Are there any Landau poles in wavelet-based quantum field theory?

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Following previous work by one of the authors [M. V. Altaisky, Unifying renormalization group and the continuous wavelet transform, Phys. Rev. D **93**, 105043 (2016).], we develop a new approach to the renormalization group, where the effective action functional $\Gamma_A[\phi]$ is a sum of all fluctuations of scales from the size of the system L down to the scale of observation A. It is shown that the renormalization flow equation of the type $\frac{\partial \Gamma_A}{\partial \ln A} = -Y(A)$ is a limiting case of such consideration, when the running coupling constant is assumed to be a differentiable function of scale. In this approximation, the running coupling constant, calculated at one-loop level, suffers from the Landau pole. In general, when the scale-dependent coupling constant is a nondifferentiable function of scale, the Feynman loop expansion results in a difference equation. This keeps the coupling constant finite for any finite value of scale A. As an example, we consider Euclidean ϕ^4 field theory.

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I. INTRODUCTION

The renormalization group (RG) was discovered by Stueckelberg and Petermann as a group of parametrizations of the S matrix emerging after cancellation of UV divergences in quantum electrodynamics [1]. The RG method has become popular in high-energy physics since Gell-Mann and Low, who used the functional equation to study the renormalized photon propagator in QED, have shown that the charge distribution surrounding a test charge in vacuum does not depend on the coupling constant at small scales, except for a scale factor, i.e., it possesses a kind of self-similarity [2]. The breakthrough in the RG approach has been achieved by Wilson, who applied it to statistical mechanics, where continuously many degrees of freedom are correlated over long distances. It was found that if there are many degrees of freedom within the correlation length ξ, the behavior of the system is primarily determined by cooperative behavior and the number of degrees of freedom, rather than by the type of Hamiltonian interaction [3]. The core of the Wilson formulation was to successively integrate out the fluctuations of small scales to obtain progressively coarse-grained descriptions of fluctuations at larger scales [4,5]. By doing so, the RG approach unifies the theory of phase transitions, quantum field theory, turbulence, and many other branches of physics. The RG approach not only provides an explanation of critical phenomena, but also renders a practical tool for calculation of second-order phase transitions [3,6].

Having started from the weak interaction limit, where one-loop corrections to the correlation functions have tractable physical meaning, the RG approach has gradually evolved into the scaling equations for the exact ("dressed") correlation functions. In this paper, following the previous papers [7–9], we sum up the fluctuations starting from the IR edge and go down to the observation scale. If the fluctuations are summed up in a thin shell of scales, the beta function coincides with the known results, regardless if the summation starts from the IR or from the UV edge [8]. We have shown that summing up the fluctuations from the IR edge (from size of the system) down to the observation scale in a finite range of scales renders a finite renormalization of the coupling constant without any Landau poles.

The use of *continuous* wavelet transform is not the only way the wavelet regularization in quantum field

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theory aimed to sum up the fluctuations of different scales. Historically, the first attempts to use wavelets in quantum field theory were related to the numerical simulation of quantum field theory (QFT) models. For instance, Ref. [10] presents a simple 2D model with local ϕ^4 interaction simulated using discrete wavelet transform (DWT) of the field $\phi(x)$ performed with orthogonal Daubechies wavelets. The Daubechies wavelets [11] form an orthogonal basis of the compactly supported functions $\psi^j_k(x) \coloneqq 2^{-\frac{j}{2}}\psi(2^{-j}x-k)$, where $\psi(x)$ is a function with compact support, obeying certain recursive equation. The advantage found in wavelet simulations was that the coefficients of different scales j were varied *independently* when searching for the ground state of the field configuration. This effectuates the simulation if compared to the usual Metropolis algorithm.

The main advantages of DWT in quantum field theory simulations are more or less the same as its advantages in signal and image processing, where independence of coefficients of different scales provides fast and efficient algorithms for data processing [12,13]. As it concerns the quantum field theory itself, the technique based on DWT was later generalized to the multiscale entanglement renormalization ansatz, closely related to the real space renormalization group [14,15]. Wavelet technique is also closely related to the RG, since the wavelet basis is generated by a single basic function, which is shifted and dilated to form the bases of different scales. This scale hierarchy of wavelet bases provides a natural framework for renormalization on a lattice [16–18].

There is an essential difference between usual renormalization procedures—Kadanoff blocking procedure [19], Wilson's RG [4], etc.—and the wavelet technique. In the usual approach there is only one universal operator D, which makes the blocks of different sizes behave like each other, although with different values of the coupling constant. In discrete wavelet transform there are two distinct operators: the low- and high-pass filters. The former averages degrees of freedom of the small scales into the coarser degrees of freedom of the block—exactly as in the usual RG approach—the latter processes the details lost in block averaging $\hat{l}+\hat{h}=\hat{\mathbb{I}}$ [12,16]. For the case of discrete wavelet transform, the problem of lattice renormalization was described in detail in the monograph [20].

In this paper, we are not going to dig into the details of DWT methods in quantum field theory, but we do mention that these methods have gradually evolved into an effective numerical technique for finding the ground state of many-body quantum field theory models [21]. Their promising implementation is arising due to the analogy between DWT and the tensor networks [22,23] as expected on quantum computers [24]. Discrete wavelet representation of field theory models with the orthogonal Daubechies wavelets also provides an interesting approach to the evolution of lattice Hamiltonian systems [21,25,26].

To the authors knowledge, the first attempt to use wavelets for an analytic study of a continuous field theory of practical importance—the quantum chromodynamics beyond the lattice approximation—was made by Federbush [27], but has turned out to be technically complicated. In this study the basis was not restricted to the orthogonal wavelets, and the general type of DWT was considered, with the examples of Meyer wavelets. The mathematical idea itself was expressed even earlier [28].

In our approach, following the previous papers [8,29,30], we use continuous wavelet transform, rather than *discrete* wavelet transform. By doing so, we lose orthogonality of the basis, but harness the capability of analytical calculations in the perturbation theory and shape our model into the framework of the group field theory [31,32], defined on the affine group $G\colon\, x' = ax+b$. The use of continuous wavelet transform provides a quantum field theory model finite by construction, if the scale argument of wavelet transform (a) is considered as a physical scale of measurement of a quantum field performed with a certain aperture function–mother wavelet and the causality restriction being imposed on scale arguments, as the absence of scales in the internal lines of Feynman diagrams less than the best scale of measurement, given as minimal scale of all external lines [29,33]. In this setting, the scale dependence of the observed Green's functions [field correlators $\langle \phi_{a_1}(x_1)\ldots\phi_{a_n}(x_n)\rangle$] is completely expressed in terms of the wavelet scale arguments a_i, no external renormalization scale μ is required, and the role of renormalization group symmetry is merely to relate the fluctuations of different scales to each other: there is no need for removal of divergences [8].

The remainder of this paper is organized as follows. In Sec. II we briefly remind the reader of some definitions of the Euclidean field theory in its statistical interpretation. Section III presents the formalism of continuous wavelet transform in Euclidean QFT. In Sec. IV we present the one-loop contribution to the vertex in the ϕ^4 model and show that accurate summation of contributions of all scales from the size of the system down to the observation scale does not produce any singularities such as Landau poles. A few concluding remarks are given in the last section.

II. STATISTICAL MECHANICS VIEW ON QUANTUM FIELD THEORY

Let us briefly remind the reader of the statistical view on the formalism of Euclidean quantum field theory. At the state of thermodynamic equilibrium, the distribution of a continuous field $\phi(x)$, say a magnetization, is given by the canonical partition function

$$Z = \mathrm{Tr}e^{-\beta H},$$

where $H = H[\phi]$ is the Hamiltonian, $\beta = \frac{1}{T}$ is the inverse temperature, and the trace operator assumes the summation over all degrees of freedom. The trace can be expressed in terms of the Feynman integral,

$$Z[J] = \int \mathcal{D}\phi \exp\left(-S[\phi] + \int J(x)\phi(x)d^dx\right), \quad (1)$$

where the formal source $J(x)$ can be understood as an external magnetic field.

The Euclidean action functional $S[\phi]$ is proportional to the Hamiltonian of the field $\phi(x)$,

$$S[\phi] = \frac{1}{T}\int d^dx \left[\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right], \quad (2)$$

in the Ginzburg-Landau theory of phase transitions [34]. The correlation functions of the field $\phi(x)$ can be derived as functional derivatives,

$$G^{(n)}(x_1, \ldots, x_n) = \left.\frac{\delta^n W[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right|_{J=0}, \quad (3)$$

where $W[J] = \ln Z[J]$ is the connected Green's function generating functional, which is proportional to the Helmholtz free energy $F[J] = -T\ln Z[J]$. The effective action functional $\Gamma[\phi]$ is defined via the Legendre transform of $W[J]$,

$$\Gamma[\phi] = -W[J] + \int J(x)\phi(x)d^dx. \quad (4)$$

(Here we keep the notation of [35].)

The functional derivatives of $W[J]$ with respect to the external source $J(x)$ determine the mean field $\phi = \phi[J]$,

$$\frac{\delta W[J]}{\delta J(x)} = \phi(x).$$

The functional derivatives of the effective action $\Gamma[\phi]$ are the "vertex functions" $\Gamma^{(n)}[\phi]$. In the above considered ϕ^4 model, the (renormalized) vertex function $\Gamma^{(4)}[\phi]$ accounts for the value of the coupling constant calculated at some reference scale; the $\Gamma^{(2)}[\phi]$ function is the renormalized inverse propagator, which defines the renormalization of mass at the same reference scale.

The most instructive case of the locally known microscopic interaction is the Ising model, described by the microscopic Hamiltonian

$$H = -J\sum_{\langle ij\rangle} S_i S_j - B\sum_i S_i, \quad (5)$$

where J is the coupling constant of interaction between the neighboring spins, B is the external magnetic field, and the Ising spins, with the values $S_i = \pm 1$, are located on some regular lattice. In the continuous limit, the Hamiltonian of the Ising model (5) with the nearest-neighbor interaction is transferred into the Euclidean QFT model with ϕ^4 interaction (2), which meets the Ginzburg-Landau theory [3,34].

In many cases, the interaction Hamiltonian or the bare action functional is known at some *macroscopic* scale μ, but the microscopic theory at smaller scales (higher momentum transfer) should be unveiled. The typical cases are the QED and the quantum gravity—both having $1/r$ asymptotic behavior at macroscopic scales, but different behavior at smaller scales [36,37].

The renormalization group method displays its best merits when the microscopic fluctuations of atomic scales cooperate their behavior into large-scale fluctuations, which are well described by classical mean-field equations. This happens in the theory of phase transitions, critical behavior, kinetic description of gases, etc. [3,38,39]. However, if fluctuations of all scales do matter equally, the averaging of fluctuations from the atomic scales up to the larger scales (say, by Bogoliubov's chain) becomes notoriously difficult. It turns out to be easier, say in hydrodynamics, to start with the laminar large-scale motion and to sum up all fluctuations arising from instabilities down to the atomic scales, where these fluctuations are completely damped by viscosity [38].

III. USING CONTINUOUS WAVELET TRANSFORM IN QUANTUM FIELD THEORY MODELS

A. Continuous wavelet transform

To separate fluctuations of different scales in quantum field theory, it is convenient to use the formalism of continuous wavelet transform, as described, e.g., in [29,33]. Let us briefly remind the reader of the basics of wavelet transform; see the monographs [12,40] for a detailed introduction.

Let $\phi(x) \in L^2(\mathbb{R}^d)$ be a square-integrable function. Let $\chi(x) \in L^2(\mathbb{R}^d)$ be a suitably well-localized function, which satisfies the "admissibility condition,"

$$C_\chi = \int |\tilde{\chi}(k)|^2 \frac{d^dk}{S_d|k|^d} < \infty, \quad (6)$$

where the tilde denotes the Fourier transform,

$$\tilde{\chi}(k) \coloneqq \int_{\mathbb{R}^d} e^{\imath kx}\chi(x)d^dx,$$

and $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the area of the unit sphere in $\mathbb{R}^d$. Then it is possible to decompose the function ϕ with respect to the basis, provided by shifted, dilated, and rotated copies of $\chi(x)$. This decomposition is known as the continuous wavelet transform [41,42],

$$\phi(x) = \frac{1}{C_\chi}\int \frac{1}{a^d}\chi\left(R^{-1}(\theta)\frac{x-b}{a}\right)\phi_{a\theta}(b)\frac{da d^d b}{a}d\mu(\theta), \quad (7)$$

where $R(\theta)$ is the rotation matrix, $d\mu(\theta)$ is the left-invariant measure on the $SO(d)$ rotation group, usually written in terms of the Euler angles,

$$d\mu(\theta) = 2\pi \prod_{k=1}^{d-2}\int_0^\pi \sin^k\theta_k d\theta_k.$$

The functions

$$\phi_{a,\theta}(b) \coloneqq \int_{\mathbb{R}^d}\frac{1}{a^d}\bar{\chi}\left(R^{-1}(\theta)\frac{x-b}{a}\right)\phi(x)d^d x \quad (8)$$

are known as wavelet coefficients of the function ϕ with respect to the mother wavelet χ.

The decomposition (8) and the reconstruction formula (7) represent a particular case of the "partition of unity" in Hilbert space $\mathcal{H}$ with respect to representation $U(g)$ of a Lie group G acting transitively on $\mathcal{H}$ [43,44],

$$\hat{\mathbb{I}} = \frac{1}{C_\chi}\int_G U(g)|\chi\rangle d\mu(g)\langle\chi|U^*(g), \quad (9)$$

with G being the group of affine transformations,

$$G\colon\ x' = aR(\theta)x + b, x, b\in\mathbb{R}^d, a\in\mathbb{R}_+, \theta\in SO(d). \quad (10)$$

Wavelet coefficients (8) have clear physical meaning: The convolution of the analyzed function ϕ with a well-localized function χ at a fixed window width a comprise only the fluctuations with typical scales close to a and is insensitive to all other fluctuations.

B. Scale-dependent fields

The reconstruction (7) of the function ϕ from the set of its wavelet coefficients is generally nonorthogonal, and the wavelet basis is overcomplete [40]. Although the integration $\int_0^\infty \frac{da}{a}\ldots$ in (7) provides a formally exact reconstruction formula, depending on the physics of the considered problem, we can restrict the integration by the minimal scale A from below (lattice size, in the case of ferromagnetism) and by the system size L from above: $\int_A^L \frac{da}{a}\ldots$.

Moreover, as we know from the Heisenberg uncertainty principle, the value of a *quantum* field ϕ sharp at a point x is physically meaningless, since any measurement with $\Delta x \to 0$ implies an infinite momentum transfer $\Delta p \to \infty$, which definitely drives us out of the applicability of the model. For this reason, we have to consider A as the best available scale of measurement (observation).

There is a distinction between quantum field theory models considered as an effective large-scale description of a microscopic theory with a fundamental microscopic length scale—say, a ferromagnetic model—and fundamental models of quantum field theory. Our approach is oriented for the latter case. In the former case, we have physical evidence of what is happening at the fundamental scale, and the goal of the RG or any other technique is to construct an effective large-scale theory by a certain averaging procedure. In the latter case, our physical evidence is related to some large-scale processes—say, the interaction of charged particles in classical electrodynamics—and our goal is to construct a microscopic theory capable of describing physical phenomena at a given microscopic scale of observation. We know from theoretical calculations in QED, which harness the RG technique in the space of local square-integrated functions $\mathrm{L}^2(\mathbb{R}^d)$, that some experimental results, such as Lamb shift and anomaly in magnetic momentum, can be explained in this way [45,46].

Nevertheless, we do not have any proof that the description of quantum fields in terms of local square-integrable fields and constant charges is the unique way to describe quantum phenomena. The techniques of continuous wavelet transform, as presented in [29] and other papers, suggests an alternative description: quantum fields may be defined on affine group $G\colon\ x' = ax + b$, rather than on Euclidean or Minkowski space: $\phi = \phi_a(b)$. In this case, the charges may be explicitly dependent on scale $Q = Q(a)$. In such a theory the no-scale ("bare") coupling constant may have no physical meaning, but we still keep it to link with the known results. There is a counterpart for this situation in classical statistical mechanics and turbulence theory. The definition of viscosity and other kinetic coefficients is explicitly dependent on the size of the averaging domain; there is nothing strange in it: if the averaging size is less than the mean-free path, neither viscosity nor hydrodynamic velocity is defined, which drives us out of the model applicability. The continuous wavelet transform can be also applied to such problems [47].

In the remainder of this paper, following the previous papers [7,29], we will assume the mother wavelet $\chi(x)$ to be an isotropic function of x and thus ignore the rotation factor $R(\theta)$. In this setting, the scale component of the field ϕ, measured in a point b at the scale a with respect to the mother wavelet χ (considered as an aperture function by the analogy from optics [48]) is given by wavelet coefficient,

$$\phi_a(b) \equiv \langle a, b; \chi|\phi\rangle = \int \frac{1}{a^d}\bar{\chi}\left(\frac{x-b}{a}\right)\phi(x)d^d x. \quad (11)$$

However, the space of scale-dependent functions $\{\phi_a(b)\}$ is more general than the space of point-dependent functions $\phi(x)\in L^2(\mathbb{R}^d)$. Even if all fields $\phi_a(b)$ are well defined $\forall\ a\in\mathbb{R}_+, b\in\mathbb{R}^d$, the limit

$$\phi(x) = \lim_{A\to 0}\int_A^\infty \frac{da}{a}\int_{\mathbb{R}^d}\frac{1}{a^d}\chi\left(\frac{x-b}{a}\right)\phi_a(b)d^d x$$

does not necessarily exist. The divergence of the sum of all scale components happens in UV-divergent theories, where the value of a physical field ϕ sharp at a point x is meaningless.

If $\phi(x)$ is understood as a wave function of a physical particle, its normalization

$$\langle\phi|\phi\rangle = \int \bar{\phi}(x)\phi(x)d^dx = 1$$

is the statement of existence: the probability of finding this particle anywhere in space $\mathbb{R}^d$ is exactly 1. The wavelet approach, based on the affine group (10), generalizes the statement of existence in the form

$$\frac{1}{C_\chi}\int_{g\in G} |\langle\phi|U(g)|\chi\rangle|^2 d\mu(g) = 1, \qquad g = (a, b, \theta). \quad (12)$$

The latter equation states that sweeping the measurement parameters, position b, resolution a, and direction θ, over all possible values will necessarily imply the registration of the particle.

Technically, the use of the scale-dependent functions $\phi_a(b)$ in a local quantum field theory is rather straightforward: one can express local fields in terms of their wavelet transform,

$$\phi(x) = \frac{1}{C_\chi}\int \frac{da}{a}\int \frac{d^dk}{(2\pi)^d} e^{-\imath kx}\tilde{\chi}(ak)\tilde{\phi}_a(k), \quad (13)$$

where $\tilde{\phi}_a(k)$ are the Fourier images of the wavelet coefficients (11). This defines an easy rule to redefine the Feynman diagram technique,

$$\tilde{\phi}(k) \to \tilde{\phi}_a(k) = \overline{\tilde{\chi}(ak)}\,\tilde{\phi}(k). \quad (14)$$

Doing so, we have the following modification of the Feynman diagram technique [7,33]:

(1) Each field $\tilde{\phi}(k)$ is substituted by the scale component: $\tilde{\phi}(k) \to \tilde{\phi}_a(k) = \overline{\tilde{\chi}(ak)}\,\tilde{\phi}(k)$.

(2) Each integration in the momentum variable is accompanied by the corresponding scale integration,

$$\frac{d^dk}{(2\pi)^d} \to \frac{d^dk}{(2\pi)^d}\frac{da}{a}\frac{1}{C_\chi}.$$

(3) Each interaction vertex is substituted by its wavelet transform; for the Nth power interaction vertex, this gives multiplication by the factor $\prod_{i=1}^N \tilde{\chi}(a_ik_i)$.

According to these rules, the bare Green's function of a massive scalar field in wavelet representation takes the form

$$G_0^{(2)}(a_1, a_2, p) = \frac{\tilde{\chi}(a_1p)\tilde{\chi}(-a_2p)}{p^2 + m^2}.$$

The finiteness of loop integrals is provided by the following rule: There should be no scales a_i in internal lines smaller than the minimal scale of all external lines [29,33]. Therefore, the integration in a_i variables is performed from the minimal scale of all external lines up to infinity or up to the system size. This corresponds to the summation of all fluctuations of all scales from the system size down to the finest scale of observation.

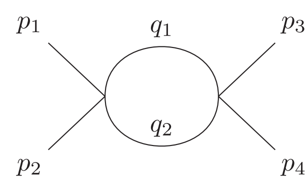
The cutoff in scale variables a is a milder assumption than momentum cutoff Λ in a usual theory. Since the scale a is a setting of observation, rather than a measurable quantity like momentum, the cutoff in it results neither in violation of momentum conservation, nor in violation of other important symmetries.

For a theory with local $\phi^N(x)$ interaction, the presence of two conjugated factors $\tilde{\chi}(ak)$ and $\overline{\tilde{\chi}(ak)}$ on each diagram line connected to an interaction vertex simply means that each internal line of the Feynman diagram carrying momentum p is supplied by the cutoff factor $f^2(Ap)$, where

$$f(x) \coloneqq \frac{1}{C_\chi}\int_x^\infty |\tilde{\chi}(a)|^2\frac{da}{a}, \qquad f(0) = 1, \quad (15)$$

with A being the minimal scale of all external lines of this diagram.

The mildness of the cutoff in scale argument A, if compared to the momentum cutoff $\Lambda \sim A^{-1}$, is quite easy to understand. Let us take the ϕ^4 model and consider a "fish" diagram



(16)

with the loop momenta q_1 and q_2 satisfying the momentum conservation in both vertices. In the case of the Fourier decomposition of fields, the restriction of momentum integration by cutoff momentum Λ results in low-frequency fields,

$$\phi_\Lambda^<(x) \coloneqq \int_{|q|<\Lambda} e^{-\imath qx}\tilde{\phi}(q)\frac{d^dq}{(2\pi)^d}.$$

If both q_1 and q_2 are above the cutoff value Λ, the contribution of both components to the loop integral will be discarded. More than that, since the Fourier transform is a decomposition with respect to the representations of the translation group, the application of cutoff violates translational invariance. Since the momentum basis is *orthogonal* basis,

$$\hat{\mathbb{I}} = \int |k\rangle \frac{d^d k}{(2\pi)^d} \langle k|, \qquad \langle k|k'\rangle = (2\pi)^d \delta(k - k'),$$

some information is lost after cutting high momenta.

In contrast to that, the wavelet basis is generally nonorthogonal and overcomplete and, in the discrete case, forms a frame; see, e.g., [40] for a general introduction to wavelets. In the continuous case, wavelet transform is a decomposition of a function with respect to representations of affine group $G\colon x' = ax + b$, see Eq. (9). When we restrict the integration over a scale domain $A \le a < \infty$, the translation subgroup (b) is not affected, and the translation invariance is preserved. For the fish diagram (16), after application of scale cutoff A, both components, with momenta q_1 and q_2, will contribute, but their contributions will be modulated by $f^2(Aq_1)$ and $f^2(Aq_2)$, respectively. The analogs of these contributions are completely lost in the usual Fourier method.

This is a typical story in information theory, when introduction of a new dimension—scale a in our case—enables one to preserve more information than available for usual methods. In machine learning, this stimulates the use of higher-dimensional feature maps [49]. For the same reasons, wavelets benefit in signal processing, where they are capable of distinguishing the low frequencies coming from the differences of two high-frequency harmonics from those coming from a natural low-frequency source [41].

The summation of all fluctuations from the system size down to the finest observation scale, but not below it, seems quite natural, for the integration over infinitely small scales is often beyond the applicability range of a particular physical model. This happens in ferromagnetic theories below the grid spacing, in turbulence below the mean-free path, etc.

C. Mother wavelets

In our calculations, we use different derivatives of the Gaussian function as mother wavelets. The admissibility condition (6) is rather loose: practically any well-localized function with the Fourier image vanishing at zero momentum [$\tilde{\chi}(0) = 0$] obeys this requirement. As for the Gaussian functions,

$$\chi_n(x) = (-1)^{n+1} \frac{d^n}{dx^n} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}, \quad n > 0, \tag{17}$$

where x is a dimensionless argument, they are easy to integrate in Feynman diagrams. The graphs of first two wavelets of the (17) family,

$$\chi_1(x) = -\frac{xe^{-\frac{x^2}{2}}}{\sqrt{2\pi}}, \qquad \chi_2(x) = \frac{(1 - x^2)e^{-\frac{x^2}{2}}}{\sqrt{2\pi}},$$

are shown in Fig. 1. Their Fourier images are

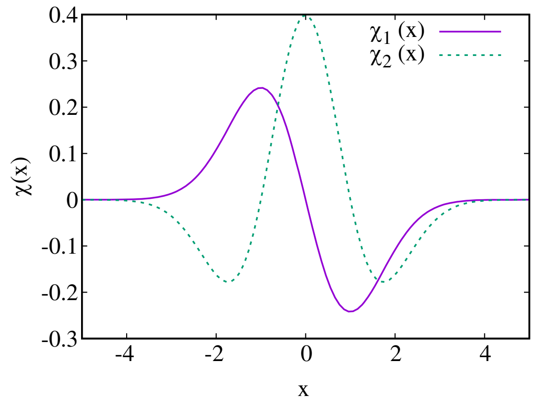


FIG. 1. First two wavelets of the wavelet family (17).

$$\tilde{\chi}_n(k) = -(\imath k)^n e^{-\frac{k^2}{2}}. \tag{18}$$

Respectively, the normalization constants and the wavelet cutoff functions are

$$C_{\chi_n} = \frac{\Gamma(n)}{2}, \qquad f_{\chi_n}(x) = \frac{\Gamma(n, x^2)}{\Gamma(n)},$$

where $\Gamma(\cdot)$ is the Euler gamma function, and $\Gamma(\cdot,\cdot)$ is the incomplete gamma function. For the first two wavelets of the family (17), the cutoff functions are

$$f_{\chi_1}(x) = e^{-x^2}, \qquad f_{\chi_2}(x) = (1 + x^2)e^{-x^2}. \tag{19}$$

IV. AN EXAMPLE OF ϕ^4 MODEL

Let us consider an Euclidean action functional of a massive scalar field with the local ϕ^4 interaction (2),

$$S[\phi] = \int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right].$$

This model is an extrapolation of a classical interacting spin model to the continual limit [50]. Known as the Ginzburg-Landau model [34], it describes phase transitions in superconductors and other magnetic systems fairly well, but it produces divergences when the correlation functions are evaluated from the generating functional (1) by perturbation expansion; see, e.g., [51] for a discussion.

The parameter λ in the action functional (2) is a phenomenological coupling constant, which knows nothing about the scale of observation and becomes the running coupling constant only because of renormalization or the cutoff application. The straightforward way to introduce scale dependence into the local model (2) is to express local field $\phi(x)$ in terms of its scale components $\phi_a(b)$ using the inverse wavelet transform (7). This leads to the generating functional for scale-dependent fields,

$$
\begin{aligned}
Z_W[J_a] = \mathcal{N}\int \mathcal{D}\phi_a(x)\exp\Bigg[-\frac{1}{2}\int \phi_{a_1}(x_1)\\
\times D(a_1,a_2,x_1-x_2)\phi_{a_2}(x_2)\frac{da_1 d^d x_1}{C_\chi a_1}\frac{da_2 d^d x_2}{C_\chi a_2}\\
-\frac{\lambda}{4!}\int V^{a_1,\ldots,a_4}_{x_1,\ldots,x_4}\phi_{a_1}(x_1)\cdots\phi_{a_4}(x_4)\\
\times\frac{da_1 d^d x_1}{C_\chi a_1}\frac{da_2 d^d x_2}{C_\chi a_2}\frac{da_3 d^d x_3}{C_\chi a_3}\frac{da_4 d^d x_4}{C_\chi a_4}\\
+\int J_a(x)\phi_a(x)\frac{da d^d x}{C_\chi a}\Bigg],
\end{aligned}
\tag{20}
$$

where $D(a_1, a_2, x_1 - x_2)$ is the wavelet transform of the ordinary propagator, and $\mathcal{N}$ is a formal normalization constant [33].

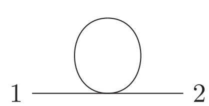
The functional (20)—if integrated over all scale arguments in infinite limits $\int_0^\infty \frac{da_i}{a_i}$—will certainly drive us back to the known divergent theory (1). All scale-dependent fields $\phi_a(x)$ in Eq. (20) still interact with each other with the same coupling constant λ, but their interaction is now modulated by the wavelet factor $V^{a_1a_2a_3a_4}_{x_1x_2x_3x_4}$, which is the Fourier transform of $\prod_{i=1}^4 \tilde{\chi}(a_i k_i)$.

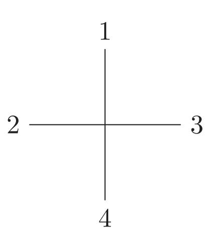
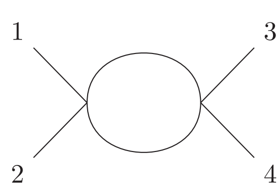
As usual in the functional renormalization group technique [52], we can introduce the effective action functional (4), the functional derivatives of which are the vertex functions $\Gamma^{(n)}_{(A)}$,

$$
\begin{aligned}
\Gamma_{(A)}[\phi_a] = \Gamma^{(0)}_{(A)} + \sum_{n=1}^{\infty}\int \Gamma^{(n)}_{(A)}(a_1,b_1,\ldots,a_n,b_n)\phi_{a_1}\\
\times(b_1)\ldots\phi_{a_n}(b_n)\frac{da_1 d^d b_1}{C_\chi a_1}\ldots\frac{da_n d^d b_n}{C_\chi a_n}.
\end{aligned}
$$

The subscript (A) indicates the presence in the theory of some minimal scale—the observation scale.

In one-loop approximation, the two- and four-point vertex functions, $\Gamma^{(2)}$ and $\Gamma^{(4)}$, respectively, are given by the following diagrams:

$\Gamma^{(2)} = \Delta_{12} - \frac{1}{2}$  (21)

$\Gamma^{(4)} = -$  $- \frac{3}{2}$ 

(22)

Each vertex of the Feynman diagram corresponds to $-\lambda$, and each external line of the one-particle irreducible diagram contains wavelet factor $\tilde{\chi}(ak)$. In one-loop approximation, we have the following expressions, for the renormalized inverse propagator $\Gamma^{(2)}_{(A)}$ and renormalized vertex function $\Gamma^{(4)}_{(A)}$, respectively:

$$
\frac{\Gamma^{(2)}_{(A)}(a_1,a_2,p)}{\tilde{\chi}(a_1p)\tilde{\chi}(-a_2p)} = p^2 + m^2 + \frac{\lambda}{2}T^d_\chi(A), \tag{23}
$$

$$
\frac{\Gamma^{(4)}_{(A)}}{\tilde{\chi}(a_1p_1)\tilde{\chi}(a_2p_2)\tilde{\chi}(a_3p_3)\tilde{\chi}(a_4p_4)} = \lambda - \frac{3}{2}\lambda^2 X^d_\chi(A), \tag{24}
$$

where A is the minimal scale of all external lines of the corresponding diagram.

The tadpole integral in Eq. (23),

$$
T^d_\chi(A) = \int \frac{d^d q}{(2\pi)^d}\frac{f^2_\chi(Aq)}{q^2+m^2},
$$

determines the contribution of all fluctuations with scales from A to ∞ to the "dressed mass" at the observation scale A. In the local theory with ϕ^4 interaction, the natural length scale is the bare mass, the parameter of the action (2). Expressing the momenta in the units of mass m, we get

$$
T^d_\chi(A) = \frac{S_d m^{d-2}}{(2\pi)^d}\int_0^\infty f^2_\chi(Amx)\frac{x^{d-1}dx}{x^2+1}, \tag{25}
$$

where x is dimensionless, and $\alpha = Am$ is the dimensionless scale of observation.

For the $n = 1$ wavelet, we get

$$
\begin{aligned}
T^4_{\chi_1}(A) &= \frac{m^2}{8\pi^2}\int_0^\infty e^{-2\alpha^2x^2}\frac{x^3dx}{x^2+1}\\
&= \frac{m^2}{32\pi^2}\left(\frac{1}{\alpha^2} - 2e^{2\alpha^2}\mathrm{Ei}_1(2\alpha^2)\right),
\end{aligned}
\tag{26}
$$

where $\alpha = Am$, and

$$
\mathrm{Ei}_1(z) \coloneqq \int_1^\infty \frac{e^{-xz}}{x}dx
$$

is the exponential integral of the first kind.

Similarly, for the $n = 2$ wavelet, we get

$$
\begin{aligned}
T^4_{\chi_2}(A) &= \frac{m^2}{8\pi^2}\int_0^\infty e^{-2\alpha^2x^2}(1+\alpha^2x^2)^2\frac{x^3dx}{x^2+1}\\
&= \frac{m^2}{32\pi^2}\Bigg(\frac{5}{2\alpha^2} - \frac{5}{2} + \alpha^2 + 2e^{2\alpha^2}\mathrm{Ei}_1(2\alpha^2)\\
&\quad\times[2\alpha^2 - \alpha^4 - 1]\Bigg).
\end{aligned}
\tag{27}
$$

For small scales ($Am \ll 1$) the one-loop contribution to the effective mass in (23) is dominated by the square term $\propto \frac{\lambda}{A^2}$.

Similarly, the one-loop contribution to the vertex function is given by the fish integral,

$$X_\chi^d(A) = \int \frac{d^d q}{(2\pi)^d} \frac{f_\chi^2(qA) f_\chi^2((q-s)A)}{[q^2+m^2][(q-s)^2+m^2]}. \quad (28)$$

Let us consider one-loop integrals (25) and (28) in $d=4$ dimension, where the coupling constant λ is dimensionless, for different mother wavelets $n=1$, 2 of the family (17).

The fish integral contribution (28) to the vertex function (24) can be evaluated by symmetrization of loop momenta $q \to q + s/2$, where $s = p_1 + p_2$ is the sum of the incoming momenta. In terms of the dimensionless momentum $\mathbf{y} = \mathbf{q}/|s|$, the integral (28) takes the form

$$X_\chi^d(A) = \frac{S_{d-1} s^{d-4}}{(2\pi)^d} \int_0^\pi d\theta \sin^{d-2}\theta \int_0^\infty dy y^{d-3} \times \frac{f_\chi^2\left(As\sqrt{y^2 + y\cos\theta + \frac{1}{4}}\right) f_\chi^2\left(As\sqrt{y^2 - y\cos\theta + \frac{1}{4}}\right)}{\left[\frac{y^2+\frac{1}{4}+\frac{m^2}{s^2}}{y} + \cos\theta\right]\left[\frac{y^2+\frac{1}{4}+\frac{m^2}{s^2}}{y} - \cos\theta\right]}, \quad (29)$$

where θ is the angle between the loop momentum q and the total momentum s.

The integral (29) can be evaluated in the relativistic limit $s^2 \gg 4m^2$. In logarithmic dimension $d=4$, where the coupling constant λ is dimensionless, relativistic approximation drastically simplifies the integral: the dependence on the total momentum s is manifested only through the dimensionless scale As in wavelet cutoff factors f_χ^2.

For $n=1$ this gives

$$X_{\chi_1}^4(A) = \frac{1}{16\pi^2}\left[2\mathrm{Ei}_1(2\alpha^2) - \mathrm{Ei}_1(\alpha^2) + e^{-\alpha^2}\frac{1-e^{-\alpha^2}}{\alpha^2}\right], \quad (30)$$

where $\alpha = As$. Similarly, for $n=2$ we have

$$X_{\chi_2}^4(A) = \frac{1}{16\pi^2}\left[2\mathrm{Ei}_1(2\alpha^2) - \mathrm{Ei}_1(\alpha^2) - e^{-2\alpha^2}\left(\frac{5}{2\alpha^2} + \frac{1}{2}\right) + e^{-\alpha^2}\left(\frac{67}{128} + \frac{9}{128}\alpha^2 + \frac{1}{256}\alpha^4 + \frac{5}{2\alpha^2}\right)\right]. \quad (31)$$

The details of integral evaluation can be found in the Appendix of [9].

Since Eq. (24) gives an exact (in one-loop approximation) contribution of all fluctuations with scales from A to infinity to the dependence of the effective coupling constant on the observation scale A, this dependence can be written as a function of the dimensionless scale $\alpha = As$. These dependences, calculated with χ_1 wavelet (30) and with χ_2 wavelet (31), respectively, are

$$\lambda_{\mathrm{eff}}^{(1)}(\alpha^2) = \lambda + \frac{3}{2}\frac{\lambda^2 e^{-\alpha^2}}{16\pi^2} \times \left[e^{\alpha^2}(2\mathrm{Ei}_1(2\alpha^2) - \mathrm{Ei}_1(\alpha^2)) + \frac{1-e^{-\alpha^2}}{\alpha^2}\right],$$

$$\lambda_{\mathrm{eff}}^{(2)}(\alpha^2) = \lambda + \frac{3}{2}\frac{\lambda^2 e^{-\alpha^2}}{16\pi^2}\left[e^{\alpha^2}(2\mathrm{Ei}_1(2\alpha^2) - \mathrm{Ei}_1(\alpha^2)) + \frac{1-e^{-\alpha^2}}{\alpha^2} + \frac{\alpha^6 + 18\alpha^4 + 134\alpha^2 + 384 - e^{-\alpha^2}(128\alpha^2 + 384)}{256\alpha^2}\right], \quad (32)$$

where we have changed sign in (24) to invert it from $\lambda = \lambda_{\mathrm{bare}}$ to $\lambda = \lambda_{\mathrm{phys}}$.

Let us consider the contribution of a finite shell of scales (A, L), when a classical field is known at certain finite scale L, in contrast to the previous construction [(30), (32)], where we have integrated out all fluctuations in the semi-infinite range $(A, L = \infty)$. The value of the effective coupling constant of the type (32) does not diverge for any finite scale $A > 0$ [in contrast to its differential analog (40), presented below, which suffers from the Landau pole]. The reason for this can be understood physically, if we assume a system of size L in equilibrium, with well-defined coupling constant λ_L. Any measurements on such system can be executed at scales $A < L$. The effective coupling constant relevant to a measurement at the scale A is λ_A. Its particular value is determined by all fluctuations in the range of scales $[A, L]$. In one-loop approximation for the ϕ^4 theory, this effective coupling constant is

$$\lambda_A = \lambda_L + \frac{3}{2}\lambda_L^2[X_\chi^d(A) - X_\chi^d(L)], \quad (33)$$

where the function $X(A)$ is the fish integral of the type (30). Analogously, we can express effective mass at scale α, using the value of physical mass m_L at some large scale L and the tadpole correction to mass,

$$m_A^2 = m_L^2 - \frac{\lambda_L}{2}(T_\chi^d(A) - T_\chi^d(L)). \quad (34)$$

The renormalization of the coupling constant (33) and the mass (34), due to the integration over fluctuations within the range $[A, L]$, can be considered as a counterpart of the RG flow of the standard ϕ_4^4 model. An example, calculated from the scale $L = 4$ down to the scale $A = 0.0625$ with a $\sqrt{2}$ step in scale, is shown in Fig. 2. We cannot give any comments concerning the dimension of stable and unstable manifolds of the RG flow for a number of reasons. First of all, in contrast to the usual RG, the case of wavelet field theory contains explicit dependence on the scale argument A. That is why we

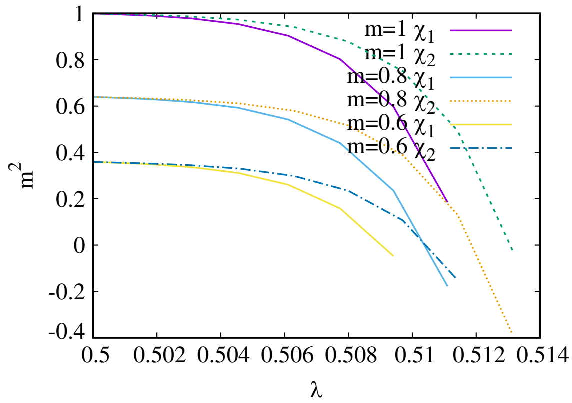


FIG. 2. Iteration of the finite shell renormalization (33), (34). The iteration process goes from $L = 4$ with the value of coupling constant $\lambda = \frac{1}{2}$ at the left of the picture by setting $L_{i+1} = \sqrt{2}L_i \equiv A_i$. The right side of the picture corresponds to the UV limit of iteration. The graphs are shown for different values of mass and different wavelets. An arbitrary value of $s = 2$ was chosen.

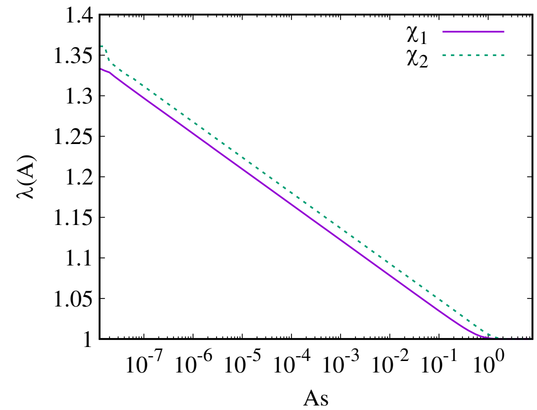


FIG. 3. Dependence of the coupling constant on the logarithm of the dimensionless scale $\alpha = As$ calculated in one-loop approximation, with χ_1 and χ_2 wavelets, respectively. The value of the coupling constant is normalized to $\lambda_L = 1$ at infinity. The parameter $s = 2$ was taken.

cannot draw a β function as a function of coupling constants only and find a critical point $\beta(\lambda_*) = 0$ to estimate the RG flow near these points. What we can do, instead, is to calculate the logarithmic derivatives $A\frac{\partial}{\partial A}$ of the coupling constants and, starting from some large-scale value λ_L of the coupling constant, iterate the model parameters to the smaller scales, exactly as was shown in Fig. 2 according to Eqs. (33) and (34). Needless to say, since the physical fields in our model $\phi_a(x)$ explicitly depend on scale, there is no field renormalization in our model. Perhaps, the RG flow shown in Fig. 2 can be considered only "above the critical temperature" $m^2 > 0$ where the wavelet evaluation of integrals is valid. Instead of considering the fixed points $\beta(\lambda_*) = 0$, we have the explicit dependence of the coupling constant on the scale A. If the size of the system tends to infinity, and the physical mass m_L and coupling constant λ_L are defined in this limit, the scale dependence of the coupling constant $\lambda = \lambda(A)$ can be explicitly calculated from (33) at $X^d_\chi(L) = 0$. The examples of this dependence are shown in Fig. 3.

If the scales A and L are sufficiently close to each other, the difference equation (33),

$$-\frac{\Delta\lambda}{\lambda^2} = -\frac{3}{2}\Delta X,$$

can be transformed to the differential equation $d\frac{1}{\lambda} = -\frac{3}{2}dX$, which has the solution

$$\lambda(A) = \frac{\lambda_L}{1 - \frac{3}{2}\lambda_L(X(A) - X(L))}, \qquad (35)$$

which coincides with the solution of the original Eq. (33) only for *small* values of λ_L; otherwise, it suffers from the pole.

The formal differentiation of the effective coupling constant (32) with respect to the logarithmic scale argument gives the scaling equation

$$\alpha^2\frac{\partial\lambda}{\partial\alpha^2} = \frac{3}{2}\lambda^2\alpha^2\frac{\partial X^4_{\chi_1}}{\partial\alpha^2} = \frac{3\lambda^2}{32\pi^2}\frac{e^{-\alpha^2}-1}{\alpha^2}e^{-\alpha^2} \qquad (36)$$

for the first wavelet χ_1. The asymptote of (36) for small values $\alpha \ll 1$ coincides with the standard result,

$$\frac{\partial\lambda_{\rm eff}}{\partial\mu} \approx \frac{3\lambda^2}{16\pi^2}, \qquad \mu = -\ln\alpha.$$

Similar equations can be obtained for other wavelets of the family (17). For either of these wavelets, the asymptote of logarithmic derivatives for small scales $\alpha^2 \ll 1$ gives the same slope,

$$d_i \coloneqq \alpha^2\frac{\partial X^4_{\chi_i}}{\partial\alpha^2} = -\frac{1}{16\pi^2} + O(\alpha^2)$$

for the dependence of coupling constant λ on the logarithm of scale. For the first two wavelets, the small-scale Taylor series gives

$$d_1 = -\frac{1}{16\pi^2} + \frac{3\alpha^2}{32\pi^2} - \frac{7\alpha^4}{96\pi^2} + O(\alpha^6),$$
$$d_2 = -\frac{1}{16\pi^2} - \frac{13\alpha^2}{1024\pi^2} + \frac{139\alpha^4}{3072\pi^2} + O(\alpha^6).$$

The shape of the mother wavelet works as an aperture of the microscope used to study the details of different scales [48]. Different apertures can see different values when the scale of aperture is comparable to size of the object, but the asymptote at small scales is certainly the same—since the cutoff function (15) calculated for the family (17) is an exponent multiplied by a polynomial $1 + O(x^2)$—and coincides with the standard RG result; see also [9] for a similar result in quantum electrodynamics.

In the limit of small α, the RG equation for the coupling constant,

$$\frac{\partial \lambda}{\partial \ln \alpha} = -\frac{3\lambda^2}{16\pi^2}, \tag{37}$$

has a well-known solution,

$$\lambda(\alpha) = \frac{\lambda_1}{1 + \frac{3\lambda_1}{16\pi^2}\ln\frac{\alpha}{\alpha_1}}, \tag{38}$$

where $\lambda_1 \equiv \lambda(\alpha_1)$ is a reference value of the coupling constant at a certain reference value α_1. The solution (38) suffers from a Landau pole.

In the full form, the ordinary differential equation (36) can be solved as an RG-type equation,

$$d\left(\frac{1}{\lambda}\right) = -\frac{3}{32\pi^2}\frac{e^{-\alpha^2}(e^{-\alpha^2}-1)}{\alpha^4}d\alpha^2. \tag{39}$$

If the value of the effective coupling constant λ is known at certain squared dimensionless scale $\lambda_1 = \lambda(x_1 = (A_1 s)^2)$, its value at other scales $x = (As)^2$ is given by the explicit solution,

$$\lambda(x) = \frac{1}{\frac{1}{\lambda_1} + \frac{3}{32\pi^2}[F(x) - F(x_1)]}, \tag{40}$$

where $F(x) \coloneqq 2\Gamma(-1, 2x) - \Gamma(-1, x)$, with

$$\Gamma(a, z) = \int_z^\infty t^{a-1}e^{-t}dt$$

being the incomplete gamma function. Similar to the small-scale case, the solution (40) also suffers from the Landau pole.

In the actual sense of the Ginzburg-Landau model, we cannot really assert that $\phi^4(x)$ interaction is a realistic large-scale interaction from which one can derive the small-scale interaction of fields at $A \to 0$ by means of the RG and loop corrections to the large-scale theory. Instead, what we can do is to approximate some medium-scale interaction from the known parameters of the Hamiltonian at microscopic scale, i.e., at the UV-cutoff scale. In the case of a ferromagnetic model, this is the grid size. From this microscopic interaction, we can infer the interaction strength λ for bigger (Kadanoff's) blocks, but not for the ground state of the whole crystal of finite size [4]. In this case, the Ginzburg-Landau model is not a fairly good approximation.

However, there are QFT models in which the large-scale fields provide a good approximation for the measured physical fields, and the renormalization group with the loop corrections provides a good estimation of field interactions at smaller scales. Quantum electrodynamics is a well-known example.

V. CONCLUSIONS

We have shown in this paper that the summation of all fluctuations with scales from the size of the system down to the observation scale by means of continuous wavelet transform results in a finite renormalization of the coupling constant without any Landau poles. It was demonstrated on a simple example of ϕ^4 field theory in $d = 4$ dimensions. Our conclusion seems rather general, since the same technique can be applied to QED [9], QCD [7], and other models. The same method of summing up the fluctuations from IR scale—the size of the system—down to the observation scale can be also applied in other dimensions, different from the dimension of physical spacetime, but the calculations may be more difficult technically. The one-loop integrals for the case of ϕ^4 theory in $d = 3$ dimensions are presented in the Appendix for completeness.

The essence of this paper is to show that the extension of the functional space of quantum fields from the space of square-integrable local functions $\phi(x)$ to the space of scale-dependent functions $\phi_a(x)$, defined on the affine group with the help of continuous wavelet transform, leads to a theory finite by construction. The singularities—Landau poles and UV and IR divergences—turn out to be the artifacts of the $L^2(\mathbb{R}^d)$ space of functions, which is too poor to provide a correct account of the dependence of physical fields on the observer's settings, i.e., on the observation scale. Especially, the Landau poles then remain the artifacts of approximation of the results in a *finite* range of scales by the results obtained from the differential equation defined in an infinitesimally thin shell.

In a probabilistic sense, the summation of all fluctuations from large scales down to smaller scales may be related to the probabilities of small-scale fluctuations constrained by the fluctuations of larger scales [53].

ACKNOWLEDGMENTS

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APPENDIX: EVALUATION OF ONE-LOOP CORRECTIONS IN 3D

The one-loop corrections (23) and (24) can be evaluated for arbitrary dimension d, although the calculations may be more difficult than in $d = 4$, where the coupling constant is dimensionless. Here we present the results for $d = 3$ calculated for the χ_1 wavelet.

The tadpole integral (25) can be evaluated explicitly for $d=3$ with the cutoff function (19) $f_{\chi_1}(x)=e^{-x^2}$,

$$T^3_{\chi_1}(A)=\frac{m}{2\pi^2}\int_0^\infty e^{-2A^2m^2x^2}\frac{x^2dx}{x^2+1}=\frac{m}{2\pi^2}\left[\frac{\pi}{2}e^{2A^2m^2}(\mathrm{erf}(\sqrt{2}Am)-1)+\frac{\sqrt{2\pi}}{4Am}\right]. \tag{A1}$$

The fish integral (24) in $d=3$ takes the form

$$X^3_\chi(A)=\frac{1}{(2\pi)^2s}\int_0^\pi d\theta\sin\theta\int_0^\infty dy\frac{f^2_\chi\left(As\sqrt{y^2+y\cos\theta+\frac{1}{4}}\right)f^2_\chi\left(As\sqrt{y^2-y\cos\theta+\frac{1}{4}}\right)}{\left[\frac{y^2+\frac{1}{4}+\frac{m^2}{s^2}}{y}+\cos\theta\right]\left[\frac{y^2+\frac{1}{4}+\frac{m^2}{s^2}}{y}-\cos\theta\right]}, \tag{A2}$$

where $s=p_1+p_2$ is the sum of the incident momenta, and $f_\chi(\cdot)$ is the cutoff function corresponding to the chosen wavelet.

For the simplest case of the χ_1 wavelet, the product of the two squared Gaussian cutoff functions in the numerator gives the factor $\exp(-\alpha^2(4y^2+1))$, where $\alpha=As$. The whole integral (A2), after the change of variables $u=-\cos\theta$, takes the form

$$X^3_{\chi_1}(A)=\frac{1}{(2\pi)^2s}\int_{-1}^1 du\int_0^\infty dy\frac{e^{-4\alpha^2(y^2+\frac{1}{4})}}{\beta^2(y)-u^2}, \tag{A3}$$

where

$$\beta(y)=\frac{y^2+\frac{1}{4}+\frac{m^2}{s^2}}{y}>1$$

in the domain of integration in the dimensionless momentum y. Having integrated over the angular variable u, we get

$$X^3_{\chi_1}(A)=\frac{1}{(2\pi)^2s}\int_0^\infty\frac{dy\,ye^{-4\alpha^2(y^2+\frac{1}{4})}}{y^2+\frac{1}{4}+\epsilon}\times\left(\ln\left[\left(y+\frac{1}{2}\right)^2+\epsilon\right]-\ln\left[\left(y-\frac{1}{2}\right)^2+\epsilon\right]\right), \tag{A4}$$

where we have introduced a presumably small parameter $\epsilon\equiv\frac{m^2}{s^2}$. In the IR limit $\epsilon\to0$ the integral (A4) is divergent at $y=\frac{1}{2}$, otherwise its value can be calculated approximately, using the Laplace method.

Changing the integration variable $y=\frac{1}{2}+t$, we get the integral

$$X^3_{\chi_1}(A)=\frac{1}{(2\pi)^2s}\int_{-\frac{1}{2}}^\infty\frac{dt(\frac{1}{2}+t)e^{-4\alpha^2(t^2+t+\frac{1}{2})}}{t^2+t+\frac{1}{2}+\epsilon}\ln\frac{(t+1)^2+\epsilon}{t^2+\epsilon}\equiv\frac{1}{(2\pi)^2s}\int_{-\frac{1}{2}}^\infty dt\exp(S(t,\epsilon)). \tag{A5}$$

The "action,"

$$S(t,\epsilon)\equiv\ln\left(t+\frac{1}{2}\right)-\ln\left(t^2+t+\frac{1}{2}+\epsilon\right)-4\alpha^2\left(t^2+t+\frac{1}{2}\right)+\ln\ln\frac{(t+1)^2+\epsilon}{t^2+\epsilon}, \tag{A6}$$

has a sharp maximum at $t\approx0$. The exact value of the extremal point t_0 is given by the equation

$$\left.\frac{\partial S}{\partial t}\right|_{t=t_0}=0,$$

where

$$\frac{\partial S}{\partial t}=\frac{1}{t+\frac{1}{2}}-\frac{2t+1}{t^2+t+\frac{1}{2}+\epsilon}-4\alpha^2(2t+1)+\frac{\frac{2t+2}{(t+1)^2+\epsilon}-\frac{2t}{t^2+\epsilon}}{\ln\frac{(t+1)^2+\epsilon}{t^2+\epsilon}}. \tag{A7}$$

The decomposition of the latter equation in a power series in t, omitting the $O(t^2)$ order terms, leads to

$$t_0\approx\frac{-2\alpha^2+\frac{1}{\ln\frac{1}{\epsilon}}}{2+4\alpha^2+\frac{1+\frac{1}{\epsilon}+\frac{2}{\ln\frac{1}{\epsilon}}}{\ln\frac{1}{\epsilon}}}. \tag{A8}$$

The estimation (A8) tends to zero for $\epsilon\to0$. The second derivative of the action,

$$\frac{\partial^2S}{\partial t^2}=-\frac{1}{(t+\frac{1}{2})^2}-\frac{2}{t^2+t+\frac{1}{2}+\epsilon}+\left(\frac{2t+1}{t^2+t+\frac{1}{2}+\epsilon}\right)^2-8\alpha^2-\left(\frac{\frac{2t+2}{(t+1)^2+\epsilon}-\frac{2t}{t^2+\epsilon}}{\ln\frac{(t+1)^2+\epsilon}{t^2+\epsilon}}\right)^2+\frac{\frac{2}{(t+1)^2+\epsilon}-\frac{(2t+2)^2}{((t+1)^2+\epsilon)^2}-\frac{2}{t^2+\epsilon}+\frac{4t^2}{(t^2+\epsilon)^2}}{\ln\frac{(t+1)^2+\epsilon}{t^2+\epsilon}}. \tag{A9}$$

The value of second derivative near the extremal point is

$$S_{tt}(0,\epsilon) = -4 - 8\alpha^2 - \frac{2}{\frac{1}{2}+\epsilon} + \frac{1}{\left(\frac{1}{2}+\epsilon\right)^2} + \frac{\frac{2}{1+\epsilon} - \frac{4}{(1+\epsilon)^2} - \frac{2}{\epsilon}}{\ln\frac{1+\epsilon}{\epsilon}} - \frac{4}{(1+\epsilon)^2\left[\ln\frac{1+\epsilon}{\epsilon}\right]^2}. \quad \text{(A10)}$$

Thus, the fish integral for ϕ_3^4 theory with the χ_1 wavelet can be estimated as

$$X^3_{\chi_1}(A) \approx \frac{1}{(2\pi)^2 s}\sqrt{\frac{2\pi}{S_{tt}(0,\epsilon)}}\exp(S(0,\epsilon)), \quad \text{where } S(0,\epsilon) = \ln\frac{\frac{1}{2}}{\frac{1}{2}+\epsilon} - 2\alpha^2 + \ln\ln\frac{1+\epsilon}{\epsilon}. \quad \text{(A11)}$$

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