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# Conformal gravitational theories in Barthel–Kropina-type Finslerian geometry, and their cosmological implications

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**Abstract** We consider dark energy models obtained from the general conformal transformation of the Kropina metric, representing an $(\alpha, \beta)$-type Finslerian geometry, constructed as the ratio of the square of a Riemannian metric $\alpha$ and the one-form $\beta$. Conformal symmetries appear in many fields of physics, and they may play a fundamental role in our understanding of the Universe. We investigate the possibility of obtaining conformal theories of gravity in the osculating Barthel–Kropina geometric framework, where gravitation is described by an extended Finslerian-type model, with the metric tensor depending on both the base space coordinates and a vector field. We show that it is possible to formulate a family of conformal Barthel–Kropina theories in an osculating geometry with second-order field equations, depending on the properties of the conformal factor, whose presence leads to the appearance of an effective scalar field of geometric origin in the gravitational field equations. The cosmological implications of the theory are investigated in detail by assuming a specific relation between the component of the one-form of the Kropina metric and the conformal factor. The cosmological evolution is thus determined by the initial conditions of the scalar field and a free parameter of the model. We analyze in detail three cosmological models corresponding to different values of the theory parameters. Our results show that the conformal Barthel–Kropina model can provide an acceptable description of the observational data, and may represent a theoretically attractive alternative to the standard ΛCDM cosmology.

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## 1 Introduction

Einstein's theory of general relativity (GR) represents an impressive scientific achievement of the last century. Its exceptional success is mainly due to its remarkable geometric description of gravity [1,2]. GR is a far-reaching theory of spacetime, matter, and gravity which gives a very precise account of the dynamics of the solar system. However, when extended to very large and very small scales, the theory faces several important problems, mainly deriving from cosmology and quantum field theory. Moreover, several recent cosmological observations have raised serious concerns about the validity of GR as the theoretical foundation of cosmology.

The discovery of the recent acceleration of the Universe [3–5] can be explained very well by reinserting into the Einstein gravitational field equations the old and much debated cosmological constant $\Lambda$ [6], together with a mysterious (and not yet understood) matter-type component, called dark matter. Dark matter is supposed to be pressureless, and cold. In recent years, the $\Lambda$CDM ($\Lambda$ Cold Dark Matter) cosmological model has become the standard paradigmatic approach for the interpretation of cosmological data, and in this respect it is extremely successful. However, there are a few important open questions that may suggest that $\Lambda$CDM is just a first approximation of a more realistic model yet to be found [7]. Firstly, it lacks a firm theoretical basis (no generally accepted explanation of the geometrical or physical character of the cosmological constant is known), and secondly, despite the extremely intensive experimental and observational effort, the particles assumed to form dark matter have not been discovered yet in terrestrial experiments or astrophysical observations.

The tremendous increase in the precision of recent cosmological observations and the technological advances in the field have led to another important challenge the $\Lambda$CDM standard paradigm must face. There are significant deviations between the Hubble expansion rates of the Universe as measured by the Planck satellite experiment using cosmic microwave background radiation (CMBR), originating from the early Universe as a result of the decoupling of matter and radiation, and the low redshift (local) measurements. These differences in the values of the present-day Hubble constant $H_0$ are generally called Hubble tension, which could represent a fundamental crisis in cosmology [8–14]. The difference in the determination of the numerical values of H0 as obtained by the Planck satellite, $H_0 = 66.93 \pm 0.62$ km/s/Mpc [13,14], and the values of $H_0 = 73.24 \pm 1.74$ km/s/Mpc [10] determined by the SH0ES collaboration exceeds $3\sigma$ [14]. The Hubble tension, if it indeed exists, points strongly towards the need to find new gravitational theories, as well as to replace the $\Lambda$CDM model with an alternative one.

The Big Bang singularity, so important for the understanding of the nature of the Universe, is still unexplained in the framework of the $\Lambda$CDM cosmology, and it seems that GR cannot describe the Universe at extremely high-density phases and in the presence of very strong gravitational fields. On the other hand, there is very little progress, if any, in the quantization of spacetime, geometry, and gravity [15]. Since a quantum description of the gravitational interaction is (still) missing, and there is no complete quantum formulation of gravity, GR cannot yet be considered as a fundamental physical theory similar to the other theories of physics so successfully describing elementary particle interactions.

The solution of these fundamental problems may require the consideration of novel theories of gravity, which contain GR as a particular weak field limit. There are a large number of attempts to construct gravitational theories as alternatives to GR, and they are constructed by using various, and sometimes very different, mathematical and physical perspectives (for comprehensive and detailed reviews of modified gravity theories and their astrophysical and cosmological applications, see [16–19].

One of the interesting approaches to gravitational phenomena is related to the use of the conformal transformations (rescalings) [20–29]. The important role that conformal transformations and structures may play in gravitational physics and cosmology was suggested by Penrose [20], who developed an interesting cosmological model called conformal cyclic cosmology (CCC). This theoretical model originated from consideration of the fact that when the de Sitter exponentially accelerating stage, triggered by the existence of the positive $\Lambda$, ends, the spacetime is conformally flat and space-like. This geometry coincides with the initial boundary of the very early Universe immediately after the Big Bang. In the CCC model, one assumes that the Universe is made up of eons, which represent time-oriented manifolds, with the conformally invariant compactifications possessing space-like null infinities. The CCC model has been studied in detail in [21–27].

The important role of the local conformal symmetry transformations was pointed out by 't Hooft [28], where it was shown that conformal symmetry is an exact symmetry of nature that is broken spontaneously. The breaking of the conformal symmetry could reveal a physical process explaining the small-scale properties of gravity. Conformal symmetry could be of equal importance as the Lorentz symmetry of the fundamental equations and laws of the elementary particle physics, and it may contribute significantly to the understanding of the physics of the Planck scale. By supposing that local conformal symmetry is an exact but spontaneously broken symmetry of the physical world, a theory of the gravitational interaction was proposed in [29]. In this theory, the conformal part of the metric is interpreted as a dilaton field. The theory has interesting physical consequences, with the black holes transformed into topologically trivial, regular solitons, without singularities, firewalls, or horizons.

By using conformal transformations of the metric and of the physical and geometrical quantities, it is possible to reduce gravitational theories containing higher-order and non-minimally coupled terms to GR plus some minimally coupled scalar fields [30–32]. Hence, in the GR framework, it is possible to change frames via conformal transformations. Two frames obtained from each other by conformal transformations are called conformal frames. It is important to point out that conformal frames are mathematically equivalent [30–32]. However, their physical equivalence is a topic that has generated strong debate among physicists [31,32]. Among the many possible conformal frames, two are of special interest: the Einstein frame and the Jordan frame. In the Einstein frame, there are only minimal coupling terms in the action. On the other hand, in the Jordan frame, non-minimal couplings between the gravitational fields, described by equivalent geometric quantities, and the other fields are present [32].

The conformal transformations can also be interpreted, both mathematically and physically, as local unit transformations (rescaling of the lengths and distances). They were first considered by Hermann Weyl in his proposal for a unified theory of gravitation and electromagnetism [33–35], in which he also introduced the first generalization of the Riemann geometry. Weyl called the conformal transformations gauge transformations. Gauge transformations have become the standard theoretical tool in elementary particle physics, and gauge field theories are the basis of our present-day understanding of the properties of the elementary particles. On the other hand, in the field of gravitational theories, the Weyl gauge transformations are called conformal transformations, and the invariance of physical laws or geometric quantities under them is called conformal invariance [30–32]. The electromagnetic field equations satisfy the local scale invariance. At this moment, one should note that local scale transformations do not keep the magnitudes of the vectors constant, as they are parallelly displaced in the spacetime manifold. In Riemannian geometry, a nonvanishing curvature implies that the direction of a vector parallelly transported around a closed path is modified with respect to the direction of the initial vector, while its length remains unchanged. On the other hand, in Weyl geometry, the length of a vector is modified when parallelly transported around a closed loop, with the change being a function of the spacetime position. Gravitational models and theories based on the Weyl geometry have been extensively investigated and studied in the mathematical and physical literature [36–69].

In the same that year Weyl proposed his beautiful generalization of Riemann geometry, another important geometric theory was published. This is called Finsler geometry [70], and it also represents an important extension of Riemann geometry. Even if Chern stated that Finsler geometry is "…just Riemannian geometry without the quadratic restriction," in the following, we will still refer to Finsler geometry as a generalization of Rimeann geometry. Actually, the geometry of Finsler had already been predicted by Riemann [71], who defined a geometric structure in a general space as given by the expression $ds = F\left(x^1, \ldots, x^n; \mathrm{d}x^1, \ldots, \mathrm{d}x^n\right) = F(x, \mathrm{d}x)$. In this definition, for a nonzero $y$, $y \neq 0$, the function $F(x, y)$, called the Finsler metric function, must be a positive function defined on the tangent bundle $TM$. Moreover, $F(x, y)$ must satisfy the important requirement of being homogeneous of degree one in $y$, thus satisfying the condition $F(x, \nu \mathrm{d}x) = \nu F(x, \mathrm{d}x)$, where $\nu$ is a positive constant. For $F^2 = g_{ij}(x)\mathrm{d}x^i \mathrm{d}x^j$, we obtain the important limiting case of the Riemann geometry [72].

The Finsler metric function $F$ can be expressed using the canonical coordinates $(x, y) = \left(x^I, y^I\right)$ of the tangent bundle $TM$, where $y = y^I \left(\partial/\partial x^I\right)$ denotes the tangent vector at the point $x$ of the base manifold. Thus, the arc element in a general Finsler space takes the form $ds^2 = g_{IJ}\mathrm{d}x^I \mathrm{d}x^J$. Finsler spaces possess a much more general mathematical and geometrical structure than Riemann spaces. For example, in a Finsler space, it is possible to define three kinds of curvature tensors $\left(R^{\kappa}_{\nu\lambda\mu}, S^{\kappa}_{\nu\lambda\mu}, P^{\kappa}_{\nu\lambda\mu}\right)$, while there are five torsion tensors [73].

Even though the first physical applications of Finsler geometry were proposed a relatively long time after its birth, physical and gravitational theories based on Finsler geometries have been intensively investigated as alternatives or extensions of standard GR [74–100]. One of the interesting possibilities for modeling gravitational phenomena is the dark gravity approach, which goes beyond the geometrical and mathematical formalism of Riemann spaces. In this direction, Finsler theories and cosmological models are important and interesting alternatives to the standard $\Lambda$CDM model, since they can provide a geometric explanation, or replacement, of dark energy, and perhaps even of dark matter. A large number of studies have been devoted to the investigation of possible applications of Finsler geometry in gravitational physics and cosmology, with the main goal of understanding from a new geometric perspective the evolution and the dynamics of the cosmic components [101–130].

In this respect, the cosmological applications of a special class of Finsler geometries, called Barthel–Kropina geometries, were investigated in detail in [129] and subsequently in [130]. The Kropina spaces are $(\alpha, \beta)$-type Finsler spaces, in which the Finsler metric function is defined by $F = \alpha^2/\beta$, where $\alpha$ is a Riemannian metric, $\alpha(x, y) = g_{IJ}(x)y^I y^J$, and $\beta(x, y) = A_I(x)y^I$ is a one-form. To simplify the mathematical approach, one can use the theory of the osculating Riemann spaces of Finsler geometries [132,133]. In the osculating space approach, one associates to a complicated

Finsler geometric object a simpler mathematical one, such as a Riemann metric. Hence, with the help of the osculating approach, a simpler mathematical formalism can be obtained. In the case of the Kropina metric, one can choose the field $Y(x)$ as $Y(x) = A(x)$, and one can define the A-osculating Riemannian manifold $\left(M, \hat{g}_{IJ}(x, A(x)\right)$. One can associate to this structure the Barthel connection, which is nothing but the Levi–Civita connection of the Riemann metric $\hat{g}_{IJ}(x) = \hat{g}_{IJ}(x, A(x))$.

For a cosmological metric of the Friedmann–Lemaître–Robertson–Walker type, the generalized Friedmann equations in the Barthel–Kropina geometry were obtained in [129]. These equations lead to a dark energy model which can explain the accelerating expansion of the Universe and other observational cosmological features. The predictions of the Barthel–Kropina dark energy model were compared with observational data in [130]. The model parameters were constrained using 57 Hubble data points and the Pantheon supernovae type Ia data sample. The statistical analysis was performed using Markov chain Monte Carlo (MCMC) numerical simulations. An in-depth comparison with the standard $\Lambda$CDM model was also considered, and the Akaike information criterion (AIC) and Bayesian information criterion (BIC) were considered as selection tools for the two models. The statefinder diagnostics were also considered. The results show that the Barthel–Kropina dark energy model can provide an excellent explanation of the cosmological observational data, and hence it may represent an interesting and viable alternative to the $\Lambda$CDM model.

It is the goal of the present work to introduce and develop another view on the Barthel–Kropina cosmology. Namely, we will consider the effects of a conformal transformation on the Finsler function $F(x, y)$ of the form $F(x, y) \rightarrow e^{\sigma(x)}F(x, y)$. From a physical point of view, such a transformation represents a change from the Finslerian Einstein frame to a Finslerian Jordan-type frame. As a result of the conformal transformation, the Levi–Civita connection becomes a Weyl-type connection, and a new scalar degree of freedom associated to the conformal factor $\sigma(x)$ does appear in the mathematical structure of the Einstein gravitational field equations. In order to obtain a consistent description of the gravitational phenomena, we adopt as Finslerian metric the Kropina-type $(\alpha, \beta)$ metric and the osculating geometrical approach, in which we assume $y = Y(x)$. Moreover, we introduce the Barthel connection, which is the Levi–Civita connection associated to the Riemannian metric $g(x, Y(x))$. As a first step in our analysis, we obtain the expression of the Einstein tensor in the conformal Barthel–Kropina geometry. The vector $Y$ is assumed to be the coefficient of the one-form $\beta = A_I dx^I$, $Y = A$. We also investigate the conformal transformation properties of the matter action. After formulating the gravitational field equations in a general form, by assuming the usual proportionality of the Einstein tensor with the matter energy–momentum tensor, the cosmological applications of the conformal Finsler-type geometry are considered. By adopting the Riemannian metric for the Friedmann–Lemaître–Robertson–Walker form, we obtain the generalized Friedmann equations, which also contain a new scalar degree of freedom coming from the conformal factor, assumed to be a function of the cosmological time only. By assuming a specific relation between the coefficient of the one-form $\beta$ and the conformal factor, we obtain a consistent cosmological model, formulated in the redshift space. The model is dependent on a single free parameter. A de Sitter-type solution of the conformal Barthel–Kropina field equations also exists. Several solutions of the generalized Friedmann equations corresponding to different values of the model parameter are obtained using numerical methods, and the redshift evolution of the Hubble function, deceleration parameter, matter density, and conformal factor are obtained. A comparison with the Hubble observational data and with the $\Lambda$CDM model is also performed. As a result of these investigations, we find that the conformal Barthel–Kropina model can give an acceptable description of the observational data, and it may represent an attractive alternative to standard cosmology.

The present paper is organized as follows. The conformal transformations in Riemann geometry, the Barthel–Kropiona cosmological model, and the conformal properties of the matter energy–momentum tensor are briefly reviewed in Sect. 2. The conformal transformation properties of the cosmological Kropina metric are considered in Sect. 3, where the expression of the Einstein tensor is also obtained. The generalized Friedmann equations of the conformal Barthel–Kropina cosmological model are written down in Sect. 4, where a dark energy model is also presented, built upon a specific form of the scalar conformal factor. Specific cosmological models are investigated numerically in Sect. 5, where a comparison of the models with a limited set of observational data and with the $\Lambda$CDM model is also performed. We discuss our results and present conclusions in Sect. 7.

## 2 From conformal transformation in Riemann geometry to the Barthel–Kropina cosmology

In the following, we will summarize some of the basic mathematical results on the conformal transformations in Riemann geometry to be used in the sequel. We will also present the basics of the Barthel–Kropina cosmological theory, and write down the generalized Friedmann equations for this model.

### 2.1 Conformal transformations and Riemann geometry

Let $\nabla$ be a connection on $M$. We introduce on $M$ a symmetric metric $g$ with the components of the metric tensor denoted by $g_{ij}$ The global formula of the Levi–Civita connection is given by

$$(\nabla_X g)(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z). \tag{1}$$

Locally, we have

$$\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} = \gamma^i_{jk}(x) \frac{\partial}{\partial x^i}, \tag{2}$$

where $\gamma^i_{jk}$ are called the *Christoffel symbols*, and they are defined according to

$$\gamma^i_{jk} = \frac{1}{2} g^{is} \left( \frac{\partial g_{js}}{\partial x^k} + \frac{\partial g_{ks}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s} \right). \tag{3}$$

The global expression of the curvature operator is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{4}$$

Locally, we have

$$R_{j\ kl}^{\ i} = \frac{\partial \gamma^i_{jl}}{\partial x^k} - \frac{\partial \gamma^i_{jk}}{\partial x^l} + \gamma^s_{jl}\gamma^i_{sk} - \gamma^s_{jk}\gamma^i_{sl}. \tag{5}$$

We now consider the conformal transformation of the metric given by

$$\tilde{g}_{ij}(x) = \Omega^2(x) g_{ij}(x) = e^{2\sigma(x)} g_{ij}(x), \tag{6}$$

where $\sigma(x)$ is an arbitrary function of the coordinates $x$ defined on the spacetime manifold $M$, and we have denoted the conformally transformed metric by $\tilde{g}_{ij}$. Then we obtain the first result in the conformal Riemann geometry in the form of

**Lemma 1** *The relation of Riemannian–Christoffel of $\tilde{g}_{ij}$ and $g_{ij}$ is*

$$\tilde{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma^i g_{jk}, \tag{7}$$

*where*

$$\sigma_j := \frac{\partial \sigma(x)}{\partial x^j}, \sigma^i = g^{ij} \sigma_j. \tag{8}$$

*Next, for the definition of the covariant derivative, we obtain the expression*

$$\tilde{\nabla}_X Y = \nabla_X Y + \mathrm{d}\sigma(X) Y + \mathrm{d}\sigma(Y) X - g(X, Y) \nabla\sigma, \tag{9}$$

*where we have denoted*

$$\nabla\sigma = \sigma^i \frac{\partial}{\partial x^i}, \mathrm{d}\sigma(x) = \frac{\partial \sigma(x)}{\partial x^i} X^i, \tag{10}$$

*for any $X = X^i \frac{\partial}{\partial x^i}$. In local coordinates, the proof of this relation is as follows*

$$\tilde{\gamma}^i_{jk} \frac{\partial}{\partial x^i} = \gamma^i_{jk} \frac{\partial}{\partial x^i} + \frac{\partial \sigma}{\partial x^k} \frac{\partial}{\partial x^j} + \frac{\partial \sigma}{\partial x^j} \frac{\partial}{\partial x^k} - \sigma^i g_{jk} \frac{\partial}{\partial x^i}$$
$$\tilde{\nabla}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} + \mathrm{d}\sigma \left( \frac{\partial}{\partial x^k} \right) \frac{\partial}{\partial x^j} + \mathrm{d}\sigma \left( \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k} - g \left( \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) \sigma^i \frac{\partial}{\partial x^i}.$$

*If we denote $X = \frac{\partial}{\partial x^k}$ and $Y = \frac{\partial}{\partial x^j}$, then*

$$\tilde{\nabla}_X Y = \nabla_X Y + \mathrm{d}\sigma(X) Y + \mathrm{d}\sigma(Y) X - g(Y, X) \sigma^i \frac{\partial}{\partial x^i} = \nabla_X Y + \mathrm{d}\sigma(X) Y + \mathrm{d}\sigma(Y) X - g(X, Y) \nabla\sigma.$$

*Next, we consider the curvature properties of the conformally transformed metric.*

**Lemma 2** *If we consider the conformal transformation $\tilde{g}_{ij} = e^{2\sigma(x)} g_{ij}(x)$, then the relation of the Riemannian curvature of $\tilde{g}_{ij}$ and $g_{ij}$ is*

$$\tilde{R}_{ijkl} = e^{2\sigma} R_{ijkl} - e^{2\sigma} \left( g_{ik} T_{jl} + g_{jl} T_{ik} - g_{il} T_{jk} - g_{jk} T_{il} \right),$$
$$\tilde{R}_{j\ kl}^{\ i} = \tilde{g}^{is} \tilde{R}_{sjkl} = R_{j\ kl}^{\ i} - \left( \delta^i_k T_{jl} + g_{jl} T^i_k - \delta^i_l T_{jk} - g_{jk} T^i_l \right),$$

*where*

$$\nabla_i \sigma = \frac{\partial \sigma}{\partial x^i} = \sigma_i, \nabla_i \nabla_j \sigma = \frac{\partial \sigma_i}{\partial x^j} - \sigma_p \gamma^p_{ij} = \sigma_{ij} - \sigma_p \gamma^p_{ij},$$
$$T_{ij} = \nabla_i \nabla_j \sigma - \nabla_i \sigma \nabla_j \sigma + \frac{1}{2} |\mathrm{d}\sigma|^2 g_{ij} = \sigma_{ij} - \sigma_p \gamma^p_{ij} - \sigma_i \sigma_j + \frac{1}{2} |\mathrm{d}\sigma|^2 g_{ij},$$
$$T^i_j = g^{is} T_{sj} = g^{is} \left( \sigma_{sj} - \sigma_p \gamma^p_{sj} - \sigma_s \sigma_j + \frac{1}{2} |\mathrm{d}\sigma|^2 g_{sj} \right),$$

*and*

$$|\mathrm{d}\sigma|^2 = \sigma^i \sigma_i, \sigma_{ij} = \frac{\partial^2 \sigma}{\partial x^i \partial x^j}, \sigma^i_j = g^{is} \sigma_{sj}. \tag{11}$$

*Here, $\nabla_i$ or $|i$ is the covariant derivative with respect to the Levi–Civita connection of $g$.*

*Remark* Please pay attention to the notation above. We have defined

$$\sigma_{ij} := \frac{\partial^2 \sigma}{\partial x^i \partial x^j} \text{ and } \sigma^k_j := g^{ik} \sigma_{ij},$$

which is different from

$$\frac{\partial \sigma^k}{\partial x^j} = \frac{\partial}{\partial x^j} \left( g^{ik} \sigma_i \right) = \frac{\partial g^{ik}}{\partial x^j} \sigma_i + g^{ik} \frac{\partial \sigma_i}{\partial x^j} = \frac{\partial g^{ik}}{\partial x^j} \sigma_i + g^{ik} \frac{\partial \sigma}{\partial x^i \partial x^j} = \frac{\partial g^{ik}}{\partial x^j} \sigma_i + g^{ik} \sigma_{ij}.$$

**Lemma 3** *If we consider the conformal transformation* $\tilde{g}_{ij} = e^{2\sigma(x)} g_{ij}(x)$*, then the relation of the Ricci tensor of* $\tilde{g}_{ij}$ *and* $g_{ij}$ *is*

$$\tilde{R}_{ij} = R_{ij} + (n-2)(\sigma_{ij} - \sigma_i\sigma_j - \sigma_m\gamma^m_{ij}) + (\Delta\sigma + (n-2)|\mathrm{d}\sigma|^2)g_{ij},$$

*where*

$$\Delta\sigma = g^{jk}\frac{\partial^2\sigma}{\partial x^j\partial x^k} - g^{jk}\gamma^l_{jk}\frac{\partial\sigma}{\partial x^l} = g^{jk}\left(\sigma_{jk} - \gamma^l_{jk}\sigma_l\right).$$

**Lemma 4** *If we consider the conformal transformation* $\tilde{g}_{ij} = e^{2\sigma(x)} g_{ij}(x)$*, then the relation between the scalar curvatures of* $\tilde{g}_{ij}$ *and* $g_{ij}$ *is*

$$\tilde{R} = e^{-2\sigma}\left[R + 2(n-1)\Delta\sigma + (n-2)(n-1)|\mathrm{d}\sigma|^2\right].$$

**Lemma 5** *If we consider the conformal transformation* $\tilde{g}_{ij} = e^{2\sigma(x)} g_{ij}(x)$*, then the Einstein tensor of* $\tilde{g}_{ij}$ *is*

$$\tilde{G}_{ij} = G_{ij} + (n-2)(\sigma_{ij} - \sigma_m\gamma^m_{ij} - \sigma_i\sigma_j) - (n-2)\left\{\Delta\sigma + \frac{(n-3)}{2}|\mathrm{d}\sigma|^2\right\}g_{ij}, \tag{12}$$

*where*

$$\tilde{G}_{ij} := \tilde{R}_{ij} - \frac{1}{2}\tilde{R}\tilde{g}_{ij},\ G_{ij} := R_{ij} - \frac{1}{2}Rg_{ij}.$$

#### 2.1.1 Conformal transformation of matter

In the previous Section, we considered only the conformal transformation properties of the geometrical quantities. We now turn to the matter part of gravitational action. The matter action can be generally written as

$$S_m = \int L_m(g,\psi)\sqrt{-g}d^4x, \tag{13}$$

where the matter Lagrangian $L_m$ is assumed to be a function of the metric tensor and of the (bosonic or fermionic) matter fields $\psi$. To investigate the conformal properties of the matter Lagrangian density, we first assume that under conformal transformations, the matter Lagrangian is transformed according to the rule

$$\tilde{L}_m = e^{-4\sigma(x)}L_m. \tag{14}$$

Then the conformally transformed action becomes

$$\tilde{S}_m = \int \tilde{L}_m\sqrt{-\tilde{g}}d^4x = \int e^{-4\sigma(x)}L_m e^{4\sigma(x)}\sqrt{-g}d^4x = S_m. \tag{15}$$

Hence, it follows that the action of the ordinary baryonic matter is invariant under the considered conformal transformations (14). This result implies that the baryonic matter component of the gravitational interaction can be described in all conformally related frames by the same expression, since it is an invariant quantity.

The matter energy–momentum tensor is defined according to the expression

$$T_{IJ} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{IJ}}\left(\sqrt{-g}L_m\right). \tag{16}$$

After performing a conformal transformation of the metric, we obtain

$$\tilde{T}_{IJ} = e^{-2\sigma(x)}T_{IJ}. \tag{17}$$

For the trace $\tilde{T} = \tilde{T}^I_I$ of the ordinary matter energy–momentum tensor, we obtain the expression $\tilde{T} = e^{-4\sigma(x)}T$, where $T = T^I_I$.

### 2.2 The Barthel–Kropina cosmological model

In [129,130] we considered the Kropina metric [131]

$$F = \frac{\alpha^2}{\beta} = \frac{g_{IJ}(x)y^Iy^J}{A_I(x)y^I},\ \ I,J = \{0,1,2,3\}$$

with the fundamental tensor

$$\hat{g}_{IJ}(x,y) = \frac{2\alpha^2}{\beta^2}g_{IJ}(x) + \frac{3\alpha^4}{\beta^4}A_IA_J - \frac{4\alpha^2}{\beta^3}(y_IA_J + y_JA_I) + \frac{4}{\beta^2}y_Iy_J, \tag{18}$$

where $y_I := g_{IJ}y^J$.

Let us consider for the Riemannian metric the expression

$$(g_{IJ}(x)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(x^0) & 0 & 0 \\ 0 & 0 & -a^2(x^0) & 0 \\ 0 & 0 & 0 & -a^2(x^0) \end{pmatrix},$$

representing the homogeneous and isotropic, flat FLRW model, and $\beta = A_I(x)y^I = A_0(x)y^0$, where

$$(A_I(x)) = (A_0,0,0,0) = (a(x^0)\eta(x^0),0,0,0)$$

is a covariant vector field on $M$.

Moreover, we consider the preferred direction

$$Y = Y^I\frac{\partial}{\partial x^I} = A^I\frac{\partial}{\partial x^I},$$

where $A^I := g^{IJ}A_J$. In the case of the FLRW metric, it follows that

$$(Y^I) \equiv (A^I) = (Y_I) = (A_I) = (a(x^0)\eta(x^0),0,0,0).$$

We evaluate

$$\beta|_{y=A} = [a(x^0)\eta(x^0)]^2$$

$$(h_{IJ})|_{y=A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -a^2(x^0) & 0 & 0 \\ 0 & 0 & -a^2(x^0) & 0 \\ 0 & 0 & 0 & -a^2(x^0) \end{pmatrix},$$

where $h_{IJ} := g_{IJ}(x) - \frac{y_I}{\alpha}\frac{y_J}{\alpha}$, $y_I := g_{IJ}(x)y^J$.

By substitution in (18), we obtain the osculating Riemannian metric

$$\hat{g}_{IJ}(x) = \hat{g}_{IJ}(x, y = A) = \begin{pmatrix} \frac{1}{a^2\eta^2} & 0 & 0 & 0 \\ 0 & -\frac{2}{\eta^2} & 0 & 0 \\ 0 & 0 & -\frac{2}{\eta^2} & 0 \\ 0 & 0 & 0 & -\frac{2}{\eta^2} \end{pmatrix}, \quad (19)$$

and the nonvanishing components of the Christoffel symbols of the second kind of (19) are

$$\hat{\gamma}^I_{JK} = \begin{cases} \hat{\gamma}^0_{00} &= -\dfrac{\eta\mathcal{H} + \eta'}{\eta}, \\ \hat{\gamma}^0_{ij} &= -\dfrac{2a^2\eta'}{\eta}\delta_{ij}, \\ \hat{\gamma}^i_{0j} &= -\dfrac{\eta'}{\eta}\delta^i_j, \end{cases} \quad (20)$$

where $\mathcal{H} = \frac{a'}{a}$.

Recall the formula of the Ricci tensor from [129]

$$\hat{R}_{IJ} = \begin{cases} \hat{R}_{00} = \dfrac{3}{\eta^2}\left[\eta\eta'' + \eta\eta'\mathcal{H} - (\eta')^2\right], \\ \hat{R}_{ij} = \dfrac{2a^2}{\eta^2}\left[3(\eta')^2 - \eta\eta'' - \eta\eta'\mathcal{H}\right]\delta_{ij}, \end{cases}$$

and the Ricci scalar

$$\hat{R} = 6a^2\left(\eta\eta'' + \eta\eta'\mathcal{H} - 2(\eta')^2\right).$$

The Einstein field equations, given by $\hat{G}_{00} = \left(8\pi G/c^4\right)\hat{g}_{00}\rho c^2$ and $\hat{G}_{ii} = -\left(8\pi G/c^4\right)\hat{g}_{ii}\,p$, where $\rho$ and $p$ denote the matter energy density and pressure, respectively, give the system of the generalized Friedmann equations

$$\frac{3(\eta')^2}{\eta^2} = \frac{8\pi G}{c^4}\frac{1}{a^2\eta^2}\rho c^2, \quad (21)$$

and

$$a^2\left[-3(\eta')^2 + 2\eta\eta'' + 2\mathcal{H}\eta\eta'\right] = \frac{8\pi G}{c^4}p, \quad (22)$$

respectively. By substituting the term $-3\left(\eta'\right)^2$ using Eq. (21), Eq. (22) takes the simple form

$$2a\eta\frac{\mathrm{d}}{\mathrm{d}x^0}\left(\eta' a\right) = \frac{8\pi G}{c^4}\left(\rho c^2 + p\right). \quad (23)$$

The cosmological implications of this model were investigated in detail in [129,130].

## 3 Conformal transformation and the Kropina metric

In this section, we will consider the conformal transformation properties of the general $(\alpha, \beta)$ metrics, with a special emphasis on the Kropina case. From a mathematical point of view, the role of the conformal transformations in Finsler geometry, including the case of the $(\alpha, \beta)$ metrics, was investigated in [134–140]. From a physical point of view, the role of the gauge transformations in Finsler geometry was considered in [141].

### 3.1 Conformal transformation of an $(\alpha, \beta)$ metric

For any $(\alpha, \beta)$ metric $F = F(\alpha, \beta)$, we can consider its conformal transformation

$$\tilde{F}(x, y) := e^{\sigma(x)}F(x, y) = \tilde{F}(\tilde{\alpha}, \tilde{\beta}), \quad (24)$$

which is an $(\tilde{\alpha}, \tilde{\beta})$-metric, where

$$\tilde{\alpha} = e^{\sigma(x)}\alpha, \ \tilde{\beta} = e^{\sigma(x)}\beta. \quad (25)$$

The fundamental tensor of $\tilde{F}$ is given by the Hessian

$$\tilde{g}_{IJ} := \frac{1}{2}\frac{\partial^2\tilde{F}^2}{\partial y^I \partial y^J}. \quad (26)$$

In the case of the general $(\alpha, \beta)$ metric, we obtain

$$\hat{g}_{IJ}(x, y) = \rho g_{IJ}(x) + \rho_0 b_I b_J + \rho_1\left(b_I\frac{y_J}{\alpha} + b_J\frac{y_I}{\alpha}\right) - s\rho_1\frac{y_I}{\alpha}\frac{y_J}{\alpha}, \quad (27)$$

where $y_I = g_{IJ}y^J$, i.e., $\frac{y_I}{\alpha} = \frac{\partial\alpha}{\partial y^I}$, and

$$\rho = \phi^2 - s\phi\phi',\ \rho_0 = \phi\phi'' + \phi'^2,\ \rho_1 = -s(\phi\phi'' + \phi'^2) + \phi\phi', \quad (28)$$

as function of $s = \frac{\beta}{\alpha}$. Here, $I, J = \{0, 1, 2, 3\}$.

Now, by considering a conformal transformation of $\tilde{F}$, we get

$$\hat{\tilde{g}}_{IJ}(x,y) = \tilde{\rho}\tilde{g}_{IJ}(x) + \tilde{\rho}_0\tilde{b}_I\tilde{b}_J + \tilde{\rho}_1\left(\tilde{b}_I\frac{\tilde{y}_J}{\tilde{\alpha}} + \tilde{b}_J\frac{\tilde{y}_I}{\tilde{\alpha}}\right)$$
$$-\tilde{s}\tilde{\rho}_1\frac{\tilde{y}_I}{\tilde{\alpha}}\frac{\tilde{y}_J}{\tilde{\alpha}} = \rho e^{2\sigma}g_{IJ}(x) + \rho_0 e^{2\sigma}b_I b_J$$
$$+\rho_1 e^{2\sigma}\left(b_I\frac{y_J}{\alpha} + b_J\frac{y_I}{\alpha}\right) - s\rho_1 e^{2\sigma}\frac{y_I}{\alpha}\frac{y_J}{\alpha}$$
$$= e^{2\sigma}\hat{g}_{IJ}(x,y), \tag{29}$$

where we have used the relations

$$\tilde{s} = \frac{\tilde{\beta}}{\tilde{\alpha}} = \frac{\beta}{\alpha},\ \tilde{\rho} = \rho,\ \tilde{\rho}_0 = \rho_0,\ \tilde{\rho}_1 = \rho_1,$$

since they are just derivatives with respect to $s$.

Moreover,

$$\frac{\tilde{y}_I}{\tilde{\alpha}} = \frac{\tilde{a}_{IJ}y^J}{\tilde{\alpha}} = \frac{e^{2\sigma}a_{IJ}y^J}{e^{\sigma}\alpha} = e^{\sigma}\frac{y_I}{\alpha}.$$

### 3.2 The Kropina case

Likewise, we can extend the case above by taking the conformal transform of the Kropina metric

$$\tilde{F} := e^{\sigma(x)}\frac{\alpha^2}{\beta} = \frac{\tilde{\alpha}^2}{\tilde{\beta}},$$

where $\tilde{\alpha} = e^{\sigma(x)}\alpha$, $\tilde{\beta} = e^{\sigma(x)}\beta$.

Similarly, with the standard Kropina case, the osculating Riemannian metric is obtained as

$$\hat{\tilde{g}}_{IJ}(x) = e^{2\sigma(x)}\hat{g}_{IJ}(x), \tag{30}$$

where $\hat{g}_{IJ}(x)$ is given by Eq. (19). Explicitly, the metric $\hat{g}_{IJ}(x)$ has the expression

$$\hat{\tilde{g}}_{IJ}(x) = e^{2\sigma(x)}\begin{pmatrix} \frac{1}{a^2\eta^2} & 0 & 0 & 0 \\ 0 & -\frac{2}{\eta^2} & 0 & 0 \\ 0 & 0 & -\frac{2}{\eta^2} & 0 \\ 0 & 0 & 0 & -\frac{2}{\eta^2} \end{pmatrix}. \tag{31}$$

Now we can compute the Christoffel symbols, curvatures, and the Einstein tensor for this conformal Riemannian metric.

#### 3.2.1 *The generalized Einstein tensor in the conformal Barthel–Kropina model*

From Lemma 1, and by using Eq. (20), we obtain the components of the Christoffel symbols of the second kind of the conformal metric (30) as

$$\hat{\tilde{\gamma}}^I_{JK} = \begin{cases} \hat{\tilde{\gamma}}^0_{00} &= -\dfrac{\eta\mathcal{H}+\eta'}{\eta} + \delta^0_0\sigma_0 + \delta^0_0\sigma_0 - \sigma^0\hat{g}_{00} \\ &= -\dfrac{\eta\mathcal{H}+\eta'}{\eta} + \sigma_0, \\ \hat{\tilde{\gamma}}^0_{ij} &= -\dfrac{2a^2\eta'}{\eta}\delta_{ij} + \delta^0_i\sigma_j + \delta^0_j\sigma_i - \sigma^0\hat{g}_{ij} \\ &= -2a^2\left(\dfrac{\eta'}{\eta} - \sigma_0\right)\delta_{ij}, \\ \hat{\tilde{\gamma}}^i_{0j} &= -\dfrac{\eta'}{\eta}\delta^i_j + \delta^i_0\sigma_j + \delta^i_j\sigma_0 - \sigma^i\hat{g}_{0j} \\ &= \left(-\dfrac{\eta'}{\eta} + \sigma_0\right)\delta^i_j, \\ \hat{\tilde{\gamma}}^i_{jk} &= \delta^i_j\sigma_k + \delta^i_k\sigma_j - \delta^{im}\sigma_m\delta_{jk}, \\ \hat{\tilde{\gamma}}^0_{0i} &= \sigma_i, \\ \hat{\tilde{\gamma}}^i_{00} &= -\dfrac{1}{2a^2}\delta^{im}\sigma_m, \end{cases} \tag{32}$$

where $\mathcal{H} = \frac{a'}{a}$ and $\sigma^0 = \hat{g}^{00}\sigma_0 = a^2\eta^2\sigma_0$.

We now successively obtain

$$|\mathrm{d}\sigma|^2 = \sigma^I\sigma_I = \hat{g}^{IJ}\sigma_J\sigma_I = a^2\eta^2\sigma_0^2 - \frac{\eta^2}{2}\sum_{i=1}^{3}\sigma_i^2 \tag{33}$$

and

$$\Delta\sigma = a^2\eta^2\sigma_{00} - \frac{\eta^2}{2}\sum_{i=1}^{3}\sigma_{ii} - a^2\eta\left(2\eta' - \eta\mathcal{H}\right)\sigma_0, \tag{34}$$

respectively. Let us now consider the four-dimensional case with $n = 4$. From Lemma 3, we obtain

$$\hat{\tilde{R}}_{IJ} = \hat{R}_{IJ} + 2(\sigma_{IJ} - \sigma_I\sigma_J - \sigma_M\hat{\gamma}^M_{IJ})$$
$$+(\Delta\sigma + 2|\mathrm{d}\sigma|^2)\hat{g}_{IJ},$$

where $I, J, M = 0, 1, 2, 3$. We can now formulate the following.

**Lemma 6** *1.* $\sigma_M\hat{\gamma}^M_{00} = \sigma_0\hat{\gamma}^0_{00} + \sigma_i\hat{\gamma}^i_{00} = -\frac{\eta\mathcal{H}+\eta'}{\eta}\sigma_0$.
*2.* $\sigma_M\hat{\gamma}^M_{ij} = \sigma_0\hat{\gamma}^0_{ij} + \sigma_t\hat{\gamma}^t_{ij} = -\frac{2a^2\eta'}{\eta}\sigma_0\delta_{ij}$,
*where* $i, j = 1, 2, 3$.

We now proceed to the computation of the components of the Ricci tensor. We first obtain

$$\hat{\tilde{R}}_{00} = 3\left[\frac{\eta'' + \eta'\mathcal{H}}{\eta} - \frac{(\eta')^2}{\eta^2} + \sigma_{00} + \mathcal{H}\sigma_0\right]$$
$$-\frac{1}{2a^2}\left(\sum_{k=1}^{3}\sigma_{kk} + 2\sum_{k=1}^{3}\sigma_k^2\right) \tag{35}$$

and

$$\hat{\tilde{R}}_{ij} = 2\left(\sigma_{ij} - \sigma_i\sigma_j\right) + \psi_1\delta_{ij}, \tag{36}$$

respectively, where

$$\psi_1 := 2a^2\left\{\frac{1}{\eta^2}[3(\eta')^2-\eta\eta''-\eta\eta'\mathcal{H}]+\frac{1}{\eta}(4\eta'-\eta\mathcal{H})\sigma_0 -\sigma_{00}-2\sigma_0^2\right\}+\sum_{k=1}^{3}\sigma_{kk}+2\sum_{k=1}^{3}\sigma_k^2. \quad (37)$$

On the other hand, if we consider $n=4$, from Lemma 4 we get

$$\hat{\hat{R}} = e^{-2\sigma}\left[\hat{R}+6\Delta\sigma+6|\mathrm{d}\sigma|^2\right]. \quad (38)$$

Hence, we immediately obtain for $\hat{\hat{R}}$ the expression

$$\hat{\hat{R}} = 6e^{-2\sigma}\left\{a^2\left(\eta\eta''+\eta\eta'\mathcal{H}-2(\eta')^2\right) +a^2\eta^2(\sigma_{00}+(\sigma_0)^2)-\frac{\eta^2}{2}\left(\sum_{k=1}^{3}\sigma_{kk}+\sum_{k=1}^{3}\sigma_k^2\right) - a^2\eta\left(2\eta'-\eta\mathcal{H}\right)\sigma_0\right\}. \quad (39)$$

Next, we consider the expression of the Einstein tensor of $\tilde{g}_{IJ}(x)=e^{2\sigma(x)}\hat{g}_{IJ}(x)$ in the four-dimensional case, $I,J=0,1,2,3$.

From Lemma 5, we obtain the general expression of the Einstein tensor

$$\hat{\hat{R}}_{IJ}-\frac{1}{2}\hat{\hat{R}}\hat{\hat{g}}_{IJ} = \hat{R}_{IJ}-\frac{1}{2}\hat{R}\hat{g}_{IJ}+2(\sigma_{IJ}-\sigma_I\sigma_J-\sigma_M\hat{\gamma}^M_{IJ}) -2(\Delta\sigma+\frac{1}{2}|\mathrm{d}\sigma|^2)\hat{g}_{IJ}. \quad (40)$$

Then, in the case of $(I,J)=(0,0)$, we get

$$\hat{\hat{R}}_{00}-\frac{1}{2}\hat{\hat{R}}\tilde{g}_{00} = \hat{R}_{00}-\frac{1}{2}\hat{R}\hat{g}_{00}+2\left[\sigma_{00}-(\sigma_0)^2-\sigma_M\hat{\gamma}^M_{00}\right] -2\left(\Delta\sigma+\frac{1}{2}|\mathrm{d}\sigma|^2\right)\hat{g}_{00}. \quad (41)$$

Explicitly, for $\hat{\hat{G}}_{00}$ we obtain

$$\hat{\hat{G}}_{00} = \frac{3(\eta')^2}{\eta^2}-3\sigma_0^2+\frac{6\eta'}{\eta}\sigma_0 +\frac{1}{a^2}\left(\sum_{i=1}^{3}\sigma_{ii}+\frac{1}{2}\sum_{i=1}^{3}\sigma_i^2\right). \quad (42)$$

For the spatial components of the Einstein tensor, we find the expression

$$\hat{\hat{G}}_{ij} = 2(\sigma_{ij}-\sigma_i\sigma_j)+\psi_2\delta_{ij}, \quad (43)$$

where

$$\psi_2 := \frac{2a^2}{\eta^2}[-3(\eta')^2+2\eta\eta'\mathcal{H}+2\eta\eta'']+4a^2\left(\sigma_{00}+\frac{1}{2}\sigma_0^2\right) +\frac{a^2}{\eta}\left(-\eta'+\eta\mathcal{H}\right)\sigma_0-\frac{1}{2}\sum_{k=1}^{3}\sigma_{kk}-\sum_{k=1}^{3}\sigma_k^2. \quad (44)$$

## 4 The generalized Friedmann equations, and their cosmological implications

We now *postulate* that the Einstein gravitational field equations can be formulated in the conformal Barthel–Kropina geometry in the form

$$\hat{\hat{G}}_{IJ} = \frac{8\pi G}{c^4}\hat{\hat{T}}_{IJ}, \quad (45)$$

where $\hat{\hat{T}}_{IJ}$ is the matter energy–momentum tensor in the conformal frame. Similarly to the previous investigations, we assume that the thermodynamic properties of the cosmological matter in the conformal Barthel–Kropina geometry are characterized by the energy density $\hat{\hat{\rho}}c^2$ and the thermodynamic pressure $\hat{\hat{p}}$ only. Moreover, *we assume the existence of a frame comoving with matter*. Hence, *we postulate* that the energy–momentum tensor of the matter in the conformal frame takes the form

$$\hat{\hat{T}}_I^J = \begin{pmatrix} \hat{\hat{\rho}}c^2 & 0 & 0 & 0\\ 0 & -\hat{\hat{p}} & 0 & 0\\ 0 & 0 & -\hat{\hat{p}} & 0\\ 0 & 0 & 0 & -\hat{\hat{p}}\end{pmatrix}, \quad (46)$$

and

$$\hat{\hat{T}}_{IJ} = e^{-2\sigma(x)}\begin{pmatrix} \frac{e^{2\sigma(x)}}{a^2\eta^2}\hat{\hat{\rho}}c^2 & 0 & 0 & 0\\ 0 & \frac{2e^{2\sigma(x)}}{\eta^2}\hat{\hat{p}} & 0 & 0\\ 0 & 0 & \frac{2e^{2\sigma(x)}}{\eta^2}\hat{\hat{p}} & 0\\ 0 & 0 & 0 & \frac{2e^{\sigma(x)}}{\eta^2}\hat{\hat{p}}\end{pmatrix}, \quad (47)$$

respectively.

Due to the homogeneity and isotropy of the cosmological spacetime, all physical and geometrical quantities can depend only on the time coordinate $x^0$. As for the conformal factor, we assume first that it has the form

$$\sigma(x) = \phi\left(x^0\right)+\gamma_1 x+\gamma_2 y+\gamma_3 z, \quad (48)$$

where $\gamma_i$, $i=1,2,3$ are arbitrary constants. For this form of the conformal factor, we have $\sigma_0=\phi'\left(x^0\right)$, $\sigma_{00}=\phi''\left(x^0\right)$, $\sigma_i=\gamma_i$, and $\sigma_{ij}\equiv 0$, $i,j=1,2,3$. With this choice, Eq. (43) gives $\hat{\hat{G}}_{ij}=-\gamma_i\gamma_j=0$, which implies $\gamma_i=0$, $i=1,2,3$. Hence, we will choose the conformal factor as $\sigma(x)=\phi\left(x^0\right)$, and thus *we will restrict our analysis to* *only the time-dependent conformal transformations of the Kropina metric*.

### 4.1 The generalized Friedmann equations

Then, the generalized Friedmann equations describing the cosmological evolution in the conformal Barthel–Kropina geometry take the form

$$\frac{3(\eta')^2}{\eta^2}=\frac{8\pi G}{c^2}\frac{1}{a^2\eta^2}\hat{\hat{\rho}}+3\left(\phi'\right)^2-6\frac{\eta'}{\eta}\phi', \tag{49}$$

and

$$\frac{2}{\eta^2}[-3(\eta')^2+2\eta\eta'\mathcal{H}+2\eta\eta'']=\frac{16\pi G}{c^4}\frac{1}{a^2\eta^2}\hat{\hat{p}} -4\left[\phi''+\frac{1}{2}\left(\phi'\right)^2\right]+\left(\frac{\eta'}{\eta}-\mathcal{H}\right)\phi', \tag{50}$$

respectively. By eliminating the term $-3\left(\eta'\right)^2/\eta^2$ between Eqs. (49) and (50), we obtain the relation

$$2\frac{1}{a\eta}\frac{\mathrm{d}}{\mathrm{d}x^0}\left(a\eta'\right)=\frac{4\pi G}{c^4}\frac{1}{a^2\eta^2}\left(\hat{\hat{\rho}}c^2+\hat{\hat{p}}\right)-\left(\phi''-\left(\phi'\right)^2\right) -\frac{11}{4}\frac{\eta'}{\eta}\phi'-\frac{1}{4}\phi'\mathcal{H}. \tag{51}$$

We now consider the limiting case of the system (49) and (50), corresponding to $\eta\rightarrow 1/a$, $(A_I(x))=(1,0,0,0)$, and $\beta=y^0$. Hence, the generalized Friedmann equations of the conformal Barthel–Kropina model take the form

$$3\mathcal{H}^2=\frac{8\pi G}{c^4}\hat{\hat{\rho}}c^2+3\left(\phi'\right)^2+6\mathcal{H}\phi', \tag{52}$$

and

$$2\mathcal{H}'+3\mathcal{H}^2=-\frac{8\pi G}{c^4}\hat{\hat{p}}+2\left[\phi''+\frac{1}{2}\left(\phi'\right)^2\right]+\mathcal{H}\phi', \tag{53}$$

respectively. For $\phi=0$, we fully recover the standard Friedmann equations of general relativity.

#### *4.1.1 The de Sitter solution*

In the case of the standard Barthel–Kropina cosmological model, with $\phi=0$, there is no vacuum de Sitter-type solution of the field equations, since for $\rho=p=0$ all field equations are satisfied by the simple case $\eta'=0$, $\eta=$ constant, a solution independent of the concrete form of $\mathcal{H}$. The situation is different in the conformal Barthel–Kropina model. By assuming $\hat{\hat{\rho}}=\hat{\hat{p}}=0$, Eq. (49) can be reformulated as

$$\left(\phi'-\frac{\eta'}{\eta}\right)^2=2\frac{\left(\eta'\right)^2}{\eta^2}, \tag{54}$$

giving

$$\phi'=\left(1\pm\sqrt{2}\right)\frac{\eta'}{\eta}=\varsigma\frac{\eta'}{\eta}, \tag{55}$$

where we have denoted $\varsigma=1\pm\sqrt{2}$. Then, for $\mathcal{H}=\mathcal{H}_0=$ constant, Eq. (50) takes the form

$$4\varsigma(\varsigma+1)\phi''\left(x^0\right)+\left(2\varsigma^2+\varsigma-2\right)\phi'\left(x^0\right)^2 -(\varsigma-4)\varsigma\mathcal{H}_0\phi'\left(x^0\right)=0, \tag{56}$$

having for $\varsigma=1+\sqrt{2}$ the general solution given by

$$\phi\left(x^0\right)=c_2-\frac{8\left(3-\sqrt{2}\right)}{5\left(8-5\sqrt{2}\right)} \times\left\{\ln\left[1+5e^{\left(4\sqrt{2}-5\right)c_1\mathcal{H}_0}e^{\left(\frac{5\sqrt{2}}{8}-1\right)\mathcal{H}_0x^0}\right] -5c_1\mathcal{H}_0\right\}, \tag{57}$$

where $c_1$ and $c_2$ are arbitrary constants of integration. In the limit of large times, $\phi\left(x^0\right)$ tends to a constant. During the vacuum de Sitter phase of expansion, the function $\eta$ is given by $\eta\propto e^{\phi/\varsigma}$.

### 4.2 Dark energy in the conformal Barthel–Kropina model

We will now investigate the possibility of the dark energy description as a geometric effect in the Barthel–Kropina cosmological model. We have already seen that in the limit $\eta\rightarrow 1/a$ and $\phi=0$, the general relativistic model without a cosmological constant is recovered. We will now assume that the departures from general relativity can be described by a small variation of $\eta$, which depends on the conformal factor. Hence, we tentatively propose a cosmological model in which $\eta$ has the form

$$\eta=\frac{e^{\gamma\phi}}{a}, \tag{58}$$

where $\gamma$ is a constant. Thus, we immediately obtain

$$\frac{\eta'}{\eta}=\gamma\phi'-\mathcal{H},\ \frac{\eta''}{\eta}=\gamma\phi''-\mathcal{H}'+\left(\gamma\phi'-\mathcal{H}\right)^2. \tag{59}$$

Hence, the generalized Friedmann Eqs. (49) and (50) of the conformal Barthel–Kropina model become

$$3\mathcal{H}^2=\frac{8\pi G}{c^2}e^{-2\gamma\phi}\hat{\hat{\rho}}+3\left(1-2\gamma-\gamma^2\right)\left(\phi'\right)^2 +6(1+\gamma)\phi'\mathcal{H} =\frac{8\pi G}{c^2}e^{-2\gamma\phi}\hat{\hat{\rho}}+\hat{\hat{\rho}}_\phi, \tag{60}$$

and

$$2\mathcal{H}' + 3\mathcal{H}^2 = -\frac{8\pi G}{c^4} e^{-2\gamma\phi} \hat{\hat{p}} + 2(1+\gamma)\phi'' - \left(1 - \frac{1}{2}\gamma - \gamma^2\right)(\phi')^2 + (1+4\gamma)\phi'\mathcal{H} = -\frac{8\pi G}{c^4} e^{-2\gamma\phi} \hat{\hat{p}} - \hat{\hat{p}}_\phi, \tag{61}$$

respectively, where we have denoted

$$\hat{\hat{\rho}}_\phi = 3\left(1 - 2\gamma - \gamma^2\right)(\phi')^2 + 6(1+\gamma)\phi'\mathcal{H}, \tag{62}$$

and

$$\hat{\hat{p}}_\phi = -2(1+\gamma)\phi'' + \left(1 - \frac{1}{2}\gamma - \gamma^2\right)(\phi')^2 - (1+4\gamma)\phi'\mathcal{H}, \tag{63}$$

respectively.

From Eqs. (60) and (61) we immediately obtain

$$\mathcal{H}' = -\frac{4\pi G}{c^4} e^{-2\gamma\phi}\left(\hat{\hat{\rho}} c^2 + \hat{\hat{p}}\right) + (1+\gamma)\phi'' + \frac{1}{2}\left(4\gamma^2 + \frac{13}{2}\gamma - 4\right)(\phi')^2 - \frac{1}{2}(2\gamma+5)\phi'\mathcal{H}. \tag{64}$$

Equations (60) and (61) give the total conservation equation for matter and the conformally induced scalar field as

$$\frac{8\pi G}{c^4}\left[\hat{\hat{\rho}}' + 3\mathcal{H}\left(\hat{\hat{\rho}} + \frac{\hat{\hat{p}}}{c^2}\right) - 2\gamma\phi'\hat{\hat{\rho}}\right] e^{-2\gamma\phi} + \hat{\hat{\rho}}_\phi' + 3\mathcal{H}\left(\hat{\hat{\rho}}_\phi + \hat{\hat{p}}_\phi\right) = 0. \tag{65}$$

We split Eq. (65) into two balance equations, for matter and the conformal scalar field, respectively, given by

$$\hat{\hat{\rho}}' + 3\mathcal{H}\left(\hat{\hat{\rho}} + \frac{\hat{\hat{p}}}{c^2}\right) - 2\gamma\phi'\hat{\hat{\rho}} = 0 \tag{66}$$

and

$$\hat{\hat{\rho}}_\phi' + 3\mathcal{H}\left(\hat{\hat{\rho}}_\phi + \hat{\hat{p}}_\phi\right) = 0, \tag{67}$$

respectively.

Equation (67) gives the dynamical evolution of the conformal scalar field as

$$6\left(1 - 2\gamma - \gamma^2\right)\phi'' + 3\left(4 - \frac{13}{2}\gamma - 4\gamma^2\right)\mathcal{H}\phi' + 6(1+\gamma)\mathcal{H}' + 3(2\gamma+5)\mathcal{H}^2 = 0. \tag{68}$$

Hence, we have obtained the basic equations describing the cosmological dynamics in the conformal Barthel–Kropina model as given by Eqs. (64), (66), and (68).

## 5 Cosmological models in the conformal Barthel–Kropina theory

In this section, we will investigate the cosmological viability of the conformal Barthel–Kropina cosmological model. Specifically, we will consider the late-time evolution of the model, and we will compare its predictions with a selected set of cosmological data for the Hubble function. We will assume that the matter content of the present-day Universe can be well described by a pressureless fluid, with zero pressure, and hence in the following we will take $\hat{\hat{p}} = 0$.

### 5.1 Cosmological evolution equations

From Eqs. (64) and (68) we obtain $\mathcal{H}'$ and $\phi''$ in the form

$$\mathcal{H}' = \frac{1}{4}\left(\gamma^2 + 2\gamma - 1\right)\frac{8\pi G}{c^2}\hat{\hat{\rho}} e^{-2\gamma\phi} - \frac{1}{8}\left(8\gamma^4 + 29\gamma^3 + 10\gamma^2 - 29\gamma + 8\right)(\phi')^2 - \frac{1}{4}\left(2\gamma^2 + 7\gamma + 5\right)\mathcal{H}^2 + \frac{3}{8}\left(4\gamma^3 + 13\gamma^2 + 7\gamma - 6\right)\mathcal{H}\phi', \tag{69}$$

and

$$\phi'' = \frac{1}{4}(\gamma+1)\frac{8\pi G}{c^2}\hat{\hat{\rho}} e^{-2\gamma\phi} - \frac{1}{8}\left(8\gamma^3 + 21\gamma^2 + 5\gamma - 8\right)(\phi')^2 - \frac{1}{4}(2\gamma+5)\mathcal{H}^2 + \frac{1}{8}\left(12\gamma^2 + 27\gamma + 2\right)\mathcal{H}\phi', \tag{70}$$

respectively.

#### *5.1.1 The dimensionless form of the generalized Friedmann equations*

We will now replace the coordinate $x^0 = ct$ in the evolution equations with the time $t$, and the Hubble function $\mathcal{H}$ with the time-dependent Hubble function $H = \dot{a}/a$, where a dot denotes the derivative with respect to the time $t$, so that $\mathcal{H} = H/c$. We will also introduce a set of dimensionless variables $(h, \tau, r_m)$, defined according to

$$H = H_0 h, \tau = H_0 t, \hat{\hat{\rho}} = \frac{3H_0^2}{8\pi G} r_m, \tag{71}$$

where $H_0$ is the present-day value of the Hubble function. Hence, we obtain the full set of the evolution equations of the conformal Barthel–Kropina cosmological model as

$$\frac{\mathrm{d}r_m}{\mathrm{d}\tau} + 3hr_m = 2\gamma\frac{\mathrm{d}\phi}{\mathrm{d}\tau} r_m, \tag{72}$$

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} = \frac{3}{4}\left(\gamma^2 + 2\gamma - 1\right) r_m e^{-2\gamma\phi}$$

$$-\frac{1}{8}\left(8\gamma^4+29\gamma+10\gamma^2-29\gamma+8\right)\left(\frac{d\phi}{d\tau}\right)^2$$
$$-\frac{1}{4}\left(2\gamma^2+7\gamma+5\right)h^2$$
$$+\frac{3}{8}\left(4\gamma^3+13\gamma^2+7\gamma-6\right)h\frac{d\phi}{d\tau}, \tag{73}$$

and

$$\frac{d^2\phi}{d\tau^2}=\frac{3}{4}(\gamma+1)r_m e^{-2\gamma\phi} -\frac{1}{8}\left(8\gamma^3+21\gamma^2+5\gamma-8\right)\left(\frac{d\phi}{d\tau}\right)^2 -\frac{1}{4}(2\gamma+5)h^2+\frac{1}{8}\left(12\gamma^2+27\gamma+2\right)h\frac{d\phi}{d\tau}, \tag{74}$$

respectively.

Equation (72) for the matter density can be immediately integrated to give

$$r_m(\tau)=r_{m0}\frac{e^{2\gamma\phi}}{a^3}=r_{m0}\frac{\eta^2}{a}, \tag{75}$$

where $r_{m0}$ is an arbitrary constant of integration.

### 5.1.2 *The redshift representation*

In order to facilitate the comparison with the observational data, we introduce the redshift variable $z$, defined as $1+z=1/a$, giving $d/d\tau=-(1+z)h(z)d/dz$. Hence, we can reformulate the cosmological evolution equations in the redshift space as

$$-(1+z)h\frac{d\phi}{dz}=u, \tag{76}$$

$$(1+z)\frac{dr_m}{dz}-3r_m=2\gamma(1+z)\frac{d\phi}{dz}r_m, \tag{77}$$

$$-(1+z)h\frac{dh}{dz}=\frac{3}{4}\left(\gamma^2+2\gamma-1\right)r_m e^{-2\gamma\phi} -\frac{1}{8}\left(8\gamma^4+29\gamma^3+10\gamma^2 -29\gamma+8\right)u^2 -\frac{1}{4}\left(2\gamma^2+7\gamma+5\right)h^2 +\frac{3}{8}\left(4\gamma^3+13\gamma^2+7\gamma-6\right)hu, \tag{78}$$

and

$$-(1+z)h\frac{du}{dz}=\frac{3}{4}(\gamma+1)r_m e^{-2\gamma\phi} -\frac{1}{8}\left(8\gamma^3+21\gamma^2+5\gamma-8\right)u^2 -\frac{1}{4}(2\gamma+5)h^2 +\frac{1}{8}\left(12\gamma^2+27\gamma+2\right)hu, \tag{79}$$

respectively. The system of Eqs. (76)–(79) must be integrated with the initial conditions $h(0)=1$, $\phi(0)=\phi_0$, $u(0)=u_0$, and $r_m(0)=r_{m0}$, respectively. We also introduce the deceleration parameter $q$, an indicator of the decelerating/accelerating nature of the cosmological expansion, defined as

$$q=\frac{d}{d\tau}\frac{1}{h}-1=-(1+z)h\frac{d}{dz}\frac{1}{h}-1=(1+z)\frac{1}{h}\frac{dh}{dz}-1. \tag{80}$$

In order to test the viability of the conformal Barthel–Kropina cosmological model, we will compare its predictions with the results of the ΛCDM standard model. In the ΛCDM model, the redshift evolution of the Hubble function $H$ is given by

$$H(z)=H_0\sqrt{\Omega_m^{(cr)}(1+z)^3+\Omega_\Lambda}, \tag{81}$$

where $\Omega_m^{(cr)}=\Omega_b^{(cr)}+\Omega_{DM}^{(cr)}$, where $\Omega_b^{(cr)}$ and $\Omega_{DM}^{(cr)}$ denote the critical densities of the baryonic and dark matter, respectively, generally defined as $\Omega_i^{(cr)}=\rho_i/\rho_{cr}$, $i=b, DM$, where $\rho_{cr}=3H_0^2/8\pi G$. The density parameter of the dark energy (a cosmological constant) is defined as $\Omega_\Lambda=\Lambda/\rho_{cr}$. The deceleration parameter is given by

$$q(z)=\frac{3(1+z)^3\Omega_m^{(cr)}}{2\left[\Omega_\Lambda+(1+z)^3\Omega_m^{(cr)}\right]}-1. \tag{82}$$

For the matter and dark energy density parameters, we adopt the numerical values $\Omega_{DM}^{(cr)}=0.2589$, $\Omega b^{(cr)}=0.0486$, and $\Omega_\Lambda=0.6911$, respectively [142,143]. This gives the value $\Omega_m^{(cr)}=0.3089$ for the total matter density parameter $\Omega_m^{(cr)}=\Omega_b^{(cr)}+\Omega_{DM}^{(cr)}$. The present-day value of the deceleration parameter is $q(0)=-0.5381$, a value that indicates that the Universe is currently in an accelerating epoch.

## 5.2 The case $\gamma=1$

We will begin the analysis of the cosmological implications of the conformal Barthel–Kropina model by considering the simple case corresponding to $\gamma=1$. For this particular choice of model parameter, the field equations take the form

$$-(1+z)h\frac{d\phi}{dz}=u, \tag{83}$$

$$(1+z)\frac{dr_m}{dz}-3r_m=2(1+z)\frac{d\phi}{dz}r_m, \tag{84}$$

$$-(1+z)h\frac{dh}{dz}=r_m e^{-2\phi}-\frac{7}{2}h^2+\frac{27}{4}hu-\frac{13}{4}u^2, \tag{85}$$

$$-(1+z)h\frac{du}{dz}=r_m e^{-2\phi}-\frac{7}{4}h^2+\frac{41}{8}hu-\frac{13}{4}u^2. \tag{86}$$

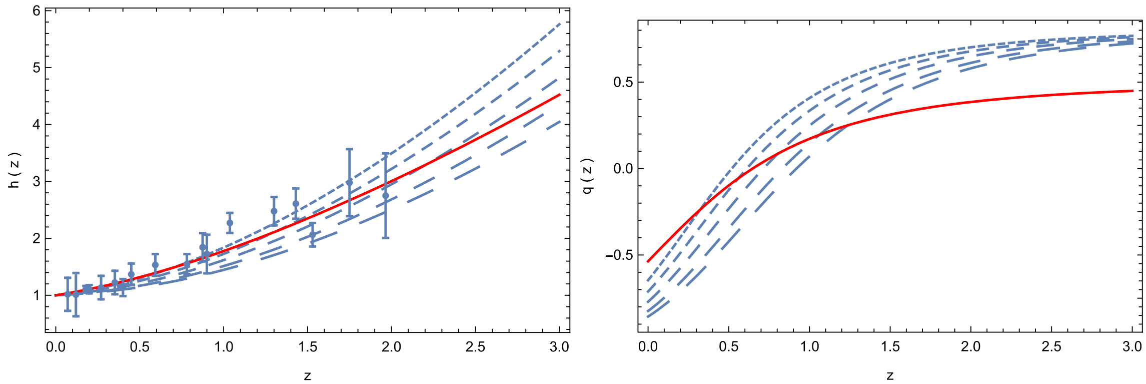


**Fig. 1** Variation in the dimensionless Hubble function $h$ (left panel) and the deceleration parameter (right) panel in the conformal Barthel–Kropina cosmological model for $\gamma = 1$, and for $u(0) = 0.69$ (dotted curve), $u(0) = 0.72$ (short dashed curve), $u(0) = 0.75$ (dashed curve), $u(0) = 0.78$ (long-dashed curve), and $u(0) = 0.80$ (ultra-long dashed curve). The initial conditions used to integrate the cosmological evolution equations are $\phi(0) = 0.16$, $r_m(0) = 0.05$, and $h(0) = 1$, respectively. The observational data are presented with their error bars, while the red curve depicts the predictions of the $\Lambda$CDM model

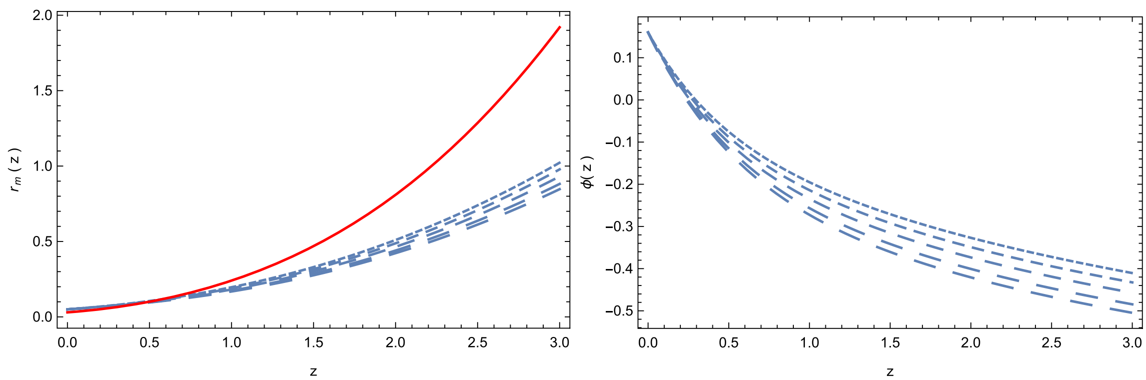


**Fig. 2** Variation in the dimensionless matter density $r$ (left panel) and the conformal factor $\phi$ (right) panel in the conformal Barthel–Kropina cosmological model for $\gamma = 1$, and for $u(0) = 0.69$ (dotted curve), $u(0) = 0.72$ (short dashed curve), $u(0) = 0.75$ (dashed curve), $u(0) = 0.78$ (long-dashed curve), and $u(0) = 0.80$ (ultra-long dashed curve). The initial conditions used to integrate the cosmological evolution equations are $\phi(0) = 0.16$, $r_m(0) = 0.05$, and $h(0) = 1$, respectively. The red curve shows the predictions of the $\Lambda$CDM model

The system of Eqs. (83)–(86) must be integrated with the initial conditions $\phi(0) = \phi_0$, $u(0) = u_0$, $r_m(0) = r_{m0}$, and $h(0) = 1$, respectively. The variations with respect to the redshift of the dimensionless Hubble function and the deceleration parameter are presented in Fig. 1.

The variations in the matter energy density and the conformal factor $\phi$ are presented in Fig. 2.

As one can see from Fig. 1, the conformal Barthel–Kropina model gives a good description of the observational data for the Hubble function, and it overlaps almost perfectly with the prediction of the $\Lambda$CDM model up to a redshift of $z = 2.5$. For higher redshifts, some important differences between models may appear. On the other hand, significant differences do exist in the behavior of the deceleration parameter, with the conformal Barthel–Kropina model predicting higher deceleration values at higher redshifts, and lower values at small redshifts. The baryonic matter content, presented in Fig. 2, appears to be much higher in the $\Lambda$CDM model. In fact, in the conformal Barthel–Kropina model, less baryonic matter is predicted to exist at higher redshifts. The conformal factor $\phi(z)$ evolves from positive values at small redshifts to larger negative values, increasing rapidly with $z$.

### 5.3 The case $\gamma = -1$

For $\gamma = -1$, the variations in the Hubble function and the deceleration parameter are presented in Fig. 3.

For negative values of $\gamma$, the conformal Barthel–Kropina model generally does not give a particularly good description of the observational data or closely reproduce the $\Lambda$CDM model. As one can see from the left panel of Fig. 3, important differences may appear between observations at both small and high redshifts. The differences are even more important in the case of the deceleration parameter, with the conformal

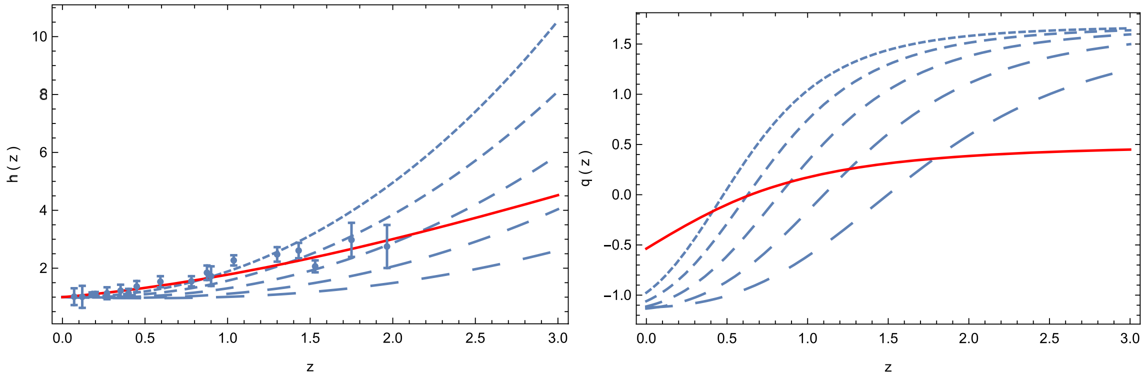


**Fig. 3** Variation in the dimensionless Hubble function $h$ (left panel) and the deceleration parameter (right) panel in the conformal Barthel–Kropina cosmological model for $\gamma = -1$, and for $u(0) = -0.01$ (dotted curve), $u(0) = -0.08$ (short dashed curve), $u(0) = -0.15$ (dashed curve), $u(0) = -0.22$ (long-dashed curve), and $u(0) = -0.29$ (ultra-long dashed curve). The initial conditions used to integrate the cosmological evolution equations are $\phi(0) = 0.025$, $r_m(0) = 0.05$, and $h(0) = 1$, respectively. The observational data are represented with their error bars, while the red curve depicts the predictions of the $\Lambda$CDM model

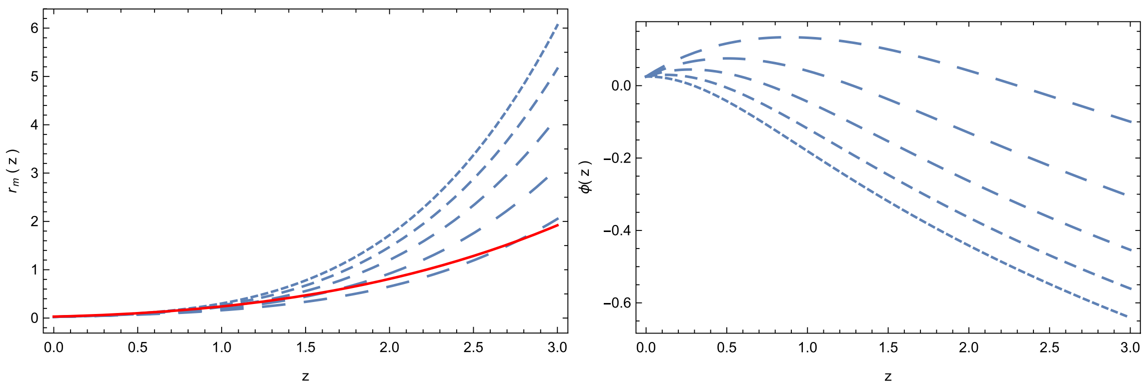


**Fig. 4** Variation in the dimensionless matter density $r$ (left panel) and the conformal factor $\phi$ (right) panel in the conformal Barthel–Kropina cosmological model for $\gamma = -1$, and for $u(0) = -0.01$ (dotted curve), $u(0) = -0.08$ (short dashed curve), $u(0) = -0.15$ (dashed curve), $u(0) = -0.22$ (long-dashed curve), and $u(0) = -0.29$ (ultra-long dashed curve). The initial conditions used to integrate the cosmological evolution equations are $\phi(0) = 0.025$, $r_m(0) = 0.05$, and $h(0) = 1$, respectively. The red curve shows the predictions of the $\Lambda$CDM model

Barthel–Kropina cosmology always ending at the present time in a de Sitter-type accelerating or super-accelerating phase, with $q$ equal to or smaller than $-1$. At high redshifts, the deceleration parameter takes much higher values than those predicted by the $\Lambda$CDM model.

The variations in the matter density and the conformal factor $\phi$ are presented in Fig. 4. Whereas at low redshifts up to the order of $z \approx 1.5$, the predicted matter density of the conformal Barthel–Kropina model has similar values as in the $\Lambda$CDM model, at higher redshifts, a much higher matter density is predicted as compared to standard cosmology. This different behavior can be traced back to the matter density evolution equation (77), which indicates the possibility of the non-conservation of the baryonic content of the Universe. The conformal factor $\phi$, shown in the right panel of Fig. 4), increases initially with the redshift, reaching a maximum value at $z \in (0.4, 1)$, after which it decreases and acquires negative values at higher redshifts.

### 5.4 The case $\gamma = 4/3$

Finally, we consider the cosmological evolution of the homogeneous and isotropic conformal Barthel–Kropina cosmological model for $\gamma = 4/3$. The variations in the Hubble function and the deceleration parameter are presented in Fig. 5, with the observational data and the predictions of the $\Lambda$CDM cosmological model. The conformal Barthel–Kropina model can give a satisfactory description of the observational data, and for a certain range of the numerical values of the model parameters (the initial conditions for the evolution equation of the conformal factor), it can almost exactly reproduce the predictions of the $\Lambda$CDM paradigm. However, important differences in the predictions of the behavior of the deceleration

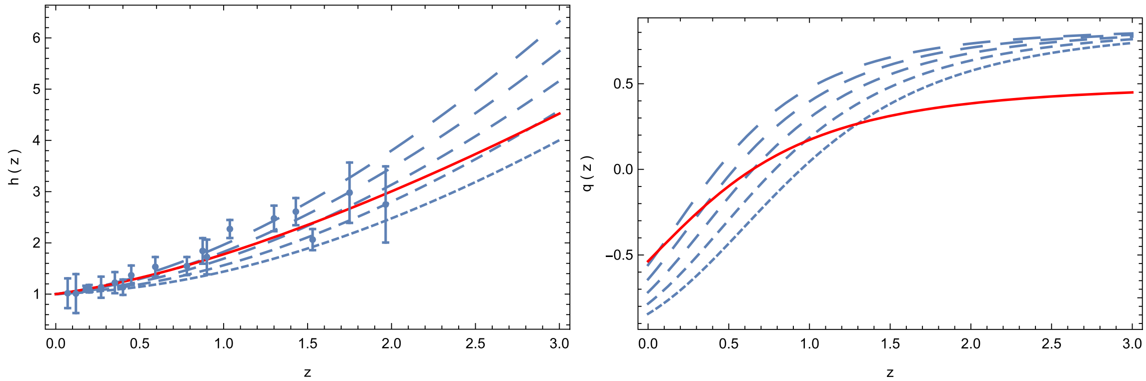


**Fig. 5** Variation in the dimensionless Hubble function $h$ (left panel) and the deceleration parameter (right) panel in the conformal Barthel–Kropina cosmological model for $\gamma = 4/3$, and for $u(0) = 0.80$ (dotted curve), $u(0) = 0.82$ (short dashed curve), $u(0) = 0.84$ (dashed curve), $u(0) = 0.86$ (long-dashed curve), and $u(0) = 0.88$ (ultra-long dashed curve). The initial conditions used to integrate the cosmological evolution equations are $\phi(0) = 0.56$, $r_m(0) = 0.05$, and $h(0) = 1$, respectively. The observational data are presented with their error bars, while the red curve represents the predictions of the $\Lambda$CDM model

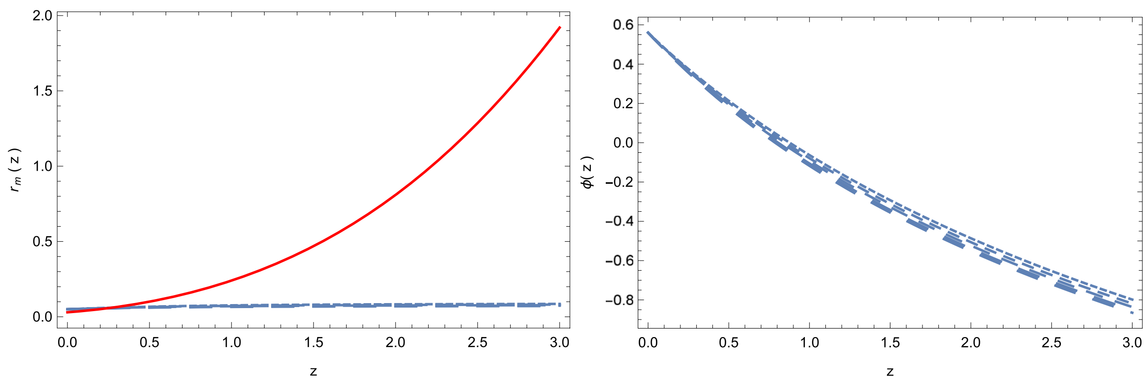


**Fig. 6** Variation in the dimensionless matter density $r$ (left panel) and the conformal factor $\phi$ (right panel) in the conformal Barthel–Kropina cosmological model for $\gamma = 4/3$, and and for $u(0) = 0.80$ (dotted curve), $u(0) = 0.82$ (short dashed curve), $u(0) = 0.84$ (dashed curve), $u(0) = 0.86$ (long-dashed curve), and $u(0) = 0.88$ (ultra-long dashed curve). The initial conditions used to integrate the cosmological evolution equations are $\phi(0) = 0.56$, $r_m(0) = 0.05$, and $h(0) = 1$, respectively. The red curve represents the predictions of the $\Lambda$CDM model

parameter persist, indicating lower values at small redshifts and higher values at high redshifts. A de Sitter-type evolution with $q(0) = -1$ is also obtained for this value of $\gamma$. Parameter values that give the same evolution of $q$ as in the $\Lambda$CDM model at low redshifts still predict much higher values at higher redshifts.

The redshift evolution of the baryonic matter energy density and the conformal factor $\phi$ is presented for the $\gamma = 4/3$ conformal Barthel–Kropina model in Fig. 6.

At redshifts $z > 0.5$, predictions of the baryonic matter content of the $\Lambda$CDM model largely exceed the predictions of the conformal Barthel–Kropina model, whose matter content is relatively constant, and equal to the present-day value. This aspect may be again explained by the conformal dependence of the matter density on the conformal factor, as can be seen in Eq. (77). The conformal factor $\phi$ decreases monotonically from its initial positive value, to negative values reached at higher redshifts. The variations in the basic model quantities $(r_m, h, \phi)$ are not significantly dependent on the initial values of the conformal factor $\phi(0)$, but show a strong dependence on the initial values of its derivative $(d\phi(z)/dz)|_{z=0}$, which gives the rate of the variation in the conformal factor with the cosmological redshift.

## 6 Testing the conformal Barthel–Kropina cosmology

## 7 Discussion and final remarks

In this paper, we have considered an extension of the Barthel–Kropina dark energy model, as introduced in [129,130], by introducing the mathematical and physical perspective of the conformal transformations. Conformal transformations are the basis of Weyl geometry [33,34], interestingly proposed in 1918, the same year as Finsler geometry [70]. However, the relations between these two fundamental geometries have been explored very little, if at all, in a physical context. Nevertheless, there are some mathematical studies trying to fill the gap between these geometries, and investigating the effects of the conformal transformations on various Finsler-type geometries [134–141].

The starting point of our investigation is Eq. (24), in which we have assumed that the fundamental function $F(x, y)$ of a Finsler geometry is conformally transformed into a new function $\tilde{F}(x, y)$. In the case of an $(\alpha, \beta)$ metric, a conformal transformation acts on both the Riemannian metric $\alpha$ and the one-form $\beta$, as indicated by Eq. (25). We can interpret, by analogy with standard Riemannian general relativity, such a conformal transformation as a transformation from a Finsler-type Einstein frame to a Finsler-type Jordan frame. Mathematically, such a transformation is always possible, and leads to some classes of Finsler geometries with interesting properties. From a mathematical point of view, one can define a conformal change in the Finsler geometry as follows. Let $F^n = (M^n.L)$ and $\tilde{F}^n = \left(M^n.\tilde{L}\right)$ be two Finsler spaces defined on the same underlying base manifold $M^n$, where $L$ denotes the Finsler metric function. If the angle between any two tangent vectors in $F^n$ is equal to the angle in $\tilde{F}^n$, then $F^n$ is called conformal to $\tilde{F}^n$, and the transformation $L \to \tilde{L}$ is called a conformal transformation of the metric. In other words, if there exists a scalar function $\sigma(x)$ such that $\tilde{L} = e^{\sigma(x)}L$, then the transformation is called a conformal transformation [138,139].

For the case of an $(\alpha, \beta)$ metric, $\tilde{L} = e^{\sigma(x)}L$ is equivalent to $\tilde{L} = \left(e^{\sigma(x)}\alpha, e^{\sigma(x)}\beta\right)$, or equivalently, $\tilde{g}_{IJ} = e^{2\sigma(x)}g_{IJ}$, and $\tilde{A}_I = e^{\sigma(x)}A_I$. The conformal transformation of the connection in Riemann geometry is given by Eq. (7), and it adds to the standard Levi–Civita connection three new terms determined by the derivatives of the conformal factor $\sigma$.

The original Barthel–Kropina cosmological model was built upon three fundamental mathematical assumptions. Firstly, we assume that the Finsler metric function is a Kropina-type $(\alpha, \beta)$ metric. Secondly, the osculating approach to the Finsler-Kropina metric was adopted, in which $g(x, y)$ becomes a Riemannian metric $g(x, Y(x))$. Thirdly, the first two assumptions lead to the result that the connection of this Riemann metric is nothing but the corresponding Levi–Civita connection, called the Barthel connection in Finsler geometry. With the help of these three assumptions, one can systematically construct the corresponding Einstein gravitational field equations and, in a cosmological setting, the generalized Friedmann equations that lead to a consistent cosmological model, which can be successfully tested against the cosmological observations. In the present study we have considered the conformal transformation of this model from its Einstein frame to the conformally related Jordan frame, which results in the introduction of a new scalar degree of freedom.

The corresponding cosmological model, constructed by assuming the homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker metric, leads to a set of generalized Friedmann equations, from which a dark energy effective geometric model can be constructed. There are two essential assumptions in the formulation of the model, the first being the adoption of the specific form (58) for the coefficient of the one-form $\beta$, which is considered to be related to the time-dependent conformal factor $\phi$ by an exponential relation, also involving the scale factor. As the second assumption in formulating the dark energy model, we considered the splitting of the global energy balance equations into two independent equations, with the equation of conservation of the conformal scalar field formulated in a form similar to that of ordinary scalar fields in cosmology. However, it is important to point out that in the present approach, the scalar field is of purely geometric origin. Once these two assumptions are adopted, one can obtain a cosmological model, representing a generalization of the standard $\Lambda$CDM model, in which the cosmological evolution is described by the set of the Friedmann equations, plus an evolution equation for the scalar field. The model has one free parameter $\gamma$, and must be studied numerically once the initial conditions for the scalar field and its derivative at the time origin are given. Once these parameters are fixed, a large number of cosmological models can be obtained.

In our investigation, we restricted our analysis to three classes of models, determined by three distinct numerical values of the parameter $\gamma$. The initial conditions for the scalar field were varied slightly. To simplify the comparison with the observational data, the redshift representation of the cosmological evolution equations was used. Moreover, we compared the predictions of the conformal Barthel–Kropina model with the similar predictions of the $\Lambda$CDM model, and with a small set of observational cosmological data. The parameter $\gamma$ was given rather different values, both positive and negative, and integer and fractional, respectively. The comparison with the observational data was performed using a trial-and-error method, and no formal mathematical fitting procedures were used. As a first conclusion of our study, we can infer that positive values of $\gamma$ give a better description of the observational data for the Hubble function and of $\Lambda$CDM cosmology as compared with the negative $\gamma$ values. However, even negative values of $\gamma$ are not completely ruled out by the observations. On the other hand, important differences between the predictions of various cosmological parameters in the conformal Barthel–Kropina model and of the $\Lambda$CDM model do appear. These differences are especially significant for the case of the deceleration parameter and the matter density. Even though the conformal Barthel–Kropina model describes well the transition to an accelerating expansion, and can even reproduce the observational present-day value of the deceleration parameter, significant differences appear, especially at higher redshifts, where the conformal Barthel–Kropina predicts a much higher rate of deceleration than the $\Lambda$CDM model. The existence of direct observational data on the deceleration parameter at higher redshifts would allow for precise discrimination between the cosmological models predicted by the conformal Barthel–Kropina model and other standard or modified models. Other important differences do appear at the level of the ordinary matter density. For $\gamma = 1$ and $\gamma = -1$, the conformal Barthel–Kropina model predicts higher matter densities, especially at redshifts $z > 2$. On the other hand, for $\gamma = 4/3$, the matter content of the Universe is much higher in the $\Lambda$CDM model than in the conformal Barthel–Kropina model, which predicts an almost constant matter density, showing a very small variation with the redshift.

To conclude, in this study we have investigated an interesting feature of a specific Finsler-type cosmology related to the impact of a conformal transformation on the Kropina metric. The cosmological implications of the model were considered in detail, and we have shown that this type of model may represent an attractive alternative to standard cosmologies based on Riemann geometries. We have developed the basic mathematical and physical tools that would allow an in-depth comparison of the predictions of the conformal Barthel–Kropina model with the observational data and with standard cosmological approaches. Hopefully, the results obtained will lead to a better understanding of the physical applications of Finsler geometry and its relevance for the description of large-scale cosmic phenomena and processes.

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**Data availability** This manuscript has no associated data or the data will not be deposited. [Authors' comment: There are no external data associated with this work.]



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