac{\sqrt{(1-\delta^2)(1+\gamma^2t_X^2)[1+(\beta^2+\gamma^2)t_X^2]}[\beta(\delta+c\gamma t_X t_N)t_X - b(1+\gamma^2t_X^2)t_N]w^2}{[\beta(\delta+c\gamma t_X t_N)t_X - b(1+\gamma^2t_X^2)t_N]^2w^2 + 12(1+\gamma^2t_X^2)^2t_N^2\Lambda^2}, \quad (3.41)$$

which may be large.

We diagonalize the mass matrix $M_{2\times2}'^2$ to get two remaining physical gauge bosons, such that

$$Z_3 = c_\xi\mathcal{Z}_3 - s_\xi\mathcal{C}, \quad Z_4 = s_\xi\mathcal{Z}_3 + c_\xi\mathcal{C}. \quad (3.42)$$

The $\mathcal{Z}_3 - \mathcal{C}$ mixing angle for the case $w \ll V, \Lambda$ is given by

$$t_{2\xi} \simeq \frac{6\sqrt{1-\delta^2}\sqrt{1+\gamma^2t_X^2}(\delta\gamma t_X - ct_N)V^2}{3[(\delta\gamma t_X - ct_N)^2 - (1-\delta^2)(1+\gamma^2t_X^2)]V^2 + 32t_N^2\Lambda^2}, \quad (3.43)$$

which may be large. For the case $w, \Lambda \ll V$, the $\mathcal{Z}_3 - \mathcal{C}$ mixing angle is defined similarly to (3.43), but the term associated to Λ should be omitted. In particular, all the two cases imply $\xi = 0$ when $\delta = ct_N/\gamma t_X$, the condition by which the kinetic mixing and symmetry breaking effects cancels out. Besides, the Z_3, Z_4 masses are given by

$$m_{Z_3,Z_4}^2 = \frac{1}{2}\left[m_{Z_3'}^2 + m_{C'}^2 \mp \sqrt{(m_{Z_3'}^2 - m_{C'}^2)^2 + 4m_{Z_3'C'}^4}\right]. \quad (3.44)$$

In summary, the original fields are related to the mass eigenstates by $(A_3\, A_8\, A_{15}\, B\, C)^T = U(A\, Z_1\, Z_2\, Z_3\, Z_4)^T$. For the first case, $w, V \ll \Lambda$, we have $U = U_\delta U_1 U_2 U_3 U_\varphi \simeq U_\delta U_1 U_2 U_\varphi$. For the second case, $w \ll V, \Lambda$, we obtain $U = U_\delta U_1 U_2 U_3' U_\xi \simeq U_\delta U_1 U_2 U_\xi$. For the last case, $w, \Lambda \ll V$, the mixing matrix is $U = U_\delta U_1 U_2 U_3' U_\xi$. Here we define

$$U_\delta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{\delta}{\sqrt{1-\delta^2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{1-\delta^2}} \end{pmatrix}, \quad U_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c_\varphi & s_\varphi & 0 \\ 0 & 0 & -s_\varphi & c_\varphi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad U_\xi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & c_\xi & s_\xi \\ 0 & 0 & 0 & -s_\xi & c_\xi \end{pmatrix}.$$

$$(3.45)$$

The fields A, Z_1 are identical to the standard model, whereas Z_2, Z_3 and Z_4 are new, heavy gauge bosons. The mixings of the standard model gauge bosons with the new gauge bosons are very small, while the mixing within the new gauge bosons may be large.

4 Electroweak precision test

4.1 ρ parameter

The new physics that contributes to the ρ-parameter starts from the tree-level. This is caused by the mixing of the Z boson with the new neutral gauge bosons. We evaluate

$$\Delta\rho = \frac{m_W^2}{c_W^2 m_{Z_1}^2} - 1 = \frac{m_Z^2}{m_Z^2 - \epsilon_1 m_{ZZ_2'}^2 - \epsilon_2 m_{ZZ_3'}^2 - \epsilon_3 m_{ZC'}^2} - 1 \simeq \frac{\epsilon_1 m_{ZZ_2'}^2 + \epsilon_2 m_{ZZ_3'}^2 + \epsilon_3 m_{ZC'}^2}{m_Z^2}$$
$$\equiv (\Delta\rho)^0 + (\Delta\rho)^\delta, \tag{4.1}$$

where

$$(\Delta\rho)^0 \simeq \frac{1}{4[1+(\beta^2+\gamma^2)t_X^2]^2} \left\{ \frac{[\beta_2 u^2 + (2\sqrt{3}\beta t_X^2 - \beta_2)v^2]^2}{(u^2+v^2)w^2} \right.$$
$$+ \frac{\{(\beta+2\sqrt{2}\gamma)t_X^2(u^2+v^2) + \sqrt{3}[1+(\beta^2+\gamma^2)t_X^2](u^2-v^2)\}^2}{3(u^2+v^2)V^2}$$
$$\left. + \frac{(b\beta+c\gamma)^2 t_X^4(u^2+v^2)}{4\Lambda^2} \right\}, \tag{4.2}$$

$$(\Delta\rho)^\delta \simeq \frac{\delta[\delta+2(b\beta+c\gamma)t_X t_N]t_X^2(u^2+v^2)}{16[1+(\beta^2+\gamma^2)t_X^2]^2 t_N^2 \Lambda^2}. \tag{4.3}$$

This tree-level contribution is appropriately suppressed due to $u, v \ll w, V, \Lambda$.

Further, the ρ parameter at one-loop level is important because its deviation from the standard model prediction is proportional to $\Delta\rho = \rho - 1 \simeq \alpha T$, where the self-energy parameter T includes (sensitive to) all new sources that are used to break the vector part of weak SU(2), such as heavy non-degenerate multiplets including non-standard Higgs representations, which often lead to a decrease in the value of m_Z/m_W, i.e. $\rho > 1$. However,

such contributions are only significant if the new physics scale is not too high, compared to the weak scale [3].

In this model, the ρ deviation may receive one-loop corrections dominantly by non-degenerate vector multiplets, such as (W_{13}, W_{23}) and (W_{14}, W_{24}, W_{34}), similar to the 3-3-1 model [53–56]. However, this source can be neglected if the new gauge bosons are heavy at TeV. In this analysis, we consider only the tree-level contribution.

Let us note that $\beta^2 + \gamma^2 < 1/s_W^2 - 1 \simeq 3.329$, which fixes $|\gamma|, |\beta| < 1.82456$. The condition $q = -(1 + \sqrt{3}\beta)/2$ leads to $-2.08012 < q < 1.08012$. Considering q to be integer implies $q = -2, -1, 0$, and 1. When $q = -2$, i.e. $\beta = \sqrt{3}$, we obtain $|\gamma| < 0.57359$. The condition $p = -(1 + \sqrt{6}\gamma - q)/3$ provides $-1.46833 < p < -0.53167$, thus $p = -1$, i.e. $\gamma = 0$, given that p is integer. When $q = -1$, i.e. $\beta = 1/\sqrt{3}$, we get $p = -2, -1, 0$, thus $\gamma = 4/\sqrt{6}, 1/\sqrt{6}, -\sqrt{2}/\sqrt{3}$, respectively. When $q = 0$, i.e. $\beta = -1/\sqrt{3}$, we gain $p = -1, 0, 1$, thus $\gamma = \sqrt{2}/\sqrt{3}, -1/\sqrt{6}, -4/\sqrt{6}$, respectively. When $q = 1$, i.e. $\beta = -\sqrt{3}$, we have $p = 0$, thus $\gamma = 0$.

However, we are interested in the four models for dark matter, such that $q = p = -1$ (or $\beta = 1/\sqrt{3}, \gamma = 1/\sqrt{6}$), $q = -1, p = 0$ (or $\beta = 1/\sqrt{3}, \gamma = -\sqrt{2}/\sqrt{3}$), $q = 0, p = -1$ (or $\beta = -1/\sqrt{3}, \gamma = \sqrt{2}/\sqrt{3}$), and $q = p = 0$ (or $\beta = -1/\sqrt{3}, \gamma = -1/\sqrt{6}$).[3] Besides, we take $n = m = 0$, thus $b = -2/\sqrt{3}, c = -\sqrt{2}/\sqrt{3}$, for brevity. This case implies that the wrong particles are old. On the other hand, the W mass, $m_W^2 = \frac{g^2}{4}(u^2 + v^2)$, implies $u^2 + v^2 = (246 \text{ GeV})^2$. We will take u in the range $(0, 246)$ GeV, while v is related to u.

The ρ deviation is given from the global fit by $0.0002 < \Delta\rho < 0.00058$ [3]. For the cases, $w, V \ll \Lambda$ and $w \ll V, \Lambda$, $\Delta\rho$ is independent of δ. Additionally, in the latter case $(w \ll V, \Lambda)$, $\Delta\rho$ is independent of γ. However, in the case $w, \Lambda \ll V$, all the parameters contribute to $\Delta\rho$, except for V. Without loss of generality, we impose $V = 2w$ for the case $w, V \ll \Lambda$ while $\Lambda = 2w$ for the case $w, \Lambda \ll V$. Besides, we put $t_N = 0.5$.

In figure 1, we make a contour of $\Delta\rho$ as the function of (u, w) concerning the first case of VEV arrangement. Here, the panels arranging from left to right correspond to the four dark matter models such as $(\beta = 1/\sqrt{3}, \gamma = 1/\sqrt{6})$, $(\beta = 1/\sqrt{3}, \gamma = -\sqrt{2}/\sqrt{3})$, $(\beta = -1/\sqrt{3}, \gamma = \sqrt{2}/\sqrt{3})$, and $(\beta = -1/\sqrt{3}, \gamma = -1/\sqrt{6})$, respectively.

In figure 2, we make a contour of $\Delta\rho$ as the function of (u, w) for the second case of VEV arrangement. Here, we have only two viable cases, the left panel for $\beta = 1/\sqrt{3}$ and the right panel for $\beta = -1/\sqrt{3}$.

The third case depending on the kinetic mixing parameter is given in figures 3, 4, 5, and 6 according to the dark matter models $(\beta = 1/\sqrt{3}, \gamma = 1/\sqrt{6})$, $(\beta = 1/\sqrt{3}, \gamma = -\sqrt{2}/\sqrt{3})$, $(\beta = -1/\sqrt{3}, \gamma = \sqrt{2}/\sqrt{3})$, and $(\beta = -1/\sqrt{3}, \gamma = -1/\sqrt{6})$, respectively. It is clear that the new physics scale bound is increased, when $|\delta|$ increases. The effect of δ is strong, when u reaches values near 145 GeV for the first dark matter model. By contrast, when u approaches 0 or 246 GeV, the effect is negligible. In summary, the kinetic mixing effect is important when the new physics is considered.
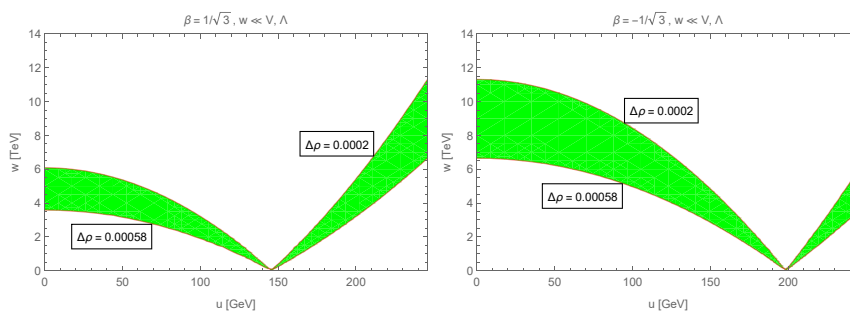
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[3]In these cases, the Landau pole is high enough, such that the new physics is viable [57, 58].
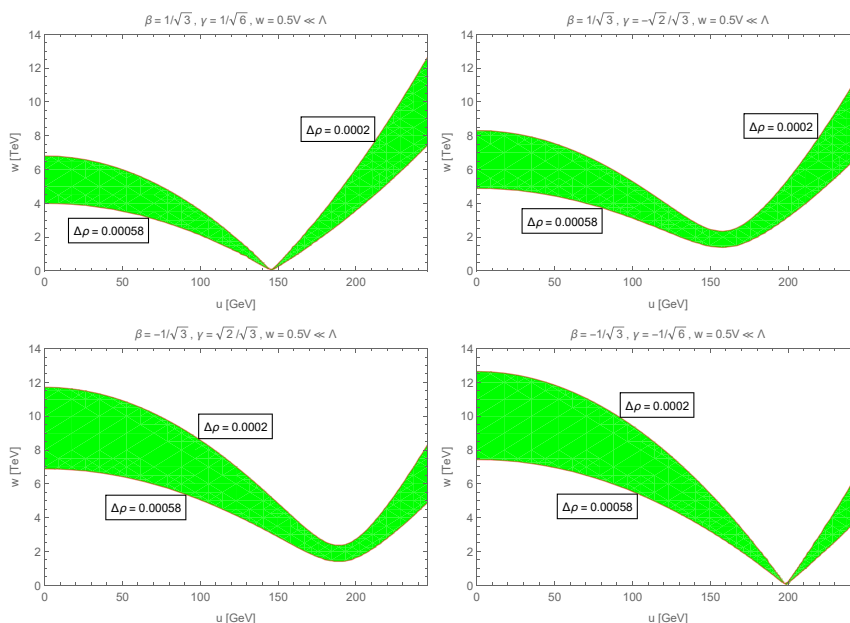
Figure 1. The (u, w) regime that is bounded by the ρ parameter for $w = 0.5V \ll \Lambda$, where the panels from left to right and top to bottom correspond to the four dark matter models.
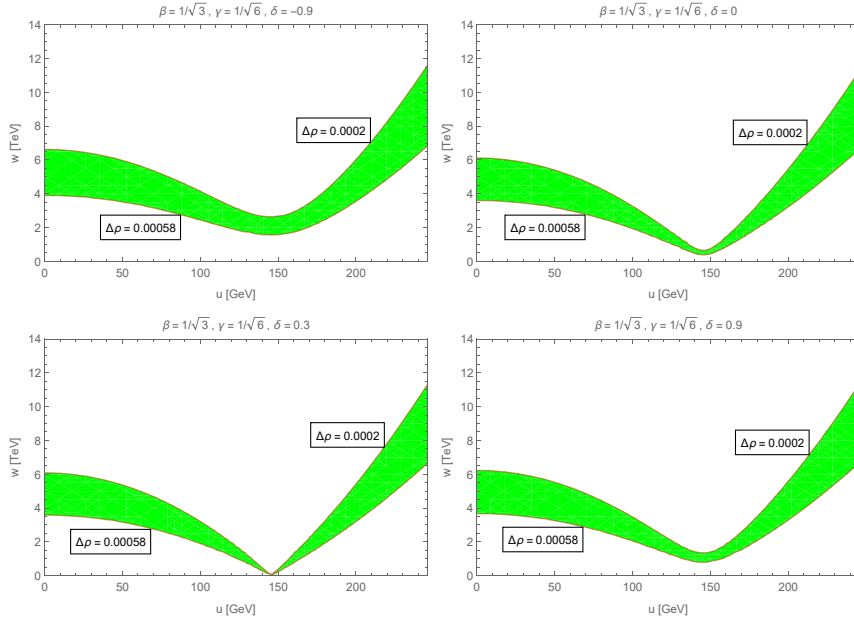

Figure 2. The (u, w) regime that is bounded by the ρ parameter for $w \ll V, \Lambda$, where the left and right panels correspond to $\beta = 1/\sqrt{3}$ and $\beta = -1/\sqrt{3}$.

4.2 $Z_1 \bar{f} f$ couplings

As stated, the considering model has the mixing of the Z boson with the new neutral gauge bosons. From (3.11) and (3.12), we get $Z = Z_1 + \epsilon_1 \mathcal{Z}'_2 + \epsilon_2 \mathcal{Z}'_3 + \epsilon_3 \mathcal{C}'$, $Z'_2 = -\epsilon_1 Z_1 + \mathcal{Z}'_2$, $Z'_3 = -\epsilon_2 Z_1 + \mathcal{Z}'_3$, and $C' = -\epsilon_3 Z_1 + \mathcal{C}'$. Hence, the couplings of Z_1 to fermions are modified by the mixing parameters $\epsilon_{1,2,3}$. Fitting the standard model precision test, the room for the mixing parameters is only 10^{-3} order. Hence, we impose the bound $|\epsilon_{1,2,3}| = 10^{-3}$.

It is observed that in the first case $(w, V \ll \Lambda)$, $\epsilon_3 = 0$ while $\epsilon_{1,2}$ are independent of δ, Λ. In the second case $(w \ll V, \Lambda)$, $\epsilon_{2,3} = 0$ while ϵ_1 is independent of δ, V, Λ. In the last case $(w, \Lambda \ll V)$, all the parameters contribute to $\epsilon_{1,2,3}$, except for V. Hence, we consider only the sensitivity of the new physics scales in terms of the kinetic mixing parameter for the last case. Since the effect of kinetic mixing does not depend on the u, v relation, we
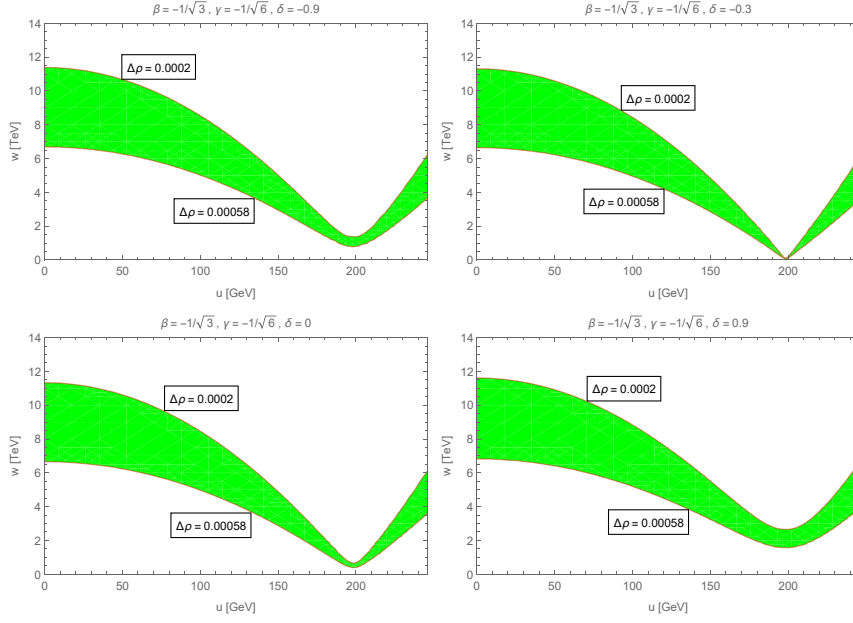
Figure 3. The (u, w) regime that is bounded by the ρ parameter for $(\beta = 1/\sqrt{3}, \gamma = 1/\sqrt{6}, b = -2/\sqrt{3}, c = -\sqrt{2}/\sqrt{3})$ and $w = 0.5\Lambda \ll V$, where the panels from left to right and top to bottom correspond to $\delta = -0.9, \ 0, \ 0.3$, and 0.9.
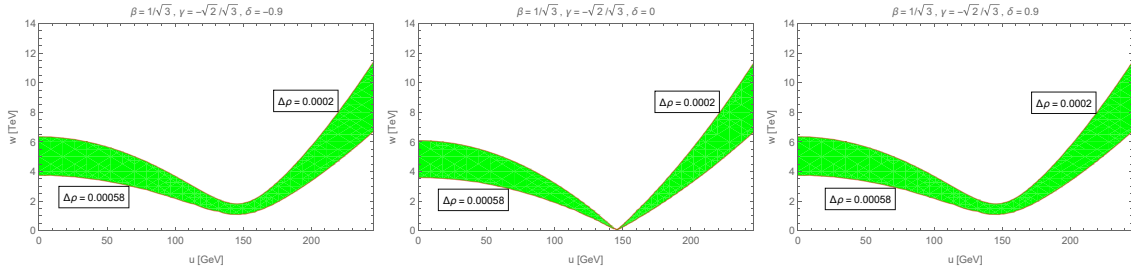
Figure 4. The (u, w) regime that is bounded by the ρ parameter for $(\beta = 1/\sqrt{3}, \gamma = -\sqrt{2}/\sqrt{3}, b = -2/\sqrt{3}, c = -\sqrt{2}/\sqrt{3})$ and $w = 0.5\Lambda \ll V$, where the panels from left to right are for $\delta = -0.9, \ 0$, and 0.9, respectively.
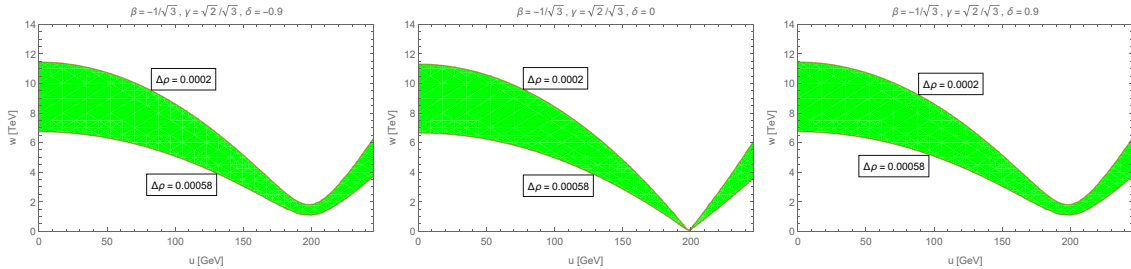
Figure 5. The (u, w) regime that is bounded by the ρ parameter for $(\beta = -1/\sqrt{3}, \gamma = \sqrt{2}/\sqrt{3}, b = -2/\sqrt{3}, c = -\sqrt{2}/\sqrt{3})$ and $w = 0.5\Lambda \ll V$, where the panels from left to right are for $\delta = -0.9, \ 0$, and 0.9, respectively.

Figure 6. The (u, w) regime that is bounded by the ρ parameter for $(\beta = -1/\sqrt{3}, \gamma = -1/\sqrt{6}, b = -2/\sqrt{3}, c = -\sqrt{2}/\sqrt{3})$ and $w = 0.5\Lambda \ll V$, where the panels ordered correspond to $\delta = -0.9, -0.3, 0,$ and 0.9, respectively.

impose $u = v = 246/\sqrt{2}$ GeV and use also the previous inputs. The results are given in figure 7. It indicates that the new physics regime changes when δ varies.

5 FCNCs

Because the fermion generations transform differently under the gauge symmetry $\mathrm{SU}(4)_L \otimes \mathrm{U}(1)_X \otimes \mathrm{U}(1)_N$, the tree-level FCNCs are present. Indeed, the neutral currents arise from

$$
\begin{aligned}
\mathcal{L}_{\mathrm{NC}} &= -g\bar{F}\gamma^\mu[T_3 A_{3\mu} + T_8 A_{8\mu} + T_{15} A_{15\mu} + t_X X B_\mu + t_N N C_\mu]F \\
&= -g\bar{F}\gamma^\mu[T_3 A_{3\mu} + T_8 A_{8\mu} + T_{15} A_{15\mu} \\
&\quad + t_X(Q - T_3 - \beta T_8 - \gamma T_{15})B_\mu + t_N(B - L - bT_8 - cT_{15})C_\mu]F. \quad (5.1)
\end{aligned}
$$

It is clear that the leptons and exotic quarks do not flavor-change. Furthermore, the terms of T_3, Q, and $B-L$ also conserve flavors. Hence, the FCNCs couple only the ordinary quarks to $T_{8,15}$, such that

$$
\mathcal{L}_{\mathrm{NC}} \supset -g[\bar{q}_L\gamma^\mu T_8^q q_L(A_{8\mu} - \beta t_X B_\mu - b t_N C_\mu) + \bar{q}_L\gamma^\mu T_{15}^q q_L(A_{15\mu} - \gamma t_X B_\mu - c t_N C_\mu)], \quad (5.2)
$$

where q is denoted either $q = (u_1, u_2, u_3)$ or $q = (d_1, d_2, d_3)$, $T_8^q = \frac{1}{2\sqrt{3}}\mathrm{diag}(-1, -1, 1)$, and $T_{15}^q = \frac{1}{2\sqrt{6}}\mathrm{diag}(-1, -1, 1)$. Changing to the mass basis, $q_{L,R} = V_{qL,qR}q'_{L,R}$ where either $q' = (u, c, t)$ or $q' = d, s, b$, and $(A_3\, A_8\, A_{15}\, B\, C)^T = U(A\, Z_1\, Z_2\, Z_3\, Z_4)$, this yields

$$
\mathcal{L}_{\mathrm{FCNC}} = -\bar{q}'_{iL}\gamma^\mu q'_{jL}(V^*_{qL})_{3i}(V_{qL})_{3j}(g_0 A_\mu + g_1 Z_{1\mu} + g_2 Z_{2\mu} + g_3 Z_{3\mu} + g_4 Z_{4\mu}) \quad (i \neq j). \quad (5.3)
$$
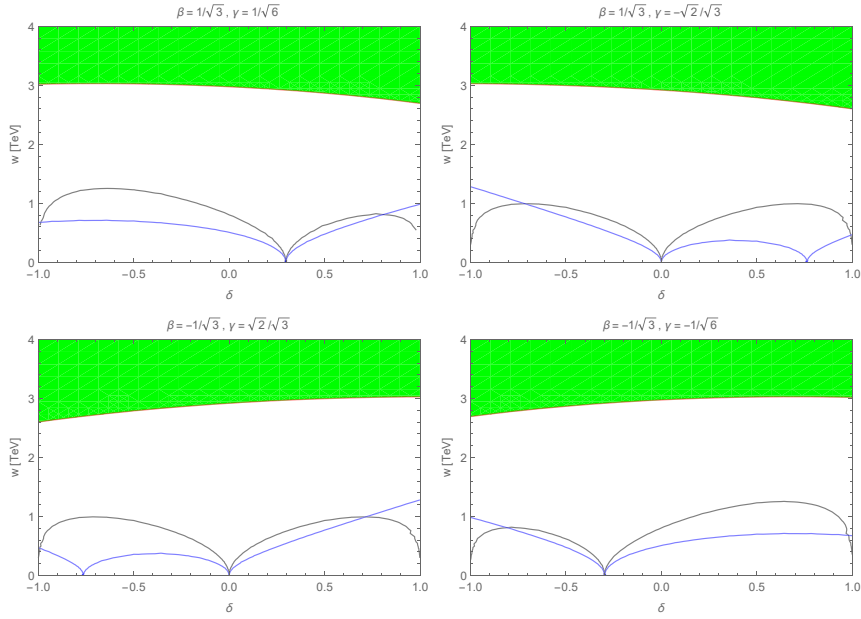
Figure 7. The bounds on new physics scales as the functions of δ for $|\epsilon_{1,2,3}| = 10^{-3}$, where the red, blue, and black lines correspond to ϵ_1, ϵ_2, ϵ_3 for the four kinds of dark matter models $(\beta = 1/\sqrt{3}, \gamma = 1/\sqrt{6})$, $(\beta = 1/\sqrt{3}, \gamma = -\sqrt{2}/\sqrt{3})$, $(\beta = -1/\sqrt{3}, \gamma = \sqrt{2}/\sqrt{3})$, and $(\beta = -1/\sqrt{3}, \gamma = -1/\sqrt{6})$, respectively.

It is noted that the photon always conserves flavors, $g_0 = 0$. In the first case $(w, V \ll \Lambda)$, the couplings $g_{1,2,3,4}$ are

$$g_1 = -\frac{g}{\sqrt{6}}\left[\frac{\sqrt{2}}{\sqrt{1-\beta^2 t_W^2}}\epsilon_1 + \frac{1+\gamma(\sqrt{2}\beta+\gamma)t_X^2}{\sqrt{1+\gamma^2 t_X^2}}\epsilon_2 + \frac{\delta(\sqrt{2}\beta+\gamma)t_X - (\sqrt{2}b+c)t_N}{\sqrt{1-\delta^2}}\epsilon_3\right], \quad (5.4)$$

$$g_2 = \frac{g}{\sqrt{6}}\left[\frac{\sqrt{2}}{\sqrt{1-\beta^2 t_W^2}}c_\varphi - \frac{1+\gamma(\sqrt{2}\beta+\gamma)t_X^2}{\sqrt{1+\gamma^2 t_X^2}}s_\varphi\right], \quad (5.5)$$

$$g_3 = g_2(c_\varphi \to s_\varphi, s_\varphi \to -c_\varphi), \quad (5.6)$$

$$g_4 = \frac{g}{\sqrt{6}}\frac{\delta(\sqrt{2}\beta+\gamma)t_X - (\sqrt{2}b+c)t_N}{\sqrt{1-\delta^2}}. \quad (5.7)$$

In the third case $(w, \Lambda \ll V)$, the coupling g_1 is identical to (5.4), while

$$g_2 = -\frac{g}{\sqrt{6}}\left[\frac{\sqrt{2}}{\sqrt{1-\beta^2 t_W^2}} + \frac{1+\gamma(\sqrt{2}\beta+\gamma)t_X^2}{\sqrt{1+\gamma^2 t_X^2}}\mathcal{E}_1 + \frac{\delta(\sqrt{2}\beta+\gamma)t_X - (\sqrt{2}b+c)t_N}{\sqrt{1-\delta^2}}\mathcal{E}_2\right], \quad (5.8)$$

$$g_3 = \frac{g}{\sqrt{6}} \left\{ c_\xi \left[\frac{\sqrt{2}}{\sqrt{1-\beta^2 t_W^2}} \mathcal{E}_1 + \frac{1+\gamma(\sqrt{2}\beta+\gamma)t_X^2}{\sqrt{1+\gamma^2 t_X^2}} \right] \right.$$

$$\left. - s_\xi \left[\frac{\sqrt{2}}{\sqrt{1-\beta^2 t_W^2}} \mathcal{E}_2 + \frac{\delta(\sqrt{2}\beta+\gamma)t_X - (\sqrt{2}b+c)t_N}{\sqrt{1-\delta^2}} \right] \right\}, \tag{5.9}$$

$$g_4 = g_3(c_\xi \to s_\xi, s_\xi \to -c_\xi). \tag{5.10}$$

In the second case ($w \ll V, \Lambda$), the couplings can be obtained from those in the third case by $\mathcal{E}_{1,2} \to 0$.

The contribution of the new physics to the meson mixing is given after integrating $Z_{1,2,3,4}$ out,

$$\mathcal{L}_{\text{FCNC}}^{\text{eff}} = (\bar{q}'_{iL} \gamma^\mu q'_{jL})^2 [(V_{qL}^*)_{3i}(V_{qL})_{3j}]^2 \left(\frac{g_1^2}{m_{Z_1}^2} + \frac{g_2^2}{m_{Z_2}^2} + \frac{g_3^2}{m_{Z_3}^2} + \frac{g_4^2}{m_{Z_4}^2} \right)$$

$$\simeq (\bar{q}'_{iL} \gamma^\mu q'_{jL})^2 [(V_{qL}^*)_{3i}(V_{qL})_{3j}]^2 \left(\frac{g_2^2}{m_{Z_2}^2} + \frac{g_3^2}{m_{Z_3}^2} + \frac{g_4^2}{m_{Z_4}^2} \right), \tag{5.11}$$

where the Z_1 contribution is small and omitted.

The strongest bound comes from $B_s^0 - \bar{B}_s^0$ mixing, implying [3]

$$[(V_{dL}^*)_{32}(V_{dL})_{33}]^2 \left(\frac{g_2^2}{m_{Z_2}^2} + \frac{g_3^2}{m_{Z_3}^2} + \frac{g_4^2}{m_{Z_4}^2} \right) < \frac{1}{(100 \text{ TeV})^2}. \tag{5.12}$$

We assume the sector of up quarks to be flavor diagonal, i.e. $V_{\text{CKM}} \equiv V_{uL}^\dagger V_{dL} = V_{dL}$. We have $|(V_{dL}^*)_{32}(V_{dL})_{33}| \simeq 3.9 \times 10^{-2}$ [3], which leads to

$$\sqrt{\frac{g_2^2}{m_{Z_2}^2} + \frac{g_3^2}{m_{Z_3}^2} + \frac{g_4^2}{m_{Z_4}^2}} < \frac{1}{3.9 \text{ TeV}}. \tag{5.13}$$

Our remark is that since $u, v \ll w, V, \Lambda$, the l.h.s. of (5.13) depends only on the new physics scales, not on the weak scales.

In the first case ($w, V \ll \Lambda$), the Z_4 contribution is negligible. The l.h.s. of (5.13) is independent of δ. The other inputs given previously are used, implying the bound for $w > 4.36 \text{ TeV}$ for all the four dark matter models.

In the second case ($w \ll V, \Lambda$), the $Z_{3,4}$ contributions are negligible. The l.h.s. of (5.13) is independent of β, γ, and δ. The bound yields $w > 3.9 \text{ TeV}$ for all the four models.

In the third case ($w, \Lambda \ll V$), since the mixing angles $\mathcal{E}_{1,2}$ are finite, the l.h.s. of (5.13) depends on β, γ, and δ, and is depicted in figure 8. The figure yields that the new physics regime changes when δ varies. Furthermore, those bounds are obviously lower than that given by the two case above.

6 Collider bounds

Since the new neutral gauge bosons couple to leptons and quarks, they contribute to the Drell-Yan and dijet processes at colliders.
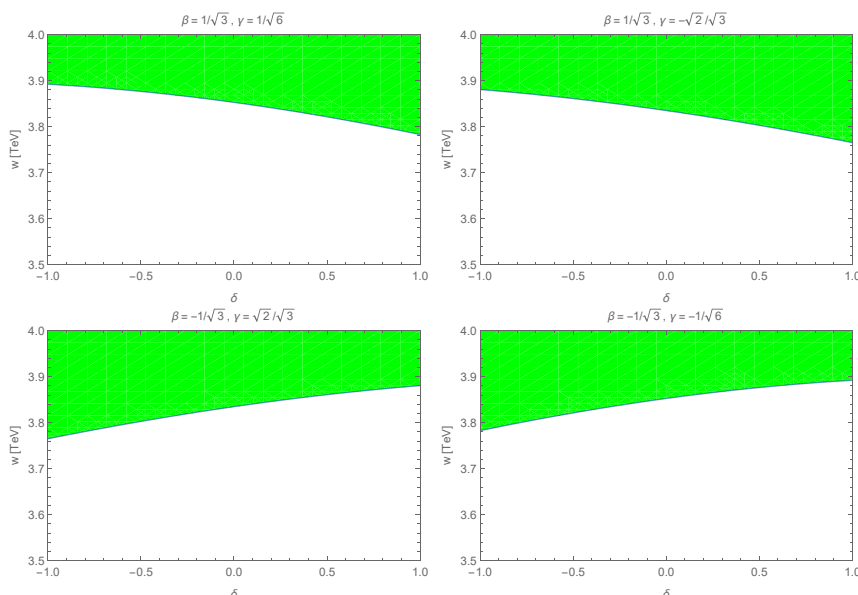
Figure 8. The bounds on the new physics scales as functions of δ from the FCNCs for $w = 0.5\Lambda \ll V$, where the panels from left to right and top to bottom are for the four dark matter models, respectively.

The LEPII searches for $e^+ e^- \to \mu^+ \mu^-$ happen similarly to the case of the 3-3-1-1 model, where all the new gauge bosons $Z_{2,3,4}$ mediate the process. Assuming that all the new physics scales are the same order, they are bounded in the TeV scale [20–24].

The LHC searches for dijet and dilepton final states can be studied. Using the above condition, the new physics scales are also in TeV, similarly to [59, 60].

Indeed, at the LHC, the cross-section for producing a lepton or quark pair as mediated by a new neutral gauge boson (commonly called Z') can be computed, using narrow width approximation [61],

$$\sigma(pp \to Z' \to f\bar{f}) = \frac{1}{3} \sum_q \frac{dL_{q\bar{q}}}{dm_{Z'}^2} \hat{\sigma}(q\bar{q} \to Z') \mathrm{Br}(Z' \to f\bar{f}), \tag{6.1}$$

where $dL_{q\bar{q}}/dm_{Z'}^2$ is parton luminosity [62], and the peak cross-section and branching ratio $\mathrm{Br}(Z' \to f\bar{f}) \equiv \Gamma(Z' \to f\bar{f})/\Gamma_{Z'}$ take the form,

$$\hat{\sigma}(q\bar{q} \to Z') = \frac{\pi g^2}{12 c_W^2}[(g_V^{Z'}(q))^2 + (g_A^{Z'}(q))^2], \tag{6.2}$$

$$\Gamma(Z' \to f\bar{f}) = \frac{g^2 m_{Z'}}{48\pi c_W^2} N_C[(g_V^{Z'}(f))^2 + (g_A^{Z'}(f))^2], \tag{6.3}$$

$$\Gamma_{Z'} = \frac{g^2 m_{Z'}}{48\pi c_W^2} \sum_f N_C[(g_V^{Z'}(f))^2 + (g_A^{Z'}(f))^2], \tag{6.4}$$

assuming that Z' decays only to fermions. [In fact, it is verified that the other Z' decay modes such as to ordinary Higgs and gauge bosons are small, compared to the Z' width.]

If $V \gg \Lambda, w$, all the Z' couplings in eqs. (6.2), (6.3) and (6.4) coincide with those in the 3-3-1-1 model, except for the last new one Z_4 that is so heavy and thus does not contribute.[4] If $V \sim \Lambda, w$, there is a finite mixing between Z_4 and $Z_{2,3}$, and in this case Z_4 contributes so that the mentioned couplings as well as the Z_4 couplings are shifted (or modified) by the cosine and sine of the mixing angle. All such contributions are necessarily proportional to that in the 3-3-1-1 model, since $\mathrm{SU}(4)_L \supset \mathrm{SU}(3)_L$ and these groups have the gauge coupling as matched. The difference from the previous 3-3-1 and 3-3-1-1 models is only the additional contribution of Z_4, which all yield equivalent amounts. Therefore, Z_4 is in or beyond TeV scale as those in the 3-3-1-1 model ($Z_{2,3}$) do under the light of the LHC experiments [63, 64].

7 The 3-3-1-1 model revisited

The 3-3-1-1 model is based upon the gauge symmetry $\mathrm{SU}(3)_C \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X \otimes \mathrm{U}(1)_N$. Thus it contains four neutral gauge bosons $A_{3,8}$, B, and C according to the last three gauge groups, in which B, C has a kinetic mixing term, $-(\delta/2)B_{\mu\nu}C^{\mu\nu}$. The kinetic mixing effect in the 3-3-1-1 model was explicitly studied in [33]. Here we present only new results beyond the previous investigation.

Changing to the canonical basis, A_3, A_8, B', and C', the corresponding mass matrix $M^2 = \{m_{ij}^2\}$ is given by

$$m_{11}^2 = \frac{g^2}{4}(u^2+v^2), \quad m_{12}^2 = \frac{g^2}{4\sqrt{3}}(u^2-v^2), \quad m_{13}^2 = -\frac{g^2 t_X}{4\sqrt{3}}[(\sqrt{3}+\beta)u^2+(\sqrt{3}-\beta)v^2],$$

$$m_{14}^2 = \frac{g^2}{4\sqrt{3(1-\delta^2)}}\{[\delta(\sqrt{3}+\beta)t_X - bt_N]u^2 + [\delta(\sqrt{3}-\beta)t_X + bt_N]v^2\},$$

$$m_{22}^2 = \frac{g^2}{12}(u^2+v^2+4w^2), \quad m_{23}^2 = -\frac{g^2 t_X}{12}[(\sqrt{3}+\beta)u^2 - (\sqrt{3}-\beta)v^2 + 4\beta w^2],$$

$$m_{24}^2 = \frac{g^2}{12\sqrt{1-\delta^2}}\{[\delta(\sqrt{3}+\beta)t_X - bt_N]u^2 - [\delta(\sqrt{3}-\beta)t_X + bt_N]v^2 + 4(\delta\beta t_X - bt_N)w^2\},$$

$$m_{33}^2 = \frac{g^2 t_X^2}{12}[(\sqrt{3}+\beta)^2 u^2 + (\sqrt{3}-\beta)^2 v^2 + 4\beta^2 w^2],$$

$$m_{34}^2 = \frac{-g^2 t_X}{12\sqrt{1-\delta^2}}\{(\sqrt{3}+\beta)[\delta(\sqrt{3}+\beta)t_X - bt_N]u^2 + (\sqrt{3}-\beta)[\delta(\sqrt{3}-\beta)t_X + bt_N]v^2 + 4\beta(\delta\beta t_X - bt_N)w^2\},$$

$$m_{44}^2 = \frac{g^2}{12(1-\delta^2)}\{[\delta(\sqrt{3}+\beta)t_X - bt_N]^2 u^2 + [\delta(\sqrt{3}-\beta)t_X + bt_N]^2 v^2 + 4(\delta\beta t_X - bt_N)^2 w^2 + 48 t_N^2 \Lambda^2\}.$$

This result is similar to that in [33], except for the last element, m_{44}^2, that differs in the coefficient of Λ^2. Note that $t_X = g_X/g$, $t_N = g_N/g$, β, b, u, v, and w are those parameters belonging to the 3-3-1-1 model and in this case we have $s_W = e/g = t_X/\sqrt{1+(1+\beta^2)t_X^2}$.

Changing to the electroweak basis, $(A_3\, A_8\, B'\, C')^T = U_1(A\, Z\, Z'\, C')^T$, where C' is orthogonal to $A = s_W A_3 + c_W\left(\beta t_W A_8 + \sqrt{1-\beta^2 t_W^2}\, B'\right)$, $Z = c_W A_3 -$

$s_W \left(\beta t_W A_8 + \sqrt{1 - \beta^2 t_W^2} B' \right)$, and $Z' = \sqrt{1 - \beta^2 t_W^2} \left(A_8 - \beta t_W B' \right)$, thus

$$U_1 = \begin{pmatrix} s_W & c_W & 0 & 0 \\ \beta s_W & -\beta s_W t_W & \sqrt{1 - \beta^2 t_W^2} & 0 \\ c_W \sqrt{1 - \beta^2 t_W^2} & -s_W \sqrt{1 - \beta^2 t_W^2} & -\beta t_W & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.1}$$

the mass matrix M^2 changes to

$$M'^2 = U_1^T M^2 U_1 = \begin{pmatrix} 0 & 0 \\ 0 & M_s'^2 \end{pmatrix}, \quad M_s'^2 \equiv \begin{pmatrix} m_Z^2 & m_{ZZ'}^2 & m_{ZC'}^2 \\ m_{ZZ'}^2 & m_{Z'}^2 & m_{Z'C'}^2 \\ m_{ZC'}^2 & m_{Z'C'}^2 & m_{C'}^2 \end{pmatrix}, \tag{7.2}$$

which has the elements as given in [33], in which $m_{C'}^2 = m_{44}^2$.

The light state Z can be separated by using the seesaw approximation,

$$\left(Z\, Z'\, C' \right)^T = U_2 \left(Z_1\, \mathcal{Z}'\, \mathcal{C}' \right)^T, \quad M''^2 = U_2^T M_s'^2 U_2 = \begin{pmatrix} m_{Z_1}^2 & 0 \\ 0 & M_{2\times 2}^2 \end{pmatrix}, \tag{7.3}$$

where

$$U_2 \simeq \begin{pmatrix} 1 & \epsilon_1 & \epsilon_2 \\ -\epsilon_1 & 1 & 0 \\ -\epsilon_2 & 0 & 1 \end{pmatrix}, \quad M_{2\times 2}^2 \simeq \begin{pmatrix} m_{Z'}^2 & m_{Z'C'}^2 \\ m_{Z'C'}^2 & m_{C'}^2 \end{pmatrix}. \tag{7.4}$$

$$m_{Z_1}^2 \simeq m_Z^2 - \epsilon_1 m_{ZZ'}^2 - \epsilon_2 m_{ZC'}^2. \tag{7.5}$$

We separate $\epsilon_{1,2} \equiv \epsilon_{1,2}^0 + \epsilon_{1,2}^\delta$, where $\epsilon_{1,2}^0$ are the mixing parameters due to the symmetry breaking [20–25], while $\epsilon_{1,2}^\delta$ determine the kinetic mixing effect,

$$\epsilon_1^0 = \frac{\sqrt{1 - \beta^2 t_W^2}}{4 c_W} \left\{ \frac{\sqrt{3}[u^2 - v^2 + \sqrt{3} \beta t_W^2 (u^2 + v^2)]}{w^2} + \frac{b^2 \beta t_W^2 (u^2 + v^2)}{4 \Lambda^2} \right\}, \tag{7.6}$$

$$\epsilon_2^0 = \frac{b \beta t_W^2 (u^2 + v^2)}{16 c_W t_N \Lambda^2}, \tag{7.7}$$

$$\epsilon_1^\delta = \frac{\delta t_W [b(1 - 2\beta^2 t_W^2) t_N - \delta \beta t_W \sqrt{1 - \beta^2 t_W^2}](u^2 + v^2)}{16 c_W t_N^2 \Lambda^2}, \tag{7.8}$$

$$\epsilon_2^\delta = \frac{\delta t_W (u^2 + v^2)}{16 c_W t_N \Lambda^2} \left(\frac{\sqrt{1 - \delta^2} \sqrt{1 - \beta^2 t_W^2}}{t_N} - \frac{\delta b \beta t_W}{1 + \sqrt{1 - \delta^2}} \right), \tag{7.9}$$

where $\epsilon_{1,2}^\delta$ differ from those in [33].

We diagonalize $M_{2\times 2}^2$ to obtain mass eigenstates,

$$Z_2 = c_\xi \mathcal{Z}' - s_\xi \mathcal{C}', \quad Z_3 = s_\xi \mathcal{Z}' + c_\xi \mathcal{C}', \tag{7.10}$$

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in which the $\mathcal{Z}' - \mathcal{C}'$ mixing angle and masses are

$$t_{2\xi} \simeq \frac{2\sqrt{1-\delta^2}(\delta\beta t_W - bt_N\sqrt{1-\beta^2 t_W^2})w^2}{[(\delta\beta t_W - bt_N\sqrt{1-\beta^2 t_W^2})^2 - (1-\delta^2)]w^2 + 12(1-\beta^2 t_W^2)t_N^2\Lambda^2}, \quad (7.11)$$

$$m_{Z_2,Z_3}^2 = \frac{1}{2}\left[m_{Z'}^2 + m_{C'}^2 \mp \sqrt{(m_{Z'}^2 - m_{C'}^2)^2 + 4m_{Z'C'}^4}\right]. \quad (7.12)$$

Generally, ξ is finite if $w \sim \Lambda$. The kinetic mixing and symmetry breaking effects cancel out if $\delta = bt_N/\beta t_X$, which takes place between δ and b/β — the embedding coefficients of T_8. Whereas, in the 3-4-1-1 model, it happens between δ and c/γ — the embedding coefficients of T_{15}.

Hence, the gauge states are connected to the physical states by $(A_3\, A_8\, B\, C)^T = U_\delta U_1 U_2 U_\xi (A\, Z_1\, Z_2\, Z_3)^T$, where

$$U_\delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{\delta}{\sqrt{1-\delta^2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{1-\delta^2}} \end{pmatrix}, \quad U_\xi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_\xi & s_\xi \\ 0 & 0 & -s_\xi & c_\xi \end{pmatrix}. \quad (7.13)$$

The ρ deviation starts from the tree-level contribution,

$$(\Delta\rho)_{\text{tree}} = \frac{m_W^2}{c_W^2 m_{Z_1}^2} - 1 = \frac{m_Z^2}{m_Z^2 - \epsilon_1 m_{ZZ'}^2 - \epsilon_2 m_{ZC'}^2} - 1$$

$$\simeq \frac{\epsilon_1 m_{ZZ'}^2 + \epsilon_2 m_{ZC'}^2}{m_Z^2} \equiv (\Delta\rho)_{\text{tree}}^0 + (\Delta\rho)_{\text{tree}}^\delta, \quad (7.14)$$

where

$$(\Delta\rho)_{\text{tree}}^0 \simeq \frac{[u^2 - v^2 + \sqrt{3}\beta t_W^2(u^2 + v^2)]^2}{4(u^2 + v^2)w^2} + \frac{b^2\beta^2 t_W^4(u^2 + v^2)}{16\Lambda^2}, \quad (7.15)$$

$$(\Delta\rho)_{\text{tree}}^\delta \simeq \frac{\delta\sqrt{1-\beta^2 t_W^2}(\delta\sqrt{1-\beta^2 t_W^2} + 2b\beta t_W t_N)t_W^2(u^2 + v^2)}{16 t_N^2\Lambda^2}. \quad (7.16)$$

In this computation, we also include one-loop contributions by the gauge vector doublet (X, Y), as supplied in [33].

If $\Lambda \gg w$, $\Delta\rho$ does not depend on Λ, t_N, b, and δ. If $\Lambda \sim w$, all the parameters modify $\Delta\rho$. Comparing to [33], the difference is only expressions related to δ. Hence, the first case is not investigated in this work. To finalize the result, we use the parameter values similar to those in [33], namely $\Lambda = 2w$, $t_N = 0.5$, $n = 0$ (thus $b = -2/\sqrt{3}$), and $q = -1, 0, 1$ (thus $\beta = 1/\sqrt{3}, -1/\sqrt{3}, -\sqrt{3}$, respectively).

We make a contour of $\Delta\rho$ as the function of (u, w), as depicted in figures 9, 10, and 11 for $\beta = -1/\sqrt{3}$, $\beta = 1/\sqrt{3}$, and $\beta = -\sqrt{3}$, respectively. The effect of δ is quite similar to the 3-4-1-1 model and obviously different from [33].

The new physics contribution is safe, given that $|\epsilon_{1,2}| = 10^{-3}$. Without loss of generality, we impose $u = v = 246/\sqrt{2}$ as well as the given values of $\Lambda = 2w, t_N, \beta, b$ are used.
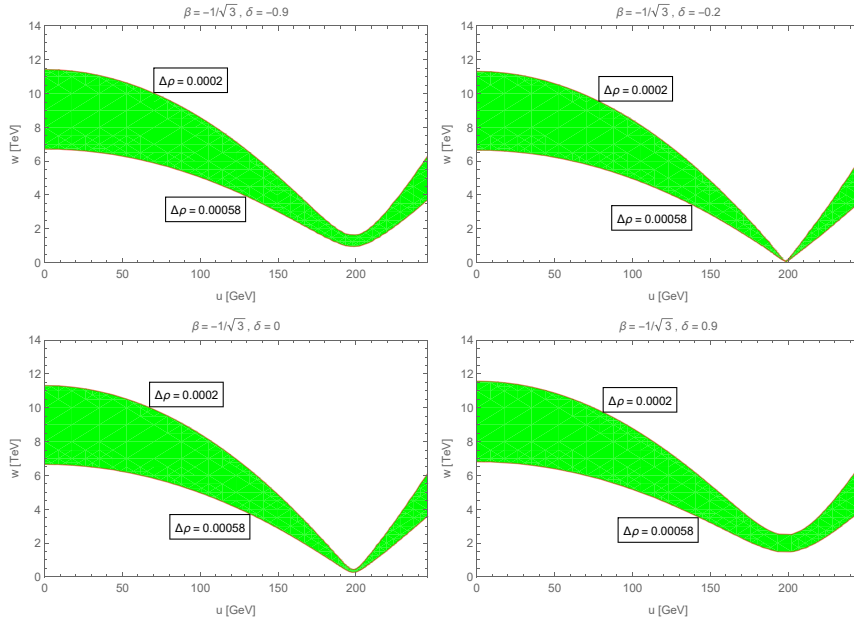
Figure 9. The (u, w) regime constrained by $\Delta \rho$ for $\beta = -1/\sqrt{3}$, $b = -2/\sqrt{3}$, $t_N = 0.5$, and $\Lambda = 2w$, where the panels correspond to $\delta = -0.9,\ -0.2,\ 0,$ and 0.9.
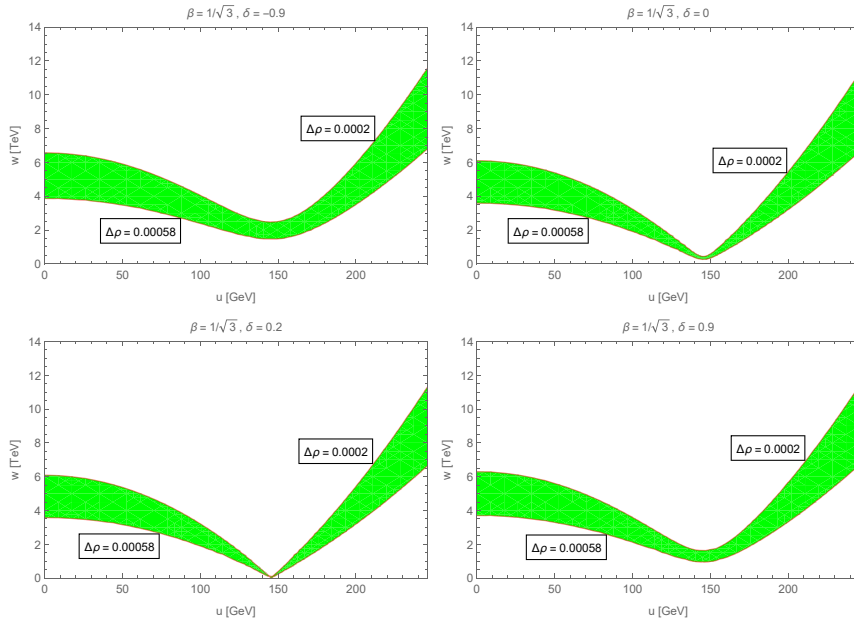
Figure 10. The (u, w) regime constrained by $\Delta \rho$ for $\beta = 1/\sqrt{3}$, $b = -2/\sqrt{3}$, $t_N = 0.5$, and $\Lambda = 2w$, where the panels correspond to $\delta = -0.9,\ 0,\ 0.2,$ and 0.9.
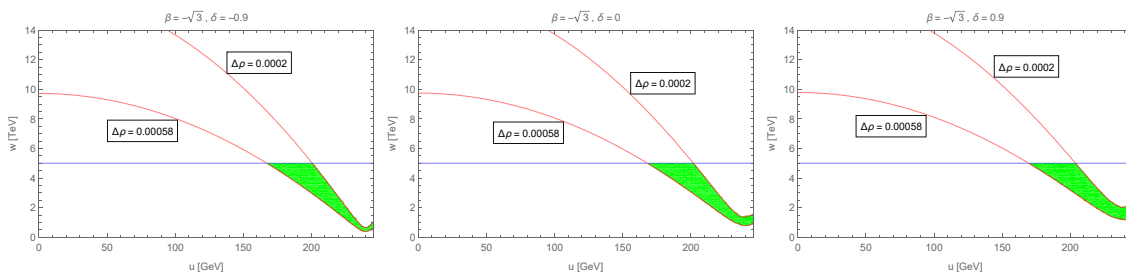
Figure 11. The (u, w) regime that is bounded by the ρ parameter for $\beta = -\sqrt{3}$, $b = -2/\sqrt{3}$, $t_N = 0.5$, and $\Lambda = 2w$, where the panels, ordering from left to right, correspond to $\delta = -0.9$, 0, and 0.9, respectively. In this case, the Landau pole, which is roundly $w = 5\,\text{TeV}$, is imposed.
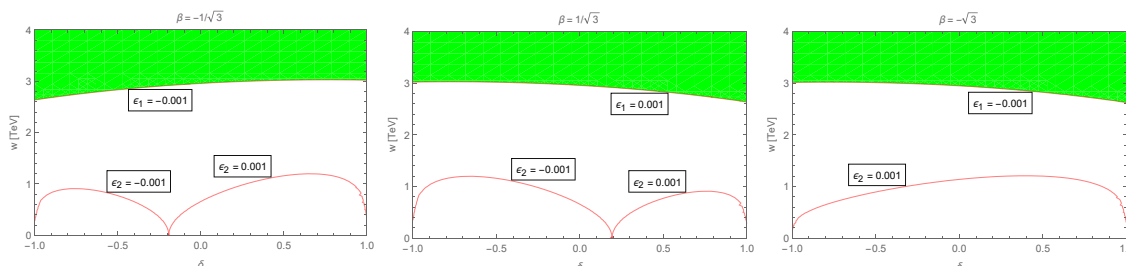

Figure 12. The bounds on the new physics scales as functions of δ, contour by $|\epsilon_{1,2}| = 10^{-3}$, for the three kinds of the models $\beta = -1/\sqrt{3}$, $\beta = 1/\sqrt{3}$, and $\beta = -\sqrt{3}$, respectively.

In figure 12, $\epsilon_{1,2}$ are contoured as the functions of (w, δ) for $\beta = -1/\sqrt{3}$, $\beta = 1/\sqrt{3}$, and $\beta = -\sqrt{3}$. It is clear that the new physics regime significantly changes when δ varies, in contradiction to [33].

The meson mixing is described via the effective interaction [33]

$$\mathcal{L}_{\text{FCNC}}^{\text{eff}} = (\bar{q}'_{iL}\gamma^\mu q'_{jL})^2[(V_{qL}^*)_{3i}(V_{qL})_{3j}]^2 \left(\frac{g_2^2}{m_{Z_2}^2} + \frac{g_3^2}{m_{Z_3}^2} \right), \tag{7.17}$$

where

$$g_2 = \frac{g}{\sqrt{3}} \left(\frac{1}{\sqrt{1 - \beta^2 t_W^2}} c_\xi + \frac{bt_N - \delta\beta t_X}{\sqrt{1 - \delta^2}} s_\xi \right), \tag{7.18}$$

$$g_3 = g_2(c_\xi \to s_\xi, s_\xi \to -c_\xi). \tag{7.19}$$

The $B_s^0 - \bar{B}_s^0$ bound leads to [33]

$$\sqrt{\frac{g_2^2}{m_{Z_2}^2} + \frac{g_3^2}{m_{Z_3}^2}} < \frac{1}{3.9\,\text{TeV}}. \tag{7.20}$$

When $w \ll \Lambda$, the above bound translates to $w > 3.9\,\text{TeV}$, independent of β, b, g, g_X, g_N, and δ. When $w \sim \Lambda$, using the existing values of parameters, the bound for both scales is similar to the previous case, which is quite in agreement with the conclusion in [33].

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8 Conclusion

We have proved that the 3-4-1-1 model provides dark matter candidates naturally, besides supplying small neutrino masses via the seesaw mechanism induced by the gauge symmetry breaking.

The kinetic mixing effects are evaluated, yielding the new physics scales at TeV scale, in agreement with the collision bound. The kinetic mixing and symmetry breaking effects are canceled out only in the new gauge sector and differs between the 3-4-1-1 and 3-3-1-1 models.

Similar to the 3-3-1-1 model [26], the 3-4-1-1 model can address the question of cosmic inflation as well as asymmetric dark and normal matter, which attracts much attention.

A Anomaly checking

The anomalies that cause troublesome include $[\mathrm{SU}(3)_C]^2\mathrm{U}(1)_X$, $[\mathrm{SU}(3)_C]^2\mathrm{U}(1)_N$, $[\mathrm{SU}(4)_L]^2\mathrm{U}(1)_X$, $[\mathrm{SU}(4)_L]^2\mathrm{U}(1)_N$, $[\mathrm{Gravity}]^2\mathrm{U}(1)_X$, $[\mathrm{Gravity}]^2\mathrm{U}(1)_N$, $[\mathrm{U}(1)_X]^2\mathrm{U}(1)_N$, $\mathrm{U}(1)_X[\mathrm{U}(1)_N]^2$, $[\mathrm{U}(1)_X]^3$, and $[\mathrm{U}(1)_N]^3$. Let us verify each of them.

$$
\begin{aligned}
[\mathrm{SU}(3)_C]^2\mathrm{U}(1)_X \sim & \sum_{\text{quarks}} (X_{q_L} - X_{q_R}) \\
= & \, 4X_{Q_3} + 2 \times 4X_{Q_\alpha} - 3X_{u_a} - 3X_{d_a} - X_{J_3} - X_{K_3} - 2X_{J_\alpha} - 2X_{K_\alpha} \\
= & \, 4\left(\frac{p+q+5/3}{4}\right) + 8\left(-\frac{p+q+1/3}{4}\right) - 3\left(\frac{2}{3}\right) - 3\left(\frac{-1}{3}\right) - \left(q+\frac{2}{3}\right) \\
& - \left(p+\frac{2}{3}\right) - 2\left(-q-\frac{1}{3}\right) - 2\left(-p-\frac{1}{3}\right) = 0.
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
[\mathrm{SU}(3)_C]^2\mathrm{U}(1)_N \sim & \sum_{\text{quarks}} (N_{q_L} - N_{q_R}) \\
= & \, 4N_{Q_3} + 2 \times 4N_{Q_\alpha} - 3N_{u_a} - 3N_{d_a} - N_{J_3} - N_{K_3} - 2N_{J_\alpha} - 2N_{K_\alpha} \\
= & \, 4\left(\frac{m+n+10/3}{4}\right) + 8\left(-\frac{m+n+2/3}{4}\right) - 3\left(\frac{1}{3}\right) - 3\left(\frac{1}{3}\right) - \left(n+\frac{4}{3}\right) \\
& - \left(m+\frac{4}{3}\right) - 2\left(-n-\frac{2}{3}\right) - 2\left(-m-\frac{2}{3}\right) = 0.
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
[\mathrm{SU}(4)_L]^2\mathrm{U}(1)_X \sim & \sum_{\text{(anti)quadruplets}} X_{F_L} = 3X_{\psi_a} + 3X_{Q_3} + 2 \times 3X_{Q_\alpha} \\
= & \, 3\left(\frac{p+q-1}{4}\right) + 3\left(\frac{p+q+5/3}{4}\right) + 6\left(-\frac{p+q+1/3}{4}\right) = 0.
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
[\mathrm{SU}(4)_L]^2\mathrm{U}(1)_N \sim & \sum_{\text{(anti)quadruplets}} N_{F_L} = 3N_{\psi_a} + 3N_{Q_3} + 2 \times 3N_{Q_\alpha} \\
= & \, 3\left(\frac{m+n-2}{4}\right) + 3\left(\frac{m+n+10/3}{4}\right) + 6\left(-\frac{m+n+2/3}{4}\right) = 0.
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
[\mathrm{Gravity}]^2\mathrm{U}(1)_X \sim & \sum_{\text{fermions}} (X_{f_L} - X_{f_R}) \\
= & \, 3 \times 4X_{\psi_a} + 3 \times 4X_{Q_3} + 2 \times 3 \times 4X_{Q_\alpha} - 3 \times 3X_{u_a} - 3 \times 3X_{d_a} \\
& - 3X_{J_3} - 3X_{K_3} - 2 \times 3X_{J_\alpha} - 2 \times 3X_{K_\alpha} - 3X_{E_a} - 3X_{F_a} - 3X_{e_a} - 3X_{\nu_a}
\end{aligned}
$$

$$= 12\left(\frac{p+q-1}{4}\right) + 12\left(\frac{p+q+5/3}{4}\right) + 24\left(-\frac{p+q+1/3}{4}\right) - 9\left(\frac{2}{3}\right)$$

$$-9\left(\frac{-1}{3}\right) - 3\left(q+\frac{2}{3}\right) - 3\left(p+\frac{2}{3}\right) - 6\left(-q-\frac{1}{3}\right) - 6\left(-p-\frac{1}{3}\right)$$

$$-3q - 3p - 3(-1) - 3(0) = 0. \tag{A.5}$$

$$[\text{Gravity}]^2 \text{U}(1)_N \sim \sum_{\text{fermions}} (N_{f_L} - N_{f_R})$$

$$= 3\times 4 N_{\psi_a} + 3\times 4 N_{Q_3} + 2\times 3\times 4 N_{Q_\alpha} - 3\times 3 N_{u_a} - 3\times 3 N_{d_a}$$

$$-3 N_{J_3} - 3 N_{K_3} - 2\times 3 N_{J_\alpha} - 2\times 3 N_{K_\alpha} - 3 N_{E_a} - 3 N_{F_a} - 3 N_{e_a} - 3 N_{\nu_a}$$

$$= 12\left(\frac{m+n-2}{4}\right) + 12\left(\frac{m+n+10/3}{4}\right) + 24\left(-\frac{m+n+2/3}{4}\right)$$

$$-9\left(\frac{1}{3}\right) - 9\left(\frac{1}{3}\right) - 3\left(n+\frac{4}{3}\right) - 3\left(m+\frac{4}{3}\right) - 6\left(-n-\frac{2}{3}\right) - 6\left(-m-\frac{2}{3}\right)$$

$$-3n - 3m - 3(-1) - 3(-1) = 0. \tag{A.6}$$

$$[\text{U}(1)_X]^2 \text{U}(1)_N = \sum_{\text{fermions}} (X_{f_L}^2 N_{f_L} - X_{f_R}^2 N_{f_R}) = 3\times 4 X_{\psi_a}^2 N_{\psi_a} + 3\times 4 X_{Q_3}^2 N_{Q_3}$$

$$+2\times 3\times 4 X_{Q_\alpha}^2 N_{Q_\alpha} - 3\times 3 X_{u_a}^2 N_{u_a} - 3\times 3 X_{d_a}^2 N_{d_a} - 3 X_{J_3}^2 N_{J_3} - 3 X_{K_3}^2 N_{K_3}$$

$$-2\times 3 X_{J_\alpha}^2 N_{J_\alpha} - 2\times 3 X_{K_\alpha}^2 N_{K_\alpha} - 3 X_{E_a}^2 N_{E_a} - 3 X_{F_a}^2 N_{F_a} - 3 X_{e_a}^2 N_{e_a} - 3 X_{\nu_a}^2 N_{\nu_a}$$

$$= 12\left(\frac{p+q-1}{4}\right)^2\left(\frac{m+n-2}{4}\right) + 12\left(\frac{p+q+5/3}{4}\right)^2\left(\frac{m+n+10/3}{4}\right)$$

$$+24\left(-\frac{p+q+1/3}{4}\right)^2\left(-\frac{m+n+2/3}{4}\right) - 9\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) - 9\left(\frac{-1}{3}\right)^2\left(\frac{1}{3}\right)$$

$$-3\left(q+\frac{2}{3}\right)^2\left(n+\frac{4}{3}\right) - 3\left(p+\frac{2}{3}\right)^2\left(m+\frac{4}{3}\right) - 6\left(-q-\frac{1}{3}\right)^2\left(-n-\frac{2}{3}\right)$$

$$-6\left(-p-\frac{1}{3}\right)^2\left(-m-\frac{2}{3}\right) - 3q^2 n - 3p^2 m - 3(-1)^2(-1) - 3(0)^2(-1) = 0. \tag{A.7}$$

$$[\text{U}(1)_X]\text{U}(1)_N^2 = \sum_{\text{fermions}} (X_{f_L} N_{f_L}^2 - X_{f_R} N_{f_R}^2) = 3\times 4 X_{\psi_a} N_{\psi_a}^2 + 3\times 4 X_{Q_3} N_{Q_3}^2$$

$$+2\times 3\times 4 X_{Q_\alpha} N_{Q_\alpha}^2 - 3\times 3 X_{u_a} N_{u_a}^2 - 3\times 3 X_{d_a} N_{d_a}^2 - 3 X_{J_3} N_{J_3}^2 - 3 X_{K_3} N_{K_3}^2$$

$$-2\times 3 X_{J_\alpha} N_{J_\alpha}^2 - 2\times 3 X_{K_\alpha} N_{K_\alpha}^2 - 3 X_{E_a} N_{E_a}^2 - 3 X_{F_a} N_{F_a}^2 - 3 X_{e_a} N_{e_a}^2 - 3 X_{\nu_a} N_{\nu_a}^2$$

$$= 12\left(\frac{p+q-1}{4}\right)\left(\frac{m+n-2}{4}\right)^2 + 12\left(\frac{p+q+5/3}{4}\right)\left(\frac{m+n+10/3}{4}\right)^2$$

$$+24\left(-\frac{p+q+1/3}{4}\right)\left(-\frac{m+n+2/3}{4}\right)^2 - 9\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 - 9\left(\frac{-1}{3}\right)\left(\frac{1}{3}\right)^2$$

$$-3\left(q+\frac{2}{3}\right)\left(n+\frac{4}{3}\right)^2 - 3\left(p+\frac{2}{3}\right)\left(m+\frac{4}{3}\right)^2 - 6\left(-q-\frac{1}{3}\right)\left(-n-\frac{2}{3}\right)^2$$

$$-6\left(-p-\frac{1}{3}\right)\left(-m-\frac{2}{3}\right)^2 - 3q n^2 - 3p m^2 - 3(-1)(-1)^2 - 3(0)(-1)^2 = 0. \tag{A.8}$$

$$[\text{U}(1)_X]^3 = \sum_{\text{fermions}} (X_{f_L}^3 - X_{f_R}^3) = 3\times 4 X_{\psi_a}^3 + 3\times 4 X_{Q_3}^3 + 2\times 3\times 4 X_{Q_\alpha}^3 - 3\times 3 X_{u_a}^3$$

$$-3\times 3 X_{d_a}^3 - 3 X_{J_3}^3 - 3 X_{K_3}^3 - 2\times 3 X_{J_\alpha}^3 - 2\times 3 X_{K_\alpha}^3 - 3 X_{E_a}^3 - 3 X_{F_a}^3 - 3 X_{e_a}^3 - 3 X_{\nu_a}^3$$

$$= 12\left(\frac{p+q-1}{4}\right)^3 + 12\left(\frac{p+q+5/3}{4}\right)^3 + 24\left(-\frac{p+q+1/3}{4}\right)^3 - 9\left(\frac{2}{3}\right)^3$$

$$-9\left(\frac{-1}{3}\right)^3 - 3\left(q+\frac{2}{3}\right)^3 - 3\left(p+\frac{2}{3}\right)^3 - 6\left(-q-\frac{1}{3}\right)^3 - 6\left(-p-\frac{1}{3}\right)^3$$
$$-3q^3 - 3p^3 - 3(-1)^3 - 3(-0)^3 = 0. \tag{A.9}$$

$$[\mathrm{U}(1)_N]^3 = \sum_{\text{fermions}} (N_{f_L}^3 - N_{f_R}^3) = 3\times 4N_{\psi_a}^3 + 3\times 4N_{Q_3}^3 + 2\times 3\times 4N_{Q_\alpha}^3 - 3\times 3N_{u_a}^3$$
$$-3\times 3N_{d_a}^3 - 3N_{J_3}^3 - 3N_{K_3}^3 - 2\times 3N_{J_\alpha}^3 - 2\times 3N_{K_\alpha}^3 - 3N_{E_a}^3 - 3N_{F_a}^3 - 3N_{e_a}^3 - 3N_{\nu_a}^3$$
$$= 12\left(\frac{m+n-2}{4}\right)^3 + 12\left(\frac{m+n+10/3}{4}\right)^3 + 24\left(-\frac{m+n+2/3}{4}\right)^3 - 9\left(\frac{1}{3}\right)^3$$
$$-9\left(\frac{1}{3}\right)^3 - 3\left(n+\frac{4}{3}\right)^3 - 3\left(m+\frac{4}{3}\right)^3 - 6\left(-n-\frac{2}{3}\right)^3 - 6\left(-m-\frac{2}{3}\right)^3$$
$$-3n^3 - 3m^3 - 3(-1)^3 - 3(-1)^3 = 0. \tag{A.10}$$

This again confirms that the embedding coefficients (β, γ, b, c) are independent of the anomalies.

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